On the motion law of fronts for scalar reaction-diffusion equations with equal depth multiple-well potentials : the degenerate case.

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On the motion law of fronts for scalar reaction-diffusion equations with equal depth multiple-well potentials: the degenerate case

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Abstract

We derive a precise motion law for fronts of solutions to *scalar* one-dimensional reaction-diffusion equations with multiple-wells, in the case the second derivative of the potential vanishes at its minimizers. We show that, *renormalizing time* in an *algebraic* way, the motion of fronts is governed by a simple system of ordinary differential equations of nearest neighbor interaction type. These interactions may be either attractive or repulsive. Our results are not constrained by the possible occurence of collisions nor splittings. They present substantial differences with the results obtained in the case the second derivative does not vanish at the wells, a case which has been extensively studied in the literature, and where fronts have been showed to move at exponentially small speed, with motion laws which are *not renormalizable*.

1 Introduction

This paper is a continuation of our previous works [4, 5] where we analyzed the behavior of solutions v of the reaction-diffusion equation of gradient type

$$(\mathrm{PGL})_\varepsilon \qquad \frac{\partial v_\varepsilon}{\partial t} - \frac{\partial^2 v_\varepsilon}{\partial x^2} = -\frac{1}{\varepsilon^2}\nabla V(v_\varepsilon),$$

where $0 < \varepsilon < 1$ is a small parameter. In [5], we considered the case where the potential V is a smooth map from $\mathbb{R}$ to $\mathbb{R}^k$ with multiple wells whose second derivative vanishes at the wells. The main result there, stated in Theorem 1 here, provides an upper bound for the speed of fronts. In the present paper we *restrict ourselves to the scalar case*, $k = 1$, and provide a precise motion law for the fronts, showing in particular that the *upper bound provided in* [5] *is sharp.* We assume throughout this paper that the potential V is a smooth function from $\mathbb{R}$ to $\mathbb{R}$ which satisfies the following assumptions:

$$(\mathrm{A}_1) \qquad \inf V = 0 \text{ and the set of minimizers } \Sigma \equiv \{y \in \mathbb{R}, V(y) = 0\} \text{ is finite,}$$

with at least two distinct elements, that is

$$\Sigma = \{\sigma_1, ..., \sigma_{\mathfrak{q}}\},\ \mathfrak{q} \geq 2,\ \sigma_1 < \cdots < \sigma_{\mathfrak{q}}.$$

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(A_2) There exists a number $\theta > 1$ such that for all i in $\{1, \cdots, \mathfrak{q}\}$, we have

$$V(u) = \lambda_i (u - \sigma_i)^{2\theta} + \underset{u \to \sigma_i}{o}((u - \sigma_i)^{2\theta}), \text{where } \lambda_i > 0.$$

(A_3) There exists constants $\alpha_\infty > 0$ and $R_\infty > 0$ such that

$$u \cdot \nabla V(u) \geq \alpha_\infty |u|^2, \text{ if } |u| > R_\infty.$$

Whereas assumption (A_1) expresses the fact that the potential possesses at least two minimizers, also termed wells, and (A_3) describes the behavior at infinity, and is of a more technical nature, assumption (A_2), which is *central in the present paper*, describes the local behavior near the minimizing wells. The number θ is of course related to the order of vanishing of the derivatives near zero. Since $\theta > 1$, then $V''(\sigma_i) = 0$, and (A_2) holds if and only if

$$\frac{d^j}{du^j} V(\sigma_i) = 0 \text{ for } j = 1, \cdots, 2\theta - 1 \text{ and } \frac{d^{2\theta}}{du^{2\theta}} V(\sigma_i) > 0,$$

with

$$\lambda_i = \frac{1}{(2\theta)!} \frac{d^{2\theta}}{du^{2\theta}} V(\sigma_i).$$

A typical example of such potentials is given by $V(u) = (1 - u^2)^{2\theta} = (1 - u)^{2\theta}(1 + u)^{2\theta}$ which has two minimizers, $+1$ and -1, so that $\Sigma = \{+1, -1\}$, minimizers vanishing at order 2θ. In this paper, the order of degeneracy is an integer assumed to be the same at all wells: fractional or site dependent orders may presumably be handled with the same tools, however at the cost of more complicated statements.

We recall that equation $(\text{PGL})_\varepsilon$ corresponds to the L^2 gradient-flow of the energy functional $\mathcal{E}$ which is defined for a function $u : \mathbb{R} \mapsto \mathbb{R}$ by the formula

$$\mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}} e_\varepsilon(u) = \int_{\mathbb{R}} \frac{\varepsilon |\dot{u}|^2}{2} + \frac{V(u)}{\varepsilon}.$$

As in [4, 5], we consider only *finite energy solutions.* More precisely, we fix an arbitrary constant $M_0 > 0$ and we consider the condition

$$(\text{H}_0) \qquad \mathcal{E}_\varepsilon(u) \leq M_0 < +\infty.$$

Besides the assumptions on the potential, the main assumption is on the initial data $v_\varepsilon^0(\cdot) = v_\varepsilon(\cdot, 0)$, assumed to satisfy ($H_0$) independently of ε. In particular, in view of the classical energy identity

$$\mathcal{E}_\varepsilon(v_\varepsilon(\cdot, T_2)) + \varepsilon \int_{T_1}^{T_2} \int_{\mathbb{R}} \left| \frac{\partial v_\varepsilon}{\partial t} \right|^2 (x, t) dx\, dt = \mathcal{E}_\varepsilon(v_\varepsilon(\cdot, T_1)) \quad \forall 0 \leq T_1 \leq T_2\,, \tag{1}$$

we have

$$\mathcal{E}_\varepsilon(v_\varepsilon(\cdot, t)) \leq M_0, \ \forall t \geq 0.$$

This implies in particular that for every given $t \geq 0$, we have $V(v_\varepsilon(x, t)) \to 0$ as $|x| \to \infty$. It is then quite straightforward to deduce from assumption (H_0), (A_1), (A_2) as well as the energy identity (1), that $v_\varepsilon(x, t) \to \sigma_\pm$ as $x \to \pm\infty$, where $\sigma_\pm \in \Sigma$ do not depend on t. In other words, for any time, our solutions connect to given minimizers of the potential.

1.1 Main results: Fronts and their speed

The notion of fronts is central in the dynamics. For a map $u : \mathbb{R} \mapsto \mathbb{R}$, the set

$$\mathcal{D}(u) \equiv \{x \in \mathbb{R},\ \mathrm{dist}(u(x), \Sigma) \geq \mu_0\},$$

is termed throughout *the front set* of u. The constant μ_0 which appears in its definition is fixed once for all, sufficiently small so that

$$\frac{\lambda_i}{2}(u - \sigma_i)^{2\theta} \leq V(u) \leq \frac{1}{\theta} V'(u)(u - \sigma_i) \leq 4V(u) \leq 8\lambda_i (u - \sigma_i)^{2\theta}, \tag{2}$$

for each $i \in \{1, \cdots, \mathfrak{q}\}$ and whenever $|u - \sigma_i| \leq \mu_0$. The front set corresponds to the set of points where u is "far" from the minimizers σ_i, and hence where transitions from one minimizer to the other may occur. A straightforward analysis yields

Lemma 1 (see e.g. [4])**.** *Assume that u verifies (H_0). Then there exists ℓ points $x_1, ..., x_\ell$ in $\mathcal{D}(u)$ such that*

$$\mathcal{D}(u) \subset \bigcup_{k=1}^{\ell} [x_k - \varepsilon, x_k + \varepsilon],$$

with a bound $\ell \leq \frac{M_0}{\eta_0}$ on the number of points, η_0 being some constant depending only on V.

In view of Lemma 1, the measure of the front sets is of order ε, and corresponds to a small neighborhood of order ε of the points x_i. Notice that if $(u_\varepsilon)_{\varepsilon>0}$ is a family of functions satisfying (H_0) then it is well-known that the family is locally bounded in $BV(\mathbb{R}, \mathbb{R})$ and hence locally compact in $L^1(\mathbb{R}, \mathbb{R})$. Passing to a subsequence if necessary, we may assert that

$$u_\varepsilon \to u^\star \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}),$$

where $u^\star$ takes values in Σ and is a step function. More precisely there exist an integer $\ell \leq \frac{M_0}{\eta_0}$, ℓ points $a_1 < \cdots < a_\ell$ and a function $\hat{\imath} : \{\frac{1}{2}, \cdots, \frac{1}{2} + \ell\} \to \{1, \cdots, \mathfrak{q}\}$ such that

$$u^\star = \sigma_{\hat{\imath}(k+\frac{1}{2})} \text{ on } (a_k, a_{k+1}),$$

for $k = 0, \cdots, \ell$, and where we use the convention $a_0 := -\infty$ and $a_{\ell+1} := +\infty$. The points a_k, for $k = 1 \cdots, \ell$, are the limits as ε shrinks to 0 of the points x_i provided by Lemma 1 (the number and the positions of which are of course ε dependent), so that the front set $\mathcal{D}(u_\varepsilon)$ shrinks as ε tends to 0 to a finite set. In the sequel, we shall refer to step functions with values into Σ as steep front chains and we will write

$$u^\star = u^\star(\ell, \hat{\imath}, \{a_k\})$$

to determine them unambigously.

We go back to equation $(\mathrm{PGL})_\varepsilon$ and consider a family of functions $(v_\varepsilon)_{\varepsilon>0}$ defined on $\mathbb{R} \times \mathbb{R}^+$ which are solutions to the equation $(\mathrm{PGL})_\varepsilon$ and satisfy the energy bound (H_0). We set

$$\mathcal{D}_\varepsilon(t) = \mathcal{D}(v_\varepsilon(\cdot, t)).$$

The evolution of the front set $\mathcal{D}_\varepsilon(t)$ when ε tends to 0 is the main focus of our paper. The following result[1] has been proved in [5]:

[1]which holds also more generally for systems.

Theorem 1 ([5])**.** *There exists constants $\rho_0 > 0$ and $\alpha_0 > 0$, depending only on the potential V and on M_0 such that if $r \geq \alpha_0\varepsilon$, then*

$$\mathcal{D}_\varepsilon(t + \Delta t) \subset \mathcal{D}_\varepsilon(t) + [-r, r], \qquad \textit{for every } t \geq 0, \tag{3}$$

provided $0 \leq \Delta t \leq \rho_0 r^2 \left(\frac{r}{\varepsilon}\right)^{\frac{\theta+1}{\theta-1}}$.

As a matter of fact, it follows from this result that the average speed of the front set at that length-scale should not exceed

$$c_{\rm ave} \simeq \frac{r}{(\Delta t)_{\max}} \leq \rho_0^{-1} r^{-(\omega+1)} \varepsilon^\omega, \tag{4}$$

where

$$\omega = \frac{\theta+1}{\theta-1}. \tag{5}$$

Notice that $1 < \omega < +\infty$ and that the upper bound provided by (4) decreases with θ, that is, the more degenerate the minimizers of V are, the higher the possible speed allowed by the bound (4). In contrast, the speed is at most exponentially small in the case of non degenerate potentials (see e.g. [9], [4] and the references therein). One aim of the present paper is to show that the *upper bound* provided by the estimate (4) is in fact optimal[2] and actually to derive a precise motion law for the fronts. An important fact, on which our results are built, is the following observation[3]:

Equation $(PGL)_\varepsilon$ is renormalizable.

This assertion means that, rescaling time in an appropriate way, the evolution of fronts in the asymptotic limit $\varepsilon \to 0$ is governed by an ordinary differential equation which *does not involve the parameter ε*. More precisely, we accelerate time by the factor $\varepsilon^{-\omega}$ and consider the new time $s = \varepsilon^\omega t$. In the accelerated time, we consider the map

$$\mathfrak{v}_\varepsilon(x, s) = v_\varepsilon(x, s\varepsilon^{-\omega}), \text{ and set } \mathfrak{D}_\varepsilon(s) = \mathcal{D}(\mathfrak{v}_\varepsilon(\cdot, s)). \tag{6}$$

It follows from Theorem 1 that for given $r \geq \alpha_0\varepsilon$,

$$\mathfrak{D}_\varepsilon(s + \Delta s) \subset \mathfrak{D}_\varepsilon(s) + [-r, r], \qquad \text{for every } s \geq 0, \tag{7}$$

provided that $0 \leq \Delta s \leq \rho_0 r^{\omega+2}$.

Concerning the initial data, we will assume that there exists a steep front chain $v^\star(\ell_0, \hat{\imath}_0, \{a_k^0\})$ such that

$$(\mathrm{H}_1) \qquad \begin{cases} v_\varepsilon^0 \longrightarrow v^\star(\ell_0, \hat{\imath}_0, \{a_k^0\}) \text{ in } L^1_{\rm loc}(\mathbb{R}), \\ \mathfrak{D}_\varepsilon(0) \longrightarrow \{a_k^0\}_{1\leq k\leq \ell_0}, \text{ locally in the sense of the Hausdorff distance }, \end{cases}$$

as $\varepsilon \to 0$. Let us emphasize that *assumption* (H_1) *is not restrictive*, since it follows assuming only the energy bound (H_0) and passing possibly to a subsequence (see above). In our first result, we will impose the additional condition

$$(\mathrm{H}_{\min}) \qquad |\hat{\imath}_0(k + \frac{1}{2}) - \hat{\imath}_0(k - \frac{1}{2})| = 1 \text{ for } 1 \leq k \leq \ell_0.$$

[2] at least in the scalar case considered here.

[3] which to our knowledge has not been observed before, even using formal arguments.

This assumption could be rephrased as a "multiplicity one" condition: it means that the jumps consist of exactly one transition between consecutive minimizers σ_i and $\sigma_{i\pm1}$. To each transition point a_k^0 we may assign a sign, denoted by $\dagger_k \in \{+,-\}$, in the following way:

$$\dagger_k = + \text{ if } \sigma_{\hat{\imath}_0(k+\frac{1}{2})} = \sigma_{\hat{\imath}_0(k-\frac{1}{2})} + 1 \text{ and } \dagger_k = - \text{ if } \sigma_{\hat{\imath}_0(k+\frac{1}{2})} = \sigma_{\hat{\imath}_0(k-\frac{1}{2})} - 1.$$

We consider next the system of ordinary differential equations

$$\mathfrak{S}_k \frac{d}{ds} a_k = \frac{\Gamma_k^+}{\left(a_k - a_{k+1}\right)^{\omega+1}} - \frac{\Gamma_k^-}{\left(a_k - a_{k-1}\right)^{\omega+1}}, \tag{$\mathcal{S}$}$$

for $1 \leq k \leq \ell_0$, where $\mathfrak{S}_k$ stands for the energy of the corresponding stationary front, namely

$$\mathfrak{S}_k = \int_{\sigma_{\hat{\imath}_0(k-\frac{1}{2})}}^{\sigma_{\hat{\imath}_0(k+\frac{1}{2})}} \sqrt{2V(u)}du, \tag{8}$$

and where we have set, for $k = 1, \cdots, \ell_0$

$$\begin{cases} \Gamma_k^+ = \quad 2^\omega \left(\lambda_{\hat{\imath}_0(k+\frac{1}{2})}\right)^{-\frac{1}{\theta-1}} \mathcal{A}_\theta \;\text{ if } \dagger_k = -\dagger_{k+1} \\ \Gamma_k^- = -2^\omega \left(\lambda_{\hat{\imath}_0(k+\frac{1}{2})}\right)^{-\frac{1}{\theta-1}} \mathcal{B}_\theta \;\text{ if } \dagger_k = \quad \dagger_{k+1}\,. \end{cases} \tag{9}$$

In (9), $\lambda_{\hat{\imath}_0(k+\frac{1}{2})}$ is defined in (A_2) and the constants $\mathcal{A}_\theta > 0$ and $\mathcal{B}_\theta > 0$, depending only on θ, are defined in (A.9) of Appendix A. Note in particular that ($\mathcal{S}$) is fully determined by the pair $(\ell_0, \hat{\imath}_0)$, and we shall therefore sometimes refer to it as $\mathcal{S}_{\ell_0,\hat{\imath}_0}$. Our first result is

Theorem 2. *Assume that the initial data $(v_\varepsilon(0))_{0<\varepsilon<1}$ satisfy conditions* (H_0), (H_1), *and* ($\mathrm{H}_{\min}$), *and let $0 < S_{\max} \leq +\infty$ denote the maximal time of existence for the system $\mathcal{S}_{\ell_0,\hat{\imath}_0}$ with initial data $a_k(0) = a_k^0$. Then, for $0 < s < S_{\max}$,*

$$\mathfrak{v}_\varepsilon(s) \longrightarrow v^\star(\ell_0, \hat{\imath}_0, \{a_k(s)\}) \tag{10}$$

in $L^\infty_{\mathrm{loc}}(\mathbb{R} \setminus \cup_{k=1}^{\ell_0}\{a_k(s)\})$, as $\varepsilon \to 0$. In particular,

$$\mathfrak{D}_\varepsilon(s) \longrightarrow \cup_{k=1}^{\ell_0}\{a_k(s)\} \tag{11}$$

locally in the sense of the Hausdorff distance, as $\varepsilon \to 0$.

We consider now the more general situation where ($\mathrm{H}_{\min}$) is not verified, and for $1 \leq k \leq \ell_0$ we denote by m_k^0 the algebraic multiplicity of a_k^0, namely we set

$$m_k^0 = \hat{\imath}(k + \frac{1}{2}) - \hat{\imath}(k - \frac{1}{2}). \tag{12}$$

The case $m_k^0 = 0$ corresponds to *ghost fronts*, whereas $|m_k^0| \geq 2$ corresponds to *multiple fronts*. The total number of fronts that will eventually emerge from such initial data is given by

$$\ell_1 = \sum_{k=1}^{\ell_0} |m_k^0|,$$

and their ordering is obtained by splitting multiple fronts according to the order in Σ. More precisely, we define the function $\hat{\imath}_1$ by

$$\begin{cases} \hat{\imath}_1(\frac{1}{2}) = \hat{\imath}_0(\frac{1}{2}), \\ \hat{\imath}_1(M_k^0 + p + \frac{1}{2}) = \hat{\imath}_0(k + \frac{1}{2}) + p, \text{ for } p = 0, \dots, |m_k^0| - 1 \text{ if } m_k^0 > 0 \\ \hat{\imath}_1(M_k^0 + p + \frac{1}{2}) = \hat{\imath}_0(k + \frac{1}{2}) - p, \text{ for } p = 0, \dots, |m_k^0| - 1 \text{ if } m_k^0 < 0, \end{cases} \tag{13}$$

where $k = 1, \cdots, \ell_0$ and $M_k^0 := \sum_{k=1}^{k-1} |m_k^0|$. We say that $(\ell_1, \hat{\imath}_1)$ is the splitting of $(\ell_0, \hat{\imath}_0)$.

Definition 1. *A splitting solution of* $(\mathcal{S})$ *with initial data* $(\ell_0, \hat{\imath}_0, \{a_k^0\})$ *on the interval* $[0, S)$ *is a solution* $a \equiv (a_1, \cdots, a_{\ell_1}) : (0, S) \to \mathbb{R}^{\ell_1}$ *of* $(\mathcal{S}_{\ell_1, \hat{\imath}_1})$ *such that*

$$\lim_{s \to 0^+} a_k(s) = a_j^0 \qquad \text{for} \qquad k = M_j^0, \cdots, M_j^0 + |m_j^0| - 1,$$

for any $j = 1, \cdots, \ell_0$*, where* $(\ell_1, \hat{\imath}_1)$ *is the splitting of* $(\ell_0, \hat{\imath}_0)$*.*

We are now in position to complete Theorem 2 by relaxing assumption $(\mathrm{H}_{\min})$.

Theorem 3. *Assume that the initial data* $(v_\varepsilon^0)_{0<\varepsilon<1}$ *satisfy conditions* (H_0) *and* (H_1)*. Then there exists a subsequence* $\varepsilon_n \to 0$*, and a splitting solution of* $(\mathcal{S})$ *with initial data* $(\ell_0, \hat{\imath}_0, \{a_k^0\})$*, defined on its maximal time of existence* $[0, S_{\max})$*, and such that for any* $0 < s < S_{\max}$

$$\mathfrak{v}_{\varepsilon_n}(s) \longrightarrow v^\star(\ell_1, \hat{\imath}_1, \{a_k(s)\}) \tag{14}$$

in $L^\infty_{\mathrm{loc}}(\mathbb{R} \setminus \cup_{k=1}^{\ell_1} \{a_k(s)\})$*, as* $n \to +\infty$*. In particular,*

$$\mathfrak{D}_{\varepsilon_n}(s) \longrightarrow \cup_{j=1}^{\ell_1} \{a_k(s)\} \tag{15}$$

locally in the sense of the Hausdorff distance, as $n \to +\infty$*.*

Remark 1. Local existence of splitting solutions can be established in different ways (including in particular using Theorem 3 !); to our knowledge, *uniqueness is not known,* unless of course if $|m_k^0| \leq 1$ for all k.

So far, our results are constrained by the maximal time of existence $S_{\max}$ of the differential equation $(\mathcal{S})$, which is related to the occurrence of collisions. To pursue the analysis past collisions, we first briefly discuss some properties of the system of equations $(\mathcal{S})$, we refer to Appendix B for more details. The system $(\mathcal{S})$ describes nearest neighbor interactions with an interaction law of the form $\pm d^{-(\omega+1)}$, d standing for the distance between fronts. The sign of the interactions is crucial, since the system may contain both repulsive forces leading to spreading and attractive forces leading to collisions, yielding the maximal time of existence $S_{\max}$. In order to take signs into account, we set

$$\epsilon_{k+\frac{1}{2}} = \mathrm{sign}\,(\Gamma_{k+\frac{1}{2}}) = -\dagger_k \dagger_{k+1}, \text{ for } k = 0, \cdots, \ell_0 - 1. \tag{16}$$

The case $\epsilon_{k+\frac{1}{2}} = -1$ corresponds to *repulsive forces* between a_k and a_{k+1}, whereas the case $\epsilon_{k+\frac{1}{2}} = +1$ corresponds to *attractive forces* between a_k and a_{k+1}, leading to collisions. As a matter of fact, in this last case a_{k+1} corresponds to the *anti-front* of a_k. In order to describe the magnitude of the forces, we introduce the subsets $J^{\pm}$ of $\{1, \cdots, \ell_0\}$ defined by $J^{\pm} = \{k \in \{1, \cdots, \ell_0 - 1\}, \text{ such that } \epsilon_{k+\frac{1}{2}} = \mp 1\}$ and the quantities

$$\begin{cases} \mathfrak{d}_a(s) = \inf\{|a_k(s) - a_{k+1}(s)|, \text{ for } k \in 1, \cdots, \ell_0 - 1\} \\ \mathfrak{d}_a^{\pm}(s) = \inf\{|a_k(s) - a_{k+1}(s)|, \text{ for } k \in J^{\pm}\} \end{cases} \tag{17}$$

Proposition 1. *There are positive constants $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$ depending only on the coefficients of the equation $(\mathcal{S})$, such that for any time $s \in [0, S_{\max})$ we have*

$$\begin{cases} \mathfrak{d}_a^+(s) \geq \left(\mathcal{S}_1 s + \mathcal{S}_2 \mathfrak{d}_a^+(0)^{\omega+2}\right)^{\frac{1}{\omega+2}}, \\ \mathfrak{d}_a^-(s) \leq \left(\mathcal{S}_3 \mathfrak{d}_a^-(0)^{\omega+2} - \mathcal{S}_4 t\right)^{\frac{1}{\omega+2}}. \end{cases} \tag{18}$$

If for every $k = 1, \cdots, \ell_0$ we have $\epsilon_{k+\frac{1}{2}} = -1$, then $S_{\max} = +\infty$. Otherwise, we have the estimate

$$S_{\max} \leq \frac{\mathcal{S}_3}{\mathcal{S}_4} \left(\mathfrak{d}_a^-(0)\right)^{\omega+2} \equiv \mathcal{K}_0 \left(\mathfrak{d}_a^-(0)\right)^{\omega+2}. \tag{19}$$

This result shows that the maximal time of existence for solutions to $(\mathcal{S})$ is related to the value of $\mathfrak{d}_a^-(0)$, the minimal distance between fronts and anti-fronts at time 0. By the semigroup property, the same can be said about $\mathfrak{d}_a^-(s)$, namely

$$S_{\max} - s \lessapprox \mathfrak{d}_a^-(s)^{\omega+2}.$$

On the other hand, in view of $(\mathcal{S})$, $\mathfrak{d}_a^-(s)$ provides an upper bound for the speeds $\dot{a}_k(s)$ in case of collision, namely

$$|\frac{d}{ds} a_k(s)| \lessapprox \mathfrak{d}_a^-(s)^{-(\omega+1)}.$$

It follows that

$$\int_0^{S_{\max}} |\frac{d}{ds} a_k(s)|\, ds \lessapprox \int_0^{S_{\max}} (S_{\max} - s)^{-\frac{\omega+1}{\omega+2}}\, ds < +\infty$$

and therefore that the trajectories are absolutely continuous up to the collision time. Also, since $\mathfrak{d}_a^+$ remains bounded from below by a positive constant, each front can only enter in collision with its anti-front (but there could be multiple copies of both). From a heuristic point of view, it is therefore rather simple to extend solutions past the collision time: it suffices to remove the colliding pairs from the collection of points, so that the total number of points has been decreased by an even number. More precisely, we have

Corollary 1. *Let $\ell_1, \hat{\imath}_1$, $a \equiv (a_1, \cdots, a_{\ell_1})$ and $S_{\max}$ be as in Theorem 3. Then, there exist $\ell_2 \in \mathbb{N}$ such that $\ell_1 - \ell_2 \in 2\mathbb{N}_*$, and there exist ℓ_2 points $b_1 < \cdots < b_{\ell_2}$ such that for all $k = 1, \cdots, \ell_1$*

$$\lim_{s \to S_{\max}^-} a_k(s) = b_{j(k)} \qquad \text{for some } j(k) \in \{1, \cdots, \ell_2\}.$$

Moreover, if we set $\hat{\imath}_2(\frac{1}{2}) = \hat{\imath}_1(\frac{1}{2})$ and

$$\hat{\imath}_2(q + \frac{1}{2}) = \hat{\imath}_1(k(q) + \frac{1}{2}) \quad \text{where} \quad k(q) = max\{k \in \{1, \cdots, \ell_1\} \text{ s.t. } j(k) = q\},$$

for $q = 1, \cdots, \ell_2$, *then*

$$\hat{\imath}_2(q + \frac{1}{2}) - \hat{\imath}_2(q - \frac{1}{2}) \in \{+1, -1, 0\}$$

for all $q = 1, \cdots, \ell_2$.

We stress than Corollary 1 is obtained from Theorem 3 using only properties of the system of ODE's $(\mathcal{S})$, in particular Proposition 1.

We are now in position to state our last result, namely

Theorem 4. *Under the assumptions of Theorem 3, we have as* $n \to +\infty$,

$$\mathfrak{v}_{\varepsilon_n}(S_{\max}) \longrightarrow v^\star(\ell_2, \hat{\imath}_2, \{b_k\}) \quad in \quad L^\infty_{\rm loc}(\mathbb{R} \setminus \cup_{k=1}^{\ell_2}\{b_k\}), \tag{20}$$

where ℓ_2, $\hat{\imath}_2$ *and* $b_1 < \cdots < b_{\ell_2}$ *are given by Corollary 1. In particular the sequence* $(\mathfrak{v}_{\varepsilon_n}(S_{\max}))_{n\in\mathbb{N}}$, *considered as initial data, satisfies the assumptions* (H_0) *and* (H_1) *with* $\ell_0 := \ell_2$ *and* $\{a_k^0\} := \{b_k^0\}$.

We may therefore apply Theorem 3 to the sequence of initial data $(\mathfrak{v}_{\varepsilon_n}(S_{\max}))_{n\in\mathbb{N}}$, and therefore, using the semi-group property of (1), extend the analysis past $S_{\max}$. Notice that since the multiplicites given by $\hat{\imath}_2$ are either equal to ±1 or 0, no further subsequences are needed to pass through the collision times. Finally, since the total number of fronts is decreased at least by 2 at each collision times, the latter are finitely many.

Some comments on the results. Motion of fronts for one-dimensional *scalar* reaction-diffusion equations has already a quite long history. Most of the efforts have been devoted until recently to the case where the potential possesses *only two wells* with non vanishing second derivative: such potentials are often referred to as *Allen-Cahn potentials.* Under suitable preparedness assumptions on the initial datum, the precise motion law for the fronts has been derived by Carr and Pego in their seminal work [9] (see also Fusco and Hale [10]). They showed that the front points are moved, up to the first collision time, according to a first order differential equation of nearest neighbor interaction type, with interactions terms proportional to $\exp(-\varepsilon^{-1}(a^\varepsilon_{k+1}(t) - a^\varepsilon_k(t)))$. These results present substantial differences with the results in the present paper, in particular we wish to emphasize the following points:

- only attractive forces leading eventually to the annihilation of *fronts* with *anti-fronts* forces are present.
- the equation is *not* renormalizable. Indeed, the various forces $\exp(-\varepsilon^{-1}(a^\varepsilon_{k+1}(t) - a^\varepsilon_k(t)))$ for different values of k may be of very different orders of magnitude, and hence not commensurable.

Besides this, the essence of their method is quite different: it relies on a careful study of the linearized problem around the stationary front, in particular from the spectral point of view. This type of approach is also sometimes termed the *geometric approach* (see e.g. [8]). At least two other methods have been applied successfully on the Allen-Cahn equation. Firstly, the method of subsolutions and supersolutions turns out to be extremely powerful and allowed to handle larger classes of initial data and also to extend the analysis past collisions: this is for instance achieved by Chen in [8]. Another direction is given by the global energy approach initiated by Bronsard and Kohn [7]. We refer to [4] for a more references on these methods.

Several ideas and concepts presented here are influenced by our earlier work on the motion of vortices in the two-dimensional parabolic Ginzburg-Landau equation [2, 3]. As a matter of fact, this equation yields another remarkable example of *renormalizable slow motion*, as proved by Lin or Jerrard and Soner ([13, 11]). Our interest in the questions studied in this paper was certainly driven by the possibility of finding an analogous situation in one space dimension.

This paper belongs to series of papers we have written on the slow motion phenomenon for reaction-diffusion equation of gradient type with mutiple-wells (see [4, 5, 6]). Common to these papers is a general approach based on the following ingredients:

- A *localized version of the energy identity* (see subsection 1.3). Fronts are then handled as concentration points of the energy, so that the evolution of local energies yields also the motion of fronts. Besides dissipation, this localized energy identity contains a flux term, involving the *discrepancy* function, which has a simple interpretation for stationary solutions. Using test functions which are *affine near the fronts*, the flux term does not see the core of the front, only its tail.
- Parabolic estimates *away* from the fronts.
- Handling the time derivative as a *perturbation* of the one-dimensional elliptic equations, allowing hence elementary tools as Gronwall's identities.

Parallel to this paper, we are also *revisiting the scalar non-degenerate case* in [6], considering in particular the case were there are more than two wells, leading as mentioned to repulsive forces which are not present in the Allen-Cahn case. Several tools are shared by the two papers, for instance we rely on related definitions and properties of regularized fronts, and the properties of the ordinary differential equations are quite similar. From a technical point of view differences appear at the level of the magnitudes of energies as well as of the parameter δ involved in the definition of regular fronts, and more crucially on the nature of the parabolic estimates off the front sets. Whereas in [6] we rely essentially on *linear* estimates, in the degenerate case considered here our estimates are *truly non-linear*, obtained mainly through an extensive use of the comparison principle.

Finally, it is presumably worthwhile to mention that the situation in higher dimension is very different: the dynamics is dominated by *mean-curvature* effects. The phenomena considered in the present paper are therefore of lower order, and do not appear in the limiting equations.

Among the problems left open in our paper, we would like to emphasize again the question of *uniqueness* of splitting solutions for $(\mathcal{S})$, as well as the possibility to interpret our convergence results in terms of *Gamma-limit* involving a renormalized energy (see e.g [15] for related results on the Ginzburg-Landau equation).

1.2 Regularized fronts

The notion of regularized fronts is central in our description of the dynamics of equation $(\mathrm{PGL})_\varepsilon$. It is aimed to describe in a quantitative way chains of stationary solutions which are well-separated and suitably glued together. It also allows to pass from *front sets* to *front points*, a notion which is more accurate and requires therefore improved estimates. Recall first that for $i \in \{1, \cdots, \mathfrak{q}-1\}$, there exist a unique (up to translations) solution ζ_i^+ to the stationary equation with $\varepsilon = 1$,

$$-v_{xx} + V'(v) = 0 \text{ on } \mathbb{R}, \tag{21}$$

with, as conditions at infinity, $v(-\infty) = \sigma_i$ and $v(+\infty) = \sigma_{i+1}$. Set, for $i \in \{1, \cdots, \mathfrak{q}-1\}$, $\zeta_i^-(\cdot) \equiv \zeta_i(-\cdot)$, so that ζ_i^- is the unique (up to translations) solution to (21) such that $v(+\infty) = \sigma_i$ and $v(-\infty) = \sigma_{i+1}$. A remarkable yet elementary fact, related to the scalar nature of the equation, is that there are no other non trivial finite energy solutions to equation (21) than the solutions $\zeta_i^\pm$ and their translates: in particular there are no solutions connecting minimizers which are not nearest neighbors. For $i = 1, \cdots, \mathfrak{q}-1$, we fix a point z_i in the interval (σ_i, σ_{i+1}) where the potential V restricted to $[\sigma_i, \sigma_{i+1}]$ achieves its maximum and we set $\mathcal{Z} = \{z_1, \cdots, z_{\mathfrak{q}-1}\}$. Again, since we consider only the one-dimensional scalar case, any solution ζ_i takes once and only once the value z_i.

We next describe a *local* notion of well-preparedness[4]. For an arbitrary $r > 0$, we denote by I_r the interval $[-r, r]$.

Definition 2. *Let* $\mathrm{L} > 0$ *and* $\delta > 0$. *We say that a map* u *verifying* (H_0) *satisfies the preparedness assumption* $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta)$ *if the following two conditions are fulfilled:*

- $(\mathrm{WPI}_\varepsilon^{\mathrm{L}}(\delta))$ *We have*

$$\mathcal{D}(u) \cap \mathrm{I}_{2\mathrm{L}} \subset \mathrm{I}_{\mathrm{L}} \tag{22}$$

and there exists a collection of points $\{a_k\}_{k \in J}$ *in* I_{L}, *with* $J = \{1, \cdots, \ell\}$, *such that*

$$\mathcal{D}(u) \cap \mathrm{I}_{2\mathrm{L}} \subset \underset{k \in J}{\cup} I_k, \quad \text{where } I_k = [a_k - \delta, a_k + \delta]. \tag{23}$$

For $k \in J$, *there exist a number* $i(k) \in \{1, \cdots, \mathfrak{q}-1\}$ *such that* $u(a_k) = z_{i(k)}$ *and a symbol* $\dagger_k \in \{+, -\}$ *such that*

$$\left\| u(\cdot) - \zeta_{i(k)}^{\dagger_k}\left(\frac{\cdot - a_k}{\varepsilon}\right) \right\|_{C_\varepsilon^1(I_k)} \leq \exp\left(-\frac{\delta}{\varepsilon}\right), \tag{24}$$

where $\|u\|_{C_\varepsilon^1(I_k)} = \|u\|_{L^\infty(I_k)} + \varepsilon \|u'\|_{L^\infty(I_k)}$.

- $(\mathrm{WPO}_\varepsilon^{\mathrm{L}}(\delta))$ *Set* $\Omega_{\mathrm{L}} = \mathrm{I}_{2\mathrm{L}} \setminus \overset{\ell}{\underset{k=1}{\cup}} I_k$. *We have the energy estimate*

$$\int_{\Omega_{\mathrm{L}}} e_\varepsilon(u(x))\, dx \leq \mathrm{C_w} M_0 \left(\frac{\varepsilon}{\delta}\right)^\omega. \tag{25}$$

In the above definition $\mathrm{C_w} > 0$ denotes a constant, whose exact value is fixed once for all by Proposition 2.1 below, and which depends only on V. Condition $\mathrm{WPI}_\varepsilon^{\mathrm{L}}(\delta)$ corresponds to an *inner matching* of the map with stationary fronts, it is only really meaningful if $\delta >> \varepsilon$. In the sequel we always assume that

$$\frac{\mathrm{L}}{2} \geq \delta \geq \alpha_1 \varepsilon, \tag{26}$$

[4]By local, we mean with respect to the interval $[-\mathrm{L}, \mathrm{L}]$. In contrast the related notion introduced in [6] is global on the whole of $\mathbb{R}$

where α_1 is larger than the α_0 of Theorem 1 and also sufficiently large so that if $\mathrm{WPI}_\varepsilon^{\mathrm{L}}(\delta)$ holds then the points a_k and the indices $i(k)$ and $\dagger_k$ are *uniquely* and therefore *unambigously* determined and the intervals I_k are disjoints. In particular, the quantity $d_{\min}^{\varepsilon,\mathrm{L}}(s)$, defined by

$$d_{\min}^{\varepsilon,\mathrm{L}}(s) := \min\left\{a_{k+1}^\varepsilon(s) - a_k^\varepsilon(s), \quad k = 1, \cdots, \ell(s) - 1\right\}$$

if $\ell(s) \geq 2$, and $d_{\min}^{\varepsilon,\mathrm{L}}(s) = 2\mathrm{L}$ otherwise, satisfies $d_{\min}^{\varepsilon,\mathrm{L}}(s) \geq 2\delta$. Condition $\mathrm{WPO}_\varepsilon^{\mathrm{L}}(\delta)$ is in some weak sense an *outer matching*: it is crucial for some of our energy estimates and its form is motivated by energy decay estimates for stationary solutions. Note that condition $\mathrm{WPI}_\varepsilon^{\mathrm{L}}(\delta)$ makes sense on its own, whereas condition $\mathrm{WPO}_\varepsilon^{\mathrm{L}}(\delta)$ only makes sense if condition $\mathrm{WPI}_\varepsilon^{\mathrm{L}}(\delta)$ is fullfilled. Note also that the larger δ is, the stronger condition $\mathrm{WPI}_\varepsilon^{\mathrm{L}}(\delta)$ is. The same is not obviously true for condition $\mathrm{WPO}_\varepsilon^{\mathrm{L}}(\delta)$, since the set of integration Ω_{L} increases with δ. As a matter of fact, the constant $\mathrm{C_w}$ in (25) is chosen sufficiently big[5] so that $\mathrm{WPO}_\varepsilon^{\mathrm{L}}(\delta)$ also becomes stronger when δ is larger. We next specify Definition 2 for the maps $x \mapsto \mathfrak{v}_\varepsilon(x, s)$.

Definition 3. *For $s \geq 0$, we say that the assumption $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta, s)$ (resp. $\mathrm{WPI}_\varepsilon^{\mathrm{L}}(\delta, s)$) holds if the map $x \mapsto \mathfrak{v}_\varepsilon(x, s)$ satisfies $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta)$ (resp. $\mathrm{WPI}_\varepsilon^{\mathrm{L}}(\delta)$).*

When assumption $\mathrm{WPI}_\varepsilon^{\mathrm{L}}(\delta, s)$ holds, then all symbols will be indexed according to s. In particular, we write[6] $\ell(s) = \ell$, $J(s) = J$, and $a_k^\varepsilon(s) = a_k$. The points $a_k^\varepsilon(s)$ for $k \in J(s)$, are now termed the *front points*. Whereas in [6] we are able, due to parabolic regularization, to establish under suitable conditions that $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta, s)$ is fulfilled for length of the same order as the minimal distance between the front points, this *is not the case* in the present situation. More precisely, two orders of magnitude for δ will be considered, namely

$$\delta_{\log}^\varepsilon = \frac{1}{\rho_{\mathrm{w}}}\varepsilon\left|\log\left(4M_0^2\frac{\varepsilon}{\mathrm{L}}\right)\right| \qquad \text{and} \qquad \delta_{\mathrm{loglog}}^\varepsilon = \frac{\omega}{\rho_{\mathrm{w}}}\varepsilon\log\left(\frac{1}{\rho_{\mathrm{w}}}\left|\log\left(4M_0^2\frac{\varepsilon}{\mathrm{L}}\right)\right|\right). \tag{27}$$

In (27), the constant ρ_{w} (given by Lemma 2.4 below) depends only on V. The main property for our purposes is that $\delta_{\mathrm{loglog}}^\varepsilon/\varepsilon$ and $\delta_{\log}^\varepsilon/\delta_{\mathrm{loglog}}^\varepsilon$ both tend to $+\infty$ as ε/L tends to 0.

In many places, it is useful to rely on a slightly stronger version of the confinement condition (22), which we assume to hold on some interval of time. More precisely, for positive L, S we consider the condition

$$(\mathcal{C}_{\mathrm{L},S}) \qquad\qquad \mathfrak{D}_\varepsilon(s) \cap \mathrm{I}_{4\mathrm{L}} \subset \mathrm{I}_{\mathrm{L}}, \qquad \forall\ 0 \leq s \leq S.$$

where the constant C_e is defined in Proposition 2 here below. For given $\mathrm{L}_0 > 0$ and $S > 0$, it follows easily from assumption (H_1) and Theorem 1 that there exists $\mathrm{L} \geq \mathrm{L}_0$ for which the first condition in $(\mathcal{C}_{\mathrm{L},S})$ is satisfied. Under condition $(\mathcal{C}_{\mathrm{L},S})$, the estimate

$$\mathcal{E}_\varepsilon(\mathfrak{v}_\varepsilon(s), \mathrm{I}_{3\mathrm{L}} \setminus \mathrm{I}_{\frac{3}{2}\mathrm{L}}) \leq \mathrm{C}_e\left(\frac{\varepsilon}{\mathrm{L}}\right)^\omega, \qquad \forall\ s \in [\varepsilon^\omega \mathrm{L}^2, S], \tag{28}$$

follows from the following regularizing effect, which was obtained in [5]:

[5] In view of $\mathrm{WPI}_\varepsilon^{\mathrm{L}}(\delta)$, how big it needs to be is indeed related to energy decay estimates for the fronts ζ_i.

[6] In principle and at this stage, all those symbols depend also upon ε. Since eventually ℓ and J will be ε-independent, at least for ε sufficiently small, we do not explicitly index them with ε.

Proposition 2 ([5]). *Let v_ε be a solution to $(PGL)_\varepsilon$, let $x_0 \in \mathbb{R}$, $r > 0$ and $0 \le s_0 < S$ be such that*

$$\mathfrak{v}_\varepsilon(y,s) \in B(\sigma_i, \mu_0) \quad \textit{for all} \quad (y,s) \in [x_0 - r, x_0 + r] \times [s_0, S], \tag{29}$$

for some $i \in \{1, \cdots, \mathfrak{q}\}$. Then we have for $s_0 < s \le S$

$$\varepsilon^{-\omega} \int_{x_0 - 3r/4}^{x_0 + 3r/4} e_\varepsilon(\mathfrak{v}_\varepsilon(x,s))\, dx \le \frac{1}{10} \mathrm{C}_e \left(1 + \left(\frac{\varepsilon^\omega r^2}{s - s_0}\right)^{\frac{\theta}{\theta - 1}}\right) \left(\frac{1}{r}\right)^\omega \tag{30}$$

as well as

$$|\mathfrak{v}_\varepsilon(y,s) - \sigma_i| \le \frac{1}{10} \mathrm{C}_e \varepsilon^{\frac{1}{\theta - 1}} \left(\left(\frac{1}{r}\right)^{\frac{1}{\theta - 1}} + \left(\frac{\varepsilon^\omega}{s - s_0}\right)^{\frac{1}{2(\theta - 1)}} \right), \tag{31}$$

for $y \in [x_0 - 3r/4, x_0 + 3r/4]$, where the constant $C > 0$ depends only on V.

Our first ingredient is

Proposition 3. *There exists $\alpha_1 > 0$, depending only on M_0 and V, such that if $L \ge \alpha_1 \varepsilon$ and if $(\mathcal{C}_{\mathrm{L},S})$ holds, then each subinterval of $[0, S]$ of length $\varepsilon^{\omega+2}(\mathrm{L}/\varepsilon)$ contains at least one time s for which $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta_{\log}^{\varepsilon}, s)$ holds.*

The idea behind Proposition 3 is that, $(\mathrm{PGL})_\varepsilon$ being a gradient flow, on a sufficiently large interval of time one may find some time where the dissipation of energy is small. Using elliptic tools, and viewing the time derivative as a forcing term, one may then establish property $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta_{\log}^{\varepsilon}, s)$ (see Section 2 and Section 3).

The next result expresses the fact that the equation preserves to some extent the well-preparedness assumption.

Proposition 4. *Assume that $(\mathcal{C}_{\mathrm{L},S})$ holds, that $\varepsilon^\omega \mathrm{L}^2 \le s_0 \le S$ is such that $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta_{\log}^{\varepsilon}, s_0)$ holds, and assume moreover that*

$$d_{\min}^{\varepsilon,\mathrm{L}}(s_0) \ge 16 \Big(\frac{\mathrm{L}}{\rho_0 \varepsilon}\Big)^{\frac{1}{\omega+2}} \varepsilon. \tag{32}$$

Then $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta_{\mathrm{loglog}}^{\varepsilon}, s)$ holds for all times $s_0 + \varepsilon^{2+\omega} \le s \le \mathcal{T}_0^\varepsilon(s_0)$, where

$$\mathcal{T}_0^\varepsilon(s_0) = \max\left\{ s \in [s_0 + \varepsilon^{2+\omega}, S] \quad s.t. \quad d_{\min}^{\varepsilon,\mathrm{L}}(s') \ge 8 \Big(\frac{\mathrm{L}}{\rho_0 \varepsilon}\Big)^{\frac{1}{\omega+2}} \varepsilon \quad \forall s' \in [s_0 + \varepsilon^{\omega+2}, s] \right\}.$$

For such s we have $J(s) = J(s_0)$ and for any $k \in J(s_0)$ we have $\sigma_{i(k \pm \frac{1}{2})}(s) = \sigma_{i(k \pm \frac{1}{2})}(s_0)$ and $\dagger_k(s) = \dagger_k(s_0)$.

Given a family of solution $(v_\varepsilon)_{0<\varepsilon<1}$, we introduce the additional condition

$$d_{\min}^{*}(s_0) \equiv \liminf_{\varepsilon \to 0} d_{\min}^{\varepsilon,\mathrm{L}}(s_0) > 0, \tag{33}$$

which makes sense if $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\alpha_1 \varepsilon, s_0)$ holds and expresses the fact that the fronts stay uniformly well-separated. The first step in our proofs, which is stated in Proposition 6 below, is to establish the conclusion of Theorem 2 under this stronger assumptions on the initial datum. From the inclusion (7) and Proposition 4 we will obtain:

Corollary 2. *Assume also that* $\mathcal{C}_{\mathrm{L},S}$ *holds, let* $s_0 \in [0, S]$ *and assume that* $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\alpha_1\varepsilon, s_0)$ *holds for all* ε *sufficiently small and that* (33) *is satisfied. Then, for* ε *sufficiently small,*

$$\mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta_{\mathrm{loglog}}^{\varepsilon}, s) \qquad \text{and} \qquad d_{\min}^{\varepsilon,\mathrm{L}}(s) \geq \frac{1}{2} d_{\min}^*(s_0) \tag{34}$$

are satisfied for any

$$s \in I^\varepsilon(s_0) \equiv \left[s_0 + 2\mathrm{L}^2\varepsilon^\omega, s_0 + \rho_0 \left(\frac{d_{\min}^*(s_0)}{8}\right)^{\omega+2}\right] \cap [0, S],$$

as well as the identities $J(s) = J(s_0)$, $\sigma_{i(k\pm\frac{1}{2})}(s) = \sigma_{i(k\pm\frac{1}{2})}(s_0)$ *and* $\dagger_k(s) = \dagger_k(s_0)$, *for any* $k \in J(s_0)$.

Hence, the collection of front points $\{a_k^\varepsilon(s)\}_{k\in J}$ is well-defined, and the approximating regularized fronts $\zeta_{i(k)}^{\dagger_k}$ do not depend on s (otherwise than through their position), on the full time interval $I^\varepsilon(s_0)$.

1.3 Paving the way to the motion law

As in [4], we use extensively the localized version of (1), a tool which turns out to be perfectly adapted to track the evolution of fronts. Let χ be an arbitrary smooth test function with compact support. Set, for $s \geq 0$,

$$\mathcal{I}_\varepsilon(s, \chi) = \int_{\mathbb{R}} e_\varepsilon\left(\mathfrak{v}_\varepsilon(x, s)\right) \chi(x) dx. \tag{35}$$

In integrated form the localized version of the energy identity writes

$$\mathcal{I}_\varepsilon(s_2, \chi) - \mathcal{I}_\varepsilon(s_1, \chi) + \int_{s_1}^{s_2} \int_{\mathbb{R}} \varepsilon^{1+\omega} \chi(x) |\partial_s \mathfrak{v}_\varepsilon(x, s)|^2 \, dx ds = \varepsilon^{-\omega} \int_{s_1}^{s_2} \mathcal{F}_S(s, \chi, v_\varepsilon) ds, \tag{36}$$

where the term $\mathcal{F}_S$ is given by

$$\mathcal{F}_S(s, \chi, \mathfrak{v}_\varepsilon) = \int_{\mathbb{R}\times\{s\}} \left(\left[\varepsilon \frac{\dot{\mathfrak{v}_\varepsilon}^2}{2} - \frac{V(\mathfrak{v}_\varepsilon)}{\varepsilon} \right] \ddot{\chi}(x) \right) dx \equiv \int_{\mathbb{R}\times\{s\}} \xi_\varepsilon(\mathfrak{v}_\varepsilon(\cdot, s)) \ddot{\chi} dx. \tag{37}$$

The last integral on the left hand side of identity (36) stands for local dissipation, whereas the right hand side second is a flux. The quantity ξ_ε is defined for a scalar function u by

$$\xi_\varepsilon(u) \equiv \varepsilon \frac{\dot{u}^2}{2} - \frac{V(u)}{\varepsilon}, \tag{38}$$

and is referred to as **the discrepancy term**. It is constant for solutions to the stationary equation $-u_{xx} + \varepsilon^{-2} V'(u) = 0$ on some given interval I and vanishes for finite energy solutions on $I = \mathbb{R}$. Notice that $|\xi_\varepsilon(u)| \leq e_\varepsilon(u)$. We set for two given times $s_2 \geq s_1 \geq 0$ and $\mathrm{L} \geq 0$

$$\mathrm{dissip}_\varepsilon^{\mathrm{L}}[s_1, s_2] = \varepsilon \int_{\mathrm{I}_{\frac{5}{3}\mathrm{L}} \times [s_1\varepsilon^{-\omega}, s_2\varepsilon^{-\omega}]} |\frac{\partial v_\varepsilon}{\partial t}|^2 dx dt = \varepsilon^{1+\omega} \int_{\mathrm{I}_{\frac{5}{3}\mathrm{L}} \times [s_1, s_2]} |\frac{\partial \mathfrak{v}_\varepsilon}{\partial s}|^2 dx ds. \tag{39}$$

Identity (36) then yields the estimate, if we assume that supp$\chi \subset \mathrm{I}_{\frac{5}{3}\mathrm{L}}$,

$$\left|\mathcal{I}_\varepsilon(s_2,\chi) - \mathcal{I}_\varepsilon(s_1,\chi) - \varepsilon^{-\omega}\int_{s_1}^{s_2} \mathcal{F}_S(s,\chi,v_\varepsilon)ds\right| \leq \mathrm{dissip}_\varepsilon^{\mathrm{L}}\,[s_1,s_2]\|\chi\|_{L^\infty(\mathbb{R})}. \tag{40}$$

We will show that under suitable assumptions, that the right hand side of (40) is small (see Step 3 in the proof of Proposition 6), so that the term $\varepsilon^{-\omega}\int_{s_1}^{s_2} \mathcal{F}_S(s,\chi,v_\varepsilon)ds$ provides a good approximation of $\mathcal{I}_\varepsilon(s_2,\chi) - \mathcal{I}_\varepsilon(s_1,\chi)$. On the other hand, it follows from the properties of regularized maps proved in Section 2.2 (see Proposition 2.1 there) that if $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta^\varepsilon_{\mathrm{loglog}}, s)$ holds then

$$\left|\mathcal{I}_\varepsilon(s,\chi) - \sum_{k\in J}\chi(a_k^\varepsilon(s))\mathfrak{S}_{i(k)}\right| \leq CM_0\left(\left(\frac{\varepsilon}{\delta^\varepsilon_{\mathrm{loglog}}}\right)^\omega\|\chi\|_\infty + \varepsilon\|\chi'\|_\infty\right), \tag{41}$$

where $\mathfrak{S}_{i(k)}$ stands for the energy of the corresponding stationary front. Set

$$\mathfrak{F}_\varepsilon(s_1,s_2,\chi) \equiv \varepsilon^{-\omega}\int_{s_1}^{s_2}\mathcal{F}_S(s,\chi,\mathfrak{v}_\varepsilon)ds \equiv \int_{s_1}^{s_2}\varepsilon^{-\omega}\xi_\varepsilon(\mathfrak{v}_\varepsilon(\cdot,s))\ddot{\chi}(\cdot)\,ds.$$

Combining (40) and (41) shows that, if $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta^\varepsilon_{\mathrm{loglog}}, s)$ holds for any $s \in (s_1,s_2)$, then we have

$$\begin{aligned} &|\sum_{k\in J}\left[\chi(a_k^\varepsilon(s_2)) - \chi(a_k^\varepsilon(s_1))\right]\mathfrak{S}_{i(k)} - \mathfrak{F}_\varepsilon(s_1,s_2,\chi)| \\ \leq\ & CM_0\left(\left(\log|\log\frac{\varepsilon}{\mathrm{L}}|\right)^{-\omega}\|\chi\|_\infty + \varepsilon\|\chi'\|_\infty\right) + \mathrm{dissip}_\varepsilon^{\mathrm{L}}\,[s_1,s_2]\|\chi\|_\infty. \end{aligned} \tag{42}$$

If the test function χ is choosen to be affine near a given front point a_{k_0} and zero near the other front points in the collection, then the first term on the left hand side yields a measure of the motion of a_{k_0} between times s_1 and s_2, whereas the second, namely $\mathfrak{F}_\varepsilon(s_1,s_2,\chi)$, is hence a good approximation of the measure of this motion, *provided we are able to estimate the dissipation* $\mathrm{dissip}_\varepsilon^{\mathrm{L}}\,(s_1,s_2)$. Our previous discussion suggests that

$$a_{k_0}^\varepsilon(s_2) - a_{k_0}^\varepsilon(s_1) \simeq \frac{1}{\chi'(a_{k_0}^\varepsilon)\mathfrak{S}_{i(k_0)}}\mathfrak{F}_\varepsilon(s_1,s_2,\chi).$$

It turns out that the computation of $\mathfrak{F}_\varepsilon(s_1,s_2,\chi)$ can be performed with satisfactory accuracy if the test function χ is affine (and hence as vanishing second derivatives) close to the front set, this is the object of the next subsections.

1.4 A first compactness result

A first step in deriving the motion law for the fronts is to obtain rough bounds from above for both $\mathrm{dissip}_\varepsilon^{\mathrm{L}}\,[s_1,s_2]$ and $\mathfrak{F}_\varepsilon(s_1,s_2,\chi)$. To obtain these, and under the assumptions of Corollary 2, notice that if $\ddot{\chi}$ vanishes on the set $\{a_k^\varepsilon(s_0)\}_{k\in J} + [-d^*_{\min}(s_0)/4, d^*_{\min}(s_0)/4]$, then from the inequality $|\xi_\varepsilon(u)| \leq e_\varepsilon(u)$, from Corollary 2 and from (30) of Proposition 2, we derive that for $s_1 \leq s_2$ in $I^\varepsilon(s_0)$,

$$|\mathfrak{F}_\varepsilon(s_1,s_2,\chi)| \leq Cd^*_{\min}(s_0)^{-\omega}\|\ddot{\chi}\|_{L^\infty(\mathbb{R})}(s_2-s_1). \tag{43}$$

Going back to (36), and choosing the test fucntion χ so that $\chi \equiv 1$ on $\mathrm{I}_{\frac{5}{3}\mathrm{L}}$ with compact support on $\mathrm{I}_{2\mathrm{L}}$, estimate (43) combined with (41) yields in turn a first rough upper bound on the dissipation $\mathrm{dissip}_\varepsilon^{\mathrm{L}}\,[s_1,s_2]$. Combining these estimates we will otain

Proposition 5. *Under the assumptions of Corollary 2, for* $s_1 \leq s_2 \in I^\varepsilon(s_0)$ *we have*

$$|a_k^\varepsilon(s_1) - a_k^\varepsilon(s_2)| \leq C \left(d^*_{\min}(s_0)^{-(\omega+1)} \ (s_2 - s_1) + M_0\big((\log|\log\frac{\varepsilon}{\mathrm{L}}|)^{-\omega} d^*_{\min}(s_0) + \varepsilon\big)\right). \quad (44)$$

As an easy consequence, we deduce the following compactness property, setting

$$I^*(s_0) = \Big(s_0, s_0 + \rho_0 \big(\frac{d^*_{\min}(s_0)}{8}\big)^{\omega+2}\Big) \cap (0, S).$$

Corollary 3. *Under the assumptions of Corollary 2, there exist a subsequence* $(\varepsilon_n)_{n\in\mathbb{N}}$ *converging to* 0 *such that for any* $k \in J$ *the function* $a_k^{\varepsilon_n}(\cdot)$ *converges uniformly on any compact interval of* $I^*(s_0)$ *to a lipschitz continuous function* $a_k(\cdot)$.

1.5 Refined estimates off the front set and the motion law

In order to derive the *precise motion law*, we have to provide an accurate asymptotic value for the discrepancy term *off the front set.* In other words, for a given index $k \in J$ we need to provide a uniform limit of the function $\varepsilon^{-\omega}\xi_\varepsilon$ near the points

$$a^\varepsilon_{k+\frac{1}{2}}(s) \equiv \frac{a^\varepsilon_k(s) + a^\varepsilon_{k+1}(s)}{2} \ \text{ and } a^\varepsilon_{k-\frac{1}{2}}(s) \equiv \frac{a^\varepsilon_{k-1}(s) + a^\varepsilon_k(s)}{2}.$$

We notice first that $\mathfrak{v}_\varepsilon$ takes values close to $\sigma_{i(k+\frac{1}{2})}$ near $a^\varepsilon_{k+\frac{1}{2}}(s)$. In view of estimate (30), we introduce the functions

$$\mathfrak{w}_\varepsilon(\cdot, s) = \mathfrak{w}^k_\varepsilon(\cdot, s) = \mathfrak{v}_\varepsilon - \sigma_{i(k+\frac{1}{2})} \text{ and } \mathfrak{W}_\varepsilon = \mathfrak{W}^k_\varepsilon \equiv \varepsilon^{-\frac{1}{\theta-1}} \mathfrak{w}^k_\varepsilon = \varepsilon^{-\frac{1}{\theta-1}} \left(\mathfrak{v}_\varepsilon - \sigma_{i(k+\frac{1}{2})}\right). \quad (45)$$

As a consequence of inequality (31) and Corollary 2 we have the uniform bound:

Lemma 2. *Under the assumptions of Corollary 2, we have*

$$|\mathfrak{W}_\varepsilon(x, s)| \leq C\big(d(x, s)\big)^{-\frac{1}{\theta-1}} \quad (46)$$

for any $x \in (a_k(s) + \delta^\varepsilon_{\mathrm{loglog}}, a_{k+1}(s) - \delta^\varepsilon_{\mathrm{loglog}})$ *and any* $s \in I^\varepsilon(s_0)$, *where we have set* $d(x, s) := \mathrm{dist}(x, \{a^\varepsilon_k(s), a^\varepsilon_{k+1}(s)\})$ *and where* $C > 0$ *depends only on* V *and* M_0. *Moreover, we also have*

$$\begin{cases} -\mathrm{sign}(\dagger_k)\mathfrak{W}_\varepsilon\left(a^\varepsilon_k(s) + \delta^\varepsilon_{\mathrm{loglog}}\right) \geq \dfrac{1}{C}\left(\delta^\varepsilon_{\mathrm{loglog}}\right)^{-\frac{1}{\theta-1}} \\ \mathrm{sign}(\dagger_{k+1})\mathfrak{W}_\varepsilon\left(a^\varepsilon_{k+1}(s) - \delta^\varepsilon_{\mathrm{loglog}}\right) \geq \dfrac{1}{C}\left(\delta^\varepsilon_{\mathrm{loglog}}\right)^{-\frac{1}{\theta-1}}. \end{cases} \quad (47)$$

We describe next on a formal level how to obtain the desired asymptotics for $\varepsilon^{-\omega}\xi_\varepsilon$, as $\varepsilon \to 0$, near the point $a_{k+\frac{1}{2}}(s)$. Going back to the limiting points $\{a_k(s)\}_{k\in J}$ defined in Proposition 5, we consider the subset of $\mathbb{R} \times \mathbb{R}^+$

$$\mathcal{V}_k(s_0) = \bigcup_{s\in I^*(s_0)} (a_k(s), a_{k+1}(s)) \times \{s\}. \quad (48)$$

It follows from the uniform bounds established in Lemma 2, that, passing possibly to a further subsequence, we may assume that

$$\mathfrak{W}_{\varepsilon_n} \rightharpoonup \mathfrak{W}_* \text{ in } L^p_{\mathrm{loc}}(\mathcal{V}_k(s_0)), \text{ for any } 1 \leq p < \infty.$$

On the other hand, thanks to estimate (46), for a given point $(x,s) \in \mathcal{V}_k(s_0)$ we expand $(\mathrm{PGL})_\varepsilon$ near (x,s) as

$$\varepsilon^\omega \frac{\partial \mathfrak{W}_\varepsilon}{\partial s} - \frac{\partial^2 \mathfrak{W}_\varepsilon}{\partial x^2} + 2\theta \lambda_{i(k+\frac{1}{2})} \mathfrak{W}_\varepsilon^{2\theta-1} = O(\varepsilon^{\frac{1}{\theta-1}}). \tag{49}$$

Passing to the limit $\varepsilon_n \to 0$, we expect that for every $s \in I^*(s_0)$, $\mathfrak{W}_*$ solves

$$\begin{cases} -\dfrac{\partial^2 \mathfrak{W}_*}{\partial x^2}(s,\cdot) + 2\theta \lambda_{i(k+\frac{1}{2})} \mathfrak{W}_*^{2\theta-1}(s,\cdot) = 0 \text{ on } (a_k(s), a_{k+1}(s)), \\ \mathfrak{W}_*(a_k(s)) = -\mathrm{sign}(\dagger_k)\infty \text{ and } \quad \mathfrak{W}_*(a_{k+1}(s)) = \mathrm{sign}(\dagger_k)\infty, \end{cases} \tag{50}$$

the boundary conditions being a consequence of the asymptotics (47). It turns out, in view of Lemma A.1 of the Appendix, that the *boundary value problem* (50) has a *unique* solution. By scaling, and setting $r_k(s) = \frac{1}{2}(a_{k+1}(s) - a_k(s))$, we obtain

$$\begin{cases} \mathfrak{W}_*(x,s) = \pm r_k(s)^{-\frac{1}{\theta-1}} \left(\lambda_{i(k+\frac{1}{2})}\right)^{-\frac{1}{2(\theta-1)}} \overset{\vee}{\mathfrak{u}}{}^{+} \left(\dfrac{x - a_{k+\frac{1}{2}}}{r_k(s)}\right), & \text{if } \dagger_k = -\dagger_{k+1}, \\ \mathfrak{W}_*(x,s) = \pm r_k(s)^{-\frac{1}{\theta-1}} \left(\lambda_{i(k+\frac{1}{2})}\right)^{-\frac{1}{2(\theta-1)}} \overset{\triangleright}{\mathfrak{u}} \left(\dfrac{x - a_{k+\frac{1}{2}}}{r_k(s)}\right), & \text{if } \dagger_k = \dagger_{k+1}, \end{cases}$$

where $\overset{\vee}{\mathfrak{u}}{}^{+}$ (resp. $\overset{\triangleright}{\mathfrak{u}}$) are the unique solutions to the problems

$$\begin{cases} -\mathcal{U}_{xx} + 2\theta \mathcal{U}^{2\theta-1} = 0 \qquad \text{on } (-1,+1), \\ \mathcal{U}(-1) = +\infty \text{ (resp. } \mathcal{U}(-1) = -\infty) \text{ and } \quad \mathcal{U}(+1) = +\infty. \end{cases} \tag{51}$$

Still on a formal level, we deduce therefore the corresponding values of the disprecancy

$$\begin{cases} \varepsilon^{-\omega} \xi_\varepsilon(\mathfrak{v}_\varepsilon) \simeq \xi(\mathfrak{W}_*) = -\lambda_{i(k+\frac{1}{2})}^{-\frac{1}{\theta-1}} r_k(s)^{-(\omega+1)} \mathcal{A}_\theta & \text{if } \dagger_k = -\dagger_{k+1}, \\ \varepsilon^{-\omega} \xi_\varepsilon(\mathfrak{v}_\varepsilon) \simeq \xi(\mathfrak{W}_*) = \lambda_{i(k+\frac{1}{2})}^{-\frac{1}{\theta-1}} r_k(s)^{-(\omega+1)} \mathcal{B}_\theta & \text{if } \dagger_k = \dagger_{k+1}, \end{cases} \tag{52}$$

where the numbers $\mathcal{A}_\theta$ and $\mathcal{B}_\theta$ are positive, depend only on θ, and correspond to the absolute value of the discrepancy of $\overset{\vee}{\mathfrak{u}}{}^{+}$ and $\overset{\triangleright}{\mathfrak{u}}$ respectively. Notice that the signs in (52) are different, the first case yields attractive forces whereas the second yields repulsive ones. Inserting this relation in (42) and arguing as for (44), we will derive the motion law.

The previous formal discussion can be put on a sound mathematical ground, relying on comparison principles and the construction of appropriate upper and lower solutions (see Section 5). This leads to the central result of this paper:

Proposition 6. *Assume that conditions* (H_0) *and* (H_1) *are fulfilled. Let* $0 < S < S_{\max}$ *be given and set*

$$\mathrm{L}_0 := 3\max\left\{|a_k^0|,\ 1 \le k \le \ell_0\,;\ (\frac{S}{\rho_0})^{\frac{1}{\omega+2}}\right\}.$$

Assume that $\mathrm{WPI}_\varepsilon^{\mathrm{L}_0}(\alpha_1\varepsilon, 0)$ *holds as well as* (33) *at time* $s = 0$. *Then* $J(s) = \{1, \cdots, \ell_0\}$ *and the functions* $a_k^\varepsilon(\cdot)$ *are well defined and converge uniformly on any compact interval of* $(0, S)$ *to the solution* $a_k(\cdot)$ *of* $(\mathcal{S})$ *supplemented with the initial condition* $a_k(0) = a_k^0$.

Notice that the combination of assumptions $\mathrm{WPI}_\varepsilon^{\mathrm{L}}(\alpha_1\varepsilon, 0)$, (H_1) and (33) at $s = 0$ implies the multiplicity one condition $(\mathrm{H}_{\min})$. Whereas the conclusion of Proposition 6 is similar to the one of Theorem 2, the assumptions of Proposition 6 are more restrictive. Indeed, on one hand we assume the well-preparedness condition $\mathrm{WPI}_\varepsilon^{\mathrm{L}}$, and on the other hand we impose (33) which is far more constraining than $(\mathrm{H}_{\min})$: it excludes in particular the possibility of having small pairs of fronts and anti-fronts. Our next efforts are hence devoted to handle this type of situation: Proposition 6, through rescaling arguments, will nevertheless be the main building block for that task.

In order to prove Theorem 2, 3 and 4 we need to relax the assumptions on the initial data, in particular we need to analyze the behavior of data with small pairs of *fronts and anti-fronts*, and show that they are going to annihilate on a short interval of time. For that purpose we will consider the following situation, corresponding to confinement of the front set at initial time. Assume that for a collection of points $\{b_q^\varepsilon\}_{q\in J_0}$ in $\mathbb{R}$ we have

$$\mathfrak{D}_\varepsilon(0) \cap \mathrm{I}_{5\mathrm{L}} \subset \bigcup_{q\in J_0} [b_q^\varepsilon - r, b_q^\varepsilon + r] \subset \mathrm{I}_{\kappa_0\mathrm{L}} \qquad \text{and} \qquad b_p^\varepsilon - b_q^\varepsilon \geq 3R \qquad \text{for } p \neq q \in J_0, \tag{53}$$

for some $\kappa_0 \leq \frac{1}{2}$ and $\alpha_1\varepsilon \leq r \leq R/2 \leq \mathrm{L}/4$. It follows from (7) that if $0 \leq s \leq \rho_0(R-r)^{\omega+2}$ then

$$\mathfrak{D}_\varepsilon(s) \cap \mathrm{I}_{4\mathrm{L}} \subset \bigcup_{k\in J_0} (b_k^\varepsilon - R, b_k^\varepsilon + R) \subset \mathrm{I}_{2\kappa_0\mathrm{L}}, \quad \text{where the union is disjoint.}$$

Consider next $0 \leq s \leq \rho_0(R-r)^{\omega+2}$ such that $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\alpha_1\varepsilon, s)$ holds, so that the front points $\{a_k^\varepsilon(s)\}_{k\in J(s)}$ are well-defined. For $q \in J_0$, consider $J_q(s) = \{k \in J(s), a_k^\varepsilon(s) \in (b_q^\varepsilon - R, b_q^\varepsilon + R)\}$, set $\ell_q = \sharp J_q$, and write $J_q(s) = \{k_q, k_{q+1}, \cdots, k_{q+\ell_q-1}\}$, where $k_1 = 1$, and $k_q = \ell_1 + \cdots + \ell_{q-1} + 1$, for $q \geq 2$. Our next result shows that, after a small time, only the repulsive forces survive at the scale given by r, provided the different lengths are sufficiently distinct.

Proposition 7. *There exists positive constants α_* and ρ_*, depending only on V and M_0, such that if* (53) *holds and*

$$\kappa_0^{-1} \geq \alpha_*, \qquad r \geq \alpha_*\varepsilon\left(\frac{\mathrm{L}}{\varepsilon}\right)^{\frac{2}{\omega+2}}, \qquad R \geq \alpha_* r, \tag{54}$$

then at time

$$s_r = \rho_* r^{\omega+2}$$

condition $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\alpha_1\varepsilon, s_r)$ holds and, for any $q \in J_0$ and any $k, k' \in J_q(s_r)$ we have $\dagger_k(s_r) = \dagger'_k(s_r)$ or equivalently for any $k \in J_q(s_r) \setminus \{k_q(s_r) + \ell_q(s_r) - 1\}$, we have

$$\epsilon_{k+\frac{1}{2}}(s_r) = \dagger_k(s_r)\,\dagger_{k+1}(s_r) = +1. \tag{55}$$

Moreover, we have

$$d_{\min}^{\varepsilon,\mathrm{L}}(s_r) \geq r, \tag{56}$$

and if $\sharp J_q(s_r) \leq 1$ for every $q \in J_0$, then we actually have $d_{\min}^{\varepsilon,\mathrm{L}}(s_r) \geq R$.

The proofs of Theorems 2, 3 and 4 are then deduced from Propositions 6 and 7.

The paper is organized as follows. We describe in Section 2 some properties of stationary fronts, as well as for solutions to some perturbations of the stationary equations. In Section 3 we describe several properties related to the well-preparedness assumption $\mathcal{WP}_\varepsilon^{\rm L}$, in particular the quantization of the energy, how it relates to dissipation, and its numerous implications for the dynamics. We provide in particular the proofs to Proposition 3, Proposition 4 and Corollary 2. In Section 4, we prove the compactness results stated in Proposition 5 and Corollary 3. Section 5, provides an expansion of the discrepancy term off the front set, from a technical point of view it is the place where the analysis differs most from the non-degenerate case. Based on this analysis, we show in Section 6 how the motion law follows from prepared datas establishing the proof to Proposition 6. In Section 7 we analyze the clearing-out of small pairs of front-antifront and more generally we present the proof of Proposition 7. Finally, in section 8 we present the proofs of the main theorems, namely Theorem 2, 3 and 4. Several results concerning the first or second order differential equations involved in the analysis of this paper are given in separate appendices, in particular the proof of Proposition 1.

2 Remarks on stationary solutions

2.1 Stationary solutions on $\mathbb{R}$ with vanishing discrepancy

Stationary solutions are described using the method of separation of variable. For u solution to (21), we multiply (21) by u and verify that ξ is constant. We restrict ourselves to solutions with vanishing discrepancy

$$\xi = \frac{1}{2}\dot{u}^2 - V(u) = 0, \tag{2.1}$$

and solve equation (2.1) by separation of variables. Let γ_i be defined on (σ_i, σ_{i+1}) by

$$\gamma_i(u) = \int_{z_i}^{u} \frac{ds}{\sqrt{2V(s)}}, \text{for } u \in (\sigma_i, \sigma_{i+1}), \tag{2.2}$$

where we recall that z_i is a fixed maximum point of V in the interval (σ_i, σ_{i+1}). The map γ_i is one-to-one from (σ_i, σ_{i+1}) to $\mathbb{R}$, so that we may define its inverse map $\zeta_i^+ : \mathbb{R} \to (\sigma_i, \sigma_{i+1})$ by

$$\zeta_i^+(x) = \gamma_i^{-1}(x) \text{ as well as } \zeta_i^-(x) = \gamma_i^{-1}(-x) \text{ for } x \in \mathbb{R}. \tag{2.3}$$

In view of the definition (2.3), we have $\zeta_i^{\pm}(0) = z_i$, $\zeta_i^{+\prime}(0) = \sqrt{2V(z_i)} > 0$, whereas a change of variable shows that ζ_i has finite energy given by the formula (8). We verify that $\zeta_i^+\left(\frac{\cdot}{\varepsilon}\right)$ and $\zeta_i^-\left(\frac{\cdot}{\varepsilon}\right)$ solve (2.1) and hence (21). The next elementary result then directly follows from uniqueness in ode's:

Lemma 2.1. *Let u be a solution to* (21) *such that* (2.1) *holds, and such that $u(x_0) \in (\sigma_i, \sigma_{i+1})$, for some $x_0 \in \mathbb{R}$, and some $i \in 1, \cdots \mathfrak{q} - 1$. Then, there exists $a \in \mathbb{R}$ such that $u(x) = \zeta_i^+(x-a)$ or $u(x) = \zeta_i^-(x-a)$, $\forall x \in \mathbb{R}$.*

We provide a few simple properties of the functions $\zeta_i^{\pm}$ which enter directly in our arguments. We expand V near σ_i for $u \geq \sigma_i$ as

$$\sqrt{V(u)} = \sqrt{\lambda_i}(u - \sigma_i)^{\theta}(1 + O(u - \sigma_i)), \quad \text{as } u \to \sigma_i.$$

Integrating, we are led to the expansion

$$\gamma_i(u) = -\frac{\theta - 1}{\sqrt{2\lambda_i}}(u - \sigma_i)^{-\theta+1}(1 + O(u - \sigma_i)), \quad \text{as } u \to \sigma_i,$$

and therefore also to the expansions

$$\zeta_i^{\pm}(x) = \sigma_i + \left(\frac{\sqrt{2\lambda_i}|x|}{\theta-1}\right)^{-\frac{1}{\theta-1}}(1+o(1)), \quad \text{as } x \to \mp\infty.$$

Similarly,

$$\zeta_i^{\pm}(x) = \sigma_{i+1} - \left(\frac{\sqrt{2\lambda_{i+1}}|x|}{\theta-1}\right)^{-\frac{1}{\theta-1}}(1+o(1)), \quad \text{as } x \to \pm\infty,$$

and corresponding asymptotics for the derivatives can be derived as well (e.g. using the fact that the discrepancy is zero).

For $0<\varepsilon<1$ given, and $i=1,\cdots,q-1$, consider the scaled function $\zeta_{i,\varepsilon}^{\pm} = \zeta_i^{\pm}\left(\frac{\cdot}{\varepsilon}\right)$ which is a solution to

$$-u_{xx} + \varepsilon^{-2}V'(u) = 0,$$

hence a stationary solution to $(\mathrm{PGL})_\varepsilon$. Straightforward computations based on the previous expansions show that

$$\begin{cases} e_\varepsilon\left(\zeta_{i,\varepsilon}^{\pm}\right)(x) = (2\lambda_i)^{-\frac{1}{\theta-1}}(\theta-1)^{\frac{2\theta}{\theta-1}}\frac{1}{\varepsilon}\left|\frac{x}{\varepsilon}\right|^{-(\omega+1)} + \underset{\frac{x}{\varepsilon}\to\mp\infty}{o}\left(\frac{1}{\varepsilon}\left|\frac{x}{\varepsilon}\right|^{-(\omega+1)}\right) \\ e_\varepsilon\left(\zeta_{i,\varepsilon}^{\pm}\right)(x) = (2\lambda_{i+1})^{-\frac{1}{\theta-1}}(\theta-1)^{\frac{2\theta}{\theta-1}}\frac{1}{\varepsilon}\left|\frac{x}{\varepsilon}\right|^{-(\omega+1)} + \underset{\frac{x}{\varepsilon}\to\pm\infty}{o}\left(\frac{1}{\varepsilon}\left|\frac{x}{\varepsilon}\right|^{-(\omega+1)}\right) \end{cases} \tag{2.4}$$

with ω defined in (5). Hence there is some constant $C>0$ independent of r and ε such that

$$\mathfrak{S}_i \geq \int_{-r}^{r} e_\varepsilon\left(\zeta_{i,\varepsilon}^{\pm}\right)dx \geq \mathfrak{S}_i - C\left(\frac{\varepsilon}{r}\right)^{\omega}. \tag{2.5}$$

2.2 On the energy of chains of stationary solutions

If u satisfies condition $\mathrm{WPI}_\varepsilon^{\mathrm{L}}(\delta)$ and (H_0), we set

$$\mathfrak{E}_\varepsilon^{\mathrm{L}}(u) = \sum_{k\in J}\mathfrak{S}_{i(k)} \text{ and } \mathcal{E}_\varepsilon^{\mathrm{L}}(u) = \int_{\mathrm{I}_{2\mathrm{L}}} e_\varepsilon(u(x))dx. \tag{2.6}$$

Proposition 2.1. *We have*

$$\begin{cases} \mathcal{E}_\varepsilon^{\mathrm{L}}(u) \geq \mathfrak{E}_\varepsilon^{\mathrm{L}}(u) - \mathrm{C_f}M_0\left(\frac{\varepsilon}{\delta}\right)^{\omega} & \text{if } \mathrm{WPI}_\varepsilon^{\mathrm{L}}(\delta) \text{ holds,} \\ \mathcal{E}_\varepsilon^{\mathrm{L}}(u) \leq \mathfrak{E}_\varepsilon^{\mathrm{L}}(u) + (\mathrm{C_w}+\mathrm{C_f})M_0\left(\frac{\varepsilon}{\delta}\right)^{\omega} & \text{if } \mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta) \text{ holds.} \end{cases} \tag{2.7}$$

Moreover, for any smooth function χ with compact support in $\mathrm{I}_{2\mathrm{L}}$ we have

$$\left|\mathcal{I}_\varepsilon(\chi) - \sum_{k\in J}\chi(a_k)\mathfrak{S}_{i(k)}\right| \leq (\mathrm{C_w}+\mathrm{C_f})M_0\left(\left(\frac{\varepsilon}{\delta}\right)^{\omega}\|\chi\|_\infty + \varepsilon\|\chi'\|_\infty\right), \qquad \textit{if } \mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta) \textit{ holds,} \tag{2.8}$$

where $\mathcal{I}_\varepsilon(\chi) = \int_{\mathrm{I}_{2\mathrm{L}}} e_\varepsilon(u)\chi(x)dx$. The constant $\mathrm{C_f}$ which appears in (2.7) and (2.8) only depends on V, and the constant $\mathrm{C_w}$ appears in the definition of condition $\mathcal{WP}_\varepsilon^{\mathrm{L}}$.

Proof. We estimate the integral of $|e_\varepsilon(u) - e_\varepsilon(\zeta^{\dagger_k}_{i(k)}(\cdot - a_k))|$ on I_k as

$$\frac{\varepsilon}{2}\int_{I_k} |\dot{u}^2 - (\dot{\zeta}^{\dagger_k}_{i(k),\varepsilon}(\cdot - a_k))^2| dx \leq \varepsilon \|\dot{u} - \dot{\zeta}^{\dagger_k}_{i(k),\varepsilon}(\cdot - a_k)\|_{L^\infty(I_k)} \left[\mathcal{E}_\varepsilon(u)^{\frac{1}{2}} + \mathcal{E}_\varepsilon(\zeta^{\dagger_k}_{i(k),\varepsilon})^{\frac{1}{2}}\right]\sqrt{\frac{\delta}{\varepsilon}}$$

and likewise we obtain

$$\varepsilon^{-1}\int_{I_k} |V(u) - V(\zeta^{\dagger_k}_{i(k),\varepsilon}(\cdot - a_k))| \leq C\frac{\delta}{\varepsilon}\|u - \zeta^{\dagger_k}_{i(k),\varepsilon}(\cdot - a_k)\|_{L^\infty(I_k)}.$$

It suffices then to invoke $\mathrm{WPI}^{\mathrm{L}}_\varepsilon(\delta)$ and $\mathrm{WPO}^{\mathrm{L}}_\varepsilon(\delta)$ as well as the decay estimates (2.5) to derive (2.7), using the fact that since $\delta \geq \alpha_1\varepsilon$, negative exponentials are readily controlled by negative powers. Estimate (2.8) is derived in a very similar way, the error in $\varepsilon\|\chi'\|_\infty$ being a consequence of the approximation of $\int \chi e_\varepsilon(\zeta^{\dagger_k}_{i(k),\varepsilon}(\cdot - a_k))$ by $\chi(a_k)\mathfrak{S}_{i(k)}$. □

This result shows that, if δ is sufficiently large, the energy is close to a set of discrete values, namely the finite sums of $\mathfrak{S}_k$. We will therefore refer to this property as *the quantization of the energy*, it will play an important role later when we will obtain estimates on the dissipation rate of energy.

2.3 Study of the perturbed stationary equation

Consider a function u defined on $\mathbb{R}$ satisfying the perturbed differential equation

$$u_{xx} = \varepsilon^{-2}V'(u) + f, \tag{2.9}$$

where $f \in L^2(\mathbb{R})$, and the energy bound (H_0). We already know, thanks to Lemma 2.1 that if $f = 0$ then u is of the form $\zeta^{\pm}_{i,\varepsilon}(\cdot - a)$. Our results below, summarized here in loose terms, show that if f is sufficiently small on some sufficiently large interval, then u is close to a chain of translations of the functions $\zeta^{\pm}_{i,\varepsilon}$ suitably glued together on that interval.

Following the approach of [4], we first recast equation (2.9) as a system of two differential equations of first order. For that purpose, we set $w = \varepsilon u_x$ so that (2.9) is equivalent to the system

$$u_x = \frac{1}{\varepsilon}w \text{ and } w_x = \frac{1}{\varepsilon}V'(u) + \varepsilon f,$$

which we may write in a more condensed form as

$$U_x = \frac{1}{\varepsilon}G(U) + \varepsilon F \text{ on } \mathbb{R}, \tag{2.10}$$

where we have set $U(x) = (u(x), w(x))$ and $F(x) = (0, f(x))$, and where G denotes the vector field $G(u, w) = (w, V'(u))$. Notice that the energy bound (H_0) and assumption (A_3) together imply a global L^∞ bound on u. In turn, this L^∞ bound imply a Lipschitz bound, denoted C_0, for the nonlinearity $G(u, w)$.

Lemma 2.2. *Let u_1 and u_2 satisfy* (2.9) *with forcing terms f_1 and f_2, and assume that both satisfy the energy bound (H_0). Denote by U_1, U_2, F_1, F_2 the corresponding solutions and forcing terms of* (2.10)*. Then, for any x, x_0 in some arbitrary interval I,*

$$|(U_1 - U_2)(x)| \leq \left(|(U_1 - U_2)(x_0)| + \frac{\varepsilon^{\frac{3}{2}}}{\sqrt{2C_0}}\|F_1 - F_2\|_{L^2(I)}\right)\exp\left(\frac{C_0|x - x_0|}{\varepsilon}\right). \tag{2.11}$$

Proof. Since $(U_1 - U_2)_x = G(U_1) - G(U_2) + \varepsilon(F_1 - F_2)$ we obtain the inequality

$$|(U_1 - U_2)_x| \leq \frac{C_0}{\varepsilon}|U_1 - U_2| + \varepsilon|F_1 - F_2|.$$

It follows from Gronwall's inequality that

$$|(U_1 - U_2)(x)| \leq \exp\left(\tfrac{C_0|x-x_0|}{\varepsilon}\right)|(U_1 - U_2)(x_0)| + |\int_{x_0}^{x} \varepsilon|(F_1 - F_2)(y)|\exp\left(\tfrac{C_0|y-x_0|}{\varepsilon}\right)dy|.$$

Claim (2.11) then follows from the Cauchy-Schwarz inequality. □

We will combine the previous lemma with

Lemma 2.3. *Let u be a solution of (2.9) satisfying (H_0). Then*

$$\sup_{x,y\in I} |\xi_\varepsilon(u)(x) - \xi_\varepsilon(u)(y)| \leq \sqrt{2M_0}\varepsilon^{\frac{1}{2}}\|f\|_{L^2(I)},$$

where $I \subset \mathbb{R}$ is an arbitrary interval.

Proof. This is a direct consequence of the equality $\frac{d}{dx}\xi_\varepsilon(u) = \varepsilon f \frac{d}{dx}u$, the Cauchy-Schwarz inequality, and the definition of the energy. □

Lemma 2.4. *Let u be a solution of (2.9) satisfying (H_0). Let $L > 0$ and assume that*

$$\mathcal{D}(u) \cap \mathrm{I}_{2\mathrm{L}} \subseteq \mathrm{I}_{\mathrm{L}}.$$

There exist a constant $0 < \kappa_{\mathrm{w}} < 1$, depending only on V, such that if

$$M_0\frac{\varepsilon}{\mathrm{L}} + M_0^{\frac{1}{2}}\varepsilon^{\frac{3}{2}}\|f\|_{L^2(\mathrm{I}_{\frac{3}{2}\mathrm{L}})} \leq \kappa_{\mathrm{w}}, \tag{2.12}$$

then the condition $\mathrm{WPI}^{\mathrm{L}}_\varepsilon(\delta)$ holds where

$$\frac{\delta}{\varepsilon} := -\frac{2}{\rho_{\mathrm{w}}}\log\left(M_0\frac{\varepsilon}{\mathrm{L}} + M_0^{\frac{1}{2}}\varepsilon^{\frac{3}{2}}\|f\|_{L^2(\mathrm{I}_{\frac{3}{2}\mathrm{L}})}\right), \tag{2.13}$$

and where the constant ρ_{w} depends only on M_0 and V. Moreover, κ_{w} is sufficiently small so that $2|\log\kappa_{\mathrm{w}}|/\rho_{\mathrm{w}} \geq \alpha_1$, where α_1 was defined in (26).

Proof. If $\mathcal{D}(u) \cap \mathrm{I}_{2\mathrm{L}} = \emptyset$ then there is nothing to prove. If not, we first claim that there exist a point $a_1 \in \mathrm{I}_{\mathrm{L}}$ such that $u(a_1) = z_{i(1)}$ for some $i(1) \in \{1, \cdots, \mathfrak{q} - 1\}$. Indeed, if not, and since the endpoints of $\mathrm{I}_{2\mathrm{L}}$ are not in the front set, the function u would have a critical point with a critical value in the complement of $\cup_j B(\sigma_j, \mu_0)$. At that point, the discrepancy would therefore be larger than C/ε for some constant $C > 0$ depending only of V (through the choice of μ_0). On the other hand, since $|\xi_\varepsilon| \leq e_\varepsilon$, by averaging there exist at least one point in $\mathrm{I}_{\frac{3}{2}\mathrm{L}}$ where the discrepancy of u is smaller in absolute value than $M_0/(3\mathrm{L})$. Combined with the estimate of Lemma 2.3 on the oscillation of the discrepancy, we hence derive our first claim, provided κ_{w} in (2.12) is choosen sufficiently small. Wet set $\dagger_1 = \mathrm{sign}(u'(a_1))$, $u_1 = u$ and $u_2 = \zeta^{\dagger_1}_{i(1),\varepsilon}(\cdot - x_1))$. Since

$$V(u_1(a_1)) = V(u_2(a_1)) = V(z_{i(1)}),$$

and since

$$|\xi_\varepsilon(u_1)(a_1) - \xi_\varepsilon(u_2)(a_1)| = |\xi_\varepsilon(u_1)(a_1)| \leq M_0/(3\mathrm{L}) + \sqrt{2M_0}\varepsilon^{\frac{1}{2}}\|f\|_{L^2(\mathrm{I}_{\frac{3}{2}\mathrm{L}})},$$

we obtain

$$\left|\varepsilon(u_1')^2(a_1) - \varepsilon(u_2')^2(a_1)\right| \leq M_0/(\mathrm{L}) + 2\sqrt{2M_0}\varepsilon^{\frac{1}{2}}\|f\|_{L^2(\mathrm{I}_{\frac{3}{2}\mathrm{L}})}.$$

Since also

$$|u_1'(a_1) + u_2'(a_1)| \geq |u_2'(a_1)| = |\sqrt{\frac{2V(z_{i(1)})}{\varepsilon^2}}| \geq C/\varepsilon,$$

it follows that

$$\left|\varepsilon(u_1' - u_2')(a_1)\right| \leq C\left(M_0\frac{\varepsilon}{\mathrm{L}} + \sqrt{M_0}\varepsilon^{\frac{3}{2}}\|f\|_{L^2(\mathrm{I}_{\frac{3}{2}\mathrm{L}})}\right),$$

for a constant $C > 0$ which depends only on V. We may then apply Lemma 2.2 to u_1 and u_2 with the choice $x_0 = a_1$, and for which we thus have, with the notations of Lemma 2.2,

$$|(U_1 - U_2)(x_0)| \leq C\left(M_0\frac{\varepsilon}{\mathrm{L}} + \sqrt{M_0}\varepsilon^{\frac{3}{2}}\|f\|_{L^2(\mathrm{I}_{\frac{3}{2}\mathrm{L}})}\right).$$

Estimate (2.11) then yields (24) on $I_1 = [a_1 - \delta, a_1 + \delta]$, for the choice of δ given by (2.13) with $\rho_{\mathrm{w}} = 4(C_0 + 1)$, where C_0 depends only on M_0 and V and was defined above Lemma 2.2.

If $\mathcal{D}(u) \cap (\mathrm{I}_{\frac{3}{2}\mathrm{L}} \setminus [a_1 - \delta, a_1 + \delta]) = \emptyset$, we are done, and if not we may repeat the previous construction (the boundary points of $[a_1 - \delta, a_1 + \delta]$ are not part of the front set), until after finitely many steps we cover the whole front set. □

We turn to the outer condition[7] $\mathrm{WPO}_\varepsilon^{\mathrm{L}}$.

Lemma 2.5. *Let u be a solution of* (2.9) *verifying* (H_0), *and assume that for some index $i \in \{1, \cdots, \mathfrak{q}\}$*

$$u(x) \in B(\sigma_i, \mu_0) \quad \forall x \in A,$$

where A is some arbitrary bounded interval. Set $R = \mathrm{length}(A)$, let $0 < \rho < R$, and set $B = \{x \in A \mid \mathrm{dist}(x, A^c) > \rho\}$. Then we have the estimate

$$\mathcal{E}_\varepsilon(u, B) \leq \mathrm{C_o}\left(\mathcal{E}_\varepsilon(u, A \setminus B)^{\frac{1}{\theta}}\left(\tfrac{\varepsilon}{\rho}\right)^{1+\frac{1}{\theta}} + R^{\frac{3}{2}}M_0^{\frac{1}{2\theta}}\left(\tfrac{\varepsilon}{R}\right)^{1+\frac{1}{2\theta}}\|f\|_{L^2(A)}\right),$$

where the constant $\mathrm{C_o}$ depends only on V.

Proof. Let $0 \leq \chi \leq 1$ be a smooth cut-off function with compact support in A and such that $\chi \equiv 1$ on B and $|\chi'| \leq 2/\rho$ on A. We multiply (2.9) by $\varepsilon(u - \sigma_i)\chi^2$ and integrate on A. This leads to

$$\int_A \varepsilon u_x^2\chi^2 + \frac{1}{\varepsilon}V'(u)(u - \sigma_i)\chi^2 = \int_{A\setminus B} 2\varepsilon u_x(u - \sigma_i)\chi\chi' - \int_A \varepsilon f(u - \sigma_i)\chi^2.$$

[7] for which several adaptations have to be carried out compared to the non-degenerate case.

We estimate the first term on the right-hand side above by

$$
\begin{aligned}
\Big|\int_{A\setminus B} 2\varepsilon u_x(u-\sigma_i)\chi\chi'\Big| &\le \Big(\int_A \varepsilon u_x^2\chi^2\Big)^{\frac12}\Big(\int_{A\setminus B}\varepsilon^\theta(u-\sigma_i)^{2\theta}\Big)^{\frac{1}{2\theta}}\Big(\int_{A\setminus B}|2\chi'|^{\frac{2\theta}{\theta-1}}\Big)^{\frac{\theta-1}{2\theta}}\\
&\le \frac12\int_A \varepsilon u_x^2\chi^2+\frac12\varepsilon^{1+\frac1\theta}\Big(\int_{A\setminus B}\frac{2}{\lambda_i}e_\varepsilon(u)\Big)^{\frac1\theta}\left(\frac{4}{\rho}\right)^2(2\rho)^{\frac{\theta-1}{\theta}}\\
&\le \frac12\int_A u_x^2\chi^2+16\lambda_i^{-\frac1\theta}\left(\frac{\varepsilon}{\rho}\right)^{1+\frac1\theta}\mathcal{E}_\varepsilon(u,A\setminus B)^{\frac1\theta},
\end{aligned}
$$

where we have used (2) and the fact that length$(A\setminus B)=2\rho$. Similarly we estimate

$$
\begin{aligned}
\Big|\int_A \varepsilon f(u-\sigma_i)\chi^2\Big| &\le \varepsilon\|f\|_{L^2(A)}\Big(\int_A (u-\sigma_i)^{2\theta}\Big)^{\frac{1}{2\theta}}R^{\frac{\theta-1}{2\theta}}\\
&\le \varepsilon^{1+\frac{1}{2\theta}}\|f\|_{L^2(A)}\Big(\frac{2}{\lambda_i}\Big)^{-1}M_0^{\frac{1}{2\theta}}R^{\frac{\theta-1}{2\theta}}.
\end{aligned}
$$

Also, by (2) we have

$$\int_A \frac1\varepsilon V'(u)(u-\sigma_i)\chi^2\ge\theta\int_B\frac1\varepsilon V(u).$$

Combining the previous inequalities the conclusion follows. □

Combining Lemma 2.4 with Lemma 2.5 we obtain

Proposition 2.2. *Let u be a solution to* (2.9) *satisfying assumption* (H_0)*, and such that* $\mathcal{D}(u)\cap \mathrm{I}_{3\mathrm{L}}\subset \mathrm{I}_{\mathrm{L}}$. *There exist positive constants*[8] $\mathrm{C_w}$ *and* α_1, *depending only on* M_0 *and* V, *such that if* $\alpha\ge\alpha_1$ *and if*

1. $M_0\dfrac{\varepsilon}{\mathrm{L}}\le\dfrac12\exp(-\frac{\rho_{\mathrm w}}{2}\alpha)$,
2. $\|f\|_{L^2(\mathrm{I}_{3\mathrm{L}})}\le\dfrac12 M_0^{-\frac12}\varepsilon^{-\frac32}\exp(-\frac{\rho_{\mathrm w}}{2}\alpha)$,
3. $\|f\|_{L^2(\mathrm{I}_{3\mathrm{L}})}\le\dfrac{\mathrm{C_w}}{2\mathrm{C_o}}M_0^{1-\frac{1}{2\theta}}\left(\dfrac{\varepsilon}{\mathrm{L}}\right)^{-1-\frac{1}{2\theta}}\mathrm{L}^{-\frac32}\alpha^{-\omega}$,

then $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\alpha\varepsilon)$ *holds.*

Proof. Direct substitution shows that assumptions *1.* and *2.* imply condition (2.12), provided α_1 is choosen sufficiently large, and also imply condition $\mathrm{WPI}_\varepsilon^{\mathrm{L}}(\delta)$ for some $\delta\ge\alpha\varepsilon$ given by (2.13). It remains to consider $\mathrm{WPO}_\varepsilon^{\mathrm{L}}(\alpha\varepsilon)$. We invoke Lemma 2.5 on each of the intervals $A=(a_k+\frac12\alpha\varepsilon, a_{k+1}-\frac12\alpha\varepsilon)$, taking $B=(a_k+\alpha\varepsilon,a_{k+1}-\alpha\varepsilon)$. In view of $\mathrm{WPI}_\varepsilon^{\mathrm{L}}(\alpha\varepsilon)$ and (2.5), we obtain

$$\mathcal{E}_\varepsilon(u,A\setminus B)\le C\alpha^{-\omega},$$

and therefore

$$\mathcal{E}_\varepsilon(u,A\setminus B)^{\frac1\theta}\alpha^{-1-\frac1\theta}\le C\alpha^{-\omega},$$

[8]Recall that $\mathrm{C_w}$ enters in the definition of condition $\mathcal{WP}_\varepsilon^{\mathrm{L}}$. A parameter named $\mathrm{C_w}$ already appears in the statement of Proposition 2.1 above: We impose that its updated value here is be larger han its original value in Proposition 2.1 (and Proposition 2.1 remains of course true with this updated value!).

where C depends only on V. Also, in view of assumption *3.* we have

$$\mathrm{C_o}\sum_k R^{\frac{3}{2}} M_0^{\frac{1}{2\theta}} \left(\tfrac{\varepsilon}{R}\right)^{1+\frac{1}{2\theta}} \|f\|_{L^2(A)} \leq \mathrm{C_o} L^{\frac{3}{2}} M_0^{\frac{1}{2\theta}} \left(\tfrac{\varepsilon}{L}\right)^{1+\frac{1}{2\theta}} \|f\|_{L^2(\mathrm{I_{3L}})} \leq \frac{1}{2}\mathrm{C_w} M_0 \alpha^{-\omega},$$

provided α_1 is sufficiently large (third requirement). It remains to estimate $e_\varepsilon(u)$ on the intervals $(-2\mathrm{L}, a_1)$ and $(a_\ell, 2\mathrm{L})$. We first use Lemma 2.5 with $A = (-3\mathrm{L}, -\mathrm{L})$ (resp. $A = (\mathrm{L}, 3\mathrm{L})$ and $B = (-\frac{5}{2}\mathrm{L}, -\frac{3}{2}\mathrm{L})$ (resp. $B = (\frac{3}{2}\mathrm{L}, \frac{5}{2}\mathrm{L})$). This yields, using the trivial bound $\mathcal{E}_\varepsilon(u, A \setminus B) \leq M_0$, the estimate

$$\mathcal{E}_\varepsilon(u, \mathrm{I}_{\frac{5}{2}\mathrm{L}} \setminus \mathrm{I}_{\frac{3}{2}\mathrm{L}}) \leq C \left(M_0^{\frac{1}{\theta}} \left(\frac{\varepsilon}{\mathrm{L}}\right)^{1+\frac{1}{\theta}} + M_0^{\frac{1}{2\theta}} \left(\frac{\varepsilon}{\mathrm{L}}\right)^{\frac{1}{2\theta}} \right) \leq C\alpha^{-\omega}, \tag{2.14}$$

in view of *1.* and provided α_1 is sufficiently large. We apply one last time Lemma 2.5, with $A = (-2\mathrm{L} - \frac{1}{2}\alpha\varepsilon, a_1 - \frac{1}{2}\alpha\varepsilon)$ (resp. $A = (a_\ell + \frac{1}{2}\alpha\varepsilon, 2\mathrm{L} + \frac{1}{2}\alpha\varepsilon)$) and $B = (-2\mathrm{L}, a_1 - \alpha\varepsilon)$ (resp. $B = (a_\ell + \alpha\varepsilon, 2\mathrm{L})$). Since $A \setminus B \subset \mathrm{I}_{\frac{5}{2}\mathrm{L}} \setminus \mathrm{I}_{\frac{3}{2}\mathrm{L}}$, it follows from (2.14) and Lemma 2.5, combined with our previous estimates, that condition $\mathrm{WPO}_\varepsilon^{\mathrm{L}}(\alpha\varepsilon)$ is satisfied provided we choose $\mathrm{C_w}$ sufficiently large. □

Remark 2.1. *Notice that condition 1. in Proposition 2.2 is always satisfied when $\alpha\varepsilon \leq \delta_{\log}^{\varepsilon}$, since $L/\varepsilon \geq 1$. Also, for $\alpha = \delta_{\log}^{\varepsilon}/\varepsilon$, assumption 3. in Proposition 2.2 is weaker than assumption 2. We therefore deduce*

Corollary 2.1. *Let u be a solution to (2.9) satisfying assumption (H_0), and such that $\mathcal{D}(u) \cap \mathrm{I_{3L}} \subset \mathrm{I_L}$. If*

$$\varepsilon\|f\|_{L^2(\mathrm{I_{3L}})} \leq \left(\frac{M_0}{\mathrm{L}}\right)^{\frac{1}{2}}, \tag{2.15}$$

then $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta_{\log}^{\varepsilon})$ holds.

3 Regularized fronts

In the whole section, we assume that v_ε is a solution of $(\mathrm{PGL})_\varepsilon$ which satisfies (H_0) and the confinement condition $\mathcal{C}_{\mathrm{L},S}$.

3.1 Finding regularized fronts

We provide here the proof to Proposition 3, which is deduced from the following:

Lemma 3.1. *Given any $s_1 < s_2$ in $[0, S]$, there exist at least one time s in $[s_1, s_2]$ for which $\mathfrak{v}_\varepsilon(\cdot, s)$ solves (2.9) with*

$$\|f\|^2_{L^2(\mathrm{I_{3L}})} \equiv \varepsilon^{\omega-1}\|\partial_s \mathfrak{v}_\varepsilon(\cdot, s)\|^2_{L^2(\mathrm{I_{3L}})} \leq \varepsilon^{\omega-1}\frac{\mathrm{dissip}_\varepsilon^{3\mathrm{L}}(s_1, s_2)}{s_2 - s_1} \leq \varepsilon^{\omega-1}\frac{M_0}{s_2 - s_1}. \tag{3.1}$$

Proof. It is a direct mean value argument, taking into account the rescaling of $(\mathrm{PGL})_\varepsilon$ according to our rescaling of time. □

Proof of Proposition 3. We invoke Lemma 3.1, and from (3.1) and the assumption $s_2 - s_1 = \varepsilon^{\omega+1}\mathrm{L}$ of Proposition 3, we derive exactly the assumption (2.15) in Corollary 2.1, from which the conclusion follows. $\square$

Following the same argument, but relying on Lemma 2.4 and Proposition 2.2 rather than on Corollary 2.1, we readily obtain

Proposition 3.1. *For* $\alpha_1 \le \alpha \le \delta^{\varepsilon}_{\log}$:

1. *Each subinterval of* $[0, S]$ *of size* $\mathfrak{q}_0(\alpha)\varepsilon^{\omega+2}$ *contains at least one time* s *at which* $\mathrm{WPI}^{\mathrm{L}}_{\varepsilon}(\alpha\varepsilon, s)$ *holds, where*
$$\mathfrak{q}_0(\alpha) = 4M_0^2 \exp(\rho_{\mathrm{w}}\alpha). \tag{3.2}$$
2. *Each subinterval of* $[0, S]$ *of size* $\mathfrak{q}_0(\alpha, \beta)\varepsilon^{\omega+2}$ *contains at least one time* s *at which* $\mathcal{WP}^{\mathrm{L}}_{\varepsilon}(\alpha\varepsilon, s)$ *holds, where*
$$\beta := \frac{\mathrm{L}}{\varepsilon} \quad \text{and} \quad \mathfrak{q}_0(\alpha, \beta) = \max\left(\mathfrak{q}_0(\alpha), \left(\frac{2\mathrm{C_o}}{\mathrm{C_w}}\right)^2\left(\frac{\beta}{M_0}\right)^{1-\frac{1}{\theta}}\alpha^{2\omega}\right). \tag{3.3}$$

3.2 Local dissipation

For $s \in [0, S]$, set $\mathcal{E}^{\mathrm{L}}_{\varepsilon}(s) = \mathcal{E}^{\mathrm{L}}_{\varepsilon}(\mathfrak{v}_\varepsilon(s))$ and, when $\mathrm{WPI}^{\mathrm{L}}_{\varepsilon}(\alpha_1\varepsilon, s)$ holds, $\mathfrak{E}^{\mathrm{L}}_{\varepsilon}(s) = \mathfrak{E}^{\mathrm{L}}_{\varepsilon}(\mathfrak{v}_\varepsilon(s))$, $\mathfrak{E}^{\mathrm{L}}_{\varepsilon}$ being defined in (2.6). We assume throughout that $s_1 \le s_2$ are contained in $[0, S]$, and in some places (in view of (28) that $s_2 \ge \mathrm{L}^2\varepsilon^\omega$.

Proposition 3.2. *If* $s_2 \ge \mathrm{L}^2\varepsilon^\omega$*, we have*
$$\mathcal{E}^{\mathrm{L}}_{\varepsilon}(s_2) + \mathrm{dissip}^{\mathrm{L}}_{\varepsilon}(s_1, s_2) \le \mathcal{E}^{\mathrm{L}}_{\varepsilon}(s_1) + 100\mathrm{C}_e\mathrm{L}^{-(\omega+2)}(s_2 - s_1) + \mathrm{C_e}(1 + M_0)\left(\frac{\mathrm{L}}{\varepsilon}\right)^{-\omega}. \tag{3.4}$$

Proof. Let $0 \le \varphi \le 1$ be a smooth function with compact support in $\mathrm{I}_{2\mathrm{L}}$, such that $\varphi(x) = 1$ on $\mathrm{I}_{\frac{5}{3}\mathrm{L}}$, $|\varphi''| \le 100\mathrm{L}^{-2}$. It follows from the properties of φ and (28) that
$$\mathcal{I}_\varepsilon(s, \varphi) \le \mathcal{E}^{\mathrm{L}}_{\varepsilon}(s) \ \text{ for } s \in (s_1, s_2) \text{ and } \mathcal{I}_\varepsilon(s_2, \varphi) \ge \mathcal{E}^{\mathrm{L}}_{\varepsilon}(s_2) - \mathrm{C_e}\left(\frac{\mathrm{L}}{\varepsilon}\right)^{-\omega},$$
which combined with (36) yields
$$\mathcal{E}^{\mathrm{L}}_{\varepsilon}(s_2) + \mathrm{dissip}^{\mathrm{L}}_{\varepsilon}(s_1, s_2) \le \mathcal{E}^{\mathrm{L}}_{\varepsilon}(s_1) + \mathrm{C_e}\left(\frac{\mathrm{L}}{\varepsilon}\right)^{-\omega} + \varepsilon^{-\omega}\int_{s_1}^{s_2} \mathcal{F}_S(s, \varphi, \mathfrak{v}_\varepsilon)ds$$
where $\mathcal{F}_S$ is defined in (37). The estimate (3.4) is then obtained invoking the inequality $|\xi_\varepsilon| \le e_\varepsilon$ to bound the term involving $\mathcal{F}_S$: combined with (28) for times $s \ge \mathrm{L}^2\varepsilon^\omega$ and with assumption (H_0) for times $s \le \mathrm{L}^2\varepsilon^\omega$. $\square$

If $\mathcal{WP}^{\mathrm{L}}_{\varepsilon}(\delta, s_1)$ and $\mathrm{WPI}^{\mathrm{L}}_{\varepsilon}(\delta', s_2)$ hold, for some $\delta, \delta' \ge \alpha_1\varepsilon$ and $s_2 \ge \mathrm{L}^2\varepsilon^\omega$, then combining inequality (3.4) with the first inequality (2.7) applied to $\mathfrak{v}_\varepsilon(s_2)$ as well as the second applied to $\mathfrak{v}_\varepsilon(s_1)$ we obtain

$$
\begin{aligned}
&\mathfrak{E}^{\mathrm{L}}_\varepsilon(s_2) + \mathrm{dissip}^{\mathrm{L}}_\varepsilon(s_1, s_2)\\
&\leq \mathcal{E}^{\mathrm{L}}_\varepsilon(s_2) + \mathrm{C_f} M_0 \left(\frac{\varepsilon}{\delta'}\right)^\omega + \mathrm{dissip}^{\mathrm{L}}_\varepsilon(s_1, s_2)\\
&\leq \mathcal{E}^{\mathrm{L}}_\varepsilon(s_1) + 100\mathrm{C}_e\mathrm{L}^{-(\omega+2)}(s_2 - s_1) + \mathrm{C_f} M_0 \left(\frac{\varepsilon}{\delta'}\right)^\omega + \mathrm{C_e}(1 + M_0)\left(\frac{\varepsilon}{\mathrm{L}}\right)^\omega\\
&\leq \mathfrak{E}^{\mathrm{L}}_\varepsilon(s_1) + (\mathrm{C_w} + \mathrm{C_f}) M_0 \left(\frac{\varepsilon}{\delta}\right)^\omega + \mathrm{C_f} M_0 \left(\frac{\varepsilon}{\delta'}\right)^\omega + 100\mathrm{C}_e\mathrm{L}^{-(\omega+2)}(s_2 - s_1) + \mathrm{C_e}(1 + M_0)\left(\frac{\varepsilon}{\mathrm{L}}\right)^\omega.
\end{aligned}
\tag{3.5}
$$

We deduce from this inequality an estimate for the dissipation between s_1 and s_2 and an upper bound on $\mathcal{E}^{\mathrm{L}}_\varepsilon(s_2)$:

Corollary 3.1. *Assume that $\mathcal{WP}^{\mathrm{L}}_\varepsilon(\delta, s_1)$ and $\mathrm{WPI}^{\mathrm{L}}_\varepsilon(\delta', s_2)$ hold, for some $\delta, \delta' \geq \alpha_1\varepsilon$ and $s_2 \geq \mathrm{L}^2\varepsilon^\omega$, and that $\mathfrak{E}^{\mathrm{L}}_\varepsilon(s_1) = \mathfrak{E}^{\mathrm{L}}_\varepsilon(s_2)$. Then*

$$
\mathrm{dissip}^{\mathrm{L}}_\varepsilon[s_1, s_2] \leq (\mathrm{C_w} + \mathrm{C_f}) M_0 \left(\frac{\varepsilon}{\delta}\right)^\omega + \mathrm{C_f} M_0 \left(\frac{\varepsilon}{\delta'}\right)^\omega + 100\mathrm{C_e}\mathrm{L}^{-(\omega+2)}(s_2 - s_1) + \mathrm{C_e}(1 + M_0)\left(\frac{\varepsilon}{\mathrm{L}}\right)^\omega,
$$

$$
\mathcal{E}^{\mathrm{L}}_\varepsilon(s_2) - \mathfrak{E}^{\mathrm{L}}_\varepsilon(s_2) \leq (\mathrm{C_w} + \mathrm{C_f}) M_0 \left(\frac{\varepsilon}{\delta}\right)^\omega + 100\mathrm{C_e}\mathrm{L}^{-(\omega+2)}(s_2 - s_1) + \mathrm{C_e}(1 + M_0)\left(\frac{\varepsilon}{\mathrm{L}}\right)^\omega.
$$

3.3 Quantization of the energy

Let $s \in [0, S]$ and $\delta \geq \alpha_1\varepsilon$, and assume that $\mathfrak{v}_\varepsilon$ satisfies $\mathcal{WP}^{\mathrm{L}}_\varepsilon(\delta, s)$. The front energy $\mathfrak{E}^{\mathrm{L}}_\varepsilon(s)$, by definition, may only take a finite number of values, and is hence *quantized*. We emphasize that, at this stage, $\mathfrak{E}^{\mathrm{L}}_\varepsilon(s)$ is only defined *assuming* condition $\mathrm{WPI}^{\mathrm{L}}_\varepsilon(\delta, s)$ holds. However, the *value* of $\mathfrak{E}^{\mathrm{L}}_\varepsilon(s)$ *does not* depend on δ, provided that $\delta \geq \alpha_1\varepsilon$, so that it suffices *ultimately* to check that condition $\mathrm{WPI}^{\mathrm{L}}_\varepsilon(\alpha_1\varepsilon, s)$ is fulfilled.

Since $\mathfrak{E}_\varepsilon(s)$ may take only a finite number of values, let $\mu_1 > 0$ be the smallest possible difference between two distinct such values. Let $\mathrm{L}_0 \equiv \mathrm{L}_0(s_1, s_2) > 0$ be such that

$$
100\mathrm{C}_e\mathrm{L}_0^{-(\omega+2)}(s_2 - s_1) = \frac{\mu_1}{4} \tag{3.6}
$$

and finally choose α_1 sufficiently large so that

$$
\Big((2\mathrm{C_f} + \mathrm{C_w})M_0 + \mathrm{C}_e(1 + M_0)\Big)\alpha_1^{-\omega} \leq \frac{\mu_1}{4}. \tag{3.7}
$$

As a direct consequence of (3.5), (3.6), (3.7) and the definition of μ_1 we obtain

Corollary 3.2. *For $s_1 \leq s_2 \in [0, S]$ with $s_2 \geq \varepsilon^\omega\mathrm{L}^2$, assume that $\mathcal{WP}^{\mathrm{L}}_\varepsilon(\alpha_1\varepsilon, s_1)$ and $\mathrm{WPI}^{\mathrm{L}}_\varepsilon(\alpha_1\varepsilon, s_2)$ hold and that $\mathrm{L} \geq \mathrm{L}_0(s_1, s_2)$. Then we have $\mathfrak{E}^{\mathrm{L}}_\varepsilon(s_2) \leq \mathfrak{E}^{\mathrm{L}}_\varepsilon(s_1)$. Moreover, if $\mathfrak{E}^{\mathrm{L}}_\varepsilon(s_2) < \mathfrak{E}^{\mathrm{L}}_\varepsilon(s_1)$, then $\mathfrak{E}^{\mathrm{L}}_\varepsilon(s_2) + \mu_1 \leq \mathfrak{E}^{\mathrm{L}}_\varepsilon(s_1)$.*

In the opposite direction we have:

Lemma 3.2. *For $s_1 \leq s_2 \in [0, S]$, assume that $\mathrm{WPI}^{\mathrm{L}}_\varepsilon(\alpha_1\varepsilon, s_1)$ and $\mathrm{WPI}^{\mathrm{L}}_\varepsilon(\alpha_1\varepsilon, s_2)$ hold and that $\mathrm{L} \geq \mathrm{L}_0(s_1, s_2)$. Assume also that*

$$
s_2 - s_1 \leq \rho_0 \left(\frac{1}{8} d^{\varepsilon,\mathrm{L}}_{\min}(s_1)\right)^{\omega+2}. \tag{3.8}
$$

Then we have $\mathfrak{E}^{\rm L}_\varepsilon(s_2) \geq \mathfrak{E}^{\rm L}_\varepsilon(s_1)$. *In case of equality, we have* $J(s_1) = J(s_2)$ *and*

$$\sigma_{i(k\pm\frac{1}{2})}(s_1) = \sigma_{i(k\pm\frac{1}{2})}(s_1), \text{ for any } k \in J(s_1) \text{ and } d^{\varepsilon,\rm L}_{\min}(s_2) \geq \frac{1}{2} d^{\varepsilon,\rm L}_{\min}(s_1). \tag{3.9}$$

Proof. It a consequence of the bound (7) in Theorem 1 on the speed of the front set combined with assumption (3.8). Indeed, this implies that for arbitrary $s \in [s_1, s_2]$, the front set a time s is contained in a neighborhood of size $d^{\varepsilon,\rm L}_{\min}(s_1)/8$ of the front set at time s_1. In view of the definition of $d^{\varepsilon,\rm L}_{\min}(s_1)$, and of the continuity in time of the solution, this implies that for all $k_0 \in J(s_1)$ the set

$$\mathcal{A}_{k_0} = \left\{k \in J(s_2) \text{ such that } a^\varepsilon_k(s_2) \in \left[a^\varepsilon_{k_0}(s_1) - \frac{1}{4} d^{\varepsilon,\rm L}_{\min}(s_1), a^\varepsilon_{k_0}(s_1) + \frac{1}{4} d^{\varepsilon,\rm L}_{\min}(s_1)\right]\right\}$$

is non empty, since it must contain a front connecting $\sigma_{i(k_0-\frac{1}{2})}(s_1)$ to $\sigma_{i(k_0+\frac{1}{2})}(s_1)$. In particular, summing over all fronts in $\mathcal{A}_{k_0}$, we obtain

$$\sum_{k\in\mathcal{A}_{k_0}} \mathfrak{S}^{\rm L}_{i(k)} \geq \mathfrak{S}^{\rm L}_{i(k_0)},$$

with equality if and only if $\sharp\mathcal{A}_{k_0} = 1$. Summing over all indices k_0, we are led to the conclusion. □

3.4 Propagating regularized fronts

We discuss in this subsection the case of equality $\mathfrak{E}^{\rm L}_\varepsilon(s_1) = \mathfrak{E}^{\rm L}_\varepsilon(s_2)$. We assume throughout that we are given $\overset{\varepsilon}{\delta}_{\log} \geq \delta > \alpha_1\varepsilon$ and two times $s_1 \leq s_2 \in [\varepsilon^\omega {\rm L}^2, S]$ such that

$$\mathcal{C}(\delta, L, s_1, s_2) \qquad \begin{cases} \mathcal{WP}^{\rm L}_\varepsilon(\delta, s_1) \quad \text{and} \quad {\rm WPI}^{\rm L}_\varepsilon(\delta, s_2) \quad \text{hold} \\ \mathfrak{E}^{\rm L}_\varepsilon(s_1) = \mathfrak{E}^{\rm L}_\varepsilon(s_2), \quad \text{with } {\rm L} \geq {\rm L}_0(s_1, s_2). \end{cases}$$

Under that assumption, our first result shows that $\mathfrak{v}_\varepsilon$ remains well-prepared on almost the whole time interval $[s_1, s_2]$, with a smaller δ though.

Proposition 3.3. *There exists* $\alpha_2 \geq \alpha_1$, *depending only on* V, M_0 *and* ${\rm C_w}$, *with the following property. Assume that* $\mathcal{C}(\delta, L, s_1, s_2)$ *holds with* $\alpha_2\varepsilon \leq \delta \leq \overset{\varepsilon}{\delta}_{\log}$, *then property* $\mathcal{WP}^{\rm L}_\varepsilon(\Lambda_{\log}(\delta), s)$ *holds for any time* $s \in [s_1 + \varepsilon^{2+\omega}, s_2]$, *where*

$$\Lambda_{\log}(\delta) = \frac{\omega}{\rho_{\rm w}}\varepsilon\left(\log\frac{\delta}{\varepsilon}\right). \tag{3.10}$$

The proof of Proposition 3.3 relies on the following.

Lemma 3.3. *Assume that* $\mathcal{C}(\delta, L, s_1, s_2)$ *holds with* $\delta \geq \alpha_1\varepsilon$. *We have the estimate, for* $s \in [s_1 + \varepsilon^{\omega+2}, s_2]$

$$\int_{{\rm I}_{\frac{3}{2}{\rm L}}} |\partial_t v_\varepsilon(x, s\varepsilon^{-\omega})|^2 dx \leq C\varepsilon^{-3}{\rm dissip}^{\rm L}_\varepsilon[s, s - \varepsilon^{\omega+2}].$$

Proof of Lemma 3.3. Differentiating equation (PGL_ε) with respect to time, we are led to

$$|\partial_t(\partial_t v_\varepsilon) - \partial_{xx}(\partial_t v_\varepsilon)\,| \leq \frac{C}{\varepsilon^2}|\partial_t v_\varepsilon|.$$

It follows from standard parabolic estimates, working for $x \in \mathrm{I}_{2\mathrm{L}}$ on the cylinder $\Lambda_\varepsilon(x) = [x-\varepsilon, x+\varepsilon] \times [t-\varepsilon^2, t]$, where $t := s\varepsilon^{-\omega}$, that for any point $y \in [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}]$ we have

$$|\partial_t v_\varepsilon(y,t)| \leq C\varepsilon^{-\frac{3}{2}} \|\partial_t v_\varepsilon\|_{L^2(\Lambda_\varepsilon(x))}.$$

Taking the square of the previous inequality, and integrating over $[x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}]$, we are led to

$$\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} |\partial_t v_\varepsilon(y,t)|^2 dy \leq C\varepsilon^{-2} \int_{[x-2\varepsilon, x+2\varepsilon]\times[t-\varepsilon^2,t]} |\partial_t v_\varepsilon(y,t)|^2 dy.$$

A elementary covering argument then yields

$$\int_{\mathrm{I}_{\frac{3}{2}\mathrm{L}}} |\partial_t v_\varepsilon(y,t)|^2 dy \leq C\varepsilon^{-2} \|\partial_t v_\varepsilon\|^2_{L^2(\mathrm{I}_{\frac{5}{3}\mathrm{L}}\times[t-\varepsilon^2,t])} \leq C\varepsilon^{-3} \mathrm{dissip}_\varepsilon^{\mathrm{L}}[s, s-\varepsilon^{\omega+2}].$$

□

Proof of Proposition 3.3. In view of Proposition 3.1, of Corollary 3.2, and of assumption $\mathcal{C}(\updelta, \mathrm{L}, s_1, s_2)$, we may assume, without loss of generality, that

$$s_2 - s_1 \leq 2\mathfrak{q}_0(\updelta/\varepsilon, \mathrm{L}/\varepsilon). \tag{3.11}$$

Let $s \in (s_1 + \varepsilon^{\omega+2}, s_2)$, and consider once more the map $u = \mathfrak{v}_\varepsilon(\cdot, s)$, so that u is a solution to (2.9), with source term $f = \partial_t v_\varepsilon(\cdot, s\varepsilon^{-\omega})$. It follows from Lemma 3.3, combined with the first of Corrolary 3.1 on the dissipation, that

$$\|f\|^2_{L^2(\mathrm{I}_{\frac{3}{2}\mathrm{L}})} \leq C\varepsilon^{-3}\left[(\mathrm{C_w} + 2\mathrm{C_f})M_0\left(\frac{\varepsilon}{\updelta}\right)^\omega + 100\mathrm{C}_e\mathrm{L}^{-(\omega+2)}(s_2 - s_1) + \mathrm{C_e}(1 + M_0)\left(\frac{\varepsilon}{\mathrm{L}}\right)^\omega\right].$$

Notice that (3.11) combined with the assumption $\updelta \leq \updelta^{\varepsilon}_{\log}$ yields

$$100\mathrm{C}_e\mathrm{L}^{-(\omega+2)}(s_2 - s_1) \leq C\left(\frac{\varepsilon}{\delta}\right)^\omega.$$

We deduce from Lemma 2.4, imposing on $\upalpha_2$ the additional condition $\dfrac{\omega}{\uprho_\mathrm{w}}(\log \upalpha_2) \geq \upalpha_1$, that $\mathrm{WPI}^{\mathrm{L}}_\varepsilon((\Lambda_{\log}(\updelta), s)$ holds. It remains to show that $\mathrm{WPO}^{\mathrm{L}}_\varepsilon(\Lambda_{\log}(\updelta), s)$ holds likewise. To that aim, we invoke (3.1) which we use with the choice $s_1 = s_1$ and $s_2 = s$. This yields, taking once more (3.11) into account,

$$\mathfrak{E}^{\mathrm{L}}_\varepsilon(s) - \mathcal{E}^{\mathrm{L}}_\varepsilon(s) \leq (C + \mathrm{C_w})\left(\frac{\varepsilon}{\updelta}\right)^\omega.$$

Combining this relation with (2.5) and the first inequality of (2.7), we deduce that

$$\int_\Omega e_\varepsilon(\mathfrak{v}_\varepsilon(s))ds \leq (C + \mathrm{C_w})\left(\frac{\varepsilon}{\updelta}\right)^\omega + C\left(\frac{\varepsilon}{\Lambda_{\log}(\updelta)}\right)^\omega \leq \mathrm{C_w}M_0\left(\frac{\varepsilon}{\Lambda_{\log}(\updelta)}\right)^\omega, \tag{3.12}$$

provided $\upalpha_2$ is choosen sufficiently large. □

In view of (3.8) and (7), we introduce the function

$$\mathfrak{q}_1(\alpha) := \left(\frac{\mathfrak{q}_0(\alpha)}{\rho_0}\right)^{\frac{1}{\omega+2}},$$

which represents therefore the maximum displacement of the front set in the interval of time needed (at most) to find two consecutive times at which $\mathrm{WPI}^{\mathrm{L}}_{\varepsilon}(\alpha\varepsilon)$ holds.

From Proposition 3.3 and Lemma 3.2 we deduce

Corollary 3.3. *Let $s \in [\varepsilon^{\omega}\mathrm{L}^2, S]$ and $\alpha_2 \leq \alpha \leq \delta^{\varepsilon}_{\log}$, and assume that $\mathcal{WP}^{\mathrm{L}}_{\varepsilon}(\alpha\varepsilon, s)$ holds as well as $d^{\varepsilon,\mathrm{L}}_{\min}(s) \geq 16\mathfrak{q}_1(\alpha)\varepsilon$. Then $\mathcal{WP}^{\mathrm{L}}_{\varepsilon}(\Lambda_{\log}(\alpha\varepsilon), s')$ holds for any $s+\varepsilon^{2+\omega} \leq s' \leq \mathcal{T}^{\varepsilon}_0(\alpha, s)$, where*

$$\mathcal{T}^{\varepsilon}_0(\alpha,s) = \max\left\{s+\varepsilon^{2+\omega} \leq s' \leq S \quad s.t. \quad d^{\varepsilon,\mathrm{L}}_{\min}(s'') \geq 8\mathfrak{q}_1(\alpha)\varepsilon \quad \forall s'' \in [s+\varepsilon^{\omega+2}, s']\right\}.$$

We complete this section presenting the

Proof of Proposition 4. This follows directly from Corollary 3.3 with the choice $\alpha = \delta^{\varepsilon}_{\log}$, noticing that $\Lambda_{\log}(\delta^{\varepsilon}_{\log}) = \delta^{\varepsilon}_{\mathrm{loglog}}$. □

Proof of Corollary 2. If we assume moreover that $s_0 \geq \varepsilon^{\omega}\mathrm{L}^2$ and that $\mathcal{WP}^{\mathrm{L}}_{\varepsilon}(\delta^{\varepsilon}_{\log}, s_0)$ holds, then it is a direct consequence of the inclusion (7) and Proposition 4, taking into account the assumption (33). If we assume only that $s_0 \geq 0$ and that $\mathcal{WP}^{\mathrm{L}}_{\varepsilon}(\alpha_1\varepsilon, s_0)$ holds, then it suffices to consider the first time $s'_0 \geq s_0 + \varepsilon^{\omega}\mathrm{L}^2$ at which $\mathcal{WP}^{\mathrm{L}}_{\varepsilon}(\delta^{\varepsilon}_{\log}, s'_0)$ holds and to rely on Proposition 4 likewise. Indeed, since $s'_0 - s_0 \leq \varepsilon^{\omega}\mathrm{L}^2 + \varepsilon^{\omega+1}\mathrm{L}$ by Proposition 3, we may apply Corollary 3.2 and Lemma 3.2 for $s_1 = s_0$ and $s_2 = s'_0$, which yields $\mathfrak{C}^{\mathrm{L}}_{\varepsilon}(s_0) = \mathfrak{C}^{\mathrm{L}}_{\varepsilon}(s'_0)$ and therefore also the same asymptotics for $d^{\varepsilon,\mathrm{L}}_{\min}$ at times s_0 and s'_0. □

4 A first compactness results for the front points

The purpose of this section is to provide the proofs of Proposition 5 and Corollary 3.

Proof of Proposition 5. As mentioned, we choose the test functions (indepently of time) so that they are affine near the front points for any $s \in I^{\varepsilon}(s_0)$. More precisely, for a given $k_0 \in J$ we impose the following conditions on the test functions $\chi \equiv \chi_{k_0}$ in (42):

$$\begin{cases} \chi \text{ has compact support in } [a^{\varepsilon}_k(s_0) - \frac{1}{3}d^*_{\min}(s_0), a^{\varepsilon}_k(s_0) + \frac{1}{3}d^*_{\min}(s_0)], \\ \chi \text{ is affine on the interval } [a^{\varepsilon}_k(s_0) - \frac{1}{4}d^*_{\min}(s_0), a^{\varepsilon}_k(s_0) + \frac{1}{4}d^*_{\min}(s_0)], \text{ with } \chi' = 1 \text{ there} \\ \|\chi\|_{L^\infty(\mathbb{R})} \leq Cd^*_{\min}(s_0), \|\chi'\|_{L^\infty(\mathbb{R})} \leq C \text{ and } \|\chi''\|_{L^\infty(\mathbb{R})} \leq Cd^*_{\min}(s_0)^{-1}. \end{cases} \tag{4.1}$$

It follows from Corollary 2 that, for ε sufficiently small, we are in position to claim (42) and (43) for arbitrary s_1 and s_2 in the full interval $I^*(s_0)$. Combined with the first estimate of Corollary 3.1, with $\delta = \delta' = \delta^{\varepsilon}_{\mathrm{loglog}}$, this yields the conclusion (44). □

Proof of Corollary 3. The family of functions $(\mathfrak{v}_{\varepsilon})_{0<\varepsilon<1}$ is equi-continuous on every compact subset of the interval $I^*(s)$, so that the conclusion follows from the Arzela-Ascoli theorem. □

5 Refined asymptotics off the front set

5.1 Relaxations towards stationary solutions

Throughout this section, we assume that we are in the situation described by Corollary 2, in particular L is fixed and ε will tend to zero. Our main purpose is then to provide rigorous mathematical statements and proofs concerning the properties of the function $\mathfrak{W}_{\varepsilon_n} = \mathfrak{W}^k_{\varepsilon_n}$ defined in (45), for given $k \in J$, which have been presented, most of them in a formal way, in Subsection 1.5. We notice first that we may expand V' near $\sigma \equiv \sigma_{i(k+\frac{1}{2})}$ as

$$V'(\sigma + u) = 2\theta\lambda u^{2\theta-1}\left(1 + ug(u)\right), \tag{5.1}$$

where g is a some smooth function on $\mathbb{R}$ and where we have set for the sake of simplicity $\lambda = \lambda_{i(k+\frac{1}{2})}$. We work on the sets $\mathcal{V}_k(s_0)$ defined in (48) and on their analogs at the ε level

$$\mathcal{V}^\varepsilon_k(s_0) = \underset{s\in I^\varepsilon(s_0)}{\cup} \mathcal{J}_\varepsilon(s) \times \{s\} \equiv \underset{s\in I^\varepsilon(s_0)}{\cup} \left(a^\varepsilon_k(s) + \delta^\varepsilon_{\rm loglog}, a^\varepsilon_{k+1}(s) - \delta^\varepsilon_{\rm loglog}\right) \times \{s\}. \tag{5.2}$$

We will therefore work only with *arbitrary* small values of u. Let $u_0 > 0$ be sufficiently small so that $|ug(u)| \leq 1/4$ on $(-u_0, u_0)$ and $V'(\sigma + u)$ is strictly increasing on $(-u_0, u_0)$, convex on $(0, u_0)$ and concave on $(-u_0, 0)$. For small values of ε, the value of u in (5.1), in view of (46) in Lemma 2, will not exceed u_0, and we may therefore assume for the considerations in this section that $ug(u) = u_0 g(u_0)$, if $u \geq u_0$ and $-ug(u) = u_0 g(u_0)$, if $u \leq -u_0$. Equation $(\mathrm{PGL})_\varepsilon$ translates into the following equation for $\mathfrak{W}_\varepsilon$

$$L_\varepsilon(\mathfrak{W}_\varepsilon) \equiv \varepsilon^\omega \frac{\partial \mathfrak{W}_\varepsilon}{\partial s} - \frac{\partial^2 \mathfrak{W}_\varepsilon}{\partial x^2} + \lambda f_\varepsilon(\mathfrak{W}_\varepsilon) = 0, \tag{5.3}$$

where we have set

$$f_\varepsilon(w) = 2\theta w^{2\theta-1}\left(1 + \varepsilon^{\frac{1}{\theta-1}} w g(\varepsilon^{\frac{1}{\theta-1}} w)\right). \tag{5.4}$$

Notice that our assumption yield in particular

$$|f_\varepsilon(w)| \geq \frac{3}{2}\theta |w|^{2\theta-1}. \tag{5.5}$$

The analysis of the parabolic equation (5.3) is the core of this section. As mentioned, our results express convergence to stationary solutions. We first provide a few properties concerning these stationary solutions: the first lemma describes stationary solutions involved in the attractive case, whereas the second lemma is used in the repulsive case.

Lemma 5.1. *Let $r > 0$ and $0 < \varepsilon < 1$. There exist unique solutions $\overset{\vee+}{\mathfrak{u}}_{\varepsilon,r}$ (resp. $\overset{\vee-}{\mathfrak{u}}_{\varepsilon,r}$) to*

$$\begin{cases} -\dfrac{d\mathcal{U}}{dx^2} + \lambda f_\varepsilon(U) = 0 \text{ on } (-r, r), \\ \mathcal{U}(-r) = +\infty \text{ (resp. } \mathcal{U}(-r) = -\infty) \text{ and } \quad \mathcal{U}(r) = +\infty \text{ (resp. } \mathcal{U}(r) = -\infty). \end{cases}$$

Moreover we have,

$$C^{-1} r^{-\frac{1}{\theta-1}} \leq \overset{\vee+}{\mathfrak{u}}_{\varepsilon,r} \leq C\left(r - |x|\right)^{-\frac{1}{\theta-1}} \text{ and } C^{-1} r^{-\frac{1}{\theta-1}} \leq -\overset{\vee-}{\mathfrak{u}}_{\varepsilon,r} \leq C\left(r - |x|\right)^{-\frac{1}{\theta-1}}, \tag{5.6}$$

for some constant $C > 0$ depending only on V.

Lemma 5.2. *Let $r > 0$ and $0 < \varepsilon < 1$ be given. There exists a unique solution $\overset{\triangleright}{\mathfrak{u}}_{\varepsilon,r}$ to*

$$-\frac{d\mathcal{U}}{dx^2} + \lambda f_\varepsilon(U) = 0 \text{ on } (-r, r),\ \mathcal{U}(-r) = -\infty \text{ and } \quad \mathcal{U}(r) = +\infty.$$

These and related results are standard and have been considered since the works of Keller [12] and Osserman [14] in the fifties, at least regarding existence. The convexity and concavity assumptions are sufficient for uniqueness. We refer to Lemma A.1 in the Appendix for a short discussion of the case a a pure power nonlinearity.

We set $r^\varepsilon(s) = r^\varepsilon_{k+\frac{1}{2}}(s) = \frac{1}{2}(a^\varepsilon_{k+1}(s) - a^\varepsilon_k(s))$. Our aim is to provide sufficiently accurate expansions of $\mathfrak{W}_\varepsilon$ and the renormalized discrepancy $\varepsilon^{-\omega}\xi_\varepsilon$ on neighborhoods of the points $a^\varepsilon_{k+\frac{1}{2}}(s)$, for instance the intervals

$$\Theta^\varepsilon_{k+\frac{1}{2}}(s) = a^\varepsilon_{k+\frac{1}{2}}(s) + [-\frac{7}{8}r^\varepsilon(s), \frac{7}{8}r^\varepsilon(s)] = [a^\varepsilon_k(s) + \frac{1}{8}r^\varepsilon(s), a^\varepsilon_{k+1}(s) - \frac{1}{8}r^\varepsilon(s)]. \tag{5.7}$$

We first turn to the the attractive case $\dagger_k = -\dagger_{k+1}$. We may assume additionally that

$$k \in \{1, \cdots, \ell - 1\} \text{ and } \dagger_k = -\dagger_{k+1} = 1, \tag{5.8}$$

the case $\dagger_k = -\dagger_{k+1} = -1$ being handled similarly.

Proposition 5.1. *If* (5.8) *hold and ε is sufficiently small, then for any $s \in I^\varepsilon(s_0)$ and every $x \in \Theta^\varepsilon_{k+\frac{1}{2}}(s)$ we have the estimate*

$$|\mathfrak{W}_\varepsilon(x, s) - \lambda^{-\frac{1}{2(\theta-1)}} \overset{\vee+}{\mathfrak{u}}_{r^\varepsilon(s)}(x)| \leq C\varepsilon^{\min(\frac{1}{\omega+2}, \frac{\omega-1}{2(\theta-1)})}. \tag{5.9}$$

The repulsive case corresponds to $\dagger_k = \dagger_{k+1}$ and we may assume as above that

$$k \in \{1, \cdots, \ell - 1\} \text{ and } \dagger_k = \dagger_{k+1} = 1. \tag{5.10}$$

Proposition 5.2. *If* (5.10) *hold and ε is sufficiently small, then for any $s \in I^\varepsilon(s_0)$ and every $x \in \Theta^\varepsilon_{k+\frac{1}{2}}(s)$ we have the estimate*

$$|\mathfrak{W}_\varepsilon(x, s) - \lambda^{-\frac{1}{2(\theta-1)}} \overset{\triangleright}{\mathfrak{u}}_{r^\varepsilon(s)}(x)| \leq C\varepsilon^{\min(\frac{1}{\omega+2}, \frac{\omega-1}{2(\theta-1)})}. \tag{5.11}$$

Combining these results with parabolic estimates, we obtain estimates for the discrepancy.

Proposition 5.3. *If ε is sufficiently small, then for any $s \in I^\varepsilon(s_0)$ and every $x \in \Theta^\varepsilon_{k+\frac{1}{2}}(s)$ we have the estimate*

$$|\varepsilon^{-\omega}\xi_\varepsilon(\mathfrak{v}_\varepsilon) - \lambda_{i(k+\frac{1}{2})}^{-\frac{1}{2(\theta-1)}} r^\varepsilon(s)^{-(\omega+1)}\gamma_{k+\frac{1}{2}}| \leq C\,\varepsilon^{\frac{1}{\theta^2}}, \tag{5.12}$$

where

$$\begin{cases} \gamma_{k+\frac{1}{2}} = A_\theta \text{ if } \quad \dagger_k = -\dagger_{k+1} \\ \gamma_{k+\frac{1}{2}} = B_\theta \text{ if } \quad \dagger_k = \dagger_{k+1}. \end{cases} \tag{5.13}$$

For the outer regions, corresponding to $k = 0$ and $k = \ell$ estimates for the discrepancy are directly deduced from the crude estimates provided by Proposition 2. Proposition 5.3 provides a rigorous ground to the formal computation (52) of the introduction, and hence allows to derive the precise motion law. The proofs of Proposition 5.1 and Proposition 5.2 however are the central part of this section. Note that by no mean the estimates provided in Propositions 5.1, 5.2 and 5.3 are optimal; our goal was only to obtain convergence estimates, valid for all ε sufficiently small, uniformly on $\cup_{s\in I^\varepsilon(s_0)}\Theta^\varepsilon_{k+\frac{1}{2}}(s) \times \{s\}$.

5.2 Preliminary results

We first turn to the proof of Lemma 2, which provides first properties of $\mathfrak{W}_\varepsilon$.

Proof of Lemma 2. Let $x \in (a_k(s) + \delta^{\varepsilon}_{\rm loglog}, a_{k+1}(s) - \delta^{\varepsilon}_{\rm loglog})$ and any $s \in I^{\varepsilon}(s_0)$, and recall that $d(x,s) := \mathrm{dist}(x, \{a^{\varepsilon}_k(s), a^{\varepsilon}_{k+1}(s)\})$. In view of Proposition 3, and in particular of estimate (31), it suffices to show that

$$\mathfrak{v}_\varepsilon(y,s) \in B(\sigma_i, \mu_0) \quad \text{for all} \quad (y,s) \in [x - \frac{d(x,s)}{2}, x + \frac{d(x,s)}{2}] \times [s - \varepsilon^{\omega} d(x,s)^2, s].$$

By Theorem 1, on such a time scale the front set moves at most by a distance

$$d := \left(\frac{\varepsilon^{\omega} d(x,s)^2}{\rho_0}\right)^{\frac{1}{\omega+2}} \leq \rho_0^{-\frac{1}{\omega+2}} (\frac{\varepsilon}{\delta^{\varepsilon}_{\rm loglog}})^{\frac{\omega}{\omega+2}} d(x,s) \leq \frac{d(x,s)}{4},$$

provided ε/L is sufficiently small. More precisely, Theorem 1 only provides one inclusion, forward in time, but its combination with Corrolary 2 provides both forward and backward inclusions (for times in the interval $I^{\varepsilon}(s_0)$), from which the conclusion then follows. □

For the analysis of the scalar parabolic equation (5.3), we will extensively use the fact that the map f_ε is *non-decreasing* on $\mathbb{R}$, allowing comparison principles. The desired estimates for $\mathfrak{W}_\varepsilon$ will be obtained using appropriate choices of sub- and super-solutions. The construction of these functions involve a number of elementary solutions. First, we use the functions $\mathcal{W}^{\pm}_\varepsilon$, independent of the space variable x and solving the ordinary differential equation

$$\begin{cases} \varepsilon^{\omega} \dfrac{\partial \mathcal{W}^{\pm}_\varepsilon}{\partial s} = -\lambda f_\varepsilon(\mathcal{W}^{\pm}_\varepsilon) \\ \mathcal{W}_\varepsilon(0) = \pm\infty. \end{cases} \tag{5.14}$$

Using separation of variables, we may construct such a solution which verifies the bounds

$$0 < \mathcal{W}^{+}_\varepsilon(s) \leq C\varepsilon^{\frac{\omega}{2(\theta-1)}} [\lambda s]^{-\frac{1}{2(\theta-1)}} \text{ and } 0 \geq \mathcal{W}^{-}_\varepsilon(s) \geq -C\varepsilon^{\frac{\omega}{2(\theta-1)}} [\lambda s]^{-\frac{1}{2(\theta-1)}}, \tag{5.15}$$

so that it relaxes quickly to zero. We will also use solutions of the standard heat equation and rely in several places on the next remark:

Lemma 5.3. *Let Φ be a non negative solution to the heat equation $\varepsilon^{\omega}\partial_s\Phi - \Phi_{xx} = 0$, and U be such that $L_\varepsilon(U) = 0$. Then $L_\varepsilon(U + \Phi) \geq 0$, and $L_\varepsilon(U - \Phi) \leq 0$.*

Proof. Notice that $L_\varepsilon(U \pm \Phi) = \lambda(f_\varepsilon(U \pm \Phi) - f_\varepsilon(U))$, so that the conclusion follows from the fact that f_ε is non-decreasing. □

Next, let s be given $I^{\varepsilon}(s_0)$. By translation invariance, we may assume without loss of generality that

$$a^{\varepsilon}_{k+\frac{1}{2}}(s) = 0. \tag{5.16}$$

We set $h_\varepsilon = (\varepsilon/2\rho_0)^{\frac{1}{\omega+2}}$, and consider the cylinders

$$\Lambda^{\rm ext}_\varepsilon(s) = \mathcal{J}^{\rm ext}_\varepsilon(s) \times [s-\varepsilon, s] \text{ and } \Lambda^{\rm int}_\varepsilon(s) = \mathcal{J}^{\rm int}_\varepsilon(s) \times [s-\varepsilon, s], \tag{5.17}$$

where $\mathcal{J}_\varepsilon^{\rm int}(s) = [-r^\varepsilon_{\rm int}(s), r^\varepsilon_{\rm int}(s)]$, $\mathcal{J}_\varepsilon^{\rm ext}(s) = [-r^\varepsilon_{\rm ext}(s), r^\varepsilon_{\rm ext}(s)]$ with

$$r^\varepsilon_{\rm ext}(s) = r^\varepsilon(s) + 2h_\varepsilon \text{ and } r^\varepsilon_{\rm int}(s) = r^\varepsilon(s) - 2h_\varepsilon.$$

If ε is sufficiently small, in view of (7) we have the inclusions, with $\mathcal{V}_k^\varepsilon(s_0)$ defined in (5.2) ,

$$\Lambda_\varepsilon^{\rm int}(s) \subset \Pi_\varepsilon(s) \equiv \mathcal{V}_k^\varepsilon(s_0) \cap ([s-\varepsilon, s] \times \mathbb{R}) \subset \Lambda_\varepsilon^{\rm ext}(s).$$

As a matter of fact, still for ε sufficiently small, we have for any $\tau \in [s-\varepsilon, s]$,

$$\begin{cases} -r^\varepsilon_{\rm ext}(s) + h_\varepsilon \le a_k^\varepsilon(\tau) + \delta^\varepsilon_{\rm loglog} \le -r^\varepsilon_{\rm int}(s) - h_\varepsilon, \\ r^\varepsilon_{\rm int} + h_\varepsilon \le a_{k+1}^\varepsilon(\tau) - \delta^\varepsilon_{\rm loglog} \le r^\varepsilon_{\rm ext}(s) - h_\varepsilon(s_0). \end{cases} \tag{5.18}$$

We also consider the parabolic boundary of $\Lambda_\varepsilon^{\rm ext}(s)$

$$\begin{aligned} \partial_p \Lambda_\varepsilon^{\rm ext}(s) &= [-r^\varepsilon_{\rm ext}(s), r^\varepsilon_{\rm ext}(s)] \times \{s-\varepsilon\} \cup \{-r^\varepsilon_{\rm ext}\} \times [s-\varepsilon, s] \cup \{r^\varepsilon_{\rm ext}\} \times [s-\varepsilon, s] \\ &= \partial \Lambda_\varepsilon^{\rm ext}(s) \setminus [-r^\varepsilon_{\rm ext}(s), r^\varepsilon_{\rm ext}(s)] \times \{s\}, \end{aligned}$$

and define $\partial_p \Lambda_\varepsilon^{\rm int}(s)$ accordingly. Finally, we set

$$\partial_p \Pi_\varepsilon(s) = \partial(\Pi_\varepsilon(s)) \setminus [a_k^\varepsilon(s) + \delta^\varepsilon_{\rm loglog}, a_{k+1}^\varepsilon(s) - \delta^\varepsilon_{\rm loglog}] \times \{s\}.$$

A first application of the comparison principle leads to the following bounds:

Proposition 5.4. *For $x \in \mathcal{J}_\varepsilon^{\rm int}(s)$*

$$\begin{cases} \mathfrak{W}_\varepsilon(x,s) \le \overset{\vee +}{\mathfrak{u}}_{\varepsilon, r^\varepsilon_{\rm int}}(x) + C\varepsilon^{\frac{\omega-1}{2(\theta-1)}} \\ \mathfrak{W}_\varepsilon(x,s) \ge \overset{\vee -}{\mathfrak{u}}_{\varepsilon, r^\varepsilon_{\rm int}}(x) - C\varepsilon^{\frac{\omega-1}{2(\theta-1)}}. \end{cases} \tag{5.19}$$

Proof. We work on the cylinder $\Lambda_\varepsilon^{\rm int}(s)$ and consider there the comparison map

$$W_\varepsilon^{\rm sup}(y,\tau) = \overset{\vee +}{\mathfrak{u}}_{\varepsilon, r^\varepsilon_{\rm int}}(y) + \mathcal{W}_\varepsilon(\tau - (s-\varepsilon)) \text{ for } (y,\tau) \in \Lambda_\varepsilon^{\rm int}(s).$$

Since the two functions on the r.h.s of the definition of $W_\varepsilon^{\rm sup}$ are positive solutions to (5.3) and since f_ε is superadditive on $\mathbb{R}^+$, that is, since

$$f_\varepsilon(a+b) \ge f_\varepsilon(a) + f_\varepsilon(b) \;\text{ provided } a \ge 0, b \ge 0, \tag{5.20}$$

we deduce that

$$L_\varepsilon\left(W_\varepsilon^{\rm sup}(y,\tau)\right) \ge 0 \text{ on } \Lambda_\varepsilon^{\rm int}(s) \;\text{ with } W_\varepsilon^{\rm sup}(y,\tau) = +\infty \text{ for } (y,\tau) \in \partial_p \Lambda_\varepsilon^{\rm int},$$

so that $W^{{\rm sup}_\varepsilon}(x,s) \ge \mathfrak{W}_\varepsilon$ on $\partial_p \Lambda_\varepsilon^{\rm int}$. It follows that $W_\varepsilon^{\rm sup}(y,\tau) \ge \mathfrak{W}_\varepsilon$ on $\Lambda_\varepsilon^{\rm int}$, which, combined with (5.15) immediately leads to the first inequality. The second is derived similarly. ☐

At this stage, the constructions are some somewhat different in the case of attractive and repulsive forces, so that we need to distinguish the two cases.

5.3 The attractive case

We assume here that $\dagger_k = -\dagger_{k+1}$. Without loss of generality, we may assume that

$$\dagger_k = -\dagger_{k+1} = 1, \tag{5.21}$$

the case $\dagger_k = -\dagger_k = -1$ being handled similarly. The purpose of this subsection is to provide *the proof to Proposition 5.1.* We split the proof into separate lemmas, the main efforts being devoted to the construction of *subsolutions*. We start with the following lower bound:

Lemma 5.4. *Assume that* (5.21) *holds. Then, for $x \in \mathcal{J}_\varepsilon(s - \frac{\varepsilon}{2})$, we have the lower bound*

$$\mathfrak{W}_\varepsilon(x, s - \frac{\varepsilon}{2}) \geq -C\varepsilon^{\frac{\omega-1}{2(\theta-1)}}.$$

Proof. In view of (47), we notice that

$$\mathfrak{W}_\varepsilon(y,\tau) \geq 0 \text{ on } \partial_p\Pi_\varepsilon(s) \setminus [a_k(s-\varepsilon) + \delta^{\varepsilon}_{\text{loglog}}, a_{k+1}(s-\varepsilon) - \delta^{\varepsilon}_{\text{loglog}}] \times \{s - \varepsilon\}.$$

We consider next the function W_ε defined for $\tau \geq s - \varepsilon$ by $W_\varepsilon(y,\tau) = \mathcal{W}^-_\varepsilon(\tau - (s-\varepsilon))$. Since $W_\varepsilon < 0$, and since $W_\varepsilon(s-\varepsilon) = -\infty$, we obtain $W_\varepsilon \leq \mathfrak{W}_\varepsilon$ on $\partial_p\Pi_\varepsilon(s)$, so that, by the comparison principle we are led to $W_\varepsilon \leq \mathfrak{W}_\varepsilon$ on $\Pi_\varepsilon(s)$ leading to the conclusion. □

Proposition 5.5. *Assume that* (5.21) *holds. We have the lower bound for $x \in \mathcal{J}_\varepsilon(s)$*

$$\mathfrak{W}_\varepsilon(x,s) \geq \overset{\vee+}{\mathfrak{u}}_{\varepsilon, r^\varepsilon_{\text{ext}}}(x) - C\varepsilon^{-\frac{1}{3\theta-1}} \exp\left(-\pi^2 \frac{\varepsilon^{-\omega+1}}{32(r^\varepsilon(s))^2}\right). \tag{5.22}$$

Proof. On $\mathcal{J}_\varepsilon(s - \frac{\varepsilon}{2})$ we consider the map φ_ε defined by

$$\varphi_\varepsilon(x) = \inf\{\mathfrak{W}_\varepsilon(x, s - \frac{\varepsilon}{2}) - \overset{\vee+}{\mathfrak{u}}_{\varepsilon, r^\varepsilon_{\text{ext}}}(x), 0\} \leq 0. \tag{5.23}$$

Invoking (5.18) and estimates (5.6) for $\overset{\vee+}{\mathfrak{u}}_{\varepsilon, r^\varepsilon_{\text{ext}}}$, we obtain, for $x \in \mathcal{J}_\varepsilon(s - \frac{\varepsilon}{2})$

$$0 \leq \overset{\vee+}{\mathfrak{u}}_{\varepsilon, r^\varepsilon_{\text{ext}}}(x) \leq Ch_\varepsilon^{-\frac{1}{\theta-1}}, \tag{5.24}$$

which combined with Lemma 5.4 yields

$$|\varphi_\varepsilon(x)| \leq Ch_\varepsilon^{-\frac{1}{\theta-1}} \text{ for } x \in \mathcal{J}_\varepsilon(s - \frac{\varepsilon}{2}). \tag{5.25}$$

Combining (5.24), estimate (5.6) of Lemma 5.1 and estimate (47) of Lemma 2, we deduce that, if ε is sufficiently small then

$$\varphi_\varepsilon(a^\varepsilon_k(s - \frac{\varepsilon}{2}) + \delta^{\varepsilon}_{\text{loglog}}) = \varphi_\varepsilon(a^\varepsilon_{k+1}(s - \frac{\varepsilon}{2}) - \delta^{\varepsilon}_{\text{loglog}}) = 0. \tag{5.26}$$

We extend φ_ε by 0 outside the set $\mathcal{J}_\varepsilon(s - \frac{\varepsilon}{2})$, and consider the solution Φ_ε to

$$\begin{cases} \varepsilon^\omega \dfrac{\partial \Phi_\varepsilon}{\partial \tau} - \dfrac{\partial \Phi_\varepsilon}{\partial x^2} = 0 \text{ on } \Lambda^{\text{ext}}_\varepsilon(s) \cap \{\tau \geq s - \dfrac{\varepsilon}{2}\} \\ \Phi_\varepsilon(x, s - \dfrac{\varepsilon}{2}) = \varphi_\varepsilon(x) \quad \text{for } x \in \mathcal{J}^{\text{ext}}_\varepsilon(s - \dfrac{\varepsilon}{2}) \\ \Phi_\varepsilon(\pm r^\varepsilon_{\text{ext}}(s), \tau) = 0 \quad \text{for } \tau \in (s - \dfrac{\varepsilon}{2}, s). \end{cases} \tag{5.27}$$

Notice that $\Phi_\varepsilon \le 0$. We consider next on $\Lambda_\varepsilon^{\rm ext}(s) \cap \{\tau \ge s - \frac{\varepsilon}{2}\}$ the function $W_\varepsilon^{\rm inf}$ defined by

$$W_\varepsilon^{\rm inf}(y,\tau) = \overset{\vee+}{\mathfrak{u}}_{\varepsilon, r^\varepsilon_{\rm ext}}(y) + \Phi_\varepsilon(y,\tau).$$

It follows from Lemma 5.3 that $L_\varepsilon(W_\varepsilon^{\rm inf}) \le 0$, so that $W_\varepsilon^{\rm inf}$ is a *subsolution.* Since $W_\varepsilon^{\rm inf} \le \mathfrak{W}_\varepsilon$ on $\partial_p\left(\Pi_\varepsilon(s) \cap \{\tau \ge s - \frac{\varepsilon}{2}\}\right)$ it follows in particular that

$$W_\varepsilon^{\rm inf} \le \mathfrak{W}_\varepsilon \text{ on } \mathcal{J}_\varepsilon(s). \tag{5.28}$$

To complete the proof, we rely on the next linear estimates for Φ_ε.

Lemma 5.5. *We have the bound, for $y \in \mathcal{J}_\varepsilon^{\rm ext}$ and $\tau \in (s - \frac{\varepsilon}{2}, s)$*

$$|\Phi_\varepsilon(y,\tau)| \le C \exp\left(-\pi^2 \varepsilon^{-\omega} \frac{(\tau - (s - \frac{\varepsilon}{2})}{16(r^\varepsilon(s))^2}\right) \|\varphi_\varepsilon\|_{L^\infty(\mathcal{J}_\varepsilon(s - \frac{\varepsilon}{2}))}.$$

We postpone the proof of Lemma 5.5 and complete the proof of Proposition 5.5.

Proof of Proposition 5.5 completed. Combining Lemma 5.5 with (5.25), we are led, for $x \in \mathcal{J}_\varepsilon(s)$, to

$$|\Phi_\varepsilon(x,s)| \le C h_\varepsilon^{-\frac{1}{\theta-1}} \exp\left(-\pi^2 \frac{\varepsilon^{-\omega+1}}{32(r^\varepsilon(s))^2}\right). \tag{5.29}$$

The conclusion then follows, invoking (5.28). □

Proof of Lemma 5.5. Consider on the interval $[-2r^\varepsilon(s), 2r^\varepsilon(s)]$ the function $\psi(x)$ defined by $\psi(x) = \cos(\frac{\pi}{4r^\varepsilon(s)} x)$, so that $-\ddot{\psi} = \frac{\pi^2}{16}(r^\varepsilon(s))^{-2}\psi$, $\psi \ge 0$, $\psi(-2r^\varepsilon(s)) = \psi(2r^\varepsilon(s)) = 0$ and $\psi(x) \ge 1/2$ for $x \in [-r^\varepsilon_{\rm ext}(s), r^\varepsilon_{\rm ext}(s)]$. Hence, we obtain

$$\varepsilon^\omega \Psi_\tau - \Psi_{xx} = 0 \text{ on } \Lambda_\varepsilon^{\rm ext}(s) \cap \{\tau \ge s - \frac{\varepsilon}{2}\}, \text{ where } \Psi(x,\tau) = \exp\left(-\pi^2 \varepsilon^{-\omega} \frac{\tau - (s - \frac{\varepsilon}{2})}{16 r^\varepsilon(s)^2}\right) \psi(x).$$

On the other hand, for $(y,\tau) \in \partial_p\left(\Lambda_\varepsilon^{\rm ext}(s) \cap \{\tau \ge s - \frac{\varepsilon}{2}\}\right)$ we have

$$|\Phi_\varepsilon(y,\tau)| \le \|\varphi_\varepsilon\|_{L^\infty(\mathcal{J}_\varepsilon(s - \frac{\varepsilon}{2}))} 2\Psi(y,\tau)$$

and the conclusion follows therefore from the comparison principle for the heat equation. □

Proof of Proposition 5.1 completed. Combining the upper bound (5.19) of Proposition 5.4 with the lower bound (5.22) of Proposition 5.5, we are led, for ε sufficiently small, to

$$\overset{\vee+}{\mathfrak{u}}_{\varepsilon, r^\varepsilon_{\rm ext}}(x) - A_\varepsilon \le \mathfrak{W}_\varepsilon(x,s) \le \overset{\vee+}{\mathfrak{u}}_{\varepsilon, r^\varepsilon_{\rm int}}(x) + A_\varepsilon, \tag{5.30}$$

where we have set

$$A_\varepsilon = C \varepsilon^{\frac{\omega - 1}{2(\theta - 1)}}. \tag{5.31}$$

The conclusion (5.9) then follows from Proposition A.1 of the Appendix combined with the definition of h_ε and (A.7). □

5.4 The repulsive case

In this subsection, we assume throughout that $\dagger_k = \dagger_{k+1}$ and may assume moreover that

$$\dagger_k = \dagger_{k+1} = 1, \tag{5.32}$$

the case $\dagger_k = \dagger_k = -1$ is handled similarly. The main purpose of this subsection is to provide *the proof of Proposition 5.2*, the central part being the construction of accurate *supersolutions*, subsolutions being provided by the same construction. We assume as before that (5.16) holds, and use as comparison map $\mathfrak{U}_\varepsilon$ defined on $\mathcal{I}_\varepsilon^{\rm trs}(s) \equiv (-r^\varepsilon_{\rm ext}(s), r^\varepsilon_{\rm int}(s))$ by

$$\mathfrak{U}_\varepsilon(\cdot) \equiv \overset{\triangleright}{\mathfrak{u}}_{\varepsilon, r^\varepsilon(s)}\left(\cdot + 2h_\varepsilon\right),$$

so that $\mathfrak{U}_\varepsilon(x) \to +\infty$ as $x \to r^\varepsilon_{\rm int}(s)$, $\mathfrak{U}_\varepsilon(x) \to -\infty$ as $x \to -r^\varepsilon_{\rm ext}(s)$ and $|\mathfrak{U}_\varepsilon(-r^\varepsilon(s))| \leq Ch_\varepsilon^{-\frac{1}{\theta-1}}$.

Proposition 5.6. *For $x \in (a_k(s) + \delta^\varepsilon_{\rm loglog}, r^\varepsilon_{\rm int}(s))$ we have the inequality, where $C > 0$ denotes some constant*

$$\mathfrak{W}_\varepsilon(x,s) \leq \mathfrak{U}_\varepsilon(x) + C\varepsilon^{-\frac{1}{3\theta-1}} \exp\left(-\pi^2 \frac{\varepsilon^{-\omega+1}}{16(r^\varepsilon(s))^2}\right). \tag{5.33}$$

Proof. As for (5.23), write for $x \in \mathcal{I}_\varepsilon^{\rm trs}(s) \cap \mathcal{J}_\varepsilon(s-\varepsilon)$

$$\psi_\varepsilon(x) = \sup\{\mathfrak{W}_\varepsilon(x, s-\varepsilon) - \mathfrak{U}_\varepsilon, 0\} \geq 0.$$

We notice that

$$\psi_\varepsilon(a_k(s-\varepsilon) + \delta^\varepsilon_{\rm loglog}) = \psi_\varepsilon(r^\varepsilon_{\rm int}(s)) = 0.$$

Indeed, for the first relation, we argue as in (5.26) whereas for the second, we have $\mathfrak{U}_\varepsilon(r^\varepsilon_{\rm int}(s)) = \overset{\triangleright}{\mathfrak{u}}_{\varepsilon, r^\varepsilon(s)}(r^\varepsilon(s)) = +\infty$. We extend ψ_ε by 0 outside the interval $\mathcal{I}_\varepsilon^{\rm trs}(s) \cap \mathcal{J}_\varepsilon(s-\varepsilon)$ and derive, arguing as for (5.25),

$$|\psi_\varepsilon(x)| \leq Ch_\varepsilon^{-\frac{1}{\theta-1}} \leq C\varepsilon^{-\frac{1}{3\theta-1}} \text{ for } x \in \mathbb{R}. \tag{5.34}$$

We introduce the cylinder $\Lambda_\varepsilon^{\rm trans}(s) \equiv (-r^\varepsilon_{\rm ext}(s), r^\varepsilon_{\rm int}(s)) \times (s-\varepsilon, s)$ and the solution Ψ_ε to

$$\begin{cases} \varepsilon^\omega \dfrac{\partial \Psi_\varepsilon}{\partial \tau} - \dfrac{\partial \Psi_\varepsilon}{\partial x^2} = 0 \text{ on } \Lambda_\varepsilon^{\rm trans}(s) \\ \Phi_\varepsilon(x, s-\varepsilon) = \psi_\varepsilon(x) \text{ for } x \in (-r^\varepsilon_{\rm ext}(s), r^\varepsilon_{\rm int}(s)) \text{ and} \\ \Psi_\varepsilon(-r^\varepsilon_{\rm ext}(s), \tau) = \Psi_\varepsilon(r^\varepsilon_{\rm int}(s), \tau) = 0 \text{ for } \tau \in (s-\varepsilon, s), \end{cases} \tag{5.35}$$

so that $\Psi_\varepsilon \geq 0$. Arguing as for (5.29), we obtain for $\tau \in (s-\varepsilon, s)$

$$|\Psi_\varepsilon(y,\tau)| \leq C\varepsilon^{-\frac{1}{3\theta-1}} \exp\left(-\pi^2 \varepsilon^{-\omega} \frac{(\tau - (s-\varepsilon))}{16(r^\varepsilon(s))^2}\right). \tag{5.36}$$

We consider on $\Lambda_\varepsilon^{\rm trans}(s)$ the function $W_\varepsilon^{\rm trans}$ defined by

$$W_\varepsilon^{\rm trans}(y,\tau) = \mathfrak{U}_\varepsilon(y) + \Psi_\varepsilon(y,\tau).$$

It follows from Lemma 5.3 that $L_\varepsilon(W^{\text{trans}})_\varepsilon \geq 0$, that is $W_\varepsilon^{\text{trans}}$ is a *supersolution* for L_ε on $\Lambda_\varepsilon^{\text{trans}}(s)$. Consider next the subset $\Pi_\varepsilon^{\text{trans}}(s)$ of $\Lambda_\varepsilon^{\text{trans}}$ defined by

$$\Pi_\varepsilon^{\text{trans}}(s) \equiv \bigcup_{\tau\in(s-\varepsilon,s)} (a_k(\tau) + \delta_{\text{loglog}}^\varepsilon, r_{\text{int}}^\varepsilon(s)) \times \{\tau\}.$$

We claim that

$$W_\varepsilon^{\text{trans}} \geq \mathfrak{W}_\varepsilon \text{ on } \partial_p \Pi_\varepsilon^{\text{trans}}(s). \tag{5.37}$$

Indeed, by construction, we have $W_\varepsilon^{\text{trans}} = +\infty$ on $r_{\text{int}}^\varepsilon(s) \times (s-\varepsilon, s)$ and $W_\varepsilon^{\text{trans}}(x, s-\varepsilon) \geq \mathfrak{W}_\varepsilon(x, s-\varepsilon)$ for $x \in (a_k(s-\varepsilon) + \delta_{\text{loglog}}^\varepsilon, r_{\text{int}}^\varepsilon(s))$. Finally on $\cup_{\tau\in(s-\varepsilon,s)}\{a_k(\tau) + \delta_{\text{loglog}}^\varepsilon\} \times \{\tau\}$, the conclusion (5.37) follows from estimate (47) of Lemma 2. Combining inequality (5.37) with the comparison principle, we are led to

$$W_\varepsilon^{\text{trans}} \geq \mathfrak{W}_\varepsilon \text{ on } \Pi_\varepsilon^{\text{trans}}(s). \tag{5.38}$$

Combining (5.38) with (5.36) we are led to (5.33). □

Our next task is to construct a *subsolution.* To that aim, we rely on the symmetries of the equation, in particular the invariance $x \to -x$ and the *almost oddness* of the nonlinearity. To be more specific, we introduce the operator

$$\tilde{L}_\varepsilon(u) \equiv \varepsilon^\omega \frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} + \lambda \tilde{f}_\varepsilon(u) = 0, \text{ with } \tilde{f}_\varepsilon(u) = 2\theta u^{2\theta-1}\left(1 - \varepsilon^{\frac{1}{\theta-1}} u g(-\varepsilon^{\frac{1}{\theta-1}} u)\right),$$

which has the same properties as L_ε and consider the stationary solution $\overset{\triangleleft}{\mathfrak{u}}_{\varepsilon,r^\varepsilon(s)}$ for L_ε defined on $(-r^\varepsilon(s), r^\varepsilon(s))$ by

$$-\frac{\partial^2 \overset{\triangleleft}{\mathfrak{u}}_{\varepsilon,r^\varepsilon(s)}}{\partial x^2} + \lambda f_\varepsilon(\overset{\triangleleft}{\mathfrak{u}}_{\varepsilon,r^\varepsilon(s)}) = 0, \quad \overset{\triangleleft}{\mathfrak{u}}_{\varepsilon,r^\varepsilon(s)}(-r^\varepsilon(s)) = +\infty \text{ and } \overset{\triangleleft}{\mathfrak{u}}_{\varepsilon,r^\varepsilon(s)}(r^\varepsilon(s)) = -\infty,$$

so that $-\overset{\triangleleft}{\mathfrak{u}}_{\varepsilon,r^\varepsilon(s)}$ is a stationary solution to $\tilde{L}_\varepsilon$. Consider the function $\tilde{\mathfrak{W}}_\varepsilon$ defined by

$$\tilde{\mathfrak{W}}_\varepsilon(x,\tau) = -\mathfrak{W}_\varepsilon(-x,\tau) \tag{5.39}$$

and observe that $\tilde{L}_\varepsilon(\tilde{\mathfrak{W}}_\varepsilon) = 0$. Finally, we define the interval $(-r_{\text{int}}^\varepsilon(s), r_{\text{ext}}^\varepsilon(s))$ the function

$$\mathfrak{V}_\varepsilon(x) \equiv \overset{\triangleleft}{\mathfrak{u}}_{\varepsilon,r^\varepsilon(s)}\left(2h_\varepsilon - x\right),$$

so that $\mathfrak{V}_\varepsilon(x) \to -\infty$ as $x \to -r_{\text{int}}^\varepsilon(s)$ and $\mathfrak{V}_\varepsilon(x) \to +\infty$ as $x \to r_{\text{ext}}^\varepsilon(s)$.

Proposition 5.7. *For $x \in (-r_{\text{int}}^\varepsilon(s), a_{k+1}(s) - \delta_{\text{loglog}}^\varepsilon)$ we have the inequality,*

$$\mathfrak{W}_\varepsilon(x,s) \geq \mathfrak{V}_\varepsilon(x) - C\varepsilon^{-\frac{1}{3\theta-1}} \exp\left(-\pi^2 \frac{\varepsilon^{-\omega+1}}{16(r^\varepsilon(s))^2}\right). \tag{5.40}$$

Proof. We argue as in the proof of Proposition 5.6, replacing L_ε by $\tilde{\varepsilon}$, $\mathfrak{W}_\varepsilon$ by $\tilde{\mathfrak{W}}_\varepsilon$, and $\mathfrak{U}_\varepsilon$ by $\tilde{\mathfrak{U}}_\varepsilon = -\overset{\triangleleft}{\mathfrak{u}}_{\varepsilon,r^\varepsilon(s)}\left(\cdot - 2h_\varepsilon(s_0)\right)$. Inequality (5.40) for $\mathfrak{W}_\varepsilon$ is then obtained inverting relation (5.39) and from the corresponding estimate on $\tilde{\mathfrak{W}}_\varepsilon$. □

Proof of Proposition 5.2 completed. Combining (5.33) with (5.40) we are led to

$$\mathfrak{U}_\varepsilon(x) - \tilde{A}_\varepsilon \leq \mathfrak{W}_\varepsilon(x,s) \leq \mathfrak{V}_\varepsilon(x) + \tilde{A}_\varepsilon, \tag{5.41}$$

where we have set $\tilde{A}_\varepsilon = C\varepsilon^{-\frac{1}{3\theta-1}} \exp(-\pi^2 \frac{\varepsilon^{-\omega+1}}{16(r^\varepsilon(s))^2})$. The proof is then completed with the same arguments as in the proof of Proposition 5.1 □

5.5 Estimating the discrepancy

5.5.1 Linear estimates

The purpose of this section is to provide the proof of Proposition 5.3. So far Proposition 5.1 and Proposition 5.2 provide a good approximation of $\mathfrak{W}_\varepsilon$ on the level of the *uniform norm.* However, the discrepancy involves also a first order derivative, for which we rely on the regularization property of the linear heat equation. To that aim, set

$$\begin{cases} \Lambda \equiv (-1,1) \times [0,1],\ \Lambda^{1/2} \equiv (-\frac{1}{2}, \frac{1}{2}) \times [\frac{3}{4}, 1], \text{ and more generally for } \varrho > 0 \\ \Lambda_\varrho \equiv (-\varrho, \varrho) \times [0, \varrho^2],\ \Lambda_\varrho^{1/2} \equiv (-\frac{1}{2}\varrho, \frac{1}{2}\varrho) \times [\frac{3}{4}\varrho^2, \varrho^2]. \end{cases}$$

The following standard result (see e.g. [2] Lemma A7 for a proof) is useful in our context.

Lemma 5.6. *Let u be a smooth real-valued function on Λ. There exists a constant $C > 0$ such that*

$$\|u_x\|_{L^\infty(\Lambda^{1/2})} \leq C(\|u_t - u_{xx}\|_{L^\infty(\Lambda)} + \|u\|_{L^\infty(\Lambda)}).$$

We deduce from this result the following scaled version.

Lemma 5.7. *Let $\varrho > 0$ and let u be defined on Λ_ϱ. Then we have for some constant $C > 0$ independent of ϱ*

$$\|u_x\|^2_{L^\infty(\Lambda_\varrho^{1/2})} \leq C\left[\|u_t - u_{xx}\|_{L^\infty(\Lambda_\varrho)}\|u\|_{L^\infty(\Lambda_\varrho)} + \varrho^{-2}\|u\|^2_{L^\infty(\Lambda_\varrho)}\right]. \tag{5.42}$$

Proof. The argument is parallel to the proof of Lemma A.1 in [1], which corresponds to its elliptic version. Set $h = u_t - u_{xx}$, let (x_0, t_0) be given in $\Lambda_\varrho^{1/2}$, and let $0 < \mu \leq \frac{\varrho}{2}$ be a constant to be determined in the course of the proof. We consider the function

$$v(y, \tau) = u\left(2\mu y + x_0,\ 4\mu^2(\tau - 1) + t_0\right),$$

so that v is defined on Λ and satisfies there

$$v_t - v_{yy} = \mu^2 h(\left(2\mu y + x_0,\ 4\mu^2(\tau - 1) + t_0\right) \text{ on } \Lambda.$$

Applying Lemma 5.6 to v we are led to

$$\begin{aligned} |v_y(0,1)| &\leq C\left(\mu^2\|h\left(2\mu y + x_0,\ 4\mu^2(\tau-1) + t_0\right)\|_{L^\infty(\Lambda)} + \|v\|_{L^\infty(\Lambda)}\right) \\ &\leq C\left(\mu^2\|h\|_{L^\infty(\Lambda_\varrho)} + \|u\|_{L^\infty(\Lambda_\varrho)}\right), \end{aligned}$$

so that, going back to u, we obtain

$$\mu|u_x(x_0, t_0)| \leq C\left(\mu^2\|h|_{L^\infty(\Lambda_\varrho)} + \|u\|_{L^\infty(\Lambda_\varrho)}\right). \tag{5.43}$$

We distinguish two cases:

Case 1: $\|u\|_{L^\infty} \leq \varrho^2\|h\|_{L^\infty}$. In this case we apply (5.43) with $\mu = \left(\frac{\|u\|_{L^\infty}}{\|h\|_{L^\infty}}\right)^{\frac{1}{2}}$. This yields

$$|u_y(x_0, t_0)| \leq 2C\|u\|^{1/2}_{L^\infty}\|h\|^{1/2}_{L^\infty}.$$

Case 2: $\|u\|_{L^\infty} \geq \varrho^2 \|h\|_{L^\infty}$. In this case we apply (5.43) with $\mu = \varrho$. We obtain

$$\begin{aligned} |u_x(x_0,t_0)| \leq& C\left(\varrho\|h\|_{L^\infty(\Lambda_\varrho)} + \varrho^{-1}\|u\|_{L^\infty(\Lambda_\varrho)}\right) \\ &\leq C\left(\|h\|^{1/2}_{L^\infty(\Lambda_\varrho)}\|u\|^{1/2}_{L^\infty(\Lambda_\varrho)} + r^{-1}\|u\|_{L^\infty(\Lambda_\varrho)}\right). \end{aligned} \tag{5.44}$$

In both cases, we obtain the desired inequality. □

5.5.2 Estimating the derivative of $\mathfrak{W}_\varepsilon$

Consider the general situation where we are given two functions U and U_ε defined for $(x,t) \in \Lambda_\varrho$ and such that $L_0(U) = 0$ and $L_\varepsilon(U_\varepsilon) = 0$, where $s := \varepsilon^{-\omega} t$, so that, in view of (5.4),

$$|\partial_t(U - U_\varepsilon) - \partial_{xx}(U - U_\varepsilon)| \leq C\left[|U - U_\varepsilon|(|U|^{2\theta-2} + |U_\varepsilon|^{2\theta-2}) + \varepsilon^{\frac{1}{\theta-1}}|U_\varepsilon|^{2\theta}\right] \text{ on } \Lambda_\varrho.$$

We deduce from (5.42) applied to the difference $U - U_\varepsilon$ that we have (we use the notation $\|\cdot\| = \|\cdot\|_{L^\infty(\Lambda_\varrho)}$ for simplicity)

$$\|(U - U_\varepsilon)_x\|^2_{L^\infty(\Lambda_\varrho^{1/2})} \leq C\,\|U - U_\varepsilon\|^2\left(\|U\|^{2\theta-2} + \|U_\varepsilon\|^{2\theta-2} + \varrho^{-2}\right) + C\,\varepsilon^{\frac{1}{\theta-1}}\|U - U_\varepsilon\|\|U_\varepsilon\|^{2\theta}.$$

Similarly applying (5.42) to U and U_ε we obtain

$$\|(U + U_\varepsilon)_x\|^2_{L^\infty(\Lambda_\varrho^{1/2})} \leq C(\|U\|^{2\theta} + \|U_\varepsilon\|^{2\theta} + \varrho^{-2}\left(\|U\|^2 + \|U_\varepsilon\|^2\right) + \varepsilon^{\frac{1}{\theta-1}}(\|U_\varepsilon\|^{2\theta+1} + \|U\|\|U_\varepsilon\|^{2\theta})),$$

so that

$$\|\left(U^2 - U_\varepsilon^2\right)_x\|^2_{L^\infty(\Lambda_\varrho^{1/2})} \leq C\left[\|U - U_\varepsilon\|^2\mathcal{R}_1^\varepsilon(U,U_\varepsilon) + \|U - U_\varepsilon\|\mathcal{R}_2^\varepsilon(U,U_\varepsilon)\right], \tag{5.45}$$

where we have set

$$\begin{cases} \mathcal{R}_1^\varepsilon(U,U_\varepsilon) = (\|U\|^{2\theta-2} + \|U_\varepsilon\|^{2\theta-2} + \varrho^{-2})(\|U\|^{2\theta} + \|U_\varepsilon\|^{2\theta} + \varrho^{-2}(\|U\|^2 + \|U_\varepsilon\|^2) \\ \qquad\qquad + \varepsilon^{\frac{1}{\theta-1}}(\|U_\varepsilon\|^{2\theta+1} + \|U\|\|U_\varepsilon\|^{2\theta})), \\ \mathcal{R}_2^\varepsilon(U,U_\varepsilon) = \varepsilon^{\frac{1}{\theta-1}}\|U_\varepsilon\|^{2\theta}(\|U\|^{2\theta} + \|U_\varepsilon\|^{2\theta} + \varrho^{-2}(\|U\|^2 + \|U_\varepsilon\|^2) \\ \qquad\qquad + \varepsilon^{\frac{1}{\theta-1}}(\|U_\varepsilon\|^{2\theta+1} + \|U\|\|U_\varepsilon\|^{2\theta})). \end{cases}$$

We specify next the discussion to our original situation. Thanks to the general inequality (5.45), we are in position to establish:

Proposition 5.8. *If* (5.21) *hold and ε is sufficiently small, then for any $s \in I^\varepsilon(s_0)$ and every $x \in \Theta^\varepsilon_{k+\frac{1}{2}}(s)$ we have the estimate*

$$|(\mathfrak{W}_\varepsilon)^2_x(x) - \lambda^{-\frac{1}{(\theta-1)}}(\overset{\vee+}{\mathfrak{u}}_{r^\varepsilon(s)})^2_x(x)| \leq C\,\varepsilon^{\frac{1}{\theta^2}}.$$

Proof. We apply inequality (5.45) on the cylinder Λ_ϱ with $\varrho = \frac{1}{16}d^*_{\min}(s_0)$ and to the functions $U(y,\tau) = \mathfrak{W}_\varepsilon(y + x, \varepsilon^\omega\tau + s)$ and $U_\varepsilon(y,\tau) = \overset{\vee+}{\mathcal{U}}_{r^\varepsilon(s)}(y + x)$. We first estimate $\mathcal{R}_1$ and $\mathcal{R}_2$. Since we have

$$|U(y,\tau)| + |U_\varepsilon| \leq C d^*_{\min}(s_0)^{-\frac{1}{\theta-1}}, \qquad \text{for } (y,\tau) \in \Lambda_\varrho,$$

it follows that

$$\mathcal{R}_1^\varepsilon(U,U_\varepsilon) \le d^*_{\min}(s_0)^{-4-\frac{2}{\theta-1}} \quad \text{and} \quad \mathcal{R}_2^\varepsilon(U,U_\varepsilon) \le \varepsilon^{\frac{1}{\theta-1}} d^*_{\min}(s_0)^{-4-\frac{4}{\theta-1}}.$$

Invoking inequality (5.42) of Lemma 5.7, and combining it with (A.7) and the conclusion of Proposition 5.1, we derive the conclusion using a crude lower bound for the power of ε. □

Similarly we obtain

Proposition 5.9. *If* (5.10) *hold and ε is sufficiently small, then for any $s \in I^\varepsilon(s_0)$ and every $x \in \Theta^\varepsilon_{k+\frac{1}{2}}(s)$ we have the estimate*

$$|(\mathfrak{W}_\varepsilon)^2_x(x) - \lambda^{-\frac{1}{(\theta-1)}}(\overset{\triangleright}{\mathfrak{u}}_{r^\varepsilon(s)})^2_x(x)| \le C\,\varepsilon^{\frac{1}{\theta^2}}. \tag{5.46}$$

Proof of Proposition 5.3 completed. The proof of Proposition 5.3 follows combining Proposition 5.8 in the attractive case and Proposition 5.9 in the repulsive case with the estimates (A.10). □

6 The motion law for prepared datas

In this section, we present the

Proof of Proposition 6. Step 1. First, by definition of L_0, assumption (H_1) and estimate (7), it follows that for fixed $\mathrm{L} \ge \mathrm{L}_0$, and for all ε sufficiently small (depending only on L),

$$\mathfrak{D}_\varepsilon(s) \cap \mathrm{I}_{4\mathrm{L}} \subset \mathrm{I}_{\mathrm{L}} \qquad \forall 0 \le s \le S,$$

so that $(\mathcal{C}_{\mathrm{L},S})$ holds.

Step 2. Since the assumptions of Corollary 3.1 are met with the choice $s_0 = 0$ and $\mathrm{L} = \mathrm{L}_0$, we obtain that for ε sufficiently small, $\mathcal{WP}^{\mathrm{L}_0}_\varepsilon(\overset{\varepsilon}{\delta}_{\mathrm{loglog}}, s)$ holds and $d^{\varepsilon,\mathrm{L}}_{\min}(s) \ge \frac{1}{2} d^*_{\min}(0) = \frac{1}{2}\min\{a^0_{k+1} - a^0_k,\ k = 1, \cdots, \ell_0 - 1\}$, for all $s \in I^\varepsilon(0)$, as well as the identities $J(s) = J(0)$, $\sigma_{i(k\pm\frac{1}{2})}(s) = \sigma_{i(k\pm\frac{1}{2})}(0)$ and $\dagger_k(s) = \dagger_k(0)$, for any $k \in J(0)$.

Step 3. We claim that for any $s_1 \le s_2 \in I^*(0)$, we have

$$\limsup_{\varepsilon\to 0}\,(\mathrm{dissip}^{\mathrm{L}}_\varepsilon(s_1, s_2)) = 0. \tag{6.1}$$

Indeed, let $\mathrm{L} \ge \mathrm{L}_0$ be arbitrary. We know from Step 1 that $(\mathcal{C}_{\mathrm{L},S})$ holds provided ε is sufficiently small. By Proposition 3, for ε sufficiently small there exists two times s^ε_1 and s^ε_2 such that $0 < s^\varepsilon_1 \le s_1 \le s_2 \le s^\varepsilon_2$, $|s_i - s^\varepsilon_i| \le \varepsilon^{\omega+1}\mathrm{L}$ and $\mathcal{WP}^{\mathrm{L}}_\varepsilon(\overset{\varepsilon}{\delta}_{\mathrm{log}}, s^\varepsilon_i)$ holds for $i = 1, 2$. From the second step and assumption (H_1) we infer that $\mathfrak{E}^{\mathrm{L}}_\varepsilon(s^\varepsilon_1) = \mathfrak{E}^{\mathrm{L}_0}_\varepsilon(s^\varepsilon_1) = \mathfrak{E}^{\mathrm{L}_0}_\varepsilon(s^\varepsilon_2) = \mathfrak{E}^{\mathrm{L}}_\varepsilon(s^\varepsilon_2)$. Invoking Corollary 3.1 we are therefore led to the inequality

$$\mathrm{dissip}^{\mathrm{L}_0}_\varepsilon(s_1, s_2) \le \mathrm{dissip}^{\mathrm{L}}_\varepsilon(s^\varepsilon_1, s^\varepsilon_2) \le CM_0\left(\frac{\varepsilon}{\overset{\varepsilon}{\delta}_{\mathrm{log}}}\right)^{\omega} + C\mathrm{L}^{-(\omega+2)}(s_2 - s_1 + 2\varepsilon^{\omega+1}\mathrm{L}).$$

Since $\mathrm{L} \ge \mathrm{L}_0$ was arbitrary the conclusion (6.1) follows letting first $\varepsilon \to 0$ and then $\mathrm{L} \to \infty$.

Step 4. In view of Corollary 3 we may find a subsequence $(\varepsilon_n)_{\in\mathbb{N}}$ tending to 0 such that the functions $a_k^{\varepsilon_n}(\cdot)_{n\in\mathbb{N}}$ converge uniformly as $n \to 0$ on compact subsets on $I^\star(0)$. Consider the cylinder

$$\mathcal{C}^*_{k+\frac{1}{2}} \equiv [a_k^0 + \frac{1}{4}d^*_{\min}(0),\, a^0_{k+1} - \frac{1}{4}d^*_{\min}(0)] \times I^*(0).$$

It follows from Step 2 and Proposition 5.3 that

$$\varepsilon_n{}^{-\omega}\xi_{\varepsilon_n}(\mathfrak{v}_{\varepsilon_n}) \to \lambda_{i(k+\frac{1}{2})}^{-\frac{1}{2(\theta-1)}} r_{k+\frac{1}{2}}(s)^{-(\omega+1)}\gamma_k \text{ as } \varepsilon_n \to 0, \text{ for } k = 1, \cdots \ell_0 - 1 \tag{6.2}$$

uniformly on every compact subset of $\mathcal{C}^*_{k+\frac{1}{2}}$, where γ_k is defined in (5.13) and where $r_{k+\frac{1}{2}}(s) = a_{k+1}(s) - a_k(s)$.

Step 5. As in (4.1), we consider a test function $\chi \equiv \chi_k$ with the following properties

$$\begin{cases} \chi \text{ has compact support in } [a_k^0 - \frac{1}{3}d^*_{\min}(0), a_k^0 + \frac{1}{3}d^*_{\min}(0)], \\ \chi \text{ is affine on the interval } [a_k^0 - \frac{1}{4}d^*_{\min}(0), a_k^0 + \frac{1}{4}d^*_{\min}(0)], \text{ with } \chi' = 1 \text{ there} \\ \|\chi\|_{L^\infty(\mathbb{R})} \le Cd^*_{\min}(0), \|\chi'\|_{L^\infty(\mathbb{R})} \le C \text{ and } \|\chi''\|_{L^\infty(\mathbb{R})} \le Cd^*_{\min}(0)^{-1}. \end{cases}$$

It follows from the definition of χ_k that $\chi''_k = 0$ outside a, and so is $\varepsilon^{-\omega}\xi_\varepsilon(\mathfrak{v}_\varepsilon)\chi''_k$. It follows from (6.2) that for $s_1 \le s_2 \in I^*(0)$,

$$\begin{aligned} \mathfrak{F}_{\varepsilon_n}(s_1, s_2, \chi_k) \to &\left(\int_{a_k^0-\frac{1}{3}d^*_{\min}(0)}^{a_k^0-\frac{1}{4}d^*_{\min}(0)} \chi''(x)dx\right)\left(\int_{s_1}^{s_2} \lambda_{k-\frac{1}{2}}^{-\frac{1}{2(\theta-1)}} r_{k-\frac{1}{2}}(s)^{-\frac{1}{\theta-1}}\gamma_{k-\frac{1}{2}}ds\right) + \\ &\left(\int_{a_k^0+\frac{1}{4}d^*_{\min}(0)}^{a_k^0+\frac{1}{3}d^*_{\min}} \chi''(x)dx\right)\left(\int_{s_1}^{s_2} \lambda_{k+\frac{1}{2}}^{-\frac{1}{2(\theta-1)}} r_{k+\frac{1}{2}}(s)^{-\frac{1}{\theta-1}}\gamma_{k+\frac{1}{2}}ds\right) \end{aligned} \tag{6.3}$$

as $\varepsilon_n \to 0$. Since the above two integrals containing χ'' are identically equal to 1 and -1 respectively, we finally deduce from (42) combined with (6.1) and (6.3), letting ε_n tend to 0, that for $s_1 \le s_2 \in I^*(0)$ we have

$$[a_k(s_1) - a_k(s_2)]\mathfrak{S}_{i(k)} = \int_{s_1}^{s_2} \left(\lambda_{i(k-\frac{1}{2})}^{-\frac{1}{2(\theta-1)}} r_{k-\frac{1}{2}}(s)^{-(\omega+1)}\gamma_{k-\frac{1}{2}} - \lambda_{i(k+\frac{1}{2})}^{-\frac{1}{2(\theta-1)}} r_{k+\frac{1}{2}}(s)^{-(\omega+1)}\gamma_{k+\frac{1}{2}}\right) ds,$$

which is nothing else than the integral formulation of the system $(\mathcal{S})$. Since the latter possesses a unique solution, the limiting points are unique and therefore convergence of the a_k^ε for $s \in I^*(s)$ holds for the full family $(\mathfrak{v}_\varepsilon)_{\varepsilon>0}$.

Step 6. We use an elementary continuation method to extend the convergence from $I^*(0)$ to the full interval $(0, S)$. Indeed, as long as $d^*_{\min}(s)$ remains bounded from below by a strictly positive constant (which holds, by definition of $S_{\max}$, as long as $s < S$) we may take s as a new origin of times (Step 2 yields $\mathcal{WP}_\varepsilon^{\mathrm{L}_0}(\upalpha_1\varepsilon, s)$) and use Steps 1 to 5 to extend the stated convergence past s. The proof is here completed. □

7 Clearing-out

The purpose of this section is to provide a proof to Proposition 7. We are led to consider the situation where for some length $\mathrm{L} \ge 0$ we have

$$\mathfrak{D}_\varepsilon(0) \cap [-5\mathrm{L}, 5\mathrm{L}] \subset [-\upkappa_0\mathrm{L}, \upkappa_0\mathrm{L}] \tag{7.1}$$

for some (small) constant $\kappa_0 \le \frac{1}{2}$. It follows from Theorem 2 that

$$\mathcal{C}_{\mathrm{L},S} \text{ holds,} \qquad \text{where } S = \rho_0 \left(\frac{\mathrm{L}}{2}\right)^{\omega+2},$$

and that for $s \in [0, S]$ we have

$$\mathfrak{D}_\varepsilon(s) \cap [-4\mathrm{L}, 4\mathrm{L}] \subset [-\kappa_0(s)\mathrm{L}, \kappa_0(s)\mathrm{L}] \tag{7.2}$$

where

$$\kappa_0(s) := \kappa_0 + \left(\frac{s}{\rho_0}\right)^{\frac{1}{\omega+2}} \frac{1}{\mathrm{L}}. \tag{7.3}$$

For those times $s \in [0, S]$ for which the preparedness assumption $\mathrm{WPI}_\varepsilon^{\mathrm{L}}(\alpha_1\varepsilon, s)$ holds we set

$$\begin{cases} d_{\min}^{\varepsilon,+}(s) = \min\{|a_{k+1}^\varepsilon(s) - a_k^\varepsilon(s)|,\ k \in J^+(s)\}, \quad \text{and} \\ d_{\min}^{\varepsilon,-}(s) = \min\{|a_{k+1}^\varepsilon(s) - a_k^\varepsilon(s)|,\ k \in J^-(s)\}, \end{cases}$$

with $J^\pm(s) = \{k \in \{1, \cdots, \ell(s)-1\},\ \text{s. t } \epsilon_{k+\frac{1}{2}} = \mp 1\}$, so that $d_{\min}^\varepsilon(s) = \min\{d_{\min}^{\varepsilon,+}(s), d_{\min}^{\varepsilon,-}(s)\}$, with the convention that the quantities are equal to L in case the defining set is empty.

At first, we will focus on the case $J^-(s) \neq \emptyset$. The following result provides an upper bound in terms of $d_{\min}^{\varepsilon,-}(s)$ for a dissipation time for the quantized function $\mathfrak{E}_\varepsilon^{\mathrm{L}}$. This phenomenon is related to the cancellation of *a front with its anti-front*, and is the main building block for the proof of Proposition 7.

Proposition 7.1. *There exist $\kappa_1 > 0$, $\alpha_3 > 0$, and $\mathcal{K}_{\mathrm{col}} > 0$, all depending only on V and M_0, with the following properties. If (7.1) holds, if $s_0 \in (\varepsilon^\omega \mathrm{L}^2, S)$ is such that $\kappa_0(s_0) \le \kappa_1$, $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\alpha_3\varepsilon, s_0)$ holds, $J^-(s_0)$ is non empty, and $s_0 + \mathcal{K}_{\mathrm{col}} d_{\min}^{\varepsilon,-}(s_0)^{\omega+2} < S$, then there exists some time $\mathcal{T}_{\mathrm{col}}^{\varepsilon,+}(s_0) \in (s_0, S)$ such that $\mathrm{WPI}_\varepsilon^{\mathrm{L}}(\alpha_3\varepsilon, \mathcal{T}_{\mathrm{col}}^{\varepsilon,+}(s_0))$ holds,*

$$\mathfrak{E}_\varepsilon^{\mathrm{L}}(\mathcal{T}_{\mathrm{col}}^{\varepsilon,+}(s_0)) \le \mathfrak{E}_\varepsilon^{\mathrm{L}}(s_0) - \mu_1, \tag{7.4}$$

where μ_1 is a constant introduced in Lemma 3.2, and

$$\mathcal{T}_{\mathrm{col}}^{\varepsilon,+}(s_0) - s_0 \le \mathcal{K}_{\mathrm{col}} \left(d_{\min}^{\varepsilon,-}(s_0)\right)^{\omega+2}. \tag{7.5}$$

We postpone the proof of Proposition 7.1 to after Section 7.1 below, where we will analyze more into details the attractive and repulsive forces at work at the ε level. We will then prove Proposition 7.1 in Section 7.2, and finally Proposition 7 in Section 7.3.

7.1 Attractive and repulsive forces at the ε level

In this subsection we consider the general situation where $\mathcal{C}_{\mathrm{L},S}$ holds, for some length $\mathrm{L} \ge 0$ and some $S > 0$.

In order to deal with the attractive and repulsive forces underlying annihilations or splittings, we set

$$\mathcal{F}_{k+\frac{1}{2}}(s) = -\omega^{-1}\mathcal{B}_{k+\frac{1}{2}} \left(a_{k+1}^\varepsilon(s) - a_k^\varepsilon(s)\right)^{-\omega}$$

and consider the positive functionals

$$\mathcal{F}^{\varepsilon}_{\rm rep}(s) = \sum_{k\in J^+(s)} \mathcal{F}_{k+\frac{1}{2}}(s),\ \mathcal{F}^{\varepsilon}_{\rm att}(s) = -\sum_{k\in J^-(s)} \mathcal{F}_{k+\frac{1}{2}}(s), \tag{7.6}$$

with the convention that the quantity is equal to $+\infty$ in case the defining set is empty. For some constants $0 < \kappa_2 \le \kappa_3$ depending only on M_0, we have and V

$$\begin{cases} \kappa_2\, \mathcal{F}^{\varepsilon}_{\rm att}(s)^{-\frac{1}{\omega}} \le d^{\varepsilon,-}_{\rm min}(s) \le \kappa_3 \mathcal{F}^{\varepsilon}_{\rm att}(s)^{-\frac{1}{\omega}}, \\ \kappa_2\, \mathcal{F}^{\varepsilon}_{\rm rep}(s)^{-\frac{1}{\omega}} \le d^{\varepsilon,+}_{\rm min}(s) \le \kappa_3 \mathcal{F}^{\varepsilon}_{\rm rep}(s)^{-\frac{1}{\omega}}. \end{cases} \tag{7.7}$$

Let $s_0 \in [\varepsilon^{\omega}\mathrm{L}^2, S]$ be such that

$$\mathcal{WP}^{\rm L}_{\varepsilon}(\alpha_2\varepsilon, s_0) \text{ holds} \qquad \text{and} \qquad d^{\varepsilon,\rm L}_{\rm min}(s_0) \ge 16\mathfrak{q}_1(\alpha_2)\varepsilon. \tag{7.8}$$

We consider as in Corollary 3.3 the stopping time

$$\mathcal{T}^{\varepsilon}_0(\alpha_2, s_0) = \max\left\{s_0 + \varepsilon^{2+\omega} \le s \le S \quad \text{s.t.} \quad d^{\varepsilon,\rm L}_{\rm min}(s') \ge 8\mathfrak{q}_1(\alpha_2)\varepsilon \quad \forall s' \in [s_0 + \varepsilon^{\omega+2}, s]\right\},$$

and for simplicity we will write $\mathcal{T}^{\varepsilon}_0(s_0) \equiv \mathcal{T}^{\varepsilon}_0(\alpha_2, s_0)$. In view of (7.8) and the statement of Corollary 3.3,

$$\mathcal{WP}^{\rm L}_{\varepsilon}(\alpha_1\varepsilon, s) \text{ holds} \qquad \forall\ s \in \mathcal{I}^{\varepsilon}_0(s_0) \equiv [s_0 + \varepsilon^{2+\omega}, \mathcal{T}^{\varepsilon}_0(s_0)].$$

The functionals $\mathcal{F}^{\varepsilon}_{\rm att}$ and $\mathcal{F}^{\varepsilon}_{\rm rep}$ are in particular well defined and continuous on the interval of time $\mathcal{I}^{\varepsilon}_0(s_0)$ with $J^+(s) = J^+(s_0)$ and $J^-(s) = J^-(s_0)$ for all s that interval. Note that the attractive forces are dominant when $d^{\varepsilon,-}_{\rm min}(s) \le d^{\varepsilon,+}_{\rm min}(s)$ and in contrario the repulsive forces are dominant when $d^{\varepsilon,+}_{\rm min}(s) \le d^{\varepsilon,-}_{\rm min}(s)$.

We first focus on the attractive case, and for $s \in \mathcal{I}^{\varepsilon}_0(s_0)$, we introduce the new stopping times

$$\mathcal{T}^{\varepsilon}_1(s) = \inf\{s \le s' \le \mathcal{T}^{\varepsilon}_0(s_0),\ F_{\rm att}(s') \ge \upsilon_1^{\omega} F_{\rm att}(s) \text{ or } s' = \mathcal{T}^{\varepsilon}_0(s_0)\},$$

where $\upsilon_1 = 10\kappa_3^2\kappa_2^{-2}$, so that $\upsilon_1 > 10$ and $\mathcal{T}^{\varepsilon}_1(s) \le \mathcal{T}^{\varepsilon}_0(s_0)$. In view of (7.7), we have

$$\frac{1}{10}\left(\frac{\kappa_2}{\kappa_3}\right)^3 d^{\varepsilon,-}_{\rm min}(s) \le d^{\varepsilon,-}_{\rm min}(\mathcal{T}^{\varepsilon}_1(s)), \tag{7.9}$$

and if $\mathcal{T}^{\varepsilon}_1(s) < \mathcal{T}^{\varepsilon}_0(s_0)$ then

$$d^{\varepsilon,-}_{\rm min}(\mathcal{T}^{\varepsilon}_1(s)) \le \frac{1}{10}\frac{\kappa_2}{\kappa_3} d^{\varepsilon,-}_{\rm min}(s) \le \frac{1}{10} d^{\varepsilon,-}_{\rm min}(s). \tag{7.10}$$

The next result provides an upper bound on $\mathcal{T}^{\varepsilon}_1(s) - s$. Central in our argument is Proposition 6, which we use combined with various arguments by contradiction. We have

Proposition 7.2. *There exists $\beta_0 > 0$, depending only on V and M_0, with the following properties. If $J^-(s_0) \neq \emptyset$, $\hat{s} \in \mathcal{I}^{\varepsilon}_0(s_0)$ and*

$$\beta_0\, \varepsilon \le d^{\varepsilon,-}_{\rm min}(\hat{s}) \le d^{\varepsilon,+}_{\rm min}(\hat{s}), \tag{7.11}$$

then we have

$$\mathcal{T}_1^\varepsilon(\hat{s}) - \hat{s} \leq \mathcal{K}_0 \left(d_{\min}^{\varepsilon,-}(\hat{s}) \right)^{\omega+2}, \tag{7.12}$$

where $\mathcal{K}_0$ *is defined in* (19), *and moreover if* $\mathcal{T}_1^\varepsilon(\hat{s}) < S$ *then*

$$d_{\min}^{\varepsilon,-}(\mathcal{T}_1^\varepsilon(\hat{s})) \leq d_{\min}^{\varepsilon,+}(\mathcal{T}_1^\varepsilon(\hat{s})). \tag{7.13}$$

Proof. Up to a translation of times we may first assume that $\hat{s} = 0$, which eases somewhat the notations. We then argue by contraction and assume that the conclusion is false, that is, there does not exist any such constant β_0, no matter how large it is chosen, such that the conclusion holds. Taking $\beta_0 = n$, this means that given any $n \in \mathbb{N}_*$ there exist some $0 < \varepsilon_n \leq 1$, a solution v_n to $(\mathrm{PGL})_{\varepsilon_n}$ such that $\mathcal{E}_{\varepsilon_n}(v_n) \leq M_0$, such that $\mathrm{WP}_{\varepsilon_n}^{\mathrm{L}_0}(\alpha_1\varepsilon_n, 0)$ holds, such that

$$n\varepsilon_n \leq d_{\min}^{\varepsilon_n,-}(0) = d_{\min}^{\varepsilon_n}(0) \leq d_{\min}^{\varepsilon_n,+}(0), \tag{7.14}$$

and such that one of the conclusion fails, that is such that either

$$\mathcal{T}_1^n \equiv \mathcal{T}_1^{\varepsilon_n}(0) > \mathcal{K}_0 \left(d_{\min}^{\varepsilon,-}(0)\right)^{\omega+2}, \tag{7.15}$$

or

$$d_{\min}^{\varepsilon,-}(\mathcal{T}_1^n) > d_{\min}^{\varepsilon,+}(\mathcal{T}_1^n). \tag{7.16}$$

Setting $S_0^n = \mathcal{K}_0(d_{\min}^{\varepsilon_n,-}(0))^{\omega+2}$, relation (7.15) may be rephrased as

$$F_{\mathrm{att}}^n(s) \leq \ \upsilon_1^\omega F_{\mathrm{att}}^n(0) \ \text{ and } \ d_{\min}^{\varepsilon_n}(s) \geq 8\mathfrak{q}_1(\alpha_2)\varepsilon_n \ \text{ for any } s \in [0, S_0^n], \tag{7.17}$$

where the superscripts n refer to the corresponding functionals computed for the map v_n. Passing possibly to a subsequence, we may therefore assume that one at least of the properties (7.17) or (7.16) holds for any $n \in \mathbb{N}_*$. Also, passing possibly to a further subsequence, we may assume that the total number of fronts of $v_n(0)$ inside $[-\mathrm{L}, \mathrm{L}]$ is constant, equal to a number ℓ, denote $a_1^n(s), \cdots, a_\ell^n(s)$ the corresponding front points, for $s \in [0, \mathcal{T}_1^n]$, and set $d_n^-(s) = d_{\min}^{\varepsilon_n,-}(s), d_n^+(s) = d_{\min}^{\varepsilon_n,+}(s), d_n(s) = d_{\min}^{\varepsilon_n}(s)$.

In order to obtain a contradiction we shall make use of *the scale invariance* of the equation: if v_ε is a solution to $(\mathrm{PGL})_\varepsilon$ then the map $\tilde{v}_{\tilde{\varepsilon}}(x,t) = v_\varepsilon(rx, r^2t)$ is a solution to $(\mathrm{PGL})_{\tilde{\varepsilon}}$ with $\tilde{\varepsilon} = r^{-1}\varepsilon$. As scaling factor r_n, we choose $r_n = d_{\min}^{\varepsilon_n,-}(0) \geq n\varepsilon_n$ and set

$$\tilde{v}_n(x,t) = v_n(r_nx, r_n^2t), \ \ \tilde{\mathfrak{v}}_n(x,\tau) = \tilde{v}_n(x, \tilde{\varepsilon}_n^{-\omega}\tau), \tag{7.18}$$

so that $\tilde{v}_n$ is a solution to $(\mathrm{PGL})_{\tilde{\varepsilon}_n}$ satisfying $\mathcal{WP}_{\tilde{\varepsilon}_n}^{L_n}(\alpha_2\tilde{\varepsilon}_n, 0)$ with $L_n = r_n^{-1}\mathrm{L}$ and

$$\tilde{\varepsilon}_n = (r_n)^{-1}\varepsilon_n = (d_{\min}^{\varepsilon_n,-}(0))^{-1}\varepsilon_n \leq \frac{1}{n}, \ \text{ hence we have } \varepsilon_n \to 0 \text{ as } n \to +\infty.$$

The points $\tilde{a}_1^n(s) = r_n^{-1}a_1^n(r_n^{-(2+\omega)}s), \cdots, \tilde{a}_\ell^n(s) = r_n^{-1}a_\ell^n(r_n^{-(2+\omega)}s)$ are the front points of $\tilde{\mathfrak{v}}_n$. Let $\tilde{d}_n^-$, $\tilde{d}_n^+$, $\tilde{d}_n$ be the quantities corresponding to $d_{\min}^{\varepsilon,-}, d_{\min}^{\varepsilon,+}, d_{\min}^\varepsilon$ for $\tilde{\mathfrak{v}}_n$, so that

$$\tilde{d}_n^-(s) = r_n^{-1}d_n^-(r_n^{-(2+\omega)}s), \ \ \tilde{d}_n^+(s) = r_n^{-1}d_n^+(r_n^{-(2+\omega)}s), \ \text{ and } \ \tilde{d}_n(s) = r_n^{-1}d_n(r_n^{-(2+\omega)}s),$$

and notice that $d_n^-(0) = d_n(0) = 1$. We next distinguish the following two complementing cases.

Case 1: (7.17) holds for all $n \in \mathbb{N}_*$. It follows from assumption (7.17) that $\mathrm{WP}^{L_n}_{\tilde{\varepsilon}_n}(\alpha_1 \tilde{\varepsilon}_n, \tau)$ holds for every $\tau \in (0, \tilde{S}_1^n)$, where $\tilde{\mathcal{S}}_1^n = r_n^{-(2+\omega)} S_0^n = \mathcal{K}_0$. Let $k_0 \in \{1, \cdots, \ell\}$ be such that

$$a^n_{k_0+1}(0) - a^n_{k_0}(0) = d^{\varepsilon_n,-}_{\min}(0).$$

Upon a translation if necessary, we may also assume that $a^n_{k_0}(0) = 0$ so that $a^n_{k_0+1}(0) = d^{\varepsilon_n,-}_{\min}(0)$. We denote by $\tilde{\mathcal{F}}^n_{\mathrm{att}}$ the functional $\mathcal{F}^{\varepsilon}_{\mathrm{att}}$ computed for the front points of $\tilde{\mathfrak{v}}_n$, so that

$$\tilde{\mathcal{F}}^n_{\mathrm{att}}(r_n^{-(2+\omega)} s) = r_n^{(2+\omega)} \mathcal{F}^n_{\mathrm{att}}(s).$$

By construction we have

$$\tilde{a}^n_{k_0}(0) = 0 \text{ and } \tilde{a}^n_{k_0+1}(0) = 1 = \tilde{d}^-_n(0). \tag{7.19}$$

Since $\tilde{\varepsilon}_n \to 0$ as $n \to \infty$, we may implement part of the already established asymptotic analysis for $(\mathrm{PGL})_\varepsilon$ on the sequence $(\tilde{v}_n)_{n\in\mathbb{N}}$. First, passing possibly to a subsequence, we may assume that for some subset $\tilde{J} \subset J(0)$ the points $\{\tilde{a}_k(0)\}_{k\in\tilde{J}}$ converge to some finite limits $\{\tilde{a}^0_k\}_{k\in\tilde{J}}$, whereas the points with indices in $J(0) \setminus \tilde{J}$ diverge either to $+\infty$ or to $-\infty$. We choose $\tilde{\mathrm{L}} \geq 1$ so that

$$\underset{k\in\tilde{J}}{\cup} \{\tilde{a}^0_k\} \subset [-\frac{\tilde{\mathrm{L}}}{2}, \frac{\tilde{\mathrm{L}}}{2}]. \tag{7.20}$$

In view of (7.19), we have $\tilde{a}_{k_0}(0) = 0, \tilde{a}_{k_0+1} = 1$ and $\inf\{|\tilde{a}_{k+1}(0) - \tilde{a}_k(0)|, k \in \tilde{J}\} = 1$. We are hence in position to apply the convergence result stated in Proposition 6 to the sequence $(\tilde{\mathfrak{v}}_n(\cdot))_{n\in\mathbb{N}}$. It states that the front points $(\tilde{a}^n_k(\tau))_{k\in J_0}$ which do not escape at infinity converge to the solution $(\tilde{a}_k(\cdot))_{k\in\tilde{J}}$ of the ordinary differential equation $(\mathcal{S})$ supplemented with the corresponding initial values $(\tilde{a}_k(0))_{k\in\tilde{J}}$, uniformly in time on every compact subset of $(0, \tilde{S}_{\max})$, where $\tilde{S}_{\max}$ denotes the maximal time of existence for the solution. In particular, we have

$$\begin{cases} \tilde{d}^-_n(\tau) \to \mathfrak{d}^-_{\tilde{a}}(\tau), \text{ uniformly on every compact subset of } (0, \tilde{S}_{\max}), \\ \limsup\limits_{n\to+\infty} \tilde{F}^n_{\mathrm{att}}(\tau) \geq F_{\mathrm{att}}(\tilde{a}(\tau)) \text{ forevery } \tau \in (0, \tilde{S}_{\max}), \end{cases}$$

the presence of the lim sup being related to the fact that some points might escape at infinity so that the limiting values of the functionals are possibly smaller. We use next the properties of the differential equation $(\mathcal{S})$ established in Appendix B. We first invoke Proposition 1 which asserts that $\tilde{S}_{\max} \leq \mathcal{K}_0$ and that

$$F_{\mathrm{att}}(\tilde{a}(\tau)) \to +\infty \text{ as } \tau \to \tilde{S}_{\max}.$$

Hence, there exists some $\tau_1 \in (0, \tilde{S}_{\max}) \subset (0, \mathcal{K}_0)$ such that, if n is sufficiently large, then

$$\tilde{\mathcal{F}}^n_{\mathrm{att}}(\tau_1) > \upsilon^\omega_1 \tilde{\mathcal{F}}^n_{\mathrm{att}}(0).$$

Going back to the original time scale, this yields $\mathcal{F}^n_{\mathrm{att}}(r_n^{\omega+2}\tau_1) > \upsilon^\omega_1 \mathcal{F}^n_{\mathrm{att}}(0)$. Since $r_n^{\omega+2}\tau_1 \in (0, r_n^{2+\omega}\mathcal{K}_0) = (0, S^n_0)$ this contradicts (7.17) and completes the proof in Case 1.

Case 2: (7.16) holds for all $n \in \mathbb{N}_*$. We consider an arbitrary index $j \in J^+$. As above, translating the origin, we may assume without loss of generally that $a^n_j(0) = 0$. We also define the map $\mathfrak{v}_n$ as in Case 1, according to the same scaling as described in (7.18), the only difference being that the origin has been shifted differently. With similar notations, we have

$$\tilde{a}_j^n(0) = 0 \text{ and } \tilde{a}_{j+1}^n(0) \geq 1 = \tilde{d}_n^-(0).$$

Passing possibly to a further subsequence, we may assume that the front points at time 0 converge to some limits in $\bar{\mathbb{R}}$ denoted $\tilde{a}_k(0)$. We are hence again in position to apply the convergence result of Proposition 6, so that the front points $(\tilde{a}_k^n(s))_{k\in J_j}$ which do not escape at infinity converge to the solution $(\tilde{a}_k(\cdot))_{k\in J_j}$ of the ordinary differential equation $(\mathcal{S})$ supplemented with the corresponding initial values $(\tilde{a}_k(0))_{k\in J_j}$, uniformly in time on every compact subset of $(0, \tilde{S}'_{\max})$, where $\tilde{S}'_{\max}$ denotes the (new) maximal time of existence for the solution. It follows from assumption (7.39), Theorem 1 and scaling that $0 < \tilde{\mathcal{T}}_1 \equiv \liminf \tilde{\mathcal{T}}_1^n \leq \tilde{S}'_{\max}$. We claim that, for any $\tau \in (0, \tilde{\mathcal{T}}_1)$, and for sufficiently large n, we have

$$|\tilde{a}_j^n(\tau) - \tilde{a}_{j+1}^n(\tau)| \geq \frac{\kappa_2}{2\kappa_3}. \tag{7.21}$$

This is actually a property of the differential equation $(\mathcal{S})$. We have indeed, in view of Proposition B.1, $0 < F_{\rm rep}(\tilde{a}(\tau)) \leq F_{\rm rep}(\tilde{a}(0))$, so that it follows from (7.7) that

$$|\tilde{a}_j(\tau) - \tilde{a}_{j+1}(\tau)| \geq \frac{\kappa_2}{\kappa_3},$$

which yields (7.21) taking the convergence into account. Since (7.21) holds for any j, we deduce that

$$d_{\min}^{\varepsilon_n,+}(\mathcal{T}_1^n) \geq \frac{\kappa_2}{2\kappa_3} d_{\min}^{\varepsilon_n,-}(0)$$

and therefore by (7.16) we have

$$d_{\min}^{\varepsilon_n}(\mathcal{T}_1^n) = d_{\min}^{\varepsilon_n,-}(\mathcal{T}_1^n) \geq d_{\min}^{\varepsilon_n,+}(\mathcal{T}_1^n) \geq \frac{\kappa_2}{2\kappa_3} d_{\min}^{\varepsilon_n,-}(0) \geq \frac{\kappa_2}{2\kappa_3} n\varepsilon. \tag{7.22}$$

For n sufficiently large, this implies that $\mathcal{T}_1^n < \mathcal{T}_0^n$, and therefore from (7.10) we have

$$d_{\min}^{\varepsilon_n,-}(\mathcal{T}_1^n) \leq \frac{1}{10}\frac{\kappa_2}{\kappa_3} d_{\min}^{\varepsilon_n,-}(0),$$

which is in contradiction with (7.22). □

We turn now to the case where $d_{\min}^{\varepsilon,+}(s) \leq d_{\min}^{\varepsilon,-}(s)$. In order to handle the repulsive forces at work, for $s \in \mathcal{I}_0^\varepsilon(s_0)$ we introduce the new stopping times

$$\mathcal{T}_2^\varepsilon(s) = \inf\{s \leq s' \leq \mathcal{T}_0^\varepsilon(s_0),\ \mathcal{F}_{\rm rep}^\varepsilon(s') \leq \mathfrak{v}_2^\omega \mathcal{F}_{\rm rep}^\varepsilon(s) \text{ or } s' = \mathcal{T}_0^\varepsilon(s_0)\},$$

where $\mathfrak{v}_2 = \frac{\kappa_2^2}{10\kappa_3^2}$, so that $\mathfrak{v}_2 < 1$. Notice that, in view of (7.7), we have, if $\mathcal{T}_2^\varepsilon(s) < \mathcal{T}_0^\varepsilon(s_0)$,

$$d_{\min}^{\varepsilon,+}(\mathcal{T}_2^\varepsilon(s)) \geq \mathfrak{v}_2^{-1}\frac{\kappa_2}{\kappa_3} d_{\min}^{\varepsilon,+}(s) \geq 10\, d_{\min}^{\varepsilon,+}(s). \tag{7.23}$$

With $\mathcal{S}_1$ introduced in Proposition 1, we set

$$\mathcal{K}_1 = \mathcal{S}_1^{-\omega}\left(\frac{2\kappa_3}{\kappa_2\mathfrak{v}_2}\right)^{\omega+2}. \tag{7.24}$$

Proposition 7.3. *There exists* $\beta_1 > 0$*, depending only on* V *and* M_0*, with the following properties. If* $J^+(s_0) \neq \emptyset$*,* $\hat{s} \in \mathcal{I}_0^\varepsilon(s_0)$ *and*

$$\beta_1 \varepsilon \leq d_{\min}^{\varepsilon,+}(\hat{s}) \leq d_{\min}^{\varepsilon,-}(\hat{s}), \tag{7.25}$$

then we have

$$\mathcal{T}_2^\varepsilon(\hat{s}) - \hat{s} \leq \mathcal{K}_1 \left(d_{\min}^{\varepsilon,+}(\hat{s})\right)^{\omega+2}, \tag{7.26}$$

and if $\mathcal{T}_2^\varepsilon(\hat{s}) < S$ *then* $\mathcal{T}_2^\varepsilon(\hat{s}) < \mathcal{T}_0^\varepsilon(s_0)$ *and for any* $s \in [\hat{s}, \mathcal{T}_2^\varepsilon(\hat{s})]$*, we have*

$$d_{\min}^\varepsilon(s) \geq \frac{1}{2}\mathcal{S}_2 d_{\min}^{\varepsilon,+}(\hat{s}), \tag{7.27}$$

and

$$\mathcal{F}_{\rm att}^\varepsilon(s)^{-\frac{1}{\omega}} \leq \mathcal{F}_{\rm att}^\varepsilon(\hat{s})^{-\frac{1}{\omega}} + \frac{1}{\kappa_3}(d_{\min}^{\varepsilon,+}(\hat{s})), \tag{7.28}$$

where $\mathcal{S}_2$ *is defined in Proposition 1 and* κ_3 *is defined in* (7.7)*.*

Proof. The argument possesses strong similarities with the proof of Proposition 7.2, we therefore just sketch its main points, in particular relying implicitly on the notations introduced there, as far as this is possible. By translation in time we also assume that $\hat{s} = 0$ and argue by contradiction assuming that for any $n \in \mathbb{N}_*$ there exist some $0 < \varepsilon_n \leq 1$, a solution v_n to $(\mathrm{PGL})_{\varepsilon_n}$ such that $\mathcal{E}_{\varepsilon_n}(v_n) \leq M_0, \mathcal{WP}_{\varepsilon_n}^{\rm L}(\alpha_1\varepsilon_n, 0)$ holds, such that $n\varepsilon_n \leq d_n^+(0)$, and such that either, we have for any $s \in (0, S_1^n)$, where $S_1^n = \mathcal{K}_1 d_n^+(0)^{\omega+2}$, $d_n(s) \geq 8\mathfrak{q}(\alpha_2)\varepsilon_n$ and

$$\kappa_3^\omega(d_n^+(s'))^{-\omega} \geq F_{\rm rep}^n(s') \geq \upsilon_2^\omega F_{\rm rep}^n(0) \geq \upsilon_2^\omega \kappa_2^\omega (d_n^+(0))^{-\omega} \tag{7.29}$$

or, there is some $\tau_n \in (0, \mathcal{T}_2^n)$ such that

$$d_{\min}^{\varepsilon,+}(\tau_n) < \frac{1}{2}\mathcal{S}_2 d_{\min}^{\varepsilon,+}(s) \tag{7.30}$$

or

$$\mathcal{F}_{\rm att}^n(\tau_n)^{-\frac{1}{\omega}} < \mathcal{F}_{\rm att}^n(0)^{-\frac{1}{\omega}} + \frac{1}{32\kappa_3}(d_n^+(s)). \tag{7.31}$$

As in (7.18), but with a different scaling r_n we set

$$r_n = d_{\min}^{\varepsilon_n,+}(0) \geq n\varepsilon_n,\ \tilde{v}_n(x,t) = v_n(r_n x, r_n^2 t), \text{ and } \tilde{\mathfrak{v}}_n(x,s) = \tilde{v}_n(x, \tilde{\varepsilon}_n^{-\omega} s). \tag{7.32}$$

We verify that $\tilde{v}_n$ is a solution to $(\mathrm{PGL})_{\tilde{\varepsilon}_n}$ with $\tilde{\varepsilon}_n = (r_n)^{-1}\varepsilon_n \to 0$ as $n \to \infty$ and that the points $\tilde{a}_k^n(\tau) = r_n^{-1} a_k^n(r_n^{-(2+\omega)}\tau)$ for $k \in J$, are the front points of $\tilde{\mathfrak{v}}_n$. We distinguish three cases, which are complementing going if necessary to subsequences.

Case 1: (7.29) *holds, for any* $n \in \mathbb{N}$. It follows $\mathcal{WP}_{\tilde{\varepsilon}_n}^{L_n}(\alpha_1\tilde{\varepsilon}_n, \tau)$ holds for every $\tau \in (0, \tilde{\mathcal{S}}_1^n)$, where $\tilde{\mathcal{S}}_1^n = r_n^{-(2+\omega)}\mathcal{S}_1^n = \mathcal{K}_1$. Let j be an arbitrary index in J^+. Translating if necessary the origin, we may assume that $a_j^n(0) = 0$ so that $a_{j+1}^n(0) \geq d_n^+(0) \geq n\varepsilon_n$ and hence $\tilde{a}_{j+1}^n(0) - \tilde{a}_j^n(0) \geq 1$. Since $\tilde{\varepsilon}_n \to 0$ as $n \to \infty$, we may implement part of the already established asymptotic analysis for $(\mathrm{PGL})_\varepsilon$ on the sequence $(\tilde{v}_n)_{n\in\mathbb{N}}$. First, passing possibly to a subsequence, we may assume that for some subset $\tilde{J} \subset J(0)$ the points $\{\tilde{a}_k(0)\}_{k\in\tilde{J}}$ converge to some finite limits $\{\tilde{a}_k^0\}_{k\in\tilde{J}}$, whereas the points with indices in $J(0) \setminus \tilde{J}$ diverge either to

$+\infty$ or to $-\infty$. We choose $\tilde{\mathrm{L}} \geq 1$ so that (7.20) holds. It follows from Proposition 6 that for $\tau \in (0, \mathcal{K}_1)$, we have

$$|\tilde{a}_{j+1}^n(\tau) - \tilde{a}_j^n(\tau)| \to |\tilde{a}_{j+1}(\tau) - \tilde{a}_j(\tau)| \geq \left(\mathcal{S}_1\tau + \mathcal{S}_2 \mathfrak{d}_{\tilde{a}}^+(0)^{\omega+2}\right)^{\frac{1}{\omega+2}} = \left(\mathcal{S}_1\tau + \mathcal{S}_2\right)^{\frac{1}{\omega+2}}$$

as $n \to \infty$, where the last inequality is a consequence of Proposition 1. Taking the infimum over J^+, we obtain, for n sufficiently large

$$\tilde{d}_n^+(\tau) = \inf_{j \in J^+} |\tilde{a}_{j+1}^n(\tau) - \tilde{a}_j^n(\tau)| \geq \frac{1}{2}\left(\mathcal{S}_1\tau + \mathcal{S}_2\right)^{\frac{1}{\omega+2}} \geq \frac{1}{2}\left(\mathcal{S}_1\tau\right)^{\frac{1}{\omega+2}}, \forall \tau \in (0, \mathcal{K}_1), \tag{7.33}$$

On the other hand, going back to (7.29), with the same notation as in Proposition 7.2, we are led to the inequality

$$\tilde{d}_n^+(\tau) \leq \kappa_3 \kappa_2^{-1} \upsilon_2^{-1} \text{ for } \tau \in (0, \mathcal{K}_1). \tag{7.34}$$

In view of our choice (7.24) of $\mathcal{K}_1$, relations (7.33) and (7.34) are contradictory for τ close to $\mathcal{K}_1$ yielding hence a contradiction in Case 1.

Case 2: (7.29) *does not hold, but* (7.30) *holds, for any* $n \in \mathbb{N}$. The argument is almost identical, we conclude again thanks to (7.33) but keeping $\mathcal{S}_2$ instead of $\mathcal{S}_1\tau$ in its last inequality.
Case 3: (7.29) *does not hold but* (7.31) *holds, for any* $n \in \mathbb{N}$. As in the proof of Proposition 7.2, we conclude that $0 < \tilde{\mathcal{T}}_2 \equiv \liminf_{n\to+\infty} \tilde{\mathcal{T}}_2^n$. This situation is slightly more delicate than the ones analyzed so far, and we have *to track also the fronts escaping possibly at infinity.* Up to a subsequence, we may assume that the set J is decomposed as a disjoint union of clusters $J = \overset{q}{\underset{i=1}{\cup}} J_p$ where each of the sets J_p is an ordered set of $m_p + 1$ consecutive points, that is $J_p = \{k_p, k_p + 1, \cdots k_p + m_p\}$ and such that the two following properties holds:

- There exists a constant $C > 0$ independent of n such that

$$|\tilde{a}_{k_p}^n(0) - \tilde{a}_{k_p+r}(0)| \leq C \text{ for any } p \in \{1, \cdots, q\} \text{ and any } r \in \{k_p, \cdots, m_p\} \tag{7.35}$$

- For $1 \leq p_1 < p_2 \leq q$, we have $\tilde{a}_{k_{p_2}}^n - \tilde{a}_{k_{p_1}}^n \to +\infty$.

For a given $p \in \{1, \cdots, q\}$, translating if necessary the origin, we may assume that $\tilde{a}_{k_p}^n(0) = 0$, and passing possibly to a further subsequence, that the front points at time 0 converge as $n \to +\infty$ to some limits denoted $\tilde{a}_{p,k}(0)$, for $k \in \{k_p, \cdots, k_p + m_p\}$. Notice that, as an effect of the scaling, all other front points diverge to infinity, in the chosen frame. We apply now Proposition 6 to this cluster of points : it yields uniform convergence, for $k \in \{k_p, \cdots, k_p + m_p\}$ of the front points $\tilde{a}_k^n(\cdot)$ to the solution $\tilde{a}_{p,k}(\cdot)$ of the differential equation $(\mathcal{S})$ supplemented with the initial time conditions $\tilde{a}_{p,k}(0)$ defined above. If F_{att}^p denotes the functional F_{att} defined in (7.6) restricted to the points of the cluster J_p, we have in view of (B.17)

$$\frac{d}{d\tau} F_{\mathrm{att}}^p(\tau) \geq 0, \qquad \text{for any } p = 1, \cdots, q, \quad \text{for any } \tau \in (0, \tilde{\mathcal{T}}_2).$$

On the other hand, since the mutual distances between the distinct clusters diverge towards infinity, and hence their mutual interactions energies tend to zero, one obtains, in view of the uniform convergence for each separate cluster, that

$$\lim_{n\to+\infty} \tilde{\mathcal{F}}_{\mathrm{att}}^n(\tau) = \sum_{p=1}^q F_{\mathrm{att}}^p(\tau) \geq \sum_{p=1}^q F_{\mathrm{att}}^p(0) = \lim_{n\to+\infty} \tilde{\mathcal{F}}_{\mathrm{att}}^n(0), \qquad \text{for } \tau \in \ (0, \tilde{\mathcal{T}}_2).$$

Therefore, for n sufficiently large we are led to

$$\tilde{\mathcal{F}}^n_{\rm att}(\tilde{\mathcal{T}}_2)^{-\frac{1}{\omega}} \leq \tilde{\mathcal{F}}^n_{\rm att}(0)^{-\frac{1}{\omega}} + \frac{1}{2\kappa_3}.$$

Scaling back to the original variables, this contradicts (7.31) and hence completes the proof. □

From Proposition 7.2 and Proposition 7.3 we obtain

Proposition 7.4. *There exists $\mathcal{K}_2 > 0$, depending only on V and M_0, with the following properties. Assume that $J^-(s_0) \neq \emptyset$ and that $s \in \mathcal{I}_0^\varepsilon(s_0)$ satisfies*

$$d^{\varepsilon,\rm L}_{\rm min}(s) \geq \max(\beta_0, \beta_1)\varepsilon, \qquad \textit{and} \qquad s + \mathcal{K}_2 d^{\varepsilon,-}_{\rm min}(s)^{\omega+2} < S. \tag{7.36}$$

Then there exists some time $\mathcal{T}^-_{\rm col}(s) \in \mathcal{I}_0^\varepsilon(s_0)$ such that

$$\mathcal{T}^{\varepsilon,-}_{\rm col}(s) - s \leq \mathcal{K}_2 d^{\varepsilon,-}_{\rm min}(s)^{\omega+2}, \tag{7.37}$$

and

$$d^{\varepsilon,\rm L}_{\rm min}(\mathcal{T}^{\varepsilon,-}_{\rm col}(s)) \leq \max\left(\beta_0, 8\mathfrak{q}_1(\alpha_2)\right)\varepsilon. \tag{7.38}$$

Proof. We distinguish two cases.

Case I:

$$d^{\varepsilon,\rm L}_{\rm min}(s) = d^{\varepsilon,-}_{\rm min}(s) \leq d^{\varepsilon,+}_{\rm min}(s). \tag{7.39}$$

In that case we will make use of Proposition 7.2 in an iterative argument. In view of (7.36), we are in position to invoke Proposition 7.2 at time $\hat{s} = s$ and set $s_1 = \mathcal{T}_1^\varepsilon(s)$, so that in particular

$$s_1 - s \leq \mathcal{K}_0 d^{\varepsilon,-}_{\rm min}(s)^{\omega+2} \qquad \text{and} \qquad d^{\varepsilon,-}_{\rm min}(s_1) \leq d^{\varepsilon,+}_{\rm min}(s_1). \tag{7.40}$$

Notice that by (7.36) and (7.40) we have $s_1 < S$.
We distinguish two sub-cases:

Case I.1: $s_1 = \mathcal{T}_0^\varepsilon(s_0)$ or $d^{\varepsilon,-}_{\rm min}(s_1) < \beta_0\varepsilon$. In that case, we simply set $\mathcal{T}^{\varepsilon,-}_{\rm col}(s) = s_1$ and we are done if we require $\mathcal{K}_2 \geq \mathcal{K}_2$, by (7.40) and the definition of $\mathcal{T}_0^\varepsilon(s_0)$.

Case I.2: $s_1 < \mathcal{T}_0^\varepsilon(s_0)$ and $d^{\varepsilon,-}_{\rm min}(s_1) \geq \beta_0\varepsilon$. In that case, we may apply Proposition 7.2 at time $\hat{s} = s_1$ and set $s_2 = \mathcal{T}_1^\varepsilon(s_1)$, so that in particular

$$s_2 - s_1 \leq \mathcal{K}_0 d^{\varepsilon,-}_{\rm min}(s_1)^{\omega+2} \qquad \text{and} \qquad d^{\varepsilon,-}_{\rm min}(s_2) \leq d^{\varepsilon,+}_{\rm min}(s_2). \tag{7.41}$$

Moreover, since in that case $s_1 = \mathcal{T}_1^\varepsilon(s) < \mathcal{T}_0^\varepsilon(s_0)$, it follows from (7.10) that

$$d^{\varepsilon,-}_{\rm min}(s_1) \leq \frac{1}{10} d^{\varepsilon,-}_{\rm min}(s), \tag{7.42}$$

and therefore from (7.41) we actually have

$$s_2 - s_1 \leq \mathcal{K}_0 10^{-(\omega+2)} d^{\varepsilon,-}_{\rm min}(s)^{\omega+2} \qquad \text{and} \qquad d^{\varepsilon,-}_{\rm min}(s_2) \leq d^{\varepsilon,+}_{\rm min}(s_2). \tag{7.43}$$

We then iterate the process until we fall into Case I.1. If we have not reached that stage up to step m, then thanks to Proposition 7.2 applied at time $\hat{s} = s_m$ we obtain, with $s_{m+1} := \mathcal{T}_1^\varepsilon(s_m)$,

$$s_{m+1} - s_m \le \mathcal{K}_0 d_{\min}^{\varepsilon,-}(s_m)^{\omega+2} \qquad \text{and} \qquad d_{\min}^{\varepsilon,-}(s_{m+1}) \le d_{\min}^{\varepsilon,+}(s_{m+1}). \tag{7.44}$$

Moreover, since Case I.1 was not reached before step m, we have $s_p = \mathcal{T}_1^\varepsilon(s_{p-1}) < \mathcal{T}_0^\varepsilon(s_0)$ for all $p \le m$, so that repeated use of (7.10) yields

$$d_{\min}^{\varepsilon,-}(s_p) \le \left(\frac{1}{10}\right)^p d_{\min}^{\varepsilon,-}(s), \qquad \forall p \le m. \tag{7.45}$$

From (7.44) we thus also have

$$s_{p+1} - s_p \le \mathcal{K}_0 10^{-p(\omega+2)} d_{\min}^{\varepsilon,-}(s)^{\omega+2}, \qquad \forall p \le m, \tag{7.46}$$

and therefore by summation

$$s_{m+1} - s \le \mathcal{K}_0 (\sum_{p=0}^{m} 10^{-p(\omega+2)}) d_{\min}^{\varepsilon,-}(s)^{\omega+2}, \tag{7.47}$$

so that in particular from (7.36) it holds $s_{m+1} < S$ if we choose $\mathcal{K}_2 \ge 2\mathcal{K}_0$. It follows from (7.45) that Case I.1 is necessarily reached in a finite number of steps, thus defining $\mathcal{T}_{\rm col}^{\varepsilon,-}(s)$, and from (7.47) we obtain the upper bound

$$\mathcal{T}_{\rm col}^{\varepsilon,-}(s) - s \le \mathcal{K}_0 (\sum_{p=0}^{\infty} 10^{-p(\omega+2)}) d_{\min}^{\varepsilon,-}(s)^{\omega+2} \le 2K_0 d_{\min}^{\varepsilon,-}(s)^{\omega+2}, \tag{7.48}$$

from which (7.37) follows.

Case II:

$$d_{\min}^{\varepsilon,\rm L}(s) = d_{\min}^{\varepsilon,+}(s) < d_{\min}^{\varepsilon,-}(s). \tag{7.49}$$

Note that this implies that $J^+(s_0) \neq \emptyset$. We will show that Case II can be reduced to Case I after some controlled interval of time necessary for the repulsive forces to push $d_{\min}^{\varepsilon,+}$ above $d_{\min}^{\varepsilon,-}$. More precisely, we define the stopping time

$$\mathcal{T}_{\rm cros}^\varepsilon(s) = \inf\{\mathcal{T}_0^\varepsilon(s) \ge s' \ge s,\, d_{\min}^{\varepsilon,-}(s') \le d_{\min}^{\varepsilon,+}(s')\}.$$

As in Case I, we implement an iterative argument, but based this time on Proposition 7.3. In view of (7.49) and (7.36), we may apply Proposition 7.3 at time $\hat{s} = s$ and set $s_1 = \mathcal{T}_2^\varepsilon(s)$, so that in particular

$$s_1 - s \le \mathcal{K}_1 d_{\min}^{\varepsilon,+}(s)^{\omega+2} \le \mathcal{K}_1 d_{\min}^{\varepsilon,-}(s)^{\omega+2}. \tag{7.50}$$

Notice that by (7.36) and (7.50) we have $s_1 < S$ and therefore $d_{\min}^{\varepsilon,+}(s_1) \ge 10 d_{\min}^{\varepsilon,+}(s) \ge \beta_1$, and by (7.28)

$$\begin{aligned} \mathcal{F}_{\rm att}^\varepsilon(s_1)^{-\frac{1}{\omega}} &\le \mathcal{F}_{\rm att}^\varepsilon(s)^{-\frac{1}{\omega}} + \frac{1}{\kappa_3} d_{\min}^{\varepsilon,+}(s) \\ &\le \mathcal{F}_{\rm att}^\varepsilon(s)^{-\frac{1}{\omega}} + \frac{1}{10\kappa_3} d_{\min}^{\varepsilon,+}(s_1). \end{aligned} \tag{7.51}$$

We distinguish two sub-cases.

Case II.1: $s_1 \geq \mathcal{T}^{\varepsilon}_{\rm cros}(s)$. In that case we proceed to Case I which we will apply starting at s_1 instead of s and we set $\mathcal{T}^{\varepsilon,-}_{\rm col}(s) := \mathcal{T}^{\varepsilon,-}_{\rm col}(s_1)$. Since, combining the first inequality of (7.51) with (7.7), we deduce that

$$d^{\varepsilon,-}_{\min}(s_1) \leq \kappa_3\kappa_2^{-1} d^{\varepsilon,-}_{\min}(s) + d^{\varepsilon,+}_{\min}(s) \leq \left(\kappa_3\kappa_2^{-1} + 1\right) d^{\varepsilon,-}_{\min}(s), \tag{7.52}$$

the equivalent of (7.48) becomes

$$\begin{aligned} \mathcal{T}^{\varepsilon,-}_{\rm col}(s_1) - s_1 &\leq \mathcal{K}_0 (\sum_{p=0}^{\infty} 10^{-p(\omega+2)}) d^{\varepsilon,-}_{\min}(s_1)^{\omega+2} \\ &\leq 2\mathcal{K}_0 d^{\varepsilon,-}_{\min}(s_1)^{\omega+2} \\ &\leq 2\mathcal{K}_0 \left(\kappa_3\kappa_2^{-1} + 1\right)^{\omega+2} d^{\varepsilon,-}_{\min}(s)^{\omega+2}, \end{aligned} \tag{7.53}$$

and therefore it follows from (7.50) that

$$\mathcal{T}^{\varepsilon,-}_{\rm col}(s) - s \leq \mathcal{T}^{\varepsilon,-}_{\rm col}(s_1) - s_1 + (s_1 - s) \leq \left(\mathcal{K}_1 + 2\mathcal{K}_0 \left(\kappa_3\kappa_2^{-1} + 1\right)^{\omega+2}\right) d^{\varepsilon,-}_{\min}(s)^{\omega+2}, \tag{7.54}$$

and (7.37) follows if $\mathcal{K}_2 \geq \mathcal{K}_1 + 2\mathcal{K}_0 \left(\kappa_3\kappa_2^{-1} + 1\right)^{\omega+2}$.

Case II.2: $s_1 < \mathcal{T}^{\varepsilon}_{\rm cros}(s)$. In that case we proceed to construct $s_2 = \mathcal{T}^{\varepsilon}_2(s_1)$. Notice that combining the second inequality of (7.51) with (7.7), we deduce that

$$d^{\varepsilon,-}_{\min}(s_1) \leq \kappa_3\kappa_2^{-1} d^{\varepsilon,-}_{\min}(s) + \frac{1}{5} d^{\varepsilon,+}_{\min}(s_1) \leq \kappa_3\kappa_2{}^{-1} d^{\varepsilon,-}_{\min}(s) + \frac{1}{5} d^{\varepsilon,-}_{\min}(s_1), \tag{7.55}$$

so that

$$d^{\varepsilon,+}_{\min}(s_1) \leq d^{\varepsilon,-}_{\min}(s_1) \leq \frac{5}{4}\kappa_3\kappa_2^{-1} d^{\varepsilon,-}_{\min}(s). \tag{7.56}$$

We explain now the iterative argument. Assume that for some $m \geq 1$ have already constructed $s_1, \cdots, s_m$, such that for $2 \leq p \leq m$

$$s_p < S, \qquad \beta_1\varepsilon \leq d^{\varepsilon,+}_{\min}(s_p) \leq d^{\varepsilon,-}_{\min}(s_p), \qquad s_p = \mathcal{T}^{\varepsilon}_2(s_{p-1}).$$

First, repeated use of (7.23) yields

$$d^{\varepsilon,+}_{\min}(s_p) \geq 10^p d^{\varepsilon,+}_{\min}(s), \qquad \forall 1 \leq p \leq m, \tag{7.57}$$

and actually

$$d^{\varepsilon,+}_{\min}(s_p) \geq 10^{p-q} d^{\varepsilon,+}_{\min}(s), \qquad \forall 1 \leq q \leq p \leq m. \tag{7.58}$$

Hence, by repeated use of (7.28), we obtain

$$\begin{aligned} \mathcal{F}^{\varepsilon}_{\rm att}(s_m)^{-\frac{1}{\omega}} &\leq \mathcal{F}^{\varepsilon}_{\rm att}(s)^{-\frac{1}{\omega}} + \frac{1}{\kappa_3}\left(d^{\varepsilon,+}_{\min}(s) + \sum_{p=1}^{m-1} d^{\varepsilon,+}_{\min}(s_p)\right) \\ &\leq \mathcal{F}^{\varepsilon}_{\rm att}(s)^{-\frac{1}{\omega}} + \frac{1}{\kappa_3}\sum_{p=0}^{m-1} 10^{-p} d^{\varepsilon,+}_{\min}(s_{m-1}) \\ &\leq \mathcal{F}^{\varepsilon}_{\rm att}(s)^{-\frac{1}{\omega}} + \frac{2}{\kappa_3} d^{\varepsilon,+}_{\min}(s_{m-1}) \\ &\leq \mathcal{F}^{\varepsilon}_{\rm att}(s)^{-\frac{1}{\omega}} + \frac{1}{5\kappa_3} d^{\varepsilon,+}_{\min}(s_m). \end{aligned}$$

Combining the latter with (7.7), we deduce that

$$d_{\min}^{\varepsilon,-}(s_m) \leq \kappa_3\kappa_2^{-1} d_{\min}^{\varepsilon,-}(s) + \frac{1}{5} d_{\min}^{\varepsilon,+}(s_m) \leq \kappa_3{\kappa_2}^{-1} d_{\min}^{\varepsilon,-}(s) + \frac{1}{5} d_{\min}^{\varepsilon,-}(s_m),$$

so that

$$d_{\min}^{\varepsilon,+}(s_m) \leq d_{\min}^{\varepsilon,-}(s_m) \leq \frac{5}{4}\kappa_3\kappa_2^{-1} d_{\min}^{\varepsilon,-}(s). \tag{7.59}$$

Let $s_{m+1} := \mathcal{T}_2^{\varepsilon}(s_m)$. Then by (7.26) and (7.57)

$$\begin{aligned} s_{m+1} - s &\leq \mathcal{K}_1 \left(d_{\min}^{\varepsilon,+}(s)^{\omega+2} + \sum_{p=1}^{m} d_{\min}^{\varepsilon,+}(s_p)^{\omega+2} \right) \\ &\leq \mathcal{K}_1 \sum_{p=0}^{m-1} 10^{-(\omega+2)(m-p)} d_{\min}^{\varepsilon,+}(s_m)^{\omega+2} \\ &\leq 2\mathcal{K}_1 d_{\min}^{\varepsilon,+}(s_m)^{\omega+2}. \end{aligned} \tag{7.60}$$

Combining (7.60) with (7.59) we are led to

$$s_{m+1} - s \leq 2\left(\frac{5\kappa_3}{4\kappa_2}\right)^{\omega+2} \mathcal{K}_1 d_{\min}^{\varepsilon,-}(s)^{\omega+2}$$

and therefore by (7.36) we have $s_{m+1} < S$.

Combining (7.59) with (7.57), we obtain

$$0 \leq d_{\min}^{\varepsilon,-}(s_m) - d_{\min}^{\varepsilon,+}(s_m) \leq \frac{\kappa_3}{\kappa_2} d_{\min}^{\varepsilon,-}(s) - 10^m (d_{\min}^{\varepsilon,+}(s)),$$

and therefore necessarily

$$m \leq \log_{10}\left(\frac{\kappa_3 d_{\min}^{\varepsilon,-}(s)}{\kappa_2 d_{\min}^{\varepsilon,+}(s)}\right).$$

It follows that the number $m_0 = \sup\{m \in \mathbb{N}_*, d_{\min}^{\varepsilon,-}(s_m) \geq d_{\min}^{\varepsilon,+}(s_m)\}$ is finite, and at that stage we proceed to Case I as in Case II.1 above, and the conclusion follows likewise, replacing (7.52) by

$$d_{\min}^{\varepsilon,+}(s_{m_0+1}) \leq \frac{9}{4}\kappa_3\kappa_2^{-1} d_{\min}^{\varepsilon,-}(s)$$

which is obtained combining

$$d_{\min}^{\varepsilon,+}(s_{m_0+1}) \leq \kappa_3\kappa_2^{-1} d_{\min}^{\varepsilon,-}(s) + d_{\min}^{\varepsilon,+}(s_{m_0}),$$

with

$$d_{\min}^{\varepsilon,+}(s_{m_0}) \leq d_{\min}^{\varepsilon,-}(s_{m_0}) \leq \frac{5}{4}\kappa_3\kappa_2^{-1} d_{\min}^{\varepsilon,-}(s).$$

□

7.2 Proof of Proposition 7.1

We will fix the value of the constants κ_1, α_3 and $\mathcal{K}_{\rm col}$ in the course of the proof.
Let s_0 be as in the statement. We first require that

$$\alpha_3 \geq \alpha_2 \qquad \text{and that} \qquad \alpha_3 \geq 16\mathfrak{q}_1(\alpha_2),$$

so that assumption $\mathcal{WP}_\varepsilon^{\rm L}(\alpha_3, s_0)$ implies assumption 7.8 of Subsection 7.1.

Next, we set $s = s_0 + \varepsilon^{\omega+2}$ and we wish to make sure that the assumptions of Proposition 7.4 are satisfied at time s. In view of the upper bound (7) on the velocity of the front set, we deduce that

$$d_{\min}^{\varepsilon,\rm L}(s) \geq d_{\min}^{\varepsilon,\rm L}(s_0) - C\rho_0^{\frac{1}{\omega+2}}\varepsilon \geq \alpha_3\varepsilon - C\rho_0^{\frac{1}{\omega+2}}\varepsilon \geq \max(\beta_0, \beta_1)\varepsilon$$

provided we choose α_3 sufficiently large. Also,

$$\frac{1}{2}d_{\min}^{\varepsilon,-}(s_0) \leq d_{\min}^{\varepsilon,\rm L}(s_0) - C\rho_0^{\frac{1}{\omega+2}}\varepsilon \leq d_{\min}^{\varepsilon,\rm L}(s) \leq d_{\min}^{\varepsilon,\rm L}(s_0) + C\rho_0^{\frac{1}{\omega+2}}\varepsilon \leq 2d_{\min}^{\varepsilon,\rm L}(s_0), \tag{7.61}$$

and therefore provided we choose

$$\mathcal{K}_{\rm col} \geq 2^{\omega+2}\mathcal{K}_2$$

it follows from the assumption $s_0 + \mathcal{K}_{\rm col} d_{\min}^{\varepsilon,\rm L}(s_0)^{\omega+2} < S$ that $s + \mathcal{K}_2 d_{\min}^{\varepsilon,\rm L}(s)^{\omega+2} < S$. Therefore we may apply Proposition 7.4. Let $\mathcal{T}_{\rm col}^{\varepsilon,-}(s) \in \mathcal{I}_0^\varepsilon(s_0)$ be given by its statement, so that by (7.61)

$$\mathcal{T}_{\rm col}^{\varepsilon,-}(s) - s \leq 2^{\omega+2}\mathcal{K}_2 d_{\min}^{\varepsilon,-}(s_0)^{\omega+2},$$

and

$$d_{\min}^{\varepsilon,\rm L}(\mathcal{T}_{\rm col}^{\varepsilon,-}(s)) \leq \max\left(\beta_0, 8\mathfrak{q}_1(\alpha_2)\right)\varepsilon. \tag{7.62}$$

By Proposition 3.1, there exists some time $\mathcal{T}_{\rm col}^{\varepsilon,+}(s_0) \in [\mathcal{T}_{\rm col}^{\varepsilon,-}(s), \mathcal{T}_{\rm col}^{\varepsilon,-}(s) + \mathfrak{q}_0(\alpha_3)\varepsilon^{\omega+2}]$ such that $\mathrm{WPI}_\varepsilon^{\rm L}(\alpha_3\varepsilon, \mathcal{T}_{\rm col}^{\varepsilon,+}(s_0))$ holds. In view of (7.61) and since $d_{\min}^{\varepsilon,-}(s_0) \geq \alpha_3\varepsilon$, it follows that

$$\begin{aligned} 0 \leq \mathcal{T}_{\rm col}^{\varepsilon,+}(s_0) - s_0 &\leq \varepsilon^{\omega+2} + 2^{\omega+2}\mathcal{K}_2 d_{\min}^{\varepsilon,-}(s_0)^{\omega+2} + \mathfrak{q}_0(\alpha_3)\varepsilon^{\omega+2} \\ &\leq \left(2^{\omega+2}\mathcal{K}_2 + \frac{1+\mathfrak{q}_0(\alpha_3)}{\alpha_3^{\omega+2}}\right) d_{\min}^{\varepsilon,-}(s_0)^{\omega+2} \\ &\leq \mathcal{K}_{\rm col} d_{\min}^{\varepsilon,-}(s_0)^{\omega+2} \end{aligned} \tag{7.63}$$

provided we finally *fix* the value of $\mathcal{K}_{\rm col}$ as

$$\mathcal{K}_{\rm col} = \left(2^{\omega+2}\mathcal{K}_2 + \frac{1+\mathfrak{q}_0(\alpha_3)}{\alpha_3^{\omega+2}}\right).$$

[Note that at this stage $\mathcal{K}_{\rm col}$ is fixed but its definition depend on α_3 which has not yet been fixed. Of course when we will fix α_3 below we shall do it without any reference to $\mathcal{K}_{\rm col}$, in order to avoid impossible loops]
Next, we first claim that

$$\mathfrak{E}_\varepsilon^{\rm L}(s_0) \geq \mathfrak{E}_\varepsilon^{\rm L}(s) \geq \mathfrak{E}_\varepsilon^{\rm L}(\mathcal{T}_{\rm col}^{\varepsilon,+}(s_0)).$$

In view of Corollary 3.2, it suffices to check that $\mathrm{L} \geq \mathrm{L}_0(s_0, \mathcal{T}_{\mathrm{col}}^{\varepsilon,+}(s_0))$, where we recall that the function $\mathrm{L}_0(\cdot)$ was defined in (3.6). In view of (7.63), this reduces to

$$100\mathrm{C}_e \mathrm{L}^{-(\omega+2)} \mathcal{K}_{\mathrm{col}} d_{\min}^{\varepsilon,-}(s_0)^{\omega+2} \leq \frac{\mu_1}{4}.$$

Since by (7.2) we have $d_{\min}^{\varepsilon,-}(s_0) \leq 2\kappa_0(s_0)\mathrm{L}$, it suffices therefore that

$$\kappa_0(s_0) \leq \frac{1}{2}\left(\frac{\mu_1}{400\mathrm{C}_\mathrm{e}\mathcal{K}_{\mathrm{col}}}\right)^{\frac{1}{\omega+2}} \equiv \kappa_1,$$

and we have now *fixed* the value of κ_1.
Next, we claim that actually

$$\mathfrak{E}_\varepsilon^{\mathrm{L}}(\mathcal{T}_{\mathrm{col}}^{\varepsilon,+}(s_0)) \leq \mathfrak{E}_\varepsilon^{\mathrm{L}}(s_0) - \mu_1.$$

Indeed, otherwise by Corollary 3.2 we would have $\mathfrak{E}_\varepsilon^{\mathrm{L}}(\mathcal{T}_{\mathrm{col}}^{\varepsilon,+}(s_0)) = \mathfrak{E}_\varepsilon^{\mathrm{L}}(s_0)$, and therefore condition $\mathcal{C}(\alpha_3\varepsilon, \mathrm{L}, s_0, \mathcal{T}_{\mathrm{col}}^{\varepsilon,+}(s_0))$ of Subsection 3.4 would hold. Invoking Proposition 3.3, this would imply that condition $\mathcal{WP}^{\mathrm{L}}(\Lambda_{\log}(\alpha_3\varepsilon), \tau)$ holds for $\tau \in (s_0 + \varepsilon^{\omega+2}, \mathcal{T}_{\mathrm{col}}^{\varepsilon,+}(s_0))$, so that in particular

$$d_{\min}^{\varepsilon}(\mathcal{T}_{\mathrm{col}}^{\varepsilon,-}(s_0)) \geq \Lambda_{\log}(\alpha_3\varepsilon).$$

It suffices thus to choose α_3 sufficiently big so that

$$\Lambda_{\log}(\alpha_3\varepsilon) > \max(\beta_0, 8\mathfrak{q}_1(\alpha_2)))\varepsilon,$$

and the contradiction then follows from (7.62). □

7.3 Proof of Proposition 7

We will fix the values of κ_* and ρ_* in the course of the proofs, as the smallest numbers which satisfy a finite number lower bound inequalities.

First, recall that it follows from (53) and (7) that if $0 \leq s \leq \rho_0(R-r)^{\omega+2}$ then

$$\mathfrak{D}_\varepsilon(s) \cap \mathrm{I}_{4\mathrm{L}} \subset \bigcup_{k \in J_0} (b_k^\varepsilon - R, b_k^\varepsilon + R) \subset \mathrm{I}_{2\kappa_0\mathrm{L}}, \quad \text{where the union is disjoint,} \tag{7.64}$$

and in particular $\mathcal{C}_{\mathrm{L},S}$ holds where

$$S := \rho_0(R-r)^{\omega+2} \geq \rho_0\left(\frac{R}{2}\right)^{\omega+2}.$$

Having (3.6) in mind, and in view of (7.64) and (54), we estimate

$$100\mathrm{C}_\mathrm{e}\mathrm{L}^{-(\omega+2)}S \leq 100\mathrm{C}_\mathrm{e}\left(\frac{R}{2\mathrm{L}}\right)^{\omega+2} S \leq 100\mathrm{C}_\mathrm{e}\alpha_*^{-(\omega+2)} \leq \frac{\mu_1}{4},$$

where the last inequality follows provided we choose α_* sufficiently large. As a consequence, the function $\mathfrak{E}_\varepsilon^{\mathrm{L}}$ is non-increasing on the set of times s in the interval $[\varepsilon^\omega \mathrm{L}^2, S]$ where $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\alpha_1\varepsilon, s)$ holds.

For such times s, the front points $\{a_k^\varepsilon(s)\}_{k\in J(s)}$ are well-defined, and for $q\in J_0$, we have defined in the introduction $J_q(s)=\{k\in J(s), a_k^\varepsilon(s)\in[b_q^\varepsilon-R, b_q^\varepsilon+R]\}$, and we have set $\ell_q=\sharp J_q$ and $J_q(s)=\{k_q,k_{q+1},\cdots,k_{q+\ell_q-1}\}$, where $k_1=1$, and $k_q=\ell_1+\cdots+\ell_{q-1}+1$, for $q\geq 2$.

Step 1. Annihilations of all the pairs of fronts-antifronts. We claim that there exists some time $\tilde{s}\in(\varepsilon^\omega \mathrm{L}^2,\frac{1}{2}S)$ such that $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta_{\log}^\varepsilon,\tilde{s})$ holds and such that for any $q\in J_0$, $\epsilon_{k+\frac{1}{2}}(\tilde{s})=+1$, for $\;k\in J_q(\tilde{s})\setminus\{k_q(\tilde{s})+\ell_q(\tilde{s})-1\}\;$ or $\sharp J_q(\tilde{s})\leq 1$, or equivalently that $\dagger_k(\tilde{s})=\dagger_{k'}(\tilde{s})$ for k and k' in the same $J_q(\tilde{s})$. In particular, $d_{\min}^{\varepsilon,-}(\tilde{s})\geq 2R$.

Proof of the claim. If we require α_* to be sufficiently large, then by (54) we have that $\varepsilon^\omega\mathrm{L}^2+\varepsilon^{\omega+1}\mathrm{L}\leq S/2$, and therefore by Proposition 3 we may choose a first time

$$s_0\in[\varepsilon^\omega\mathrm{L}^2,\varepsilon^\omega\mathrm{L}^2+\varepsilon^{\omega+1}\mathrm{L}]\quad\text{such that}\quad \mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta_{\log}^\varepsilon,s_0)\text{ holds.}$$

Actually, we have

$$s_0\leq\varepsilon^\omega\mathrm{L}^2+\varepsilon^{\omega+1}\mathrm{L}\leq 2\varepsilon^\omega\mathrm{L}^2\leq 2\alpha_*^{-(\omega+2)}r^{\omega+2}\leq 2\alpha_*^{-2(\omega+2)}R^{\omega+2}\leq\frac{1}{\rho_0}2^{\omega+3}\alpha_*^{-2(\omega+2)}S. \tag{7.65}$$

Note that by (7) we have the inclusion

$$\mathfrak{D}_\varepsilon(s_0)\cap\mathrm{I_L}\subset\mathcal{N}(b,r_0),$$

where

$$r_0=r+\left(\frac{s_0}{\rho_0}\right)^{\frac{1}{\omega+2}}\leq 2r,$$

provided once more that α_* is sufficiently large, and where for $\rho>0$ we have set $\mathcal{N}(b,\rho)=\cup_{q\in J_0}[b_j^\varepsilon-\rho,b_j^\varepsilon+\rho]$. In view of the confinement condition (53) only two cases can occur:

$$\text{i)}\quad d_{\min}^{\varepsilon,-}(s_0)\geq 3R-2r_0\qquad\text{or}\qquad\text{ii)}\quad d_{\min}^{\varepsilon,-}(s_0)\leq 2r_0.$$

If case i) occurs, then, for any $q\in J_0$, we have $\epsilon_{k+\frac{1}{2}}=+1$, for any $\;k\in J_q(\tau_1)\setminus\{k_q(s_0)+\ell_q(s_0)-1\}$. Choosing $\tilde{s}=s_0$, Step 1 is completed in the case considered.

If instead case ii) occurs, then we will make use of Proposition 7.1 to remove the small pairs of fronts-antifronts present at small scales. More precisely, assume that for some $j\geq 0$ we have constructed $0\leq s_j\leq S$ and $r_j>0$ such that $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\delta_{\log}^\varepsilon,s_j)$ holds, such that we have

$$\mathfrak{D}_\varepsilon(s_j)\cap\mathrm{I_L}\subset\mathcal{N}(b,r_j),\qquad \mathfrak{E}_\varepsilon^{\mathrm{L}}(s_j)\leq\mathfrak{E}_\varepsilon^{\mathrm{L}}(s_0)-j\mu_1, \tag{7.66}$$

as well as the estimates,

$$r_0\leq r_j\leq\gamma^j r_0\leq\frac{R}{2},\qquad s_j\leq s_0+(2^{\omega+2}\mathcal{K}_{\mathrm{col}}+1)\frac{\gamma^{j(\omega+2)}-1}{\gamma^{\omega+2}-1}r_0^{\omega+2}\leq\frac{S}{2}, \tag{7.67}$$

where $\gamma:=\left(2+2\left(\frac{\mathcal{K}_{\mathrm{col}}}{\rho_0}\right)^{\frac{1}{\omega+2}}\right)$, and moreover that case ii) holds at step j, that is

$$d_{\min}^{\varepsilon,-}(s_j)\leq 2r_j\leq R. \tag{7.68}$$

Let $\tilde{s}_j := \mathcal{T}^{\varepsilon,+}_{\rm col}(s_j)$ be given by Proposition 7.1 (the confinement condition holds in view of (7.64) and we have $\delta^{\varepsilon}_{\rm log} \geq \alpha_3\varepsilon$ provided α_* is sufficiently large), and let then $s_{j+1} \in [\tilde{s}_j, \tilde{s}_j + \varepsilon^{\omega+1}\mathrm{L}]$ satisfying $\mathcal{WP}^{\mathrm{L}}_{\varepsilon}(\delta^{\varepsilon}_{\rm log}, s_{j+1})$ be given by Proposition 3. In particular, we have

$$\mathfrak{E}^{\mathrm{L}}_{\varepsilon}(s_{j+1}) \leq \mathfrak{E}^{\mathrm{L}}_{\varepsilon}(\tilde{s}_j) \leq \mathfrak{E}^{\mathrm{L}}_{\varepsilon}(s_j) - \mu_1 \leq \mathfrak{E}^{\mathrm{L}}_{\varepsilon}(s_0) - (j+1)\mu_1. \tag{7.69}$$

Since

$$s_{j+1} - s_j \leq \mathcal{K}_{\rm col}(2r_j)^{\omega+2} + \varepsilon^{\omega+1}\mathrm{L} \leq \left(2^{\omega+2}\mathcal{K}_{\rm col} + 1\right) r_j^{\omega+2},$$

we have, in view of (7.67)

$$\begin{aligned} s_{j+1} &\leq s_0 + \left(2^{\omega+2}\mathcal{K}_{\rm col} + 1\right)\left[\frac{\gamma^{j(\omega+2)} - 1}{\gamma^{\omega+2} - 1} + \gamma^{j(\omega+2)}\right] r_0^{\omega+2} \\ &\leq s_0 + \left(2^{\omega+2}\mathcal{K}_{\rm col} + 1\right) \frac{\gamma^{(j+1)(\omega+2)} - 1}{\gamma^{\omega+2} - 1} r_0^{\omega+2}, \end{aligned} \tag{7.70}$$

and by (7) $\mathfrak{D}_\varepsilon(s_{j+1}) \cap \mathrm{I}_{\mathrm{L}} \subset \mathcal{N}(b, r_{j+1})$, where

$$r_{j+1} = r_j + 2\left(\frac{\mathcal{K}_{\rm col}}{\rho_0}\right)^{\frac{1}{\omega+2}} r_j + \frac{1}{\rho_0}\varepsilon\left(\frac{\mathrm{L}}{\varepsilon}\right)^{\frac{1}{\omega+2}} \leq \left(2 + 2\left(\frac{\mathcal{K}_{\rm col}}{\rho_0}\right)^{\frac{1}{\omega+2}}\right) r_j = \gamma r_j. \tag{7.71}$$

In view of (7.65) and (54), we also have

$$\gamma^{j+1} r_0 \leq 2\gamma^{j+1}\alpha_*^{-1} R \tag{7.72}$$

and

$$\begin{aligned} &s_0 + \left(2^{\omega+2}\mathcal{K}_{\rm col} + 1\right) \frac{\gamma^{(j+1)(\omega+2)} - 1}{\gamma^{\omega+2} - 1} r_0^{\omega+2} \\ \leq &\left[\frac{2^{\omega+3}}{\rho_0}\alpha_*^{-(\omega+2)} + \frac{2^{2\omega+4}}{\rho_0}\left(2^{\omega+2}\mathcal{K}_{\rm col} + 1\right) \frac{\gamma^{(j+1)(\omega+2)} - 1}{\gamma^{\omega+2} - 1}\right] \alpha_*^{-(\omega+2)} S. \end{aligned} \tag{7.73}$$

It follows from (7.70), (7.71), (7.72) and (7.73) that if α_* is sufficiently large (depending only on M_0, V and j), then (7.67) holds also for s_{j+1}. As above we distinguish two cases :

$$\text{i)} \quad d^{\varepsilon,-}_{\min}(s_{j+1}) \geq 3R - 2r_{j+1} \qquad \text{or} \qquad \text{ii)} \quad d^{\varepsilon,-}_{\min}(s_{j+1}) \leq 2r_{j+1}.$$

If case i) holds then by (7.67) we have $d^{\varepsilon,-}_{\min}(s_{j+1}) \geq 2R$, we set $\tilde{s} = s_{j+1}$ which therefore satisfies the requirements of the claim, and we proceed to Step 2.

If case ii) occur then we proceed to construct s_{j+2} as above. The key fact in this recurrence construction is the second inequality in (7.66), which, since $\mathfrak{E}^{\mathrm{L}}_{\varepsilon}(s_j) \geq 0$ independently of j, implies that *the process as to reach case i) in a number of steps less than or equal to M_0/μ_1*. In particular, choosing the constant α_* sufficiently big so that the right-hand side of (7.72) is smaller than $R/2$ for all $0 \leq j \leq M_0/\mu_1$ and so that the right hand side of (7.73) is smaller than $S/2$ for all $0 \leq j \leq M_0/\mu_1$ ensures that the construction was licit and that the process necessarily reaches case i) before it could reach $j = M_0/\mu_1 + 1$, so defining $\tilde{s}$ as above. □

Step 2: Divergence of the remaining repulsing fronts at small scale. At this stage we have constructing $\tilde{s} \in [\varepsilon^{\omega}\mathrm{L}^2, \frac{1}{2}S]$ which satisfies the requirements of the claim in Step 1. Moreover, note that in view of (7.65) and (7.67) we have the upper bound

$$\tilde{s} \leq \left(2\alpha_*^{-(\omega+2)} + 2^{\omega+2}(2^{\omega+2}\mathcal{K}_{\mathrm{col}} + 1)\frac{\gamma^{\frac{M_0}{\mu_1}(\omega+2)} - 1}{\gamma^{\omega+2} - 1} \right) r^{\omega+2}. \tag{7.74}$$

In order to complete the proof, we next distinguish two cases:

$$i) \quad \sharp J_q(\tilde{s}) \leq 1, \quad \text{for any } q \in J_0. \qquad ii) \quad \sharp J_{q_0}(\tilde{s}) > 1, \quad \text{form some } q_0 \in J_0.$$

If case i) holds, then we actually have

$$d_{\min}^{\varepsilon,\mathrm{L}}(\tilde{s}) \geq 2R. \tag{7.75}$$

Since $2R \geq 16\mathfrak{q}_1(\delta_{\log}^{\varepsilon})\varepsilon$ when α_* is sufficiently large, it follows from Corollary 3.3 that $\mathcal{WP}_{\varepsilon}^{\mathrm{L}}(\delta_{\mathrm{loglog}}^{\varepsilon}, s)$ holds for any $\tilde{s} + \varepsilon^{2+\omega} \leq s \leq \mathcal{T}_0^{\varepsilon}(\delta_{\log}^{\varepsilon}, \tilde{s})$, where

$$\mathcal{T}_0^{\varepsilon}(\delta_{\log}^{\varepsilon}, \tilde{s}) = \max\left\{ \tilde{s} + \varepsilon^{2+\omega} \leq s \leq S \quad \text{s.t.} \quad d_{\min}^{\varepsilon,\mathrm{L}}(s') \geq 8\mathfrak{q}_1(\delta_{\log}^{\varepsilon})\varepsilon \quad \forall s' \in [\tilde{s} + \varepsilon^{\omega+2}, s] \right\}.$$

In particular, $\mathcal{WP}_{\varepsilon}^{\mathrm{L}}(\delta_{\mathrm{loglog}}^{\varepsilon}, s)$ holds for any s in $\tilde{s} + \varepsilon^{2+\omega} \leq s \leq \mathcal{T}_3^{\varepsilon}(\delta_{\log}^{\varepsilon}, \tilde{s})$, where

$$\mathcal{T}_3^{\varepsilon}(\delta_{\log}^{\varepsilon}, \tilde{s}) = \max\left\{ \tilde{s} + \varepsilon^{2+\omega} \leq s \leq S \quad \text{s.t.} \quad d_{\min}^{\varepsilon,\mathrm{L}}(s') \geq R \quad \forall s' \in [\tilde{s} + \varepsilon^{\omega+2}, s] \right\}.$$

In view of (7.75) and estimate (7), we obtain the lower bound

$$\mathcal{T}_3^{\varepsilon}(\delta_{\log}^{\varepsilon}, \tilde{s}) \geq \tilde{s} + \rho_0 R^{\omega+2}. \tag{7.76}$$

Note that (7.76) and (54) also yield

$$\mathcal{T}_3^{\varepsilon}(\delta_{\log}^{\varepsilon}, \tilde{s}) \geq \tilde{s} + \rho_0\alpha_*^{-1} r^{\omega+2} \geq \rho_0\alpha_*^{-1} r^{\omega+2}. \tag{7.77}$$

Combining (7.74) and (7.77) we deduce in particular that

$$\mathcal{WP}_{\varepsilon}^{\mathrm{L}}(\alpha_1\varepsilon, s_r) \text{ holds} \qquad \text{and} \qquad d_{\min}^{\varepsilon,\mathrm{L}}(s_r) \geq R \geq r,$$

which is the claim of Proposition 7, provided

$$\rho_* \geq \left(3 + 2^{\omega+2}(2^{\omega+2}\mathcal{K}_{\mathrm{col}} + 1)\frac{\gamma^{\frac{M_0}{\mu_1}(\omega+2)} - 1}{\gamma^{\omega+2} - 1} \right) \qquad \text{and} \qquad \rho_* \leq \rho_0\alpha_*^{-1}. \tag{7.78}$$

It remains to consider the situation where case ii) holds. In that case, we have

$$d_{\min}^{\varepsilon,-}(\tilde{s}) \geq 2R \qquad \text{and} \qquad d_{\min}^{\varepsilon,+}(\tilde{s}) \leq 2\gamma^{M_0/\mu_1} r \leq R,$$

so that we are in a situation suited for Proposition 7.3. We may actually apply Proposition 7.3 recursively with $s_0 := \tilde{s}$ and $\hat{s} \equiv \hat{s}_k = (\mathcal{T}_2^{\varepsilon})^k(\hat{s}_0)$ where $\hat{s}_0 = \tilde{s} + \varepsilon^{\omega+2}$, as long as $d_{\min}^{\varepsilon,+}(\hat{s}_k)$ remains sufficiently small with respect to R (say e.g. $d_{\min}^{\varepsilon,+}(\hat{s}_k) \leq \alpha_*^{-\frac{1}{2}} R$ provided α_* is chosen sufficiently large), the details are completely similar to the ones in Case II of Proposition 7.4

and are therefore not repeated here. If we denote by k_0 the first index for which $d_{\min}^{\varepsilon,+}(\hat{s}_{k_0})$ becomes larger than $\frac{2}{\mathcal{S}_2}r$ (in view of (7.27)) and k_1 the last index before $d_{\min}^{\varepsilon,+}$ reaches $\alpha_*^{-\frac{1}{2}}R$, then we have

$$\hat{s}_{k_0} \leq Cr^{\omega+2} \qquad \text{and} \qquad \hat{s}_{k_1} \geq \frac{1}{C}\alpha_*^{-(\omega+2)/2}R^{\omega+2} \geq \frac{1}{C}\alpha_*^{(\omega+2)/2}r^{\omega+2},$$

for some constant $C > 0$ depending only on M_0 and V, and the conclusion that $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\upalpha_1\varepsilon, s_r)$ holds follows as in case i) above, choosing first $\uprho_*$ sufficiently large (independently of $\upalpha_*$) and then α_* sufficiently large (given $\uprho_*$). □

8 Proofs of Theorem 2, 3 and 4

8.1 Proof of Theorem 2

Theorem 2 being essentially a special case of Theorem 3, we go directly to the proof of Theorem 3. Notice however that in Theorem 2 the solution to the limiting system is unique, so that the result is not constrained by the need to pass to a subsequence.

8.2 Proof of Theorem 3

We fix $S < S_{\max}$ and let $\mathrm{L} \geq \upkappa_*^{-1}\mathrm{L}_0$, where L_0 is defined in the statement of Proposition 6 and $\upkappa_*$ in the statement of Proposition 7. We set $R = \frac{1}{2}\min\{a_{k+1}^0 - a_k^0, k = 1, \cdots, \ell_0 - 1\}$ and consider an arbitrary $0 < r < R/\upalpha_*$. Since (H_1) holds, there exists some constant $\varepsilon_r > 0$ such that, if $0 < \varepsilon \leq \varepsilon_r$, then (53) holds with $b_k \equiv a_k^0$ for any $k \in \{1, \cdots, \ell_0\}$. We are therefore in position to make use of Proposition 7 and assert that for all such ε condition $\mathcal{WP}_\varepsilon^{\mathrm{L}}(\upalpha_1\varepsilon, s_r)$ holds as well as (55) and (56). It follows in particular from (55) and (56) that for every $k \in 1, \cdots, \ell_0$ we have $\sharp J_k(s_r) = |m_k^0|$, where m_k^0 is defined in (12), and therefore $\sharp J(s_r) = \sum_{k=1}^{\ell_0} |m_k^0| \equiv \ell_1$, in other words the number of fronts as well as their properties do not depend on ε nor on r.

We construct next the limiting splitting solution to the ordinary differential equation and the corresponding subsequence proceeding backwards in time and using a diagonal argument. For that purpose, we introduce an arbitrary decreasing sequence $\{r_m\}_{m\in\mathbb{N}_*}$ such that $0 < r_1 \leq R/\upalpha_*$, and such that $r_m \to 0$ as $m \to +\infty$. For instance, we may take $r_m = \frac{1}{m}R/\upalpha_*$, and we set $s_m = s_{r_m}$. Taking first $m = 1$, we find a subsequence $\{\varepsilon_{n,1}\}_{n\in\mathbb{N}_*}$ such that $\varepsilon_{n,1} \to 0$ as $n \to \infty$, and such that all points $\{a_k^{\varepsilon_{n,1}}(s_1)\}_{k\in J}$ converge to some limits $\{a_k^1(s_1)\}_{k\in J}$ as $n \to +\infty$. It follows from (56), passing to the limit $n \to +\infty$, that

$$d_{\min}^*(s_1) \geq r_1. \tag{8.1}$$

We are therefore in position to apply the convergence result of Proposition 6, which yields in particular that

$$\mathfrak{D}_{\varepsilon_{n,1}}(s) \cap \mathrm{I}_{4\mathrm{L}} \longrightarrow \cup_{k=1}^{\ell_1}\{a_k^1(s)\} \qquad \forall s_1 < s < S_{\max}^1, \tag{8.2}$$

as $n \to +\infty$, where $\{a_k^1(\cdot)\}_{k\in J}$ is the unique solution of $(\mathcal{S})$ with initial data $\{a_k^1(s_1)\}_{k\in J}$ on its maximal time of existence $(s_1, S_{\max}^1)$.

Taking next $m = 2$, we may extract a subsequence $\{\varepsilon_{n,2}\}_{n\in\mathbb{N}_*}$ from the sequence $\{\varepsilon_{n,1}\}_{n\in\mathbb{N}_*}$ such that all the points $\{a_k^{\varepsilon_{n,2}}(s_2)\}_{k\in J}$ converge to some limits $\{a_k^2(s_2)\}_{k\in J}$ as $n \to +\infty$. Arguing as above, we may assert that

$$\mathfrak{D}_{\varepsilon_{n,2}}(s) \cap \mathrm{I}_{4\mathrm{L}} \longrightarrow \cup_{k=1}^{\ell_1}\{a_k^2(s)\} \qquad \forall s_2 < s < S_{\max}^2, \tag{8.3}$$

as $n \to +\infty$, where $\{a_k^2(\cdot)\}_{k\in J}$ is the unique solution of $(\mathcal{S})$ with initial data $\{a_k^2(s_2)\}_{k\in J}$ on its maximal time of existence $(s_2, S_{\max}^2)$. It follows from (55), namely that only repulsive forces are present at scale smaller than R, that $S_{\max}^2 \geq s_1$. Therefore, since we have extracted a subsequence, it follows from (8.2) and (8.3) that $a_k^2(s_1) = a_k^1(s_1)$ for all $k \in J$, and therefore also that $S_{\max}^2 = S_{\max}^1 \equiv S_{\max}$ and $a_k^2(\cdot) = a_k^1(\cdot) = a_k(\cdot)$ on $(s_2, S_{\max})$.

We proceed similarly at each step $m \in \mathbb{N}_*$, extracting a subsequence $\{\varepsilon_{n,m}\}_{n\in\mathbb{N}_*}$ from the sequence $\{\varepsilon_{n,m-1}\}_{n\in\mathbb{N}^*}$ such that all the points $\{a_k^{\varepsilon_{n,m}}(s_m)\}_{k\in J}$. Finally setting, for $n \in N_*$, $\varepsilon_n = \varepsilon_{n,n}$, we obtain that

$$\mathfrak{D}_{\varepsilon_n}(s) \cap \mathrm{I}_{4\mathrm{L}} \longrightarrow \cup_{k=1}^{\ell_1}\{a_k(s)\} \qquad \forall 0 < s < S_{\max}^m,$$

where $\{a_k(\cdot)\}_{k\in J}$ is a splitting solution of $(\mathcal{S})$ with initial data $\{a_k^0\}_{k\in J_0}$, on its maximal time of existence $(0, S_{\max})$. Since $\mathrm{L} \geq \mathrm{L}_0$ was arbitrary, it follows that (15) holds.

It remains to prove that (14). This is actually a direct consequence of (15), the continuity of the trajectories $a_k(\cdot)$ and regularizing effect off the front set stated in Proposition 2 (e.g. (31) for the L^∞ norm). As a matter of fact, it is standard to deduce from this the fact that the convergence towards the equilibria σ_q, locally outside the trajectory set, holds in any $\mathcal{C}^m$ norm, since the potential V was assumed to be smooth. □

8.3 Proof of Theorem 4

As underlined in the introduction, Theorem 4 follows rather directly from Theorem 3 and more importantly its consequence Corollary 1 (whose proof is elementary and explained after Proposition 1), combined with Theorem 1 and Proposition 2.

Let thus $\mathrm{L} > \mathrm{L}_0$ and $\delta > 0$ be arbitrarily given, we shall prove that, at least for $\varepsilon \equiv \varepsilon_n$ sufficiently small,

$$\mathcal{D}_\varepsilon(S_{\max}) \cap \mathrm{I}_\mathrm{L} \subset \cup_{j\in\{1,\cdots,\ell_2\}}[b_j - \delta, b_j + \delta], \tag{8.4}$$

and

$$|\mathfrak{v}_\varepsilon(x, S_{\max}) - \sigma_{\hat{\imath}(j+\frac{1}{2})}| \leq C(\delta, \mathrm{L})\varepsilon^{\frac{1}{\theta-1}}, \tag{8.5}$$

for all $j \in \{0, \cdots, \ell_2\}$ and for all $x \in (b_j + \delta, b_{j+1} - \delta)$, where we have used the convention that $b_0 = -\mathrm{L}$ and $b_{\ell_2+1} = \mathrm{L}$. Since L can be arbitrarily big and δ arbitrarily small, this will imply that assumption (H_1) is verified at times $S_{\max}$, which is the claim of Theorem 4.

Concerning (8.4), by Corollary 1 there exists

$$s^- \in \big[S_{\max} - \rho_0 \left(\frac{\delta}{4}\right)^{\omega+2}, S_{\max}\big). \tag{8.6}$$

such that

$$\cup_{k\in\{1,\cdots,\ell_1\}}\{a_k(s^-)\} \subset \cup_{j\in\{1,\cdots,\ell_2\}}[b_j - \frac{1}{4}\delta, b_j + \frac{1}{4}\delta].$$

The latter and Theorem 3 imply that, for ε sufficiently small,

$$\mathcal{D}_\varepsilon(s^-) \cap \mathrm{I}_{2\mathrm{L}} \cap \Big(\cup_{j\in\{1,\cdots,\ell_2\}} [b_j - \frac{1}{2}\delta, b_j + \frac{1}{2}\delta] \Big)^c = \emptyset. \tag{8.7}$$

In turn, Theorem 1 (inclusion (7)) and (8.7), combined with the upper bound (8.6) on $S_{\max} - s^-$, imply that

$$\mathcal{D}_\varepsilon(s) \cap \mathrm{I}_{\frac{3}{2}\mathrm{L}} \cap \Big(\cup_{j\in\{1,\cdots,\ell_2\}} [b_j - \frac{3}{4}\delta, b_j + \frac{3}{4}\delta] \Big)^c = \emptyset, \qquad \forall s \in [s^-, S_{\max}]. \tag{8.8}$$

For $s = S_{\max}$ this is stronger than (8.4).

We proceed to (8.5). In view of (8.8), for any $x_0 \in \mathrm{I}_\mathrm{L} \setminus \big(\cup_{j\in\{1,\cdots,\ell_2\}} [b_j - \delta, b_j + \delta]\big)$ we have, for ε sufficiently small,

$$\mathfrak{v}_\varepsilon(y,s) \in B(\sigma_i, \mu_0) \qquad \forall (y,s) \in [x_0 - \frac{1}{8}\delta, x_0 + \frac{1}{8}\delta] \times [s^-, S_{\max}].$$

The latter is nothing but (29) for $r = \frac{1}{8}\delta$, $s_0 = s^-$ and $S = S_{\max}$, and therefore the conclusion (8.5) follows from (31) of Proposition 2, with $C(\delta, \mathrm{L}) = \frac{1}{5}\mathrm{C_e}(8/\delta)^{\frac{1}{\theta-1}}$ as soon as $\varepsilon^\omega/(S_{\max} - s^-) \leq \delta^2/64$. □

Appendix A

In this Appendix we establish properties concerning the stationary solutions $\overset{\vee+}{\mathfrak{u}}$, $\overset{\triangleright}{\mathfrak{u}}$, $\overset{\vee+}{\mathfrak{u}}_{\varepsilon,r}$, etc, which we have used in the course of the previous discussion, mainly in Section 5.

A.1 The operator $\mathcal{L}_\mu$

Consider for $\mu > 0$ the nonlinear operator $\mathcal{L}_\mu$, defined for a smooth functions U on $\mathbb{R}$ by

$$\mathcal{L}_\mu(U) = -\frac{d^2}{dx^2}U + 2\mu\theta\, U^{2\theta-1},$$

and set for simplicity $\mathcal{L} \equiv \mathcal{L}_1$. Most results in this section are deduced from the comparison principle: if u_1 and u_2 are two functions defined on some non empty interval I, such that

$$\mathcal{L}_\mu(u_1) \geq 0,\ \mathcal{L}_\mu(u_2) \leq 0, \text{ and } u_1 \geq u_2 \text{ on } \partial I, \tag{A.1}$$

then $u_1(x) \geq u_2(x)$ for $x \in I$. Scaling arguments are also used extensively. Given $r > 0$ and $\eta > 0$ we provide a rescaling of a given smooth function U as follows

$$\begin{cases} U_{\eta,R} = \eta\, U\left(\dfrac{U}{r}\right), \text{ and we verify that} \\ \mathcal{L}_\mu(U_{\eta,r})(x) = \dfrac{\eta}{r^2}\mathcal{L}_\gamma(U)(\dfrac{x}{r}) \text{ where } \gamma = \mu\eta^{2(\theta-1)}r^2. \end{cases} \tag{A.2}$$

In particular, if $\mathcal{L}_\mu(U) = 0$, then we have

$$\mathcal{L}_\mu(r^{-\frac{1}{(\theta-1)}}U(\frac{\cdot}{r})) = 0 \text{ and } \mathcal{L}_{\lambda\mu}(\lambda^{-\frac{1}{2(\theta-1)}}\, U) = 0, \text{ for any } r > 0 \text{ and any } \lambda > 0.$$

Notice also that U^* defined on $(0, +\infty)$ by $U^*(x) = [\sqrt{2}(\theta-1)x]^{-\frac{1}{\theta-1}}$ solves $\mathcal{L}(U^*) = 0$.

Lemma A.1. *There exists a unique smooth map* $\overset{\vee+}{\mathfrak{u}}_r$ *on* $(-r,r)$ *such that* $\mathcal{L}(\overset{\vee+}{\mathfrak{u}}_r)=0$ *and* $\overset{\vee+}{\mathfrak{u}}_r(\pm r)=+\infty$*, and a unique solution* $\overset{\triangleright}{\mathfrak{u}}_r$ *such that* $\mathcal{L}(\overset{\triangleright}{\mathfrak{u}}_r)=0$ *and* $\overset{\triangleright}{\mathfrak{u}}_r(\pm r)=\pm\infty$*. Moreover,* $\overset{\vee+}{\mathfrak{u}}_r$ *is even,* $\overset{\triangleright}{\mathfrak{u}}_r$ *is odd, and, setting* $\overset{\vee+}{\mathfrak{u}}\equiv\overset{\vee+}{\mathfrak{u}}_1$ *and* $\overset{\triangleright}{\mathfrak{u}}\equiv\overset{\triangleright}{\mathfrak{u}}_1$*, we have*

$$\overset{\vee+}{\mathfrak{u}}_r(x)=r^{-\frac{1}{\theta-1}}\overset{\vee+}{\mathfrak{u}}(\frac{x}{r}) \text{ and } \overset{\triangleright}{\mathfrak{u}}_r(x)=r^{-\frac{1}{\theta-1}}\overset{\triangleright}{\mathfrak{u}}(\frac{x}{r}). \tag{A.3}$$

Proof. For $n\in\mathbb{N}^*$, we construct on $(-r,r)$ a unique solution $\overset{\vee+}{\mathfrak{u}}_{r,n}$ that solves $\mathcal{L}(\overset{\vee+}{\mathfrak{u}}_{r,n})=0$ and $\overset{\vee+}{\mathfrak{u}}_{r,n}(\pm r)=n$, minimizing the corresponding convex energy. By the comparison principle, $\overset{\vee+}{\mathfrak{u}}_{r,n}$ is non negative, increasing with n and uniformly bounded on compact subsets of $(-r,r)$ in view of (A.5) below. Hence a unique limit $\overset{\vee+}{\mathfrak{u}}_r$ exists, solution to $\mathcal{L}(\overset{\vee+}{\mathfrak{u}}_r)=0$. We observe that $\overset{\vee+}{\mathfrak{u}}_{r,n}(\cdot)\geq U^*(r_n-\cdot)$, where $r_n=r+[\sqrt{2}(\theta-1)]^{-1}n^{-(\theta-1)}$, so that we obtain the required boundary conditions for $\overset{\vee+}{\mathfrak{u}}_r$. Uniqueness may again be established thanks to the comparison principle. We construct similarly a unique solution $\overset{\triangleright}{\mathfrak{u}}_{r,n}$ that solves $\mathcal{L}(\overset{\triangleright}{\mathfrak{u}}_{r,n})=0$ and $\overset{\vee+}{\mathfrak{u}}_{r,n}(\pm r)=\pm n$. We notice that $\overset{\triangleright}{\mathfrak{u}}_{r,n}$ is odd, its restriction on $(0,r)$ non negative and increasing with n. Moreover, on some interval (a,r), where $0<a<r$ does not depend on n, we have $\overset{\vee+}{\mathfrak{u}}_{r,n}(\cdot)\geq U^*(\tilde{r}_n-\cdot)$ where $\tilde{r}_n=r+[(\theta-1)]^{-1}n^{-(\theta-1)}$, and we conclude as for the first assertion. □

Remark A.1. Given $r>0$ and $\lambda>0$ we notice that the function $\overset{\vee}{\mathrm{U}}{}_r^\lambda$ and $\overset{\triangleright}{\mathrm{U}}{}_r^\lambda$ defined by

$$\overset{\vee}{\mathrm{U}}{}_r^\lambda(x)=\lambda^{-\frac{1}{2(\theta-1)}}\overset{\vee+}{\mathfrak{u}}_r(x) \text{ and } \overset{\triangleright}{\mathrm{U}}{}_r^\lambda(x)=\lambda^{-\frac{1}{2(\theta-1)}}\overset{\triangleright}{\mathfrak{u}}_r(x) \tag{A.4}$$

solve $\mathcal{L}_\lambda(\overset{\vee}{\mathrm{U}}{}_r^\lambda)=0$ and $\mathcal{L}_\lambda(\overset{\triangleright}{\mathrm{U}}{}_r^\lambda)=0$ with the same boundary conditions as $\overset{\vee+}{\mathfrak{u}}_r$ and $\overset{\triangleright}{\mathfrak{u}}_r$.

Lemma A.2. *i) Assume that* $\mathcal{L}(u)\leq 0$ *on* $(-r,r)$*. Then, we have, for* $x\in(-r,r)$

$$u(x)\leq\left(\sqrt{2}(\theta-1)\right)^{-\frac{1}{\theta-1}}\left[(x+r)^{-\frac{1}{\theta-1}}+(x-r)^{-\frac{1}{\theta-1}}\right]. \tag{A.5}$$

ii) Assume that $\mathcal{L}(u)\geq 0$ *on* $(-r,r)$ *and that* $u(-r)=u(r)=+\infty$*. Then we have*

$$u(x)\geq\left(\sqrt{2}(\theta-1)\right)^{-\frac{1}{\theta-1}}\max\{(x+r)^{-\frac{1}{\theta-1}},(r-x)^{-\frac{1}{\theta-1}}\}. \tag{A.6}$$

Proof. Set $\tilde{U}=U^*(\cdot+r)+U^*(r-\cdot)$. By subaddivity and translation invariance, we have $\mathcal{L}(\tilde{U})\geq 0$ on $(-r,r)$ with $\tilde{U}(\pm r)=+\infty$, so that (A.5) follows from the comparison principle (A.1) with $u_1=\tilde{U}$ and $u_2=u$. Similarly, (A.6) follows from (A.1) with $u_1=u$ and $u_2=U^*(\cdot+r)$ or $u_2=U^*(r-\cdot)$. □

Combining estimate ii) of Lemma A.2 with the scaling law of Lemma A.1 we are led to

$$|\frac{d}{dr}\overset{\vee+}{\mathfrak{u}}_r(x)|+|\frac{d}{dr}\overset{\triangleright}{\mathfrak{u}}_r(x)|\leq Cr^{-\frac{\theta}{\theta-1}},\ \text{for } x\in(-\frac{7}{8}r,\frac{7}{8}r). \tag{A.7}$$

A.2 The discrepancy for $\mathcal{L}_\upmu$

The discrepancy $\Xi_\upmu$ for $\mathcal{L}_\upmu$ relates to a given function u the function $\Xi_\upmu(u)$ defined by

$$\Xi_\upmu(u) = \frac{\dot{u}^2}{2} - \upmu u^{2\theta}. \tag{A.8}$$

This function is *constant* if u solves $\mathcal{L}_\upmu(u) = 0$. We set $\Xi = \Xi_1$,

$$A_\theta \equiv \Xi(\overset{\vee+}{\mathfrak{u}}) = -(\overset{\vee+}{\mathfrak{u}}(0))^{2\theta} < 0 \text{ and } B_\theta \equiv \Xi(\overset{\triangleright}{\mathfrak{u}}) = \frac{(\overset{\triangleright}{\mathfrak{u}}(0))_x^2}{2} > 0. \tag{A.9}$$

In view of the scaling relations (A.3) and Remark A.1, we are hence led to the identities

$$\begin{cases} \Xi_\lambda(\overset{\vee}{\mathrm{U}}{}_r^\lambda) = \lambda^{-\frac{1}{\theta-1)}} r^{-\frac{2\theta}{\theta-1}} A_\theta = \lambda^{-\frac{1}{\theta-1)}} r^{-(\omega+1)} A_\theta, \\ \Xi_\lambda(\overset{\triangleright}{\mathrm{U}}{}_r^\lambda) = \lambda^{-\frac{1}{\theta-1}} r^{-\frac{2\theta}{\theta-1}} B_\theta = \lambda^{-\frac{1}{\theta-1}} r^{-(\omega+1)} B_\theta. \end{cases} \tag{A.10}$$

A.3 The operator $\mathcal{L}^\varepsilon$

In this subsection, we consider more generally, for given $\lambda > 0$ the operator $\mathcal{L}^\varepsilon$ given by

$$\mathcal{L}^\varepsilon(U) = -\frac{d^2}{dx^2}U + 2\lambda f_\varepsilon(U),$$

with f_ε defined in (5.4), and the solutions $\overset{\vee+}{\mathfrak{u}}_{\varepsilon,r}$, $\overset{\vee-}{\mathfrak{u}}_{\varepsilon,r}$, and $\overset{\triangleright}{\mathfrak{u}}_{\varepsilon,r}$ of $\mathcal{L}^\varepsilon(U) = 0$ on $(-r, r)$ with corresponding infinite boundary conditions, whose existence and uniqueness is proved as for Lemma A.1.

Lemma A.3. *We have the estimates*

$$|\overset{\vee+}{\mathfrak{u}}_{\varepsilon,r}(x)| + |\overset{\triangleright}{\mathfrak{u}}_{\varepsilon,r}(x)| \le C\left(\lambda^2(\theta-1)\right)^{-\frac{1}{\theta-1}} \left[(x+r)^{-\frac{1}{\theta-1}} + (x-r)^{-\frac{1}{\theta-1}}\right].$$

Proof. It follows from (5.5) that $\mathcal{L}_{\frac{3}{4}\lambda}(\overset{\vee+}{\mathfrak{u}}_{\varepsilon,r}) \le 0$, so that, invoking the comparison principle as well as the scaling law (A.2) we deduce that $\overset{\vee+}{\mathfrak{u}}_{\varepsilon,r} \le (3\lambda/4)^{-2(\theta-1)} \overset{\vee+}{\mathfrak{u}}_r$. A similar estimate holds for $\overset{\triangleright}{\mathfrak{u}}_{\varepsilon,r}$ and the conclusion follows from Lemma A.2. □

We complete this appendix by comparing the solutions $\overset{\vee+}{\mathfrak{u}}_r$ and $\overset{\vee+}{\mathfrak{u}}_{\varepsilon,r}$, as well as $\overset{\triangleright}{\mathfrak{u}}_r$ and $\overset{\triangleright}{\mathfrak{u}}_{\varepsilon,r}$.

Proposition A.1. *In the interval $(-\frac{7}{8}r, \frac{7}{8}r)$ we have the estimate*

$$|\overset{\vee+}{\mathfrak{u}}_{\varepsilon,r} - \overset{\vee+}{\mathfrak{u}}_r| \le C\varepsilon^{\frac{1}{\theta}} r^{-\frac{2\theta-1}{\theta(\theta-1)}}.$$

Proof. Let $\varepsilon < \delta < r/16$ to be fixed. It follows from Lemma A.3 that, for $x \in (-r+\delta, r-\delta)$, we have

$$0 \le \varepsilon^{\frac{1}{\theta-1}} \overset{\vee+}{\mathfrak{u}}_{\varepsilon,r}(x) \le C\left(\frac{\varepsilon}{\delta}\right)^{\frac{1}{\theta-1}},$$

and therefore also

$$\left| \varepsilon^{\frac{1}{\theta-1}} \overset{\vee+}{\mathfrak{u}}_{\varepsilon,r}(x) g\left(\varepsilon^{\frac{1}{\theta-1}} \overset{\vee+}{\mathfrak{u}}_{\varepsilon,r}(x)\right)\right| \leq C\left(\frac{\varepsilon}{\delta}\right)^{\frac{1}{\theta-1}}. \tag{A.11}$$

It follows from (A.11) and the fact that $\mathcal{L}^{\varepsilon}(\overset{\vee+}{\mathfrak{u}}_{\varepsilon,r}) = 0$, that $\mathcal{L}_{\lambda_\varepsilon^-}(\overset{\vee+}{\mathfrak{u}}_{\varepsilon,r}) \leq 0$, where $\lambda_\varepsilon^{\pm} = \lambda(1 \pm C(\frac{\varepsilon}{\delta})^{\frac{1}{\theta-1}})$. On the other hand, by the scaling law (A.2), we have

$$\mathcal{L}_{\lambda_\varepsilon^-}\left((\frac{\lambda_\varepsilon^-}{\lambda})^{-\frac{1}{2(\theta-1)}} \overset{\vee+}{\mathfrak{u}}_{r-\delta}\right) = 0.$$

It follows from the comparison principle, since the second function is infinite on the boundary of the interval $[-r+\delta, r-\delta]$, that

$$\overset{\vee+}{\mathfrak{u}}_{\varepsilon,r} \leq (\frac{\lambda_\varepsilon^-}{\lambda})^{-\frac{1}{2(\theta-1)}} \overset{\vee+}{\mathfrak{u}}_{r-\delta} \quad \text{on } [-r+\delta, r-\delta].$$

Integrating the inequality (A.7) between $r-\delta$ and r, we deduce that for $x \in (-\frac{7}{8}r, \frac{7}{8}r)$, we have the inequality

$$|\overset{\vee+}{\mathfrak{u}}_{r-\delta}(x) - \overset{\vee+}{\mathfrak{u}}_{r}(x)| \leq C\delta r^{-\frac{\theta}{\theta-1}}. \tag{A.12}$$

On the other hand, it follows from estimate (A.5) of Lemma A.2 that for $x \in (-\frac{7}{8}r, \frac{7}{8}r)$,

$$|(\frac{\lambda_\varepsilon^-}{\lambda})^{-\frac{1}{2(\theta-1)}} \overset{\vee+}{\mathfrak{u}}_{r-\delta}(x) - \overset{\vee+}{\mathfrak{u}}_{r-\delta}(x)| \leq C\left(\frac{\varepsilon}{\delta}\right)^{\frac{1}{\theta-1}} r^{-\frac{1}{\theta-1}}. \tag{A.13}$$

We optimize the outcome of (A.12) and (A.13) choosing $\delta := \varepsilon^{\frac{1}{\theta}} r^{\frac{\theta-1}{\theta}}$ and we therefore obtain

$$\overset{\vee+}{\mathfrak{u}}_{\varepsilon,r} \leq \overset{\vee+}{\mathfrak{u}}_{r} + C\varepsilon^{\frac{1}{\theta}} r^{-\frac{2\theta-1}{\theta(\theta-1)}} \quad \text{on } (-\frac{7}{8}r, \frac{7}{8}r).$$

The lower bound for $\overset{\vee+}{\mathfrak{u}}_{\varepsilon,r}$ is obtained in a similar way but reversing the role of super and subsolutions: the function $(\frac{\lambda_\varepsilon^+}{\lambda})^{-\frac{1}{2(\theta-1)}} \overset{\vee+}{\mathfrak{u}}_{r+\delta}$ yields a subsolution for $\mathcal{L}^{\varepsilon}$ on $[-r, r]$ whereas $\overset{\vee+}{\mathfrak{u}}_{\varepsilon,r}$ is a solution. The conclusion then follows. □

Similarly, we have:

Proposition A.2. *In the interval $(-\frac{7}{8}r, \frac{7}{8}r)$ we have the estimate*

$$|\overset{\triangleright}{\mathfrak{u}}_{\varepsilon,r} - \overset{\triangleright}{\mathfrak{u}}_{r}| \leq C\varepsilon^{\frac{1}{\theta}} r^{-\frac{2\theta-1}{\theta(\theta-1)}}.$$

Proof. We only sketch the necessary adaptations since the argument is closely parallel to the proof of Proposition A.1. First, by the maximum principle $\overset{\triangleright}{\mathfrak{u}}_{\varepsilon,r}$ can only have negative maximae and positive minimae, so that actually $\overset{\triangleright}{\mathfrak{u}}_{\varepsilon,r}$ has no critical point and a single zero, which we call a_ε. Arguing as in Proposition A.1, one first obtains

$$\left(\lambda_\varepsilon^- / \lambda\right)^{-\frac{1}{2(\theta-1)}} \overset{\triangleright}{\mathfrak{u}}_{r-\delta-a_\varepsilon}(\cdot - a_\varepsilon) \geq \overset{\triangleright}{\mathfrak{u}}_{\varepsilon,r} \geq \left(\lambda_\varepsilon^+ / \lambda\right)^{-\frac{1}{2(\theta-1)}} \overset{\triangleright}{\mathfrak{u}}_{r+\delta-a_\varepsilon}(\cdot - a_\varepsilon) \quad \text{on } [a_\varepsilon, r-\delta],$$

and

$$-\left(\lambda_\varepsilon^- / \lambda\right)^{-\frac{1}{2(\theta-1)}} \overset{\triangleright}{\mathfrak{u}}_{r+a_\varepsilon-\delta}(\cdot - a_\varepsilon) \geq -\overset{\triangleright}{\mathfrak{u}}_{\varepsilon,r} \geq -\left(\lambda_\varepsilon^+ / \lambda\right)^{-\frac{1}{2(\theta-1)}} \overset{\triangleright}{\mathfrak{u}}_{r+a_\varepsilon+\delta}(\cdot - a_\varepsilon) \quad \text{on } [-r+\delta, a_\varepsilon].$$

Since $\overset{\triangleright}{\mathfrak{u}}_{\varepsilon,r}$ is continuously differentiable at the point a_ε (indeed it solves $\mathcal{L}^\varepsilon(\overset{\triangleright}{\mathfrak{u}}_{\varepsilon,r}) = 0$), and since the derivative at zero of the function $\overset{\triangleright}{\mathfrak{u}}_r$ is a deacreasing function of r, it first follows from the last two sets of inequalities that $|a_\varepsilon| \leq \delta$, and the conclusion then follows as in Proposition A.1. □

Appendix B

B.1 Some properties of the ordinary differential equation $(\mathcal{S})$

This Appendix is devoted to properties of the system of ordinary differential equations $(\mathcal{S})$, the result being somewhat parallel to the results in Section 2 of [5]. We assume that we are given $\ell \in \mathbb{N}^*$, and a solution, for $k \in J = \{1, \cdots, \ell\}$ $t \mapsto a(t) = (a_1(t), \cdots, a_\ell(t))$ to the system

$$\mathfrak{q}_k \frac{d}{ds} a_k(s) = -\mathcal{B}_{(k-\frac{1}{2})}[(a_k(s) - a_{k-1}(s)]^{-(\omega+1)} + \mathcal{B}_{(k+\frac{1}{2})}[(a_{k+1}(s) - a_k(s)]^{-(\omega+1)}, \quad \text{(B.1)}$$

where the numbers $\mathfrak{q}_k$ are supposed to be positive, and are actually taken in $(\mathcal{S})$ equal to $\mathfrak{S}_{i(k)}$, whereas the numbers $\mathcal{B}_{k+\frac{1}{2}}$, which may have positive or negative signs, are taken in $(\mathcal{S})$ to be equal to $\Gamma_{i(k-\frac{1}{2})}$. We also define $\mathfrak{q}_{\min} = \min\{\mathfrak{q}_i\}$ and $\mathfrak{q}_{\max} = \max\{\mathfrak{q}_i\}$. We consider a solution on its maximal interval of existence, which we call $[0, T_{\max}]$. An important feature of the equation (B.1) is its gradient flow structure. The behavior of this system is indeed related to the function F defined on $\mathbb{R}^\ell$ by

$$\begin{cases} F(U) = \displaystyle\sum_{k=0}^{\ell-1} F_{k+\frac{1}{2}}(U), \text{ where, for } k = 0, \cdots, \ell-1, \text{ and } U = (u_1, \cdots, u_\ell), \text{ we set} \\ F_{k+\frac{1}{2}}(U) = -\omega^{-1}\mathcal{B}_{k+\frac{1}{2}}\,(u_{k+1} - u_k)^{-\omega} \text{ with the convention that } u_0 = -\infty. \end{cases}$$

If $u_1 < u_2 < \cdots < u_\ell$, for $k = 1, \ldots, \ell$, then we have

$$\frac{\partial F}{\partial u_k}(U) = \mathcal{B}_{k-\frac{1}{2}}\,(u_k - u_{k-1})^{-(\omega+1)} - \mathcal{B}_{k+\frac{1}{2}}\,(u_{k+1} - u_k)^{-(\omega+1)}, \quad \text{(B.2)}$$

so that $(\mathcal{S})$ writes $\dfrac{d}{ds} a_k(s) = -\mathfrak{q}_k^{-1} \dfrac{\partial F}{\partial u_k}(a(s))$. Hence, we have

$$\frac{d}{dt} F(a(t)) = \sum_{k=1}^{\ell} \frac{\partial F}{\partial u_k}(a(t)) \frac{da_k}{dt}(t) = -\sum_{k=1}^{\ell} \mathfrak{q}_k^{-1} \left(\frac{\partial F}{\partial u_k}(a(t)) \right)^2 \leq -\mathfrak{q}_{\max}^{-1} |\nabla F(a(t))|^2, \quad \text{(B.3)}$$

hence F decreases along the flow. We also consider the positive functionals defined by

$$F_{\rm rep}(U) = \sum_{k \in J^+} F_{k+\frac{1}{2}}(U),\ F_{\rm att} = -\sum_{k \in J^-} F_{k+\frac{1}{2}}(U), \text{ for } U = (u_1, \cdots, u_\ell),$$

where $J^\pm = \{k \in \{0, \ell-1\}$ such that $\epsilon_{k+\frac{1}{2}} \equiv \mathrm{sgn}(\mathcal{B}_{k+\frac{1}{2}}) = \pm 1\}$.

Proposition B.1. *Let $a = (a_1, \cdots, a_\ell)$ be a solution to* (B.1) *on its maximal interval of existence* $[0, T_{\max}]$. *Then, we have, for any time $t \in [0, T_{\max}]$*

$$\begin{cases} (F_{\mathrm{rep}}(a(t)))^{-\frac{\omega+2}{\omega}} \geq (F_{\mathrm{rep}}(a(0)))^{-\frac{\omega+2}{\omega}} + \mathcal{S}_0 t, & \mathfrak{d}_a^+(t) \geq \left(\mathcal{S}_1 t + \mathcal{S}_2 \mathfrak{d}_a^+(0)^{\omega+2}\right)^{\frac{1}{\omega+2}} \\ (F_{\mathrm{att}}(a(t)))^{-\frac{\omega+2}{\omega}} \leq (F_{\mathrm{att}}(a(0)))^{-\frac{\omega+2}{\omega}} - \mathcal{S}_0 t, & \mathfrak{d}_a^-(t) \leq \left(\mathcal{S}_3 \mathfrak{d}_a^-(0)^{\omega+2} - \mathcal{S}_4 t\right)^{\frac{1}{\omega+2}}, \end{cases} \tag{B.4}$$

where $\mathcal{S}_0 > 0$, $\mathcal{S}_1 > 0$, $\mathcal{S}_2 > 0$, $\mathcal{S}_3 > 0$ and $\mathcal{S}_4 > 0$ depend only the coefficients of (B.1).

Since $\mathfrak{d}_a^-(s) \geq 0$, an immediate consequence of (B.4) is that

$$T_{\max} \leq \frac{\mathcal{S}_3}{\mathcal{S}_4} \mathfrak{d}_a^-(0). \tag{B.5}$$

Since the system (B.1) involves both attractive and repulsive forces, for the proof of Proposition B.1 it is convenient to divide the collection $\{a_1(t), a_2(t), \cdots, a_\ell(t)\}$ into repulsive and attractive chains. We say that a subset A of J is a *chain* if $A = \{k, k+1, k+2, \cdots, k+m\}$ is an ordered subset of m consecutive elements in J, with $m \geq 1$.

Definition 4. *A chain A is said to be* repulsive *(resp.* attractive*) if and only if $\epsilon_{j+\frac{1}{2}} = -1$ (resp. $+1$) for $j = k, \cdots, k+m$. It is said to be a* maximal repulsive chain *(resp.* maximal attractive chain*), if there does exists any repulsive (resp. attractive) chain which contains A strictly.*

It follows from our definition that repulsive or attractive chain contain at least two elements. We may decompose J, in increasing order, as

$$J = B_0 \cup A_1 \cup B_1 \cup A_2 \cup B_2 \cup \cdots \cup B_{p-1} \cup A_p \cup B_p, \tag{B.6}$$

where the chains A_i are maximal repulsive chains, the chains B_i are maximal attractive for $i = 1, \ldots, p-1$, and the sets B_0 and B_p being possibly empty or maximal attractive chains. Moreover for $i = 1 \ \cdots, p$ the sets $A_i \cap B_i$, and $B_i \cap A_{i+1}$ contain one element.

B.2 Maximal repulsive chains

In this subsection, we restrict ourselves to the study of the behavior of a *maximal repulsive chain $A = \{j, j+1, ...j+m\}$* of $m+1$ consecutive points, $m \leq \ell - 2$ *within the general system* (B.1). Setting, for $k = 0, \cdots, m$, $\mathfrak{u}_k(\cdot) = a_{k+j}(\cdot)$, we are led to study $\mathfrak{U}(\cdot) = (\mathfrak{u}_0(\cdot), \mathfrak{u}_1(\cdot), \cdots, \mathfrak{u}_m(\cdot))$. Since $B_{k+\frac{1}{2}} < 0$ in the repulsive case the chain $\mathfrak{U}$ is moved through a system of $m-1$ ode's,

$$\mathfrak{q}_k \frac{d}{ds} \mathfrak{u}_k(s) = -|\mathcal{B}_{(k-\frac{1}{2})}|[(\mathfrak{u}_k(s) - \mathfrak{u}_{k-1}(s)]^{-(\omega+1)} + |\mathcal{B}_{(k+\frac{1}{2})}|[(\mathfrak{u}_{k+1}(s) - \mathfrak{u}_k(s)]^{-(\omega+1)} \tag{B.7}$$

for $k = 1, \cdots, m-1$, whereas the end points satisfy two differential inequalities

$$\frac{d}{ds} \mathfrak{u}_m(s) \geq -\mathfrak{q}_m^{-1} \frac{\partial \tilde{F}_{\mathrm{rep}}}{\partial u_m}(\mathfrak{u}_m(s)), \quad \frac{d}{ds} \mathfrak{u}_0(s) \leq -\mathfrak{q}_0^{-1} \frac{\partial \tilde{F}_{\mathrm{rep}}}{\partial u_0}(a(s)), \tag{B.8}$$

where we have set $\tilde{F}_{\mathrm{rep}}(U) = \sum_{k=0}^{m-1} F_{k+\frac{1}{2}}(U)$. We assume that at initial time we have

$$\mathfrak{u}_0(0) < \mathfrak{u}_1(0) < \cdots < \mathfrak{u}_m(0). \tag{B.9}$$

Set $\mathfrak{d}_{\mathfrak{u}}(t) = \min\{\mathfrak{u}_{k+1}(t) - \mathfrak{u}_k(t), \quad k = 0, \cdots, m-1\}$. We prove in this subsection:

Proposition B.2. *Assume that the function* $\mathfrak{U}$ *satisfies the system* (B.7) *and* (B.8) *on* $[0, T_{\max}]$*, and that* (B.9) *hold. Then, we have, for any* $t \in [0, T_{\max}]$

$$\left(\tilde{F}_{\text{rep}}(\mathfrak{U}(t))\right)^{-\frac{\omega+2}{\omega}} - \left(\tilde{F}_{\text{rep}}(\mathfrak{U}(0))\right)^{-\frac{\omega+2}{\omega}} \geq \frac{\omega+1}{4\omega}\mathfrak{q}_{\max}^{-1}\left(\omega\mathcal{B}_{\max}\right)^{-\frac{2(\omega+1)}{\omega}} t, \quad \text{so that} \tag{B.10}$$

$$\mathfrak{d}_{\mathfrak{u}}(t) \geq \left(\mathcal{S}_1 t + \mathcal{S}_2 \mathfrak{d}_{\mathfrak{u}}(0)^{\omega+2}\right)^{\frac{1}{\omega+2}}, \tag{B.11}$$

where $\mathcal{S}_1 > 0$ *and* $\mathcal{S}_2 > 0$ *depend only on the coefficients of the equation* (B.1).

The proof relies on several elementary observations.

Lemma B.1. *Let* $\mathfrak{U}$ *be a solution to* (B.7)*,* (B.8) *and* (B.9)*. Then, we have,*

$$\frac{d}{dt}\tilde{F}_{\text{rep}}(\mathfrak{U}(t)) \leq -\mathfrak{q}_{\max}^{-1}\left|\nabla \tilde{F}_{\text{rep}}(\mathfrak{U}(t))\right|^2, \text{ for every } t \in [0, T_{\max}]. \tag{B.12}$$

The proof is similar to (B.3) and we omit it. For $U = (u_0, u_1, \cdots, u_m) \in \mathbb{R}^{m+1}$ with $u_0 < \cdots < u_m$ set $\rho_{\min}(U) = \inf\{|u_{k+1} - u_k|,\ k = 0, \cdots, m-1\}$ and $\mathcal{B}_{\min} = \inf\{|\mathcal{B}_{k+\frac{1}{2}}|\}$, $\mathcal{B}_{\max} = \sup\{|\mathcal{B}_{k+\frac{1}{2}}|\}$.

Lemma B.2. *Let* $U = (u_0, \cdots, u_m)$ *be such that* $u_0 < u_1 < \cdots < u_m$*. We have,*

$$\mathcal{B}_{\min}\omega^{-1}\rho_{\min}(U)^{-\omega} \leq \tilde{F}_{\text{rep}}(U) \leq \mathcal{B}_{\max}(m+1)(\omega)^{-1}\rho_{\min}(U)^{-\omega}. \tag{B.13}$$

$$|\nabla \tilde{F}_{\text{rep}}(U)| \leq (m+2)\mathcal{B}_{\max}((\omega-1)\mathcal{B}_{\min})^{-\frac{\omega+1}{\omega}}(\tilde{F}_{\text{rep}}(U))^{\frac{\omega+1}{\omega}}, \tag{B.14}$$

$$\left|\frac{\partial \tilde{F}_{\text{rep}}(U)}{\partial u_k}\right| \geq \frac{1}{2}\left(\omega\mathcal{B}_{\max}\right)^{-\frac{\omega+1}{\omega}}\left(\tilde{F}_{\text{rep}}(U)\right)^{\frac{\omega+1}{\omega}}, \text{ for every } k = 0, \cdots, m. \tag{B.15}$$

Proof. Inequalities (B.13) and (B.14) are direct consequences of the definition of $\tilde{F}_{\text{rep}}$. We turn therefore to estimate (B.15). In view of formula (B.2), the cases $k = 0$ and $k = m+1$ are straightforward. Next, let $k = 1, \cdots m$ and set $T_{k+\frac{1}{2}} = \mathcal{B}_{k+\frac{1}{2}}\left(u_{k+1} - u_k\right)^{-(\omega+1)}$. We distinguish two cases:
Case 1: $T_{k-\frac{1}{2}} \leq \frac{1}{2}T_{k+\frac{1}{2}}$. Then, we have, in view of (B.2) $T_{k+\frac{1}{2}} \leq 2|\frac{\partial F}{\partial u_k}(U)| \leq 2|\nabla F(U)|$, and we are done.
Case 2 : $T_{k-\frac{1}{2}} \geq \frac{1}{2}T_{k+\frac{1}{2}}$. In that case, we repeat the argument with k replaced by $k-1$. Then either $T_{k-\frac{3}{2}} \leq \frac{1}{2}T_{k-\frac{1}{2}}$, which yields as above $T_{k-\frac{1}{2}} \leq 2|\nabla F(U)|$, so that we are done, or $T_{k-\frac{3}{2}} \geq \frac{1}{2}T_{k+\frac{1}{2}}$, and we go on. Since we have to stop at $k = 0$, this leads to the desired inequality (B.15). □

Proof of Proposition B.2. Combining (B.12) with (B.15) we are led to

$$\frac{d}{dt}\tilde{F}_{\text{rep}}(\mathfrak{U}(t)) \leq -\frac{1}{4}\mathfrak{q}_{\max}^{-1}\left(\omega\mathcal{B}_{\max}\right)^{-\frac{2(\omega+1)}{\omega}}\left(\tilde{F}_{\text{rep}}(\mathfrak{u}(t))\right)^{\frac{2(\omega+1)}{\omega}}.$$

Integrating this differential equation, we obtain (B.10). Combining the last inequality of Lemma B.1 with inequality (B.13), inequality (B.11) follows. □

B.3 Maximal attractive chains

Maximal attractive chains $B = \{j, j+1, ...j+m\}$, with $m \leq \ell - 1$ within the general system (B.1), are handled similarly. Defining $\mathfrak{U}$ as above, the function $\mathfrak{U}$ still satisfies (B.7), but the inequalities (B.8) are now replaced by

$$\frac{d}{ds}\mathfrak{u}_m(s) \leq -\mathfrak{q}_m^{-1}\frac{\partial \tilde{F}_{\rm att}}{\partial u_m}(\mathfrak{u}_m(s)), \quad \frac{d}{ds}\mathfrak{u}_0(s) \geq -\mathfrak{q}_0^{-1}\frac{\partial \tilde{F}_{\rm att}}{\partial u_0}(a(s)). \tag{B.16}$$

$\tilde{F}_{\rm att}(U)$ is defined by $\tilde{F}_{\rm att} = -\tilde{F}_{\rm rep}$, so that we have in the attractive case $\tilde{F}_{\rm att} \geq 0$. Up to a change of sign, the function $\tilde{F}_{\rm att}$ verifies the properties (B.13), (B.14) and (B.15) stated in Proposition B.2. However the differential inequality (B.12) is now turned into

$$\frac{d}{dt}\tilde{F}_{\rm att}(\mathfrak{U}(t)) \geq \mathfrak{q}_{\rm max}^{-1}\left|\nabla \tilde{F}_{\rm att}(\mathfrak{U}(t))\right|^2 \geq \mathcal{C}\tilde{F}_{\rm att}(\mathfrak{U}(t))^{\frac{2(\omega+1)}{\omega}}, \tag{B.17}$$

where the last inequality follows from (B.15) and where $\mathcal{C}$ is some constant depending only on the coefficients in (B.1). Integrating (B.17) and invoking once more (B.13), we obtain

Lemma B.3. *Assume that* $\mathfrak{U}$ *satisfies the system* (B.7) *and* (B.16) *on* $[0, T_{\rm max}]$ *with* (B.9). *Then for constants* $\mathcal{S}_3 > 0$ *and* $\mathcal{S}_4 > 0$ *depending only on the coefficients of* (B.1)*, we have*

$$\mathfrak{d}_{\mathfrak{u}}(t) \leq \left(\mathcal{S}_3\mathfrak{d}_{\mathfrak{u}}(0)^{\omega+2} - \mathcal{S}_4 t\right)^{\frac{1}{\omega+2}}.$$

B.4 Proof of Proposition B.1 completed

Inequalities (B.4) of Proposition B.1 follow immediately from Proposition B.2 and Lemma B.3 applied to each separate maximal chain provided by the decomposition (B.6). □

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