Semi-discretization for Stochastic Scalar Conservation Laws with Multiple Rough Fluxes

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ABSTRACT. We develop a semi-discretization approximation for scalar conservation laws with multiple rough time dependence in inhomogeneous fluxes. The method is based on Brenier's transport-collapse algorithm and uses characteristics defined in the setting of rough paths. We prove strong L^1-convergence for inhomogeneous fluxes and provide a rate of convergence for homogeneous one's. The approximation scheme as well as the proofs are based on the recently developed theory of pathwise entropy solutions and uses the kinetic formulation which allows to define globally the (rough) characteristics.

1. INTRODUCTION

We introduce a semi-discretization scheme and prove its convergence for stochastic scalar conservation laws (with multiple rough fluxes) of the form

$$\begin{cases} du + \sum_{i=1}^{N} \partial_{x_i} A^i(x,u) \circ dz_t^i = 0 \quad \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 \in (L^1 \cap L^2)(\mathbb{R}^N). \end{cases} \tag{1.1}$$

The precise assumptions on A, z are presented in the sections 2 and 3 below. To introduce the results here we assume that $A \in C^2(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$ and z is an α-Hölder geometric rough path; for example, z may be a d-dimensional (fractional) Brownian motion or $z(t) = (t, \dots, t)$ in which case we are back in the classical deterministic setting – see Appendix A for some background on rough paths. For spatially homogeneous fluxes, the theory is simpler and $z \in C([0,T]; \mathbb{R}^N)$ is enough. In what follows we may occasionally use the term "stochastic" even when z is a continuous or a rough path.

Stochastic scalar conservation laws of the type (1.1) arise in several applications. For example, (1.1) appears in the theory of mean field games developed by Lasry and Lions [15], [16], [17]. We refer to Gess and Souganidis [11] and Cardaliaguet, Delarue, Lasry and Lions [6] for more details on the derivation of (1.1) in this case.

The semi-discretization scheme we consider here is based on first rewriting (1.1) in its kinetic form using the classical Maxwellian

$$\chi(x,\xi,t) := \chi(u(x,t),\xi) := \begin{cases} +1 \text{ for } 0 \le \xi \le u(x,t), \\ -1 \text{ for } u(x,t) \le \xi \le 0, \\ \ 0 \text{ otherwise.} \end{cases} \tag{1.2}$$

The theory of pathwise entropy solutions introduced by Lions, Perthame and Souganidis in [19] and further developed by Lions, Perthame and Souganidis in [21] and Gess and Souganidis in [11] (see

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Appendix B for the precise definition and some results) asserts that there exists a non-negative, bounded measure m on $\mathbb{R}^N \times \mathbb{R} \times [0,T]$ such that, in the sense of distributions,

$$\partial_t \chi + \sum_{i=1}^N a^i(x,\xi)\partial_{x_i}\chi \circ dz^i + \sum_{i=1}^N \partial_{x_i}A^i(x,\xi)\partial_\xi \chi \circ dz^i = \partial_\xi m, \tag{1.3}$$

where, for notational simplicity, we set

$$a^i(x,\xi) := (\partial_u A^i)(x,\xi).$$

Given a partition $0 = t_0 < t_1 < \cdots < t_K = T$ of $[0,T]$ with mesh size $\Delta t := \max_{k=0,\ldots,K-1} |t_{k+1} - t_k|$, the approximation $u_{\Delta t}$ of the pathwise entropy solution to (1.1) that we are considering here is based on the following splitting and fast relaxation scheme.

We first solve the linear "free-streaming" transport equation

$$\partial_t f_{\Delta t} + \sum_{i=1}^N a^i(x,\xi)\partial_{x_i} f_{\Delta t} \circ dz^i + \sum_{i=1}^N \partial_{x_i}A^i(x,\xi)\partial_\xi f_{\Delta t} \circ dz^i = 0 \quad \text{on } \mathbb{R}^N \times \mathbb{R} \times [t_k, t_{k+1}), \tag{1.4}$$

and then introduce a fast relaxation step setting (see section 1 for the notation)

$$u_{\Delta t}(x,t) := \int f_{\Delta t}(x,\eta,t-)d\eta \quad \text{and} \quad f_{\Delta t}(x,\xi,t_{k+1}) := \chi(u_{\Delta t}(x,t_{k+1}),\xi); \tag{1.5}$$

for future reference we note that $f_{\Delta t}$ is discontinuous at t_k while $u_{\Delta t}$ is not.

For the homogeneous stochastic scalar conservation law

$$du + \sum_{i=1}^N \partial_{x_i}A^i(u) \circ dz^i = 0 \quad \text{in } \mathbb{R}^N \times (0,T), \tag{1.6}$$

we show the strong convergence of the approximations $u_{\Delta t}$ to the pathwise entropy solution u and provide an estimate for the rate of convergence (see Theorem 2.1 below), that is, for $u_0 \in (BV \cap L^\infty \cap L^1)(\mathbb{R}^N)$, we show that there exists $C > 0$ depending only on the data such that

$$\|u(\cdot,t) - u_{\Delta t}(\cdot,t)\|_{L^1} \le C\sqrt{\Delta z}, \tag{1.7}$$

where Δz is defined by

$$\Delta z := \max_{k=0,\ldots,K-1} \sup_{t\in[t_k,t_{k+1}]} |z_t - z_{t_k}|. \tag{1.8}$$

In the general inhomogeneous case, that is, for (1.1), no bounded variation estimates are known either for the solution u or for the approximations $u_{\Delta t}$. In addition, due to the spatial dependence, we cannot use averaging techniques. To circumvent these difficulties, we devise a new method of proof based on the concept of generalized kinetic solutions and new energy estimates (see Lemma 3.3 below). The result (see Theorem 3.1) is that, if $u_0 \in (L^1 \cap L^2)(\mathbb{R}^N)$, then

$$\lim_{\Delta t \to 0} \|u(\cdot,t) - u_{\Delta t}(\cdot,t)\|_{L^1(\mathbb{R}^N \times [0,T])}.$$

The semi-discretization scheme we introduce here is a generalization of the transport-collapse scheme developed by Brenier [3, 4] and Giga and Miyakawa [13] for the deterministic homogeneous scalar conservation law

$$\partial_t u + \sum_{i=1}^N \partial_{x_i}A^i(u) = 0 \quad \text{in } \mathbb{R}^N \times (0,T). \tag{1.9}$$

In this setting, the convergence of the scheme was proven in [3, 4, 13] based on bounded variation arguments. A general methodology for this type of result as well as for error estimates was developed by Bouchut and Perthame [2]. In [26] Vasseur provided an alternative proof of the weak convergence of the transport-collapse scheme based on averaging techniques for the Burger's equation, that is for (1.9) with $N = 1$ and $A^i(u) = \frac{1}{2}u^2$.

The results we present here are new for both deterministic and stochastic settings.

Firstly, we establish a rate of convergence for the transport-collapse scheme (see (1.7)), which was previously unavailable even in the deterministic case (although maybe not too surprising in view of [2]).

Secondly, we prove the convergence of the scheme also in the inhomogeneous case. The classical averaging techniques and, thus, the method developed in [26] do not apply here, since our assumptions allow for degenerate fluxes.

Indeed we assume that there exist $\theta \in (0, 1]$ and $C > 0$ such that, for every compact interval $I \subseteq \mathbb{R}$, all $(\sigma, z) \in S^{N-1} \times \mathbb{R}, x \in \mathbb{R}^N$, and $\varepsilon > 0$,

$$|\{\xi \in I : \ |\sigma \cdot A'(x, \xi) - z| \le \varepsilon\}| \le C\varepsilon^\theta,$$

where S^{N-1} is the unit sphere in R^N and $A'(x, u) := \partial_u A(x, u)$.

The well-posedness of the pathwise entropy solutions for (1.1) has been proven in [11,19,21]. Regularity and long-time behavior has been considered by Lions, Perthame and Souganidis [20] and Gess and Souganidis [12]. For a detailed account of numerical methods for (deterministic) conservation laws we refer to LeVeque [18], Bouchut [1], Godlewski and Raviart [14], Eymard and Gallouët, Herbin [8] and the references therein.

Finally, we recall that kinetic solutions to (1.9) were constructed by Brenier and Corrias [5], Lions, Perthame and Tadmor [22] and Perthame [25] as limits of the so-called Bhatnagar, Gross, Krook (BGK) approximation, that is,

$$\partial_t f^\varepsilon + \sum_{i=1}^{N} (A^i)'(\xi)\partial_{x_i} f^\varepsilon = \frac{1}{\varepsilon}(\mathcal{M}f^\varepsilon - f^\varepsilon), \tag{1.10}$$

where the "Maxwellian" associated with a distribution f is defined by

$$\mathcal{M}f(x, \xi, t) := \chi(\int f(x, \eta, t)d\eta, \xi). \tag{1.11}$$

In comparison, the transport-collapse scheme we are considering here is based on a fast relaxation scale for the right-hand side of (1.10), that is on enforcing $\mathcal{M}f^\varepsilon = f^\varepsilon$ at the time-steps t_k.

Structure of the paper. The strong convergence and the rate for the homogeneous case is obtained in section 2. The inhomogeneous case is treated in section 3. Some background for the theory of rough paths is presented in Appendix A. The definition and fundamental properties of pathwise entropy solutions to (1.1) are recalled in Appendix B. A basic, but crucial, bounded variation estimate for indicator functions is given in Appendix C.

Notation. We set $\mathbb{R}_+ := (0, \infty)$ and δ is the "Dirac" mass at the origin in $\mathbb{R}$. The complement and closure of a set $A \subseteq \mathbb{R}^N$ are denoted respectively by A^c and $\bar{A}$, and B_R is the open ball in $\mathbb{R}^N$ centered at the origin with radius R. We write $\|f\|_{C(\mathcal{O})}$ for the sup norm of a continuous bounded function f on $\mathcal{O} \subseteq \mathbb{R}^M$ and, for $k = 1, \ldots, \infty$, we let $C_c^k(\mathcal{O})$ be the space of all k times continuously differentiable functions with compact support in $\mathcal{O}$. For $\gamma > 0$, $\mathrm{Lip}^\gamma(\mathcal{O}; \mathbb{R}^l)$ is the set of $\mathbb{R}^l$ valued functions defined on $\mathcal{O}$ with $k = 0, \ldots, \lfloor\gamma\rfloor$ bounded derivatives and $\gamma - \lfloor\gamma\rfloor$ Hölder continuous $\lfloor\gamma\rfloor$-th derivative; for simplicity, if $\gamma = 1$ and $l = 1$, we write $\mathrm{Lip}(\mathcal{O})$ and denote by $\|\cdot\|_{C^{0,1}}$ the Lipschitz constant. The subspace of L^1-functions with bounded total variation is BV. If $f \in BV$, then $\|f\|_{BV}$ is its total variation. For $u \in L^1([0,T]; L^p(\mathbb{R}^N))$ we write $\|u(t)\|_p$ for the L^p norm of $u(\cdot,t)$. To simplify the presentation, given a function $f(x,\xi)$ we write $\|f\|_{L^1_{x,\xi}} := \int |f| dx d\xi := \int |f(x,\xi)| dx d\xi$. For a measure m on $\mathbb{R}^N \times \mathbb{R} \times [0,T]$ we often write $m(x,\xi,t)dxd\xi dt$ instead of $dm(x,\xi,t)$. If $f \in L^1(\mathbb{R}^N \times [0,T])$ is such that $t \mapsto f(\cdot,t) \in L^1(\mathbb{R}^N)$ is càdlàg, that is, right-continuous with left limits, we let

$$\int f(x,t-)dx := \lim_{h \downarrow 0} \int f(x,t-h)dx.$$

The space of all càdlàg functions from an interval $[0,T]$ to a metric space M is denoted by $D([0,T];M)$. For a function $f : [0,T] \to \mathbb{R}$ and $a,b \in [0,T]$ we set $f|_a^b := f(b) - f(a)$. The negative and positive part of a function $f : \mathbb{R}^N \to \mathbb{R}$ are defined by $f^- := \max\{-f,0\}$ and $f^+ := \max\{f,0\}$. Finally, given $a,b \in \mathbb{R}$, $a \wedge b := \min(a,b)$ and for $a,b \in \mathbb{R}^N$ we set $ab := (a^i b^i)_{i=1}^N$.

2. Spatially homogeneous stochastic scalar conservation laws

We consider stochastic homogeneous scalar conservation laws, that is, the initial value problem

$$\begin{cases} du + \sum_{i=1}^N \partial_{x_i} A^i(u) \circ dz^i = 0 \quad \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 \in (BV \cap L^\infty)(\mathbb{R}^N), \end{cases} \tag{2.1}$$

where

$$z \in C([0,T];\mathbb{R}^N) \quad \text{and} \quad A \in C^2(\mathbb{R};\mathbb{R}^N). \tag{2.2}$$

Informally, in view of (1.2), the kinetic formulation yields a non-negative bounded measure m on $\mathbb{R}^N \times \mathbb{R} \times [0,T]$, where $a := A'$, such that

$$\partial_t \chi + \sum_{i=1}^N a^i(\xi) \partial_{x_i} \chi \circ dz^i = \partial_\xi m. \tag{2.3}$$

Fix $\Delta t > 0$, define $t_k := k\Delta t$ with $k = 0, \ldots, K$ and $K\Delta t \approx T$ and Δz as in (1.8), and assume that

$$\Delta z \leq 1. \tag{2.4}$$

The approximation $u_{\Delta t}$ is defined as

$$u_{\Delta t}(\cdot,0) = u_0 \quad \text{and} \quad u_{\Delta t}(\cdot,t) := \int f_{\Delta t}(\cdot,\xi,t-)d\xi, \tag{2.5}$$

where $f_{\Delta t}$ solves

$$\begin{cases} \partial_t f_{\Delta t} + \sum_{i=1}^N a^i(\xi) \partial_{x_i} f_{\Delta t} \circ dz^i = 0 \quad \text{on } (t_k, t_{k+1}) \\ f_{\Delta t}(x,\xi,t_k) = \chi(u_{\Delta t}(x,t_k),\xi), \end{cases}$$

that is, for $t \in [t_k, t_{k+1})$, $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R}$,

$$f_{\Delta t}(x,\xi,t) = f_{\Delta t}(x - a(\xi)(z_t - z_{t_k}), \xi, t_k). \tag{2.6}$$

The main result in this section is:

Theorem 2.1. *Assume* (2.2) *and* (2.4), *and, for* $u_0 \in (BV \cap L^\infty)(\mathbb{R}^N)$, *let* u *be the pathwise entropy solution to* (2.1) *and* $u_{\Delta t}$ *be defined as in* (2.5). *Then*

$$\sup_{t\in[0,T]} \|u(\cdot,t) - u_{\Delta t}(\cdot,t)\|_1 \leq \sqrt{2\|u_0\|_{BV} \|a\|_{C^{0,1}([-\|u_0\|_\infty, \|u_0\|_\infty])}} \, \|u_0\|_2 \sqrt{\Delta z}.$$

Before presenting the rigorous proof of Theorem 2.1 we give an informal overview of the argument. For the sake of this exposition we assume $z \in C^1([0,T];\mathbb{R}^N)$ for now and for simplicity we set

$$M := \|u_0\|_\infty.$$

The proof is based on the observation that the semi-discretization scheme introduced above has the kinetic interpretation

$$\partial_t f_{\Delta t} + \sum_{i=1}^N a^i(\xi)\partial_{x_i} f_{\Delta t}\dot z^i = \partial_\xi m_{\Delta t} := \sum_k \delta(t-t_k)(\mathcal{M} f_{\Delta t} - f_{\Delta t}).. \tag{2.7}$$

Recalling (1.2) and (2.6) we observe that

$$|\chi|, |f_{\Delta t}| \in \{0,1\} \quad \text{and} \quad \operatorname{sgn}(\chi(x,\xi,t)) = \operatorname{sgn}(f_{\Delta t}(x,\xi,t)) = \operatorname{sgn}(\xi). \tag{2.8}$$

It follows that

$$\begin{aligned}\int |\chi(t) - f_{\Delta t}(t)| d\xi dx = \int |\chi(t) - f_{\Delta t}(t)|^2 d\xi dx &= \int \left(|\chi(t)|^2 - 2\chi(t) f_{\Delta t}(t) + |f_{\Delta t}(t)|^2\right) d\xi dx \\ &= \int \left(|\chi(t)| - 2\chi(t) f_{\Delta t}(t) + |f_{\Delta t}(t)|\right) d\xi dx.\end{aligned}$$

Multiplying (2.3) and (2.7) by $\operatorname{sgn}(\xi)$ and integrating yields

$$\frac{d}{dt}\int |\chi(t)| d\xi dx = -2\int m(x,0,t)dx \quad \text{and} \quad \frac{d}{dt}\int |f_{\Delta t}(t)| d\xi dx = -2\int m_{\Delta t}(x,0,t)dx,$$

and, since in the sense of distributions

$$\partial_\xi \chi = \delta(\xi) - \delta(u(x,t)-\xi) \le \delta(\xi),$$

and

$$\partial_\xi f_{\Delta t} \le \delta(\xi) - D_x f_{\Delta t}(x - a(\xi)(z_t - z_{t_k}), \xi, t_k)\cdot a'(\xi)(z_t - z_{t_k}),$$

we obtain

$$\begin{aligned}-2\frac{d}{dt}\int \chi f_{\Delta t} d\xi dx &= -2\int \left(\partial_t\chi f_{\Delta t} + \chi\partial_t f_{\Delta t}\right) d\xi dx \\ &= -2\int \left(f_{\Delta t}\big(-\sum_{i=1}^N a^i(\xi)\partial_{x_i}\chi \dot z^i + \partial_\xi m\big) + \chi\big(-\sum_{i=1}^N a^i(\xi)\partial_{x_i} f_{\Delta t}\dot z^i + \partial_\xi m_{\Delta t}\big)\right) d\xi dx \\ &= 2\int \left(\partial_\xi f_{\Delta t} m + \partial_\xi \chi m_{\Delta t}\right) d\xi dx \\ &\le 2\int \left(m(x,0,t) + m_{\Delta t}(x,0,t)\right) dx \\ &\quad - \int D_x f_{\Delta t}(x - a(\xi)(z_t - z_{t_k}), \xi, t_k)\cdot a'(\xi)(z_t - z_{t_k}) m d\xi dx \\ &\le -\frac{d}{dt}\int |\chi| d\xi dx - \frac{d}{dt}\int |f_{\Delta t}| d\xi dx + \|a'\|_{C^0([-M,M])}|z_t - z_{t_k}| \int |D_x m| d\xi dx,\end{aligned}$$

and, hence,

$$\frac{d}{dt}\int |\chi(t) - f_{\Delta t}(t)| d\xi dx \le \|a'\|_{C^0([-M,M])}|z_t - z_{t_k}| \int |D_x m| d\xi dx.$$

At this point we face a difficulty. The term $\int |D_x m| d\xi dx$ may not be finite and thus an additional approximation argument is necessary.

To resolve this issue we replace χ by its space mollification χ^ε making an error of order $\varepsilon\|u_0\|_{BV}$ and we note that, if m^ε is the mollification of m with respect to the x-variable, then

$$\int_0^T \int |D_x m^\varepsilon| d\xi dx dt \leq \frac{1}{\varepsilon}\int_0^T \int m d\xi dx dt \leq \frac{\|u_0\|_2}{2\varepsilon}.$$

In conclusion, we find

$$\int |u(t) - u_{\Delta t}(t)| dx \lesssim \varepsilon \|u_0\|_{BV} + \|a'\|_{C^0([-M,M])} \Delta z \frac{\|u_0\|_2}{2\varepsilon},$$

and choosing $\varepsilon \approx \sqrt{\Delta z}$ finishes the informal proof.

For future reference we observe that, if

$$\chi_{\Delta t}(x,\xi,t) := \chi(u_{\Delta t}(x,t),\xi),$$

then

$$\chi_{\Delta t}(x,\xi,t) = \chi(\int f_{\Delta t}(x,\eta,t)d\eta, \xi) = \mathcal{M} f_{\Delta t}(x,\xi,t). \tag{2.9}$$

We continue with

The proof of Theorem 2.1. We first assume $z \in C^1([0,T];\mathbb{R}^N)$. In this case χ and $f_{\Delta t}$ solve (2.3) and (2.7) respectively. It has been shown in Theorem 3.2 in [19] that χ depends continuously on the driving signal z, in the sense that, if u^1, u^2 are two solutions driven by z^1, z^2 respectively, then

$$\sup_{t\in[0,T]} \|u^1(t) - u^2(t)\|_1 \leq C\|z^1 - z^2\|_{C([0,T];\mathbb{R}^N)}.$$

In view of (2.6), it also follows that $f_{\Delta t}$ and $u_{\Delta t}$ depend continuously on z. This can be seen by induction over k. Given two smooth smooth signals z^1, z^2 we denote by $f^{z^1}_{\Delta t}, f^{z^2}_{\Delta t}$ the corresponding free streaming functions and we note that, since $f^{z^1}_{\Delta t}(x,\xi,t) = f^{z^2}_{\Delta t}(x,\xi,t) = 0$ for all $|\xi| > M$ and

$$\begin{aligned}
&\sup_{t\in[t_k,t_{k+1})} \int |f^{z^1}_{\Delta t}(x,\xi,t) - f^{z^2}_{\Delta t}(x,\xi,t)| dx d\xi \\
&= \sup_{t\in[t_k,t_{k+1})} \int |f^{z^1}_{\Delta t}(x - a(\xi)(z^1_t - z^1_{t_k}),\xi,t_k) - f^{z^2}_{\Delta t}(x - a(\xi)(z^2_t - z^2_{t_k}),\xi,t_k)| dx d\xi \\
&\leq \sup\Big\{ \int |f^{z^1}_{\Delta t}(x+h,\xi,t_k) - f^{z^2}_{\Delta t}(x,\xi,t_k)| dx d\xi : \ h \in \mathbb{R}^N, \ |h| \leq 2\|a\|_{C^0([-M,M])}\|z^1 - z^2\|_{C^0([0,T];\mathbb{R}^N)} \Big\},
\end{aligned}$$

it follows that

$$\lim_{\|z^1 - z^2\|_{C^0([0,T];\mathbb{R}^N)}\to 0} \sup_{t\in[t_k,t_{k+1})} \int |f^{z^1}_{\Delta t}(x,\xi,t) - f^{z^2}_{\Delta t}(x,\xi,t)| dx d\xi = 0.$$

Hence, the rough case $z \in C([0,T];\mathbb{R}^N)$ can be handled by smooth approximations in the end (see step 5 below).

Step 1: The kinetic formulation. The proof is based on the kinetic interpretation of the semi-discretization scheme given by (2.7).

An important observation, used in Lemma 3.2 below, is the following L^1-contraction property

$$\|\mathcal{M}f - \mathcal{M}g\|_{L^1_{x,\xi}} \leq \|f - g\|_{L^1_{x,\xi}}, \tag{2.10}$$

which follows from the observations that $\int |\chi(u,\xi)-\chi(v,\xi)|d\xi = |u-v|$ for all $u,v\in\mathbb{R}$ and

$$\int |\mathcal{M}f(x,\xi)-\mathcal{M}g(x,\xi)|d\xi = \int \big|\chi(\int f(x,\eta)d\eta,\xi)-\chi(\int g(x,\eta)d\eta,\xi)\big|d\xi = |\int f(x,\xi)-g(x,\xi)d\xi|.$$

We note that $m_{\Delta t}$ is a non-negative measure. Indeed,

$$m_{\Delta t} = \int_0^\xi \sum_k \delta(t-t_k)(\mathcal{M}f_{\Delta t}-f_{\Delta t})d\tilde{\xi} = \sum_k \delta(t-t_k)\int_0^\xi (\mathcal{M}f_{\Delta t}-f_{\Delta t})d\tilde{\xi}$$

and, moreover,

$$\int_0^\xi \mathcal{M}f_{\Delta t}(t)d\tilde{\xi} = \int_0^\xi \chi(\int f_{\Delta t}(x,\eta,t)d\eta,\tilde{\xi})d\tilde{\xi} = \xi\wedge\int f_{\Delta t}(x,\eta,t)d\eta$$

Since $f_{\Delta t}\le 1$ we find

$$\int_0^\xi f_{\Delta t}(t)d\tilde{\xi} \le \xi\wedge\int f_{\Delta t}(t)d\tilde{\xi},$$

and, hence,

$$\int_0^\xi (\mathcal{M}f_{\Delta t}-f_{\Delta t})d\tilde{\xi}\ge 0.$$

Step 2: The approximation. We obtain here an estimate for the error at the kinetic level between the solution and the approximation. This follows using an argument introduced by Perthame in [24,25] for the kinetic formulation as an alternative to Kružkov's method.

Aiming to estimate the error

$$\int |u(t)-u_{\Delta t}(t)|dx = \int \big|\int(\chi(t)-f_{\Delta t}(t))d\xi\big|dx \le \int |\chi(t)-f_{\Delta t}(t)|d\xi dx,$$

we begin by regularizing χ using a standard Dirac sequence $\varphi^\varepsilon(x) := \frac{1}{\varepsilon^N}\varphi(\frac{x}{\varepsilon})$ with $\|\varphi\|_1 = 1$. That is, we consider the x-convolution

$$\chi^\varepsilon(x,\xi,t) := (\chi(\cdot,\xi,t)*\varphi^\varepsilon)(x),$$

which solves, for $m^\varepsilon = m*\varphi^\varepsilon$,

$$\partial_t\chi^\varepsilon + \sum_{i=1}^N a^i(\xi)\partial_{x_i}\chi^\varepsilon\dot{z}^i = \partial_\xi m^\varepsilon.$$

In fact in order to make the following calculations rigorous it also necessary to consider a regularization in time and velocity, so that the equation on χ^ε is satisfied in a classical way. For simplicity of the presentation we drop this technicality here.

We first note that, using (2.8),

$$\begin{aligned}\int |\chi(t)-f_{\Delta t}(t)|d\xi dx &= \int |\chi(t)-f_{\Delta t}(t)|^2 d\xi dx\\ &= \int |\chi(t)|-2\chi(t)f_{\Delta t}(t)+|f_{\Delta t}(t)|d\xi dx\\ &= F^\varepsilon(t)+Err^1(t),\end{aligned}\tag{2.11}$$

where

$$F^\varepsilon(t) := \int \left(|\chi^\varepsilon(t)|-2\chi^\varepsilon(t)f_{\Delta t}(t)+|f_{\Delta t}(t)|\right)d\xi dx,$$

and

$$Err^1(t) := \int (|\chi(t)| - |\chi^\varepsilon(t)| - 2(\chi(t) - \chi^\varepsilon(t)) f_{\Delta t}(t))\, d\xi dx.$$

Since $u(\cdot,0) = u_{\Delta t}(\cdot,0)$, it follows that

$$\begin{aligned}\int |\chi(t) - f_{\Delta t}(t)| d\xi dx &= \int |\chi(t) - f_{\Delta t}(t)| d\xi dx - \int |\chi(0) - \chi_{\Delta t}(0)| d\xi dx \\ &= \int_0^t \frac{d}{dt} F^\varepsilon(s) ds + Err^1|_0^t.\end{aligned} \tag{2.12}$$

Step 3: The estimate of $\frac{d}{dt}F^\varepsilon$***.*** Using again (2.8), we first note that

$$\frac{d}{dt}\int |\chi^\varepsilon| d\xi dx = -2\int m^\varepsilon(x,0,t)dx \quad \text{and} \quad \frac{d}{dt}\int |f_{\Delta t}| d\xi dx = -2\int m_{\Delta t}(x,0,t)dx. \tag{2.13}$$

Furthermore, since

$$\partial_\xi \chi^\varepsilon = (\delta(\xi) - \delta(\xi - u(x,t))) * \varphi^\varepsilon \le \delta(\xi) \text{ and } \partial_\xi \chi \le \delta(\xi), \tag{2.14}$$

and, for $t \in [t_k, t_{k+1})$, $f_{\Delta t}(x,\xi,t_k) = \chi(u_{\Delta t}(x,t_k),\xi)$, we find ,

$$\begin{aligned}\partial_\xi f_{\Delta t} &= \partial_\xi \big(f_{\Delta t}(x - a(\xi)(z_t - z_{t_k}), \xi, t_k)\big) \\ &= (\partial_\xi \chi_{\Delta t})(x - a(\xi)(z_t - z_{t_k}), \xi, t_k) - D_x f_{\Delta t}(x - a(\xi)(z_t - z_{t_k}), \xi, t_k) \cdot a'(\xi)(z_t - z_{t_k}) \\ &\le \delta(\xi) - D_x f_{\Delta t}(x - a(\xi)(z_t - z_{t_k}), \xi, t_k) \cdot a'(\xi)(z_t - z_{t_k}),\end{aligned} \tag{2.15}$$

with the above inequalities satisfied in the sense of distributions.

Combining next (2.13), (2.14), (2.15), and the facts that $|f_{\Delta t}| \le 1$ and $f_{\Delta t}(x,\xi,t) = 0$ for all $|\xi| > M$ we obtain

$$\begin{aligned}-2\frac{d}{dt}\int \chi^\varepsilon f_{\Delta t} d\xi dx &= -2\int (\partial_t \chi^\varepsilon f_{\Delta t} + \chi^\varepsilon \partial_t f_{\Delta t}) d\xi dx \\ &= -2\int \left(f_{\Delta t}\big(-\sum_{i=1}^N a^i(\xi)\partial_{x_i}\chi^\varepsilon \dot z^i + \partial_\xi m^\varepsilon\big) + \chi^\varepsilon\big(-\sum_{i=1}^N a^i(\xi)\partial_{x_i} f_{\Delta t}\dot z^i + \partial_\xi m_{\Delta t}\big)\right) d\xi dx \\ &= 2\int (\partial_\xi f_{\Delta t} m^\varepsilon + \partial_\xi \chi^\varepsilon m_{\Delta t})\, d\xi dx \\ &\le 2\int (m^\varepsilon(x,0,t) + m_{\Delta t}(x,0,t))\, dx \\ &\quad - 2\sum_k \mathbb{I}_{\{t_k < t < t_{k+1}\}} \int D_x f_{\Delta t}(x - a(\xi)(z_t - z_{t_k}), \xi, t_k) \cdot a'(\xi)(z_t - z_{t_k}) m^\varepsilon d\xi dx \\ &\le -\frac{d}{dt}\int |\chi^\varepsilon| d\xi dx - \frac{d}{dt}\int |f_{\Delta t}| d\xi dx + 2\|a'\|_{C^0([-M,M])}\Delta z \int |D_x m^\varepsilon| d\xi dx \\ &\le -\frac{d}{dt}\int |\chi^\varepsilon| d\xi dx - \frac{d}{dt}\int |f_{\Delta t}| d\xi dx + 2\frac{\|a'\|_{C^0([-M,M])}\Delta z}{\varepsilon}\int m(x,\xi,t) d\xi dx,\end{aligned}$$

and, in conclusion,

$$\frac{d}{dt}F^\varepsilon(t) \le 2\frac{\|a'\|_{C^0([-M,M])}\Delta z}{\varepsilon}\int m(x,\xi,t) d\xi dx. \tag{2.16}$$

Step 4: The estimate of Err^1***.*** We estimate $|Err^1(t)|$ in terms of the BV-norm of u_0.

Lemma 2.2. *Assume $u_0 \in (BV \cap L^\infty)(\mathbb{R}^N)$ and (2.2). Then,*

$$\int |\chi(t) - \chi^\varepsilon(t)| d\xi dx \leq \varepsilon \|u_0\|_{BV},$$

and, for all $t \in [0,T]$,

$$|Err^1(t)| \leq \varepsilon \|u_0\|_{BV}.$$

Proof. Since

$$|\chi(t)| - |\chi^\varepsilon(t)| - 2(\chi(t) - \chi^\varepsilon(t)) f_{\Delta t}(t) = (\chi(t) - \chi^\varepsilon(t)) \mathrm{sgn}(\xi)(1 - 2|f_{\Delta t}|(t))$$

and $|f_{\Delta t}| \in \{0,1\}$, we first observe that, for all $t \geq 0$,

$$|Err^1(t)| \leq \int |\chi(t) - \chi^\varepsilon(t)| d\xi dx.$$

In addition it follows from [19, Proposition 2.1] that, for all $t \geq 0$,

$$\|u(t)\|_{BV} \leq \|u_0\|_{BV}.$$

Hence, using Lemma C.1, we find

$$\int |\chi(t) - \chi^\varepsilon(t)| d\xi dx \leq \varepsilon \int \|\chi(\cdot, \xi, t)\|_{BV} d\xi = \varepsilon \|u(t)\|_{BV} \leq \varepsilon \|u_0\|_{BV}.$$

$\square$

Step 5: The conclusion. It follows from (2.12), (2.16), (B.2) and Lemma 2.2 that, for all $t \in [0,T]$,

$$\begin{aligned}\int |\chi(x,\xi,t) - f_{\Delta t}(x,\xi,t)| d\xi dx &\leq 2 \frac{\|a'\|_{C^0([-M,M])} \Delta z}{\varepsilon} \int_0^t \int m(x,\xi,r) dx d\xi dr + 2\varepsilon \|u_0\|_{BV} \\ &\leq \frac{\|a'\|_{C^0([-M,M])} \Delta z}{\varepsilon} \|u_0\|_2^2 + 2\varepsilon \|u_0\|_{BV},\end{aligned}$$

and hence, choosing $\varepsilon \approx \sqrt{\Delta z}$ to minimize the expression yields

$$\int |\chi(x,\xi,t) - f_{\Delta t}(x,\xi,t)| d\xi dx \leq \sqrt{2\|u_0\|_{BV} \|a\|_{C^{0,1}([-M,M])}} \, \|u_0\|_2 \sqrt{\Delta z}. \tag{2.17}$$

We now go back to $z \in C([0,T]; \mathbb{R}^N)$ and choose $z^n \in C^1([0,T]; \mathbb{R}^N)$ such that $z^n \to z$ in $C([0,T]; \mathbb{R}^N)$. In view of the continuity in the driving signal, we observe that, as $n \to \infty$

$$\chi^n \to \chi \quad \text{and} \quad \chi^n_{\Delta t} \to \chi_{\Delta t} \quad \text{in} \quad C([0,T]; L^1(\mathbb{R}^{N+1})).$$

It follows from (2.17) that

$$\int |\chi^n(x,\xi,t) - f^n_{\Delta t}(x,\xi,t)| d\xi dx \leq \sqrt{2\|u_0\|_{BV} \|a\|_{C^{0,1}([-M,M])}} \, \|u_0\|_2 \sqrt{\Delta z^n}$$

Passing to the limit in n completes the proof. $\square$

3. Spatially inhomogeneous stochastic scalar conservation laws

We consider here the inhomogeneous stochastic scalar conservation law

$$\begin{cases} \partial_t u + \sum_{i=1}^N \partial_{x_i} A^i(x,u) \circ dz^i = 0 \quad \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 \in (L^1 \cap L^2)(\mathbb{R}^N), \end{cases} \tag{3.1}$$

and its kinetic formulation

$$\partial_t \chi + \sum_{i=1}^N a^i(x,\xi)\partial_{x_i}\chi \circ dz^i - \sum_{i=1}^N b^i(x,\xi)\partial_\xi \chi \circ dz^i = \partial_\xi m, \tag{3.2}$$

where

$$a^i(x,\xi) := (\partial_u A^i)(x,\xi) \quad \text{and} \quad b^i(x,\xi) := \partial_{x_i} A^i(x,\xi)$$

and z is an α-Hölder geometric rough path for some $\alpha \in (0,1)$.

More precisely, we assume that

$$\begin{cases} z \in C^{0,\alpha}([0,T]; G^{[\frac{1}{\alpha}]}(\mathbb{R}^N)), \\ A \in C^2(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N), \\ a, b \in \mathrm{Lip}^{\gamma+2}(\mathbb{R}^N \times \mathbb{R}) \text{ for some } \gamma > \frac{1}{\alpha} \geq 1, \text{ and} \\ b(x,0) = 0 \text{ for all } x \in \mathbb{R}^N, \end{cases} \tag{3.3}$$

and note that it has been shown in [11] that, under these assumptions, the theory of pathwise entropy solutions to (3.1) is well posed.

Fix $\Delta t > 0$ and a partition $\{t_0, \dots, t_K\}$ of $[0,T]$ given by $t_k := k\Delta t$. The approximation scheme is given by

$$\begin{cases} \partial_t f_{\Delta t} + \sum_{i=1}^N a^i(x,\xi)\partial_{x_i} f_{\Delta t} \circ dz^i - \sum_{i=1}^N b^i(x,\xi)\partial_\xi f_{\Delta t} \circ dz^i = 0 \quad \text{on } (t_k, t_{k+1}), \\ f_{\Delta t}(x,\xi,t_k) = \chi(u_{\Delta t}(x,t_k),\xi), \end{cases} \tag{3.4}$$

where

$$u_{\Delta t}(x,0) := u_0(x) \quad \text{and} \quad u_{\Delta t}(x,t) := \int f_{\Delta t}(x,\xi,t-)d\xi. \tag{3.5}$$

We begin by expressing $f_{\Delta t}$ in terms of the characteristics of (3.4). For each final time $t_1 \geq 0$, we consider the backward characteristics

$$\begin{cases} dX^i_{(x,\xi,t_1)}(t) = a^i(X_{(x,\xi,t_1)}(t), \Xi_{(x,\xi,t_1)}(t))dz^{t_1,i}(t), \;\; X^i_{(x,\xi,t_1)}(0) = x^i, \;\; i = 1,\dots,N, \\ d\Xi_{(x,\xi,t_1)}(t) = -\sum_{i=1}^N b^i(X_{(x,\xi,t_1)}(t), \Xi_{(x,\xi,t_1)}(t))dz^{t_1,i}(t), \;\; \Xi_{(x,\xi,t_1)}(0) = \xi, \end{cases}$$

where z^{t_1} is the time-reversed rough path, that is, for $t \in [0,t_1]$,

$$z^{t_1}(t) := z(t_1 - t). \tag{3.6}$$

Note that, in view of (3.3), the flow of backward characteristics $(x,\xi) \mapsto (X_{(x,\xi,t_1)}, \Xi_{(x,\xi,t_1)})$ is volume preserving on $\mathbb{R}^{N+1}$. This fact follows from Liouville's Theorem, the stability of X, Ξ in z and the fact that, for all $z \in C^1([0,T];\mathbb{R}^N)$, $t \in [0,T]$, $(x,\xi) \in \mathbb{R}^{N+1}$,

$$\sum_{i=1}^N \partial_{x_i} a^i(x,\xi)\dot{z}^i - \partial_\xi \left(\sum_{i=1}^N b^i(x,\xi)\dot{z}^i \right) = 0.$$

In addition, since $b(\cdot,0)\equiv 0$, for all $t_1,t\in[0,T]$ and $(x,\xi)\in\mathbb{R}^{N+1}$, we have

$$\operatorname{sgn}(\Xi_{(x,\xi,t_1)}(t))=\operatorname{sgn}(\xi) \quad \text{and} \quad \Xi_{(x,0,t_1)}(t)=0. \tag{3.7}$$

Let

$$(Y_{(x,\xi,t_1)}(t),\zeta_{(x,\xi,t_1)}(t)):=(X_{(x,\xi,t_1)}(t),\Xi_{(x,\xi,t_1)})^{-1}.$$

The solution $f_{\Delta t}$ to (3.4), for $t\in[t_k,t_{k+1})$, is given by

$$f_{\Delta t}(x,\xi,t)=f_{\Delta t}\left(X_{(x,\xi,t)}(t-t_k),\Xi_{(x,\xi,t)}(t-t_k),t_k\right).$$

We have:

Theorem 3.1. *Let $u_0\in(L^1\cap L^2)(\mathbb{R}^N)$ and assume* (3.3)*. Then*

$$\lim_{\Delta t\to 0}\|u(\cdot,t)-u_{\Delta t}(\cdot,t)\|_{L^1(\mathbb{R}^N\times[0,T])}.$$

Proof. We begin with a brief outline of the proof. The first step as in the proof of of Theorem 2.1 is to rewrite the scheme in a kinetic formulation with a defect measure $m_{\Delta t}$. Then we establish uniform in Δt estimates for $f_{\Delta t}$ and $m_{\Delta t}$. This allows to extract weakly$\star$- convergent subsequences $f_{\Delta t}\overset{\star}{\rightharpoonup} f$, $m_{\Delta t}\overset{\star}{\rightharpoonup} m$. In the third step we identify the limit f as a generalized pathwise entropy solution to (3.1). Since, in view of [11, Proposition 4.9, Theorem 3.1], generalized entropy solutions are unique, it follows that $f=\chi$, and this yields the weak convergence of the $f_{\Delta t}$. In the last step we deduce the strong convergence.

Step 1: The kinetic formulation of the approximation scheme. Similarly to the homogeneous setting we observe that the semi-discretization scheme has the following kinetic representation:

$$\partial_t f_{\Delta t}+\sum_{i=1}^N a^i(x,\xi)\partial_{x_i}f_{\Delta t}\circ dz^i+\sum_{i=1}^N \partial_{x_i}A^i(x,\xi)\partial_\xi f_{\Delta t}\circ dz^i=\partial_\xi m_{\Delta t} \tag{3.8}$$

where

$$\partial_\xi m_{\Delta t}:=\sum_k \delta(t-t_k)(\mathcal{M}f_{\Delta t}-f_{\Delta t}),$$

$m_{\Delta t}$ being a non-negative measure on $\mathbb{R}^N\times\mathbb{R}\times[0,T]$, and $\mathcal{M}$ is defined as in (2.9).

We pass to the stable form of (3.8) by convolution along characteristics. For any $\varrho^0\in C_c^\infty(\mathbb{R}^{N+1})$, $t_0\in[0,T]$ and $(y,\eta)\in\mathbb{R}^{N+1}$, we consider

$$\varrho_{t_0}(x,y,\xi,\eta,t):=\varrho^0\begin{pmatrix} X_{(x,\xi,t)}(t-t_0)-y\\ \Xi_{(x,\xi,t)}(t-t_0)-\eta\end{pmatrix}. \tag{3.9}$$

Then, in the sense of distributions in $t\in[0,T]$,

$$\partial_t(f_{\Delta t}*\varrho_{t_0})(y,\eta,t)=-\int\partial_\xi\varrho_{t_0}(x,y,\xi,\eta,t)m_{\Delta t}(x,\xi,t)dxd\xi,$$

which is equivalent to

$$(f_{\Delta t}*\varrho_{t_0})(y,\eta,t)-f_{\Delta t}*\varrho_{t_0}(y,\eta,s)=-\int_{(s,t]}\int\partial_\xi\varrho_{t_0}(x,y,\xi,\eta,r)m_{\Delta t}(x,\xi,r)drdxd\xi. \tag{3.10}$$

for all $s<t$, $s,t\in[0,T]$.

Step 2: Stable apriori estimate. We establish uniform in Δt estimates for $f_{\Delta t}$ and $m_{\Delta t}$. We begin with an L^1-estimate.

Lemma 3.2. *Let $u_0 \in (L^1 \cap L^2)(\mathbb{R}^N)$ and assume* (3.3). *Then, for all $t \in [0,T]$,*

$$\int |f_{\Delta t}|(x,\xi,t)dxd\xi \leq \|u_0\|_1. \tag{3.11}$$

and, for some independent of Δt positive constant M ,

$$\frac{1}{2}\int_0^t \int m_{\Delta t}(x,\xi,r)d\xi dxdr + \int f_{\Delta t}(x,\xi,t)\xi dxd\xi \leq \frac{1}{2}\|u_0\|_2^2 + M\|u_0\|_1. \tag{3.12}$$

Proof. Since $(x,\xi) \mapsto (X_{(x,\xi,t_1)}, \Xi_{(x,\xi,t_1)})$ is volume-preserving, using (2.10) we find, for all $t \in [t_k, t_{k+1})$,

$$\begin{aligned}\int |f_{\Delta t}|(x,\xi,t)dxd\xi &= \int |f_{\Delta t}| \left(X_{(x,\xi,t)}(t-t_k), \Xi_{(x,\xi,t)}(t-t_k), t_k\right) dxd\xi = \int |f_{\Delta t}| \left(x,\xi,t_k\right) dxd\xi \\ &= \int |\mathcal{M} f_{\Delta t}| \left(x,\xi,t_k-\right) dxd\xi \leq \int |f_{\Delta t}| \left(x,\xi,t_k-\right) dxd\xi,\end{aligned}$$

which proves (3.11) by iteration.

Then (3.12) follows as in [11, Lemma 4.7]. ☐

Next we show that the approximations $f_{\Delta t}$ are uniformly tight.

Lemma 3.3. *Let $u_0 \in (L^1 \cap L^2)(\mathbb{R}^N)$ and assume* (3.3). *The family $f_{\Delta t}$ is uniformly tight, that is, for each $\varepsilon > 0$, there is an $R > 0$ (independent of Δt) such that*

$$\sup_{t\in[0,T]} \int_{B_R^c \times \mathbb{R}} |f_{\Delta t}|(x,\xi,t)dxd\xi \leq \varepsilon.$$

Proof. Choose $\varrho^{s,0} \in C_c^\infty(\mathbb{R}^N)$ non-negative and $\varrho^{v,0} \in C_c^\infty(\mathbb{R})$ and consider (3.9) with $\varrho^0(x,\xi) := \varrho^{s,0}(x)\varrho^{v,0}(\xi)$; the superscripts s, v refer to the state and velocity variables respectively.

Then

$$\begin{aligned}\partial_\xi \rho_{t_0}(x,0,\xi,0,t) &= \partial_\xi(\varrho^{s,0}(X_{(x,\xi,t)}(t-t_0))\varrho^{v,0}(\Xi_{(x,\xi,t)}(t-t_0))) \\ &= (\partial_\xi \varrho^{s,0}(X_{(x,\xi,t)}(t-t_0)))\varrho^{v,0}(\Xi_{(x,\xi,t)}(t-t_0)) \\ &+ \varrho^{s,0}(X_{(x,\xi,t)}(t-t_0))\partial_\xi(\varrho^{v,0}(\Xi_{(x,\xi,t)}(t-t_0))) \\ &= D\varrho^{s,0}(X_{(x,\xi,t)}(t-t_0)) \cdot (\partial_\xi X_{(x,\xi,t)}(t-t_0))\varrho^{v,0}(\Xi_{(x,\xi,t)}(t-t_0)) \\ &+ \varrho^{s,0}(X_{(x,\xi,t)}(t-t_0))D\varrho^{v,0}(\Xi_{(x,\xi,t)}(t-t_0))\partial_\xi \Xi_{(x,\xi,t)}(t-t_0).\end{aligned}$$

Fix $\varepsilon > 0$. It follows from Lemma A.1 that we may choose $\delta > 0$, $s < t$, $t_0 \in [s,t]$ and $|t-s|$ so small that, for all $(x,\xi) \in \mathbb{R}^{N+1}$ and $r \in [s,t]$,

$$\partial_\xi \Xi_{(x,\xi,r)}(r-t_0) \geq 0, \quad |X_{(x,\xi,r)}(r-t_0) - x| \leq \frac{1}{4} \quad \text{and} \quad |\partial_\xi X_{(x,\xi,r)}(r-t_0)| \leq \delta. \tag{3.13}$$

Hence, for all $(x,\xi) \in \mathbb{R}^{N+1}$, $r \in [s,t]$,

$$\begin{aligned}&- D\varrho^{s,0}(X_{(x,\xi,r)}(r-t_0)) \cdot (\partial_\xi X_{(x,\xi,r)}(r-t_0))\varrho^{v,0}(\Xi_{(x,\xi,r)}(r-t_0)) \\ &\leq |D\varrho^{s,0}(X_{(x,\xi,r)}(r-t_0))||\partial_\xi X_{(x,\xi,r)}(r-t_0)||\varrho^{v,0}(\Xi_{(x,\xi,r)}(r-t_0))| \\ &\leq \delta|D\varrho^{s,0}(X_{(x,\xi,r)}(r-t_0))|.\end{aligned}$$

Next we consider a sequence of $\varrho_L^{v,0}$'s such that $\varrho_L^{v,0} \to \operatorname{sgn}$ in $L^\infty(\mathbb{R})$ for $L \to \infty$, $\varrho_L^{v,0}$ non-decreasing on $[-1,1]$ and $|D\varrho_L^{v,0}(\xi)| \le 1$ for all $1 \le |\xi|$ and $D\varrho_L^{v,0}(\xi) = 0$ for all $1 \le |\xi| \le L$. Then

$$\begin{aligned}
&-\varrho^{s,0}(X_{(x,\xi,r)}(r-t_0))D\varrho_L^{v,0}(\Xi_{(x,\xi,r)}(r-t_0))\partial_\xi \Xi_{(x,\xi,r)}(r-t_0) \\
&\le -\varrho^{s,0}(X_{(x,\xi,r)}(r-t_0))D\varrho_L^{v,0}(\Xi_{(x,\xi,r)}(r-t_0))\partial_\xi \Xi_{(x,\xi,r)}(r-t_0)1_{|\Xi_{(x,\xi,r)}(r-t_0)|\ge 1}.
\end{aligned}$$

Using Lemma 3.2 and dominated convergence, we conclude

$$-\lim_{L\to\infty}\int_{(s,t]}\int \varrho^{s,0}(X_{(x,\xi,r)}(r-t_0))D\varrho_L^{v,0}(\Xi_{(x,\xi,r)}(r-t_0))\partial_\xi \Xi_{(x,\xi,r)}(r-t_0)m_{\Delta t}(x,\xi,r)dxd\xi dr \le 0$$

and, hence,

$$-\lim_{L\to\infty}\int_{(s,t]}\int \partial_\xi \rho_{t_0,L}(x,0,\xi,0,r)m_{\Delta t}(x,\xi,r)dxd\xi dr \le \int_{(s,t]}\int \delta|D\varrho^{s,0}(X_{(x,\xi,r)}(r-t_0)|m_{\Delta t}(x,\xi,r)dxd\xi dr.$$

Thus, with $(y,\eta) = (0,0) \in \mathbb{R}^{N+1}$ in (3.10), we get

$$\begin{aligned}
&\lim_{L\to\infty}\int f_{\Delta t}(x,\xi,t)\varrho_{t_0,L}(x,0,\xi,0,t)dxd\xi - \lim_{L\to\infty}\int f_{\Delta t}(x,\xi,s)\varrho_{t_0,L}(x,0,\xi,0,s)dxd\xi \\
&= -\lim_{L\to\infty}\int_{(s,t]}\int \partial_\xi \varrho_{t_0,L}(x,0,\xi,0,r)m_{\Delta t}(x,\xi,r)dxd\xi dr \\
&\le \delta\int_{(s,t]}\int |D\varrho^{s,0}(X_{(x,\xi,r)}(r-t_0))|m_{\Delta t}(x,\xi,r)dxd\xi dr.
\end{aligned}$$

We choose $t_0 = s$, use that $\varrho_L^{v,0} \to \operatorname{sgn}$ in $L^\infty(\mathbb{R})$ for $L \to \infty$ and $\operatorname{sgn}(f_{\Delta t}(x,\xi,t)) = \operatorname{sgn}(\xi)$ to find

$$\begin{aligned}
&\int |f_{\Delta t}|(x,\xi,t)\varrho^{s,0}(X_{(x,\xi,t)}(t-s))dxd\xi - \int |f_{\Delta t}|(x,\xi,s)\varrho^{s,0}(x)dxd\xi \\
&\le \delta\int_{(s,t]}\int |D\varrho^{s,0}(X_{(x,\xi,r)}(r-s))|m_{\Delta t}(x,\xi,r)dxd\xi dr.
\end{aligned} \tag{3.14}$$

Let $R > 0$ large enough to be fixed later and choose $\varrho^{s,0} : \mathbb{R}^N \to [0,1]$ such that

$$\varrho^{s,0} = \begin{cases} 1 & |x| \ge R - \frac{1}{4}, \\ 0 & |x| < R - \frac{1}{2}, \end{cases} \quad \text{and} \quad |D\varrho^{s,0}| \le 4.$$

If follows, using (3.13), that

$$\varrho^{s,0}(X_{(x,\xi,t)}(t-s)) = \begin{cases} 1 & |x| \ge R, \\ 0 & |x| \le R-1. \end{cases}$$

We employ again Lemma A.1 to choose a partition $0 = \tau_0 < \tau_1 < \cdots < \tau_{\tilde{M}} = T$ of $[0,T]$ with $\tilde{M} = \tilde{M}(\delta)$ such that (3.13) is satisfied for all intervals $[s,t] = [\tau_k, \tau_{k+1}]$. Then, using (3.14), for all $t \in [0,T]$ and $k \in \{0,\dots,\tilde{M}\}$ such that $t \in [\tau_k, \tau_{k+1})$, we find

$$\begin{aligned}
\int_{B_R^c} |f_{\Delta t}|(x,\xi,t)dxd\xi &\le \int |f_{\Delta t}|(x,\xi,t)\varrho^{s,0}(X_{(x,\xi,t)}(t-\tau_k))dxd\xi \\
&\le \int |f_{\Delta t}|(x,\xi,\tau_k)\varrho^{s,0}(x)dxd\xi + \delta\int_{(\tau_k,t]}\int |D\varrho^{s,0}(X_{(x,\xi,r)}(r-\tau_k))|m_{\Delta t}dxd\xi dr \\
&\le \int_{B_{R-1}^c} |f_{\Delta t}|(x,\xi,\tau_k)dxd\xi + 4\delta\int_{(\tau_k,t]}\int m_{\Delta t}dxd\xi dr,
\end{aligned}$$

which, after an iteration and in view of Lemma 3.2, yields

$$\begin{aligned}\int_{B_R^c} |f_{\Delta t}|(x,\xi,t)dxd\xi &\leq \int_{B^c_{R-\tilde M}} |f_{\Delta t}|(x,\xi,0)dxd\xi + 4\delta \int_{[0,t]}\int m_{\Delta t}dxd\xi dr \\ &\leq \int_{B^c_{R-\tilde M}} |u_0|(x)dx + 4\delta(\frac{1}{2}\|u_0\|_2^2 + M\|u_0\|_1).\end{aligned}$$

To conclude, we first choose $\delta < \frac{\varepsilon}{2\|u_0\|_2^2+2M\|u_0\|_1}$ and then R large enough. □

Step 3: The weak convergence. For all $t_0 \geq 0$, all test functions ϱ_{t_0} given by (3.9) with $\varrho^0 \in C_c^\infty$ and all $\varphi \in C_c^\infty([0,T))$, we have

$$\begin{aligned}&\int_0^T \partial_t\varphi(r)(\varrho_{t_0} * f_{\Delta t})(y,\eta,r)dr + \varphi(0)(\varrho_{t_0} * f_{\Delta t})(y,\eta,0) \\ &= \int_0^T\int \varphi(r)\partial_\xi\varrho_{t_0}(x,y,\xi,\eta,r)m_{\Delta t}(x,\xi,r)dxd\xi dr,\end{aligned} \tag{3.15}$$

that is,

$$\begin{aligned}&\int_0^T\int \partial_t\varphi(r)\varrho_{t_0}(x,y,\xi,\eta,r)f_{\Delta t}(x,\xi,r)dxd\xi dr + \int \varphi(0)\varrho_{t_0}(x,y,\xi,\eta,0)\chi(u^0(x),\xi)dxd\xi \\ &= \int_0^T\int \varphi(r)\partial_\xi\varrho_{t_0}(x,y,\xi,\eta,r)m_{\Delta t}(x,\xi,r)dxd\xi dr.\end{aligned} \tag{3.16}$$

Moreover, once again using Lemma A.1 we find that, for some $C > 0$ and all $t \in [0,T]$,

$$\sup_{x,\xi}\left\|\begin{pmatrix} X_{(x,\xi,t)}(t-\cdot) - x \\ \Xi_{(x,\xi,t)}(t-\cdot) - \xi\end{pmatrix}\right\|_{C^0([0,T])} \leq C. \tag{3.17}$$

Since ϱ^0 has compact support so does ϱ_{t_0} in view of (3.17). Moreover, Lemma 3.2 gives

$$\sup_{t\in[0,T]} \|f_{\Delta t}(\cdot,\cdot,t)\|_{L^1(\mathbb{R}^N\times\mathbb{R})} \leq \|u_0\|_1.$$

We use next Lemma 3.3 and $|f_{\Delta t}| \leq 1$ to find a subsequence (again denoted as $f_{\Delta t}$) such that, as $\Delta t \to 0$,

$$f_{\Delta t} \overset{*}{\rightharpoonup} f \text{ in } L^\infty(\mathbb{R}^N\times\mathbb{R}\times[0,T]) \quad\text{and}\quad f_{\Delta t} \rightharpoonup f \text{ in } L^1(\mathbb{R}^N\times\mathbb{R}\times[0,T]).$$

Moreover, Lemma 3.2 yields

$$\|f\|_{L^\infty([0,T];L^1(\mathbb{R}^N\times\mathbb{R}))} \leq \|u_0\|_1.$$

Since $\mathrm{sgn}(f_{\Delta t}(x,\xi,t)) = \mathrm{sgn}(\xi)$, the weak$\star$ convergence of the $f_{\Delta t}$'s implies

$$f(x,\xi,t)\mathrm{sgn}(\xi) = |f|(x,\xi,t) \leq 1.$$

Next, we note that

$$\begin{aligned}\partial_\xi f_{\Delta t} =& \sum_k \partial_\xi(f_{\Delta t}(X_{(x,\xi,t)}(t-t_k),\Xi_{(x,\xi,t)}(t-t_k),t_k))1_{[t_k,t_{k+1})}(t) \\ =& \sum_k (\partial_\xi\chi_{\Delta t})(X_{(x,\xi,t)}(t-t_k),\Xi_{(x,\xi,t)}(t-t_k),t_k)\partial_\xi\Xi_{(x,\xi,t)}(t-t_k)1_{[t_k,t_{k+1})}(t) \\ &+ \sum_k (D_x f_{\Delta t})(X_{(x,\xi,t)}(t-t_k),\Xi_{(x,\xi,t)}(t-t_k),t_k)\cdot\partial_\xi X_{(x,\xi,t)}(t-t_k)1_{[t_k,t_{k+1})}(t).\end{aligned}$$

Moreover, (3.7) implies that, in the sense of distributions,

$$
\begin{aligned}
&(\partial_\xi \chi_{\Delta t})(X_{(x,\xi,t)}(t-t_k), \Xi_{(x,\xi,t)}(t-t_k), t_k) \\
&= \delta(\Xi_{(x,\xi,t)}(t-t_k)) - \delta\big(\Xi_{(x,\xi,t)}(t-t_k) - u_{\Delta t}(X_{(x,\xi,t)}(t-t_k), t_k)\big) \\
&= \delta(\xi) - \delta\big(\Xi_{(x,\xi,t)}(t-t_k) - u_{\Delta t}(X_{(x,\xi,t)}(t-t_k), t_k)\big),
\end{aligned}
\tag{3.18}
$$

where, for $\varphi \in C_c^\infty(\mathbb{R}^{N+1})$,

$$
\delta(\Xi_{(x,\xi,t)}(t-t_k) - u_{\Delta t}(X_{(x,\xi,t)}(t-t_k), t_k))(\varphi) := \int \varphi(Y_{(x,\xi,t_k)}(t), \zeta_{(x,\xi,t_k)}(t))\delta(\xi - u_{\Delta t}(x,t_k))dxd\xi,
$$

and thus

$$
\begin{aligned}
\partial_\xi f_{\Delta t} =& \delta(\xi) - \nu_{\Delta t}(x,\xi,t) + \sum_k \delta(\xi)(\partial_\xi \Xi_{(x,\xi,t)}(t-t_k) - 1)1_{[t_k,t_{k+1})}(t) \\
&+ \sum_k D_x f_{\Delta t}(X_{(x,\xi,t)}(t-t_k), \Xi_{(x,\xi,t)}(t-t_k), t_k) \cdot \partial_\xi X_{(x,\xi,t)}(t-t_k)1_{[t_k,t_{k+1})}(t),
\end{aligned}
\tag{3.19}
$$

with

$$
\nu_{\Delta t}(x,\xi,t) := \sum_k \delta(\Xi_{(x,\xi,t)}(t-t_k) - u_{\Delta t}(X_{(x,\xi,t)}(t-t_k), t_k))\partial_\xi \Xi_{(x,\xi,t)}(t-t_k)1_{[t_k,t_{k+1})}(t).
$$

We use again Lemma A.1 to get for Δt small enough and all $t \in [t_k, t_{k+1})$,

$$
\partial_\xi \Xi_{(x,\xi,t)}(t-t_k) \in [0,2],
\tag{3.20}
$$

which implies that $\nu_{\Delta t}$ is a non-negative measure.

Furthermore, for all $R > 0$, (3.20) and (3.17) give, for some constants $\tilde{R}$, $C > 0$ independent of Δt,

$$
\begin{aligned}
\int_0^T \int_{B_R} \nu_{\Delta t} dxd\xi dt &= \sum_k \int_{t_k}^{t_{k+1}} \int \int_{B_R} \delta(\Xi_{(x,\xi,t)}(t-t_k) - u_{\Delta t}(X_{(x,\xi,t)}(t-t_k), t_k))\partial_\xi \Xi_{(x,\xi,t)}(t-t_k) dxd\xi dt \\
&= \sum_k \int_{t_k}^{t_{k+1}} \int \int_{B_{\tilde{R}}} \delta(\xi - u_{\Delta t}(x,t_k))\partial_\xi \Xi_{(x,\xi,t)}(t-t_k)|_{Y_{(x,\xi,t_k)}(t), \zeta_{(x,\xi,t_k)}(t)} dxd\xi dt \\
&\le 2 \sum_k \int_{t_k}^{t_{k+1}} \int \int_{B_{\tilde{R}}} \delta(\xi - u_{\Delta t}(x,t_k)) dxd\xi dt \le C.
\end{aligned}
$$

Hence, there exists a non-negative measure ν so that, along a subsequence,

$$
\nu_{\Delta t} \overset{*}{\rightharpoonup} \nu.
$$

Observe that, for each $\varphi \in C_c^\infty(\mathbb{R}^N \times \mathbb{R} \times [0,T])$,

$$
\begin{aligned}
&\sum_k \int \varphi(x,\xi,t)(D_x f_{\Delta t})(X_{(x,\xi,t)}(t-t_k), \Xi_{(x,\xi,t)}(t-t_k), t_k) \cdot \partial_\xi X_{(x,\xi,t)}(t-t_k)1_{[t_k,t_{k+1})}(t) dxd\xi dt \\
&= \sum_k \int_{t_i}^{t_{i+1}} \int (D_x f_{\Delta t})(x,\xi,t_k) \cdot \varphi(Y_{(x,\xi,t_k)}(t), \zeta_{(x,\xi,t_k)}(t), t)\partial_\xi X_{(x,\xi,t)}(t-t_k)|_{Y_{(x,\xi,t_k)}(t), \zeta_{(x,\xi,t_k)}(t)} dxd\xi dt \\
&= -\sum_k \int_{t_i}^{t_{i+1}} \int f_{\Delta t}(x,\xi,t_k) \cdot D_x \left(\varphi(Y_{(x,\xi,t_k)}(t), \zeta_{(x,\xi,t_k)}(t), t)\partial_\xi X_{(x,\xi,t)}(t-t_k)|_{Y_{(x,\xi,t_k)}(t), \zeta_{(x,\xi,t_k)}(t)}\right) dxd\xi dt,
\end{aligned}
$$

and, since Lemma A.1 yields that, as $\Delta t \to 0$,

$$
\sup_{t \in [t_k, t_{k+1}]} \|\partial_\xi X_{(\cdot,\cdot,t)}(t-t_k)\|_{C^1(\mathbb{R}^{N+1})} \to 0,
$$

we find

$$\sum_k \int \varphi(x,\xi,t)(D_x f_{\Delta t})(X_{(x,\xi,t)}(t-t_k),\Xi_{(x,\xi,t)}(t-t_k),t_k)\cdot\partial_\xi X_{(x,\xi,t)}(t-t_k)1_{[t_k,t_{k+1})}(t)dxd\xi dt \to 0.$$

Moreover, again Lemma A.1 gives that, for $\Delta t \to 0$,

$$\|\partial_\xi \Xi_{(\cdot,\cdot,t)}(t-t_k)-1\|_{C(\mathbb{R}^{N+1})} \to 0,$$

and thus letting $\Delta t \to 0$ in (3.19) we find that, in the sense of distributions,

$$\partial_\xi f = \delta(\xi)-\nu.$$

Recall that (see Lemma 3.2), for all $t\in[0,T]$

$$\frac{1}{2}\int_0^t\int m_{\Delta t}(x,\xi,r)d\xi dxdr \leq \frac{1}{2}\|u_0\|_2^2+M\|u_0\|_1.$$

It follows that there exists some nonnegative measure m and a weak$\star$ convergent subsequence such that $m_{\Delta t} \stackrel{*}{\rightharpoonup} m$.

Taking the limit in (3.16) then yields

$$\begin{aligned}
&\int_0^T\int \partial_t\varphi(r)\varrho_{t_0}(x,y,\xi,\eta,r)f(x,\xi,r)dxd\xi dr+\int \varphi(0)\varrho_{t_0}(x,y,\xi,\eta,0)\chi(u^0(x),\xi)dxd\xi\\
&=\int_0^T\int \varphi(r)\partial_\xi\varrho_{t_0}(x,y,\xi,\eta,r)m(x,\xi,r)dxd\xi dr.
\end{aligned}$$

Hence, f is a generalized rough kinetic solution to (3.1). The uniqueness of generalized rough kinetic solutions (see [11, Theorem 3.1, Proposition 4.9]) yields that $f=\chi$ and thus f is the unique pathwise entropy solution to (3.1). Hence, the whole sequence $f_{\Delta t}$ converges to χ weakly$\star$ in $L^\infty(\mathbb{R}^N\times\mathbb{R}\times[0,T])$ and weakly in $L^1(\mathbb{R}^N\times\mathbb{R}\times[0,T])$.

Step 4: The strong convergence. We note that, in view of the weak convergence of $f_{\Delta t}$ to χ in $L^1(\mathbb{R}^N\times\mathbb{R}\times[0,T])$, we have, for $\Delta t\to 0$,

$$\begin{aligned}
\int_0^T\int |f_{\Delta t}-\chi|^2 dxd\xi dt &= \int_0^T\int |f_{\Delta t}|^2-2f_{\Delta t}\chi+|\chi|^2dxd\xi dt \leq \int_0^T\int |f_{\Delta t}|-2f_{\Delta t}\chi+|\chi|dxd\xi dt\\
&=\int_0^T\int f_{\Delta t}\mathrm{sgn}(\xi)-2f_{\Delta t}\chi+|\chi|dxd\xi dt \to \int_0^T\int \chi\mathrm{sgn}(\xi)-2\chi\chi+|\chi|dxd\xi dt\\
&=0.
\end{aligned}$$

The uniform tightness of $f_{\Delta t}$ then implies $\int_0^T\int|f_{\Delta t}-\chi|dxd\xi dt\to 0$ and, hence, as $\Delta t\to 0$,

$$\int_0^T\int |u_{\Delta t}-u|dxdt=\int_0^T\int\left|\int f_{\Delta t}d\xi-\int\chi d\xi\right|dxdt \leq \int_0^T\int|f_{\Delta t}-\chi|d\xi dxdt\to 0.$$

$\square$

Appendix A. Definitions and some estimates from the theory of rough paths

We briefly recall some basic facts of the Lyons' rough paths theory used in this paper. For more details we refer to Lyons and Qian [23] and Friz and Victoir [10].

Given $x \in C^{1-\mathrm{var}}([0,T];\mathbb{R}^N)$, the space of continuous paths of bounded variation, the step M signature $S_M(x)_{0,T}$ given by

$$S_M(x)_{0,T} := \left(1, \int_{0<u<T} dx_u, \dots, \int_{0<u_1<\dots<u_M<T} dx_{u_1} \otimes \dots \otimes dx_{u_M}\right),$$

takes values in the truncated step-M tensor algebra

$$T^M(\mathbb{R}^N) = \mathbb{R} \oplus \mathbb{R}^N \oplus (\mathbb{R}^N \otimes \mathbb{R}^N) \oplus \ldots \oplus (\mathbb{R}^N)^{\otimes M};$$

in fact, $S_M(x)$ takes values in the smaller set $G^M(\mathbb{R}^N) \subset T^M(\mathbb{R}^N)$ given by

$$G^M(\mathbb{R}^N) := \left\{ S_M(x)_{0,1} : \ x \in C^{1-\mathrm{var}}([0,1];\mathbb{R}^N) \right\}.$$

The Carnot-Caratheodory norm of $G^M(\mathbb{R}^N)$ given by

$$\|g\| := \inf \left\{ \int_0^1 |d\gamma| \ : \gamma \in C^{1-\mathrm{var}}([0,1];\mathbb{R}^N) \text{ and } S_M(\gamma)_{0,1} = g \right\},$$

gives rise to a homogeneous metric on $G^M(\mathbb{R}^N)$.

Alternatively, for any $g \in T^M(\mathbb{R}^N)$, we may set

$$|g| := |g|_{T^M(\mathbb{R}^N)} := \max_{k=1\dots M} |\pi_k(g)|,$$

where π_k is the projection of g onto the k-th tensor level, which is an inhomogeneous metric on $G^M(\mathbb{R}^N)$. It turns out that the topologies induced by $\|\cdot\|$ and $|\cdot|$ are equivalent.

For paths in $T^M(\mathbb{R}^N)$ starting at the fixed point $e := 1+0+\ldots+0$ and $\beta \in (0,1]$, it is possible to define β-Hölder metrics extending the usual metrics for paths in $\mathbb{R}^N$ starting at zero. The homogeneous β-Hölder metric is denoted by $d_{\beta-\text{Höl}}$ and the inhomogeneous one by $\rho_{\beta-\text{Höl}}$. A corresponding norm is defined by $\|\cdot\|_{\beta-\text{Höl}} = d_{\beta-\text{Höl}}(\cdot, 0)$, where 0 denotes the constant e-valued path.

A geometric β-Hölder rough path x is a path in $T^{\lfloor 1/\beta \rfloor}(\mathbb{R}^N)$ which can be approximated by lifts of smooth paths in the $d_{\beta-\text{Höl}}$ metric. It can be shown that rough paths actually take values in $G^{\lfloor 1/\beta \rfloor}(\mathbb{R}^N)$. The space of geometric β-Hölder rough paths is denoted by $C^{0,\beta}([0,T]; G^{[\frac{1}{\beta}]}(\mathbb{R}^N))$.

We state next a basic stability estimate for solutions to rough differential equations (RDE) of the form

$$dx = V(x) \circ dz,$$

where z is a geometric α-Hölder rough path.

It is well known (see, for example, [10]) that the RDE above has a flow ψ^z of solutions. The following is taken from Crisan, Diehl, Friz and Oberhauser [7, Lemma 13].

Lemma A.1. *Let $\alpha \in (0,1)$, $\gamma > \frac{1}{\alpha} \geq 1$, $k \in \mathbb{N}$ and assume that $V \in \mathrm{Lip}^{\gamma+k}(\mathbb{R}^N;\mathbb{R}^N)$. For all $R > 0$ there exist $C = C(R, \|V\|_{\mathrm{Lip}^{\gamma+k}})$ and $K = K(R, \|V\|_{\mathrm{Lip}^{\gamma+k}})$, which are non-decreasing in* *all arguments, such that, for all geometric α-Hölder rough paths $z^1, z^2 \in C^{0,\alpha}([0,T]; G^{[\frac{1}{\alpha}]}(\mathbb{R}^N))$ with $\|z^1\|_{\alpha-\mathrm{Höl};[0,T]}$, $\|z^2\|_{\alpha-\mathrm{Höl};[0,T]} \le R$ and all $n \in \{0,\dots,k\}$,*

$$\sup_{x\in\mathbb{R}^N} \|D^n(\psi^{z^1} - \psi^{z^2})(x)\|_{\alpha-\mathrm{Höl};[0,T]} \le C\rho_{\alpha-\mathrm{Höl};[0,T]}(z^1,z^2),$$

$$\sup_{x\in\mathbb{R}^N} \|D^n((\psi^{z^1})^{-1} - (\psi^{z^2})^{-1})(x)\|_{\alpha-\mathrm{Höl};[0,T]} \le C\rho_{\alpha-\mathrm{Höl};[0,T]}(z^1,z^2)$$

and, for all $n \in \{1,\dots,k\}$,

$$\sup_{x\in\mathbb{R}^N} \|D^n\psi^{z^1}(x)\|_{\alpha-\mathrm{Höl};[0,T]} \le K \quad \text{and} \quad \sup_{x\in\mathbb{R}^N} \|D^n(\psi^{z^1})^{-1}(x)\|_{\alpha-\mathrm{Höl};[0,T]} \le K.$$

Appendix B. Pathwise entropy solutions to stochastic scalar conservation laws

Assume (2.2) and consider the spatially homogeneous problem

$$\begin{cases} du + \sum_{i=1}^N \partial_{x_i} A^i(u)\circ dz^i = 0 \quad \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^N). \end{cases} \tag{B.1}$$

The following notion of pathwise entropy solutions to (B.1) and its well-posedness were introduced in [19].

Definition B.1. A function $u \in (L^1\cap L^\infty)(\mathbb{R}^N \times [0,T])$ is a pathwise entropy solution to (B.1), if there exists a nonnegative, bounded measure m on $\mathbb{R}^N \times \mathbb{R} \times [0,T]$ such that, for all $\varrho^0 \in C_c^\infty(\mathbb{R}^{N+1})$, all ϱ given by

$$\varrho(x,y,\xi,\eta,t) := \varrho^0(y - x + a(\xi)z(t), \xi - \eta),$$

and all $\varphi \in C_c^\infty([0,T))$,

$$\int_0^T \partial_t\varphi(r)(\varrho * \chi)(y,\eta,r)dr + \varphi(0)(\varrho*\chi)(y,\eta,0) = \int_0^T \int \varphi(r)\partial_\xi \varrho(x,y,\xi,\eta,r)m(x,\xi,r)dxd\xi dr,$$

where the convolution along characteristics $\varrho * \chi$ is defined by

$$\varrho * \chi(y,\eta,r) := \int \varrho(x,y,\xi,\eta,r)\chi(x,\xi,r)dxd\xi.$$

The following is proved in [19].

Theorem B.2. *Let $u_0 \in (L^1\cap L^\infty)(\mathbb{R}^N)$ and assume (2.2). Then there exists a unique pathwise entropy solution $u \in C([0,T]; L^1(\mathbb{R}^N))$ satisfying, for all $p \in [1,\infty]$,*

$$\sup_{t\in[0,T]} \|u(t)\|_p \le \|u_0\|_p,$$

and

$$\int_0^T \int_{[-\|u_0\|_\infty, \|u_0\|_\infty]^c} \int_{\mathbb{R}^N} m(x,\xi,t)dxd\xi dt = 0, \qquad \int_0^T \int_{\mathbb{R}^{N+1}} m(x,\xi,t)dxd\xi dt \le \frac{1}{2}\|u_0\|_2^2. \tag{B.2}$$

The notion of pathwise entropy solutions was extended in [21] and [11] to inhomogeneous stochastic scalar conservation laws of the type

$$\begin{cases} \partial_t u + \sum_{i=1}^N \partial_{x_i} A^i(x,u)\circ dz^i = 0 \quad \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 \in (L^1 \cap L^2)(\mathbb{R}^N). \end{cases} \tag{B.3}$$

Assume that A, z satisfy (3.3). For each $t_1 \geq 0$ and for $i = 1, \ldots, N$, consider the backward characteristics

$$\begin{cases} dX^i_{(x,\xi,t_1)}(t) = a^i(X_{(x,\xi,t_1)}(t), \Xi_{(x,\xi,t_1)}(t)) \circ dz^{t_1,i}(t), \\ d\Xi_{(x,\xi,t_1)}(t) = -\sum_{i=1}^N (\partial_{x_i} A^i)(X_{(x,\xi,t_1)}(t), \Xi_{(x,\xi,t_1)}(t)) \circ dz^{t_1,i}(t), \\ X^i_{(x,\xi,t_1)}(0) = x^i \text{ and } \Xi_{(x,\xi,t_1)}(0) = \xi, \end{cases}$$

where, for $t \in [0, t_1]$, z^{t_1} is the time-reversed rough path defined in (3.6).

Let ϱ_{t_0} be a test-function transported along the characteristics, that is, for some $\varrho^0 \in C_c^\infty(\mathbb{R}^{N+1})$, $t_0 \in [0, T]$, $(y, \eta) \in \mathbb{R}^{N+1}$,

$$\varrho_{t_0}(x, y, \xi, \eta, t) := \varrho^0 \begin{pmatrix} X_{(x,\xi,t)}(t - t_0) - y \\ \Xi_{(x,\xi,t)}(t - t_0) - \eta \end{pmatrix}. \tag{B.4}$$

The following definition is Definition 2.1 and Definition 4.2 of [11].

Definition B.3. Let $u_0 \in (L^1 \cap L^2)(\mathbb{R}^N)$. (i). A function $u \in L^\infty([0, T]; L^1(\mathbb{R}^N))$ is a pathwise entropy solution to (B.3), if there exists a nonnegative bounded measure m on $\mathbb{R}^N \times \mathbb{R} \times [0, T]$ such that, for all $t_0 \geq 0$, all test functions ϱ_{t_0} given by (B.4) with $\varrho^0 \in C_c^\infty$ and $\varphi \in C_c^\infty([0, T))$,

$$\begin{cases} \int_0^T \partial_t \varphi(r)(\varrho_{t_0} * \chi)(y, \eta, r)dr + \varphi(0)(\varrho_{t_0} * \chi)(y, \eta, 0) \\ \qquad = \int_0^T \int \varphi(r)\partial_\xi \varrho_{t_0}(x, y, \xi, \eta, r)m(x, \xi, r)dxd\xi dr. \end{cases} \tag{B.5}$$

(ii). A function $f \in L^\infty([0, T]; L^1(\mathbb{R}^N \times \mathbb{R}))$ is a generalized pathwise entropy solution to (B.3), if there exists a nonnegative measure ν and a nonnegative, bounded measure m on $\mathbb{R}^N \times \mathbb{R} \times [0, T]$ such that

$$f(x, \xi, 0) = \chi(u_0(x), \xi), \ |f|(x, \xi, t) = \operatorname{sgn}(\xi)f(x, \xi, t) \leq 1 \ \text{ and } \ \frac{\partial f}{\partial \xi} = \delta(\xi) - \nu(x, \xi, t), \tag{B.6}$$

and (B.5) holds with f replacing χ, for all $t_0 \geq 0$, ϱ_{t_0} as in (B.4) and $\varphi \in C_c^\infty([0, T))$.

The following well-posedness results was proved in [11]

Theorem B.4. *Let $u_0 \in (L^1 \cap L^2)(\mathbb{R}^N)$ and assume* (3.3)*. Then there exists a unique pathwise entropy solution to* (B.3) *and generalized pathwise entropy solutions to* (B.3) *are unique.*

Appendix C. Indicator functions of BV functions

We present here an observation which connects the BV-norms of u and $\chi(u(\cdot), \xi)$.

Lemma C.1. *Let $u \in L^1_{loc}(\mathbb{R}^N)$. Then*

$$\|u\|_{BV} = \int_{\mathbb{R}} \|\chi(u(\cdot), \xi)\|_{BV} d\xi.$$

Proof. It follows from Theorem 1 in Fleming and Rishel [9] that, for any $u \in L^1_{loc}$,

$$\|u\|_{BV} = \int_{\mathbb{R}} \|1_{(-\infty, u(\cdot))}(\xi)\|_{BV} d\xi.$$

Hence,

$$\begin{aligned}\|u\|_{BV} &= \|u^+\|_{BV} + \|u^-\|_{BV}\\ &= \int_{\mathbb{R}} \|1_{(-\infty,u^+(\cdot))}(\xi)\|_{BV} d\xi + \int_{\mathbb{R}} \|1_{(-\infty,u^-(\cdot))}(\xi)\|_{BV} d\xi\\ &= \int_0^\infty \|1_{(-\infty,u^+(\cdot))}(\xi)\|_{BV} d\xi + \int_0^\infty \|1_{(-\infty,u^-(\cdot))}(\xi)\|_{BV} d\xi\\ &= \int_0^\infty \|1_{(0,u^+(\cdot))}(\xi)\|_{BV} d\xi + \int_0^\infty \|1_{(0,u^-(\cdot))}(\xi)\|_{BV} d\xi\\ &= \int_0^\infty \|\chi(u^+(\cdot),\xi)\|_{BV} d\xi + \int_0^\infty \|\chi(u^-(\cdot),\xi)\|_{BV} d\xi.\end{aligned}$$

Since, for $\xi \geq 0$, $\chi(u,\xi) = 1_{(0,u)}(\xi) = 1_{(0,u^+)}(\xi) = \chi(u^+,\xi)$ and $\chi(u,\xi) = -\chi(-u,-\xi)$ we get

$$\begin{aligned}\int_{\mathbb{R}} \|\chi(u(\cdot),\xi)\|_{BV} d\xi &= \int_0^\infty \|\chi(u(\cdot),\xi)\|_{BV} d\xi + \int_{-\infty}^0 \|\chi(u(\cdot),\xi)\|_{BV} d\xi\\ &= \int_0^\infty \|\chi(u^+(\cdot),\xi)\|_{BV} d\xi + \int_{-\infty}^0 \|-\chi(-u(\cdot),-\xi)\|_{BV} d\xi\\ &= \int_0^\infty \|\chi(u^+(\cdot),\xi)\|_{BV} d\xi + \int_0^\infty \|\chi(-u(\cdot),\xi)\|_{BV} d\xi\\ &= \int_0^\infty \|\chi(u^+(\cdot),\xi)\|_{BV} d\xi + \int_0^\infty \|\chi(u^-(\cdot),\xi)\|_{BV} d\xi,\end{aligned}$$

and, hence, the claim. □

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