Two-sided space-time L^1 approximation and optimal control of polynomial systems

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TWO-SIDED SPACE-TIME L^1 APPROXIMATION AND OPTIMAL CONTROL OF POLYNOMIAL SYSTEMS

BRUNO DESPRÉS* AND EMMANUEL TRÉLAT†

Abstract. We study a two-sided space-time L^1 optimization problem and show how to reformulate the problem within the framework of optimal control theory for polynomial systems. This yields insight on the structure of the optimal solution. We prove existence and uniqueness of the optimal solution, and we characterize it by means of the Pontryagin maximum principle. The cost function and the control converge when the polynomial degree tends to $+\infty$. We illustrate the theory with numerical simulations, which show that our optimal control interpretation leads to efficient algorithms.

Key words. Two sided optimization, polynomial systems, optimal control

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1. Introduction. Let us start with several notations. Throughout the paper, the notation I stands for a generic bounded nonempty real interval (typically, $I = [0,1]$). The set of real polynomial functions of one variable, of maximal degree $n \in \mathbb{N}$, is denoted by $\mathcal{P}_n = \mathbb{R}_n[x]$. The (convex) subset of polynomial functions that are nonnegative over I is denoted by

$$\mathcal{P}_n^+ = \{p_n \in \mathcal{P}_n \quad | \quad 0 \leqslant p_n(x) \quad \forall x \in I\}.$$

We define the set

$$\mathcal{U}_n = \left\{p_n \in \mathcal{P}_n^+ \;\; | \;\; 1 - p_n \in \mathcal{P}_n^+\right\} = \{p_n \in \mathcal{P}_n \;\; | \;\; 0 \leqslant p_n(x) \leqslant 1 \quad \forall x \in I\}.$$

Quite obviously, $\mathcal{U}_n$ is a convex compact subset of $\mathcal{P}_n$, of nonempty interior. Given any $T > 0$, any $n \in \mathbb{N}$ and any $q_n \in \mathcal{P}_n^+$, we also define the (convex) set

$$\mathcal{K}_n(T, q_n) = \left\{ u_n \in L^\infty(0, +\infty; \mathcal{U}_n) \;\; \Big| \;\; \int_0^T u_n(t)\, dt = q_n, \quad u_n(t) = 0 \quad \forall t > T \right\}.$$

REMARK 1. *The set $\mathcal{K}_n(T, q_n)$ is nonempty if and only if $T \geqslant \|q_n\|_{L^\infty(I)}$. Indeed, if $T \geqslant \|q_n\|_{L^\infty(I)}$ then the nontrivial function u_n defined by $u_n(t,x) = \frac{1}{T} q_n(x)$ (extended by 0 outside of $[0,T]$) belongs to $\mathcal{K}_n(T, q_n)$. Conversely, if $u_n \in \mathcal{K}_n(T, q_n)$, since $0 \leqslant u_n(t,x) \leqslant 1$ we must have $0 \leqslant \int_0^T u_n(t,x)\, dt = q_n(x) \leqslant T$, and thus $T \geqslant \|q_n\|_{L^\infty(I)}$. Hence, in what follows we will always assume that $T \geqslant \|q_n\|_{L^\infty(I)}$.*

Let s be a strictly convex function (called an *entropy*), and let w be a nonnegative Lebesgue-integrable weight function defined on I and satisfying $\int_I w(x)\, dx > 0$. A typical example is to choose $I = [0,1]$, $w(x) = 1$ and $s'(t) = t$. We define the linear cost function

$$J(u_n) = \int_0^{+\infty} \int_I u_n(t,x) w(x)\, dx\, ds(t),$$

for every $u_n \in L^\infty(0, T; \mathcal{P}_n)$ such that $u_n(t) = 0$ when $t > T$. Note that, above, we use the notation $u_n(t)(x) = u_n(t,x)$ without any ambiguity.

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In the present paper, we consider the L^1-minimization problem consisting of minimizing the functional J over the convex set $\mathcal{K}_n(T, q_n)$, for some given $q_n \in \mathcal{P}_n^+$ and $T \geqslant \|q_n\|_{L^\infty(I)}$, that is,

$$\boxed{\overline{u}_n = \operatorname{argmin}\{J(u_n) \ \mid \ u_n \in \mathcal{K}_n(T, q_n)\}} \tag{1}$$

If $q_n = 0$ then the optimal solution is obviously $\overline{u}_n = 0$. If only one inequality is taken into account in the definition of $\mathcal{K}_n(T, q_n)$, then the problem becomes obvious. Here, we deal with a space-time L^1 minimization problem that is two-sided. In the case $n = 0$, a characterization due to Brenier (see [4, 6]) yields the optimal solution $\overline{u}_0(t) = 1$ for $0 < t < q_0$ and $u_0(t) = 0$ for $q_0 < t$: in this case, u_0 is optimal for any admissible entropy function s. The case $n = 1$ also has a trivial solution (see [12]), which is the same for different entropy functions s. The general case $n \geqslant 2$ is the subject of the present work. The set $\mathcal{K}_n(T, q_n)$ is a closed convex set, nonempty if $T \geqslant \|q_n\|_{L^\infty(I)}$, and we will establish further its compactness, implying existence of minimizers for Problem (1). The main issue is that J is not strictly convex since it is only linear, so there is no reason a priori to have a unique minimizer. Uniqueness will be established but the proof is far from being obvious.

1.1. Motivation of the study: kinetic polynomials. Problem (1) has first been formulated in [12], where the authors model uncertainties in kinetic formulations of nonlinear conservation laws (see also [23, 24]): it echoes some relationships between uncertainty quantification and modern L^1-minimization. References to L^1-minimization in compressed sensing and related problems can be found in [14, 18]. The theory of L^1-minimization in an alternative to formulations with moments for which we refer to [7, 17]. Moment methods in uncertainty modeling are called chaos polynomials (see [22]). In the context of uncertainty modeling, u_n is called a kinetic polynomial and the weight w characterizes some underlying probability law attached to the uncertainties. In the construction done in [12], it was assumed (but not proved) that Problem 1 has a unique solution, $\overline{u}_n$, which, in the context of that paper, is a polynomial modification of the usual "special" Maxwellian $M(t, x) = \mathrm{Ind}_{0<t<q_n(x)}$ (indicatrix function). Of course, such a function, used as well in [4, 6, 23, 24], takes values between 0 and 1 but is not a polynomial in x when fixing t. This is why it is required to define a convenient projection of this special Maxwellian onto the set $\mathcal{K}_n(T, q_n)$. In this sense, the results of the present paper also justify the construction proposed in [12] .

1.2. Formulation as an optimal control problem. Problem (1) can be equivalently expressed as an optimal control problem, as follows. Given any $u_n \in L^\infty(0, +\infty; \mathcal{U}_n)$, we define $y_n(t, x) = \int_0^t u_n(\tau, x)\, d\tau$, so that we have $\partial_t y_n(t) = u_n(t)$ and $y_n(0) = 0$, and so that $u_n \in \mathcal{K}_n(T, q_n)$ if and only if $y_n(T) = q_n$, where it is understood that $u_n(t)$ has been extended by 0 for $t > T$.

In this context, the function $t \mapsto u_n(t) \in \mathcal{U}_n$ is viewed as a *control*, and $y_n(t) \in \mathcal{P}_n$ as a state at time t. We have then the following equivalent formulation of Problem (1):

$$\boxed{\begin{array}{l} \textit{Given } q_n \in \mathcal{P}_n^+ \textit{ and } T \geqslant \|q_n\|_{L^\infty(I)}, \textit{ find a control } \overline{u}_n \in L^\infty(0, T; \mathcal{U}_n) \\ \textit{solution of the optimal control problem} \\ \qquad \begin{array}{ll} \partial_t y_n(t) = u_n(t), & \text{for a.e. } t \in [0, T], \\ y_n(0) = 0, \quad y_n(T) = q_n, & \\ u_n(t) \in \mathcal{U}_n, & \text{for a.e. } t \in [0, T], \\ \min J_T(u_n) = \displaystyle\int_0^T \int_I u_n(t, x) w(x)\, dx\, ds(t). & \end{array} \end{array}} \tag{2}$$

Settled as such, this is an optimal control problem in the $(n+1)$-dimensional state space $\mathcal{P}_n$, with controls taking their values in the $(n+1)$-dimensional convex set $\mathcal{U}_n$, with fixed initial and final conditions, and integral minimization criterion. Note that, if $\mathcal{P}_n$ is equipped with the Euclidean structure inherited, by restriction, from the weighted space $L^2_w(I)$ (which is $L^2(I)$ for the measure $w\,dx$), with the scalar product being denoted by $\langle\ ,\ \rangle_{L^2_w(I)}$, then

$$J_T(u_n) = \int_0^T \langle u_n(t), 1\rangle_{L^2_w(I)}\, ds(t).$$

Of course, this optimal control control is still not strictly convex.

Note that, with the above notations, we have $J_T(u_n) = J(u_n)$ for every $u_n \in L^\infty(0,T;\mathcal{P}_n)$ such that $u_n(t) = 0$ when $t > T$.

1.3. Main results. Our first main result is a generalization of some results due to Bojanic and DeVore in [3] on one-sided L^1 minimization, which is a branch of polynomial approximation theory. To our knowledge, this is the first time that Bojanic and DeVore results are extended to space-time with two-sided constraints.

THEOREM 1. *Let $q_n \in \mathcal{P}_n^+$ and let $T \geqslant \|q_n\|_{L^\infty(I)}$. We assume that $s'' \in L^1(0,T)$, and that there exists $s''_- > 0$ such that $s''(t) \geqslant s''_-$ for almost every $t \in [0,T]$. Then Problem (1) (or equivalently, Problem 2) has a unique optimal solution $\overline{u}_n \in \mathcal{K}_n(T, q_n)$. Moreover, $\overline{u}_n$ has the following properties:*

- *The function $t \mapsto \langle \overline{u}_n(t), 1\rangle_{L^2_w(I)}$ is nonincreasing.*
- *There exists $T_* \in [0, +\infty)$, depending on n and on q_n, such that if $T_* < T$ then $\overline{u}_n(t) = 0$ for $T_* < t < T$.*

As already alluded, the nontrivial part is to establish uniqueness. Its proof is based on convenient reformulations of the Bojanic and DeVore theorem (see [3]) and on space-time comparison inequalities using appropriate test functions. A numerical example provided at the end of this work in Lemma 36 shows that the critical time may be strictly larger than $\max_I q_n(x)$ and that it depends on the entropy function s.

Now, taking advantage of the formulation (2) of Problem (1) in terms of optimal control, one can apply the Pontryagin maximum principle (see [21, 26, 33]) and obtain a characterization of the unique optimal solution by means of a first-order optimality system. The study of the resulting conditions leads to the following theorem, which is our second main result.

THEOREM 2. *In the context of Theorem 1, there exists $\overline{\lambda}_n \in \mathcal{P}_n$ (adjoint state) such that the (unique) optimal solution of Problem (1) is given by*

$$\overline{u}_n(t) = \underset{p_n \in \mathcal{U}_n}{\operatorname{argmax}} \int_I (\overline{\lambda}_n(x) - s'(t)) p_n(x) w(x)\, dx = \underset{p_n \in \mathcal{U}_n}{\operatorname{argmax}} \langle \overline{\lambda}_n - s'(t), p_n\rangle_{L^2_w(I)}, \tag{3}$$

for almost every $t \in [0,T]$. In particular, $\overline{u}_n(t)$ is an extremal point of $\mathcal{U}_n$, for almost every $t \in [0,T]$.

Extremal points of $\mathcal{U}_n$ are characterized in Section 2.3. The expression (3) is useful to analyze the optimal solution $\overline{u}_n$. In Appendix B, we show how to use (3) in order to compute explicitly optimal solutions for $n = 2$. We also give additional properties in Sections 4.3 on the normal extremal flow and the shooting method, and in Section 4.4 where we take $T = +\infty$.

Denoting by $\overline{y}_n$ the trajectory corresponding to the control $\overline{u}_n$ in Problem 2, interpreted in the optimal control language, the quadruple $(\overline{y}_n, \overline{\lambda}_n, -1, \overline{u}_n)$ is a normal extremal lift of the optimal trajectory, and $(\overline{\lambda}_n, -1)$ is a normal Lagrange multiplier (see Section 4 for the proof of Theorem 2, and in particular the application of the Pontryagin maximum principle). The number of such normal Lagrange multipliers is equal to the dimension of the subdifferential at q_n of the value function associated with the optimal control problem 2 (see Remark 7 in Section 4.2).

For $n = 0$, we recover the Brenier inequality for any convex entropy (see [4, 6]), where the solution is the indicatrix function

$$\mathbf{I}_{\{0<t<q_0\}} = \operatorname{argmin}\left\{ \int_0^{+\infty} v(t)s'(t)\,dt \;\mid\; 0 \leqslant v \leqslant 1,\; \int_0^{+\infty} v(t)\,dt = q_0 \right\}. \tag{4}$$

A similar result holds true also for $n = 1$ because a function that is linear on an interval is characterized by its terminal points (see [12, page 20]). A consequence of the exact solution for $n = 2$ performed in Appendix B is the following result.

PROPOSITION 3. *In contrast to the cases $n = 0$ and $n = 1$, the optimal solution of the Problem* (1) *for $n \geqslant 2$ depends on the entropy function s.*

The equivalent formulation of Problem (1) as the optimal control problem (2) is not only interesting to derive first-order necessary conditions as settled above, but it brings also a rich point of view in order to implement efficient numerical methods for computing optimal solutions. Section 6 will be devoted to provide numerical simulations illustrating our theoretical results.

1.4. Organization of the paper. The paper is structured as follows.

Section 2 gathers several results that are useful to derive our main results but that have also thei own interest in themselves: in Section 2.1 we study a two-sided polynomial maximization problem and we define the useful concept of total order of contact; in Section 2.3, we characterize extremal points of the convex compact subset $\mathcal{U}_n$ of $\mathcal{P}_n$, and we give a precise description of the geometry of its boundary strata in terms of the total order of contact; in Section 2.4 we stress on the possible nonuniqueness of maximizers of a static polynomial maximization problem, in contrast to uniqueness of the dynamic Problem (1). Section 3 is devoted to prove Theorem 1. The delicate point is to establish uniqueness of the maximizer, which is done thanks to results given in Section 2. Theorem 2 is proved in Section 4, by applying the Pontryagin maximum principle to the optimal control problem (2). Convergence issues as $n \to +\infty$ are addressed in Section 5, such as the convergence as $n \to +\infty$ of $\overline{u}_n$ and of $J_T(\overline{u}_n)$. In Section 6, we present some numerical simulations that are based on an optimal control implementation of the problem. Section 7 is a conclusion. Appendix A is devoted to the proof of the technical Theorem 5, and in Appendix B we give a construction of exact solutions for $n = 2$.

2. Auxiliary results.

2.1. "Static" two-sided polynomial maximization problem. In order to prove uniqueness of the optimal solution in Theorem 1, we will use comparison estimates with appropriate test functions (usual approach in minimization problems) and some technical tools coming from one-sided best L^1 polynomial approximation problem, for which we refer to [3]. Since these results have their own interest, we present them hereafter separately, as a byproduct of our study.

Let $f_n \in \mathcal{U}_n$ be fixed. Recall that a given $f_n \in \mathcal{U}_n$ is a polynomial of maximal degree n that satisfies the two-sided constraint $0 \leqslant f_n(x) \leqslant 1$ for every $x \in I$. We set

$$\mathcal{Q}_n = \{p_n \in \mathcal{P}_n \;\mid\; -\min(1-f_n, f_n) \leqslant p_n \leqslant \min(1-f_n, f_n) \text{ on } I\}. \tag{5}$$

Note that $\mathcal{Q}_n$ is convex and nonempty since $0 \in \mathcal{Q}_n$. We consider the following auxiliary two-sided polynomial maximization problem:

$$\boxed{\overline{p}_n = \operatorname*{argmax}_{p_n \in \mathcal{Q}_n} \int_I p_n(x)w(x)\,dx} \tag{6}$$

Compared with Problem (1), this maximization problem is "static" in the sense that it does not involve any time evolution (nor constraint at the final time, a fortiori). This problem is central in our construction of the test functions that will be used in Section 3 to prove Theorem 1. We prove in Theorem 5 that Problem (6) is well posed. But before, we need to define *contact points* and the notion of *total order of contact*, also useful in order to characterize extremal points of $\mathcal{U}_n$.

2.2. Contact points. A contact point of f_n is an element $x \in I$ such that $f_n(x) = 0$ or $f_n(x) = 1$. Hereafter, for clarity the contact points will be denoted by $x_i \in I$ if $f_n(x_i) = 0$ and by $y_j \in I$ if $f_n(y_j) = 1$. Contact points yield important constraints on the maximization problem (6), and their impact has to be explained.

At a given isolated contact point, we first define the local order of contact as the maximal number of successive derivatives that vanish. The order of contact is necessarily even if the contact point is in the interior of the interval I. This yields a first family of contact points $x_i \in I$ on the lower part of the graph of f_n. These contact points are $x_0 < \cdots < x_p$ with contact orders $a_0, \dots, a_p$ such that $\frac{d^j}{dx^j} f_n(x_i) = 0$ for $0 \leqslant j \leqslant a_i - 1$. If x_i is in the interior of I then a_i must be even. If x_i is on the boundary of I then a_i may be odd or even. It follows that there exists $A \in \mathcal{P}_n$, with $A > 0$ on I, such that

$$f_n(x) = \prod_{i=0}^{p} |x - x_i|^{a_i} A(x), \tag{7}$$

for every $x \in I$. The second family of contact points $y_j \in I$ is $y_0 < \cdots < y_q$ with contact orders $b_0, \dots, b_q$ such that $\frac{d^j}{dx^j}(1 - f_n)(y_j) = 0$ for all $0 \leqslant j \leqslant b_i - 1$. A contact point of the first family cannot be a contact point of the second family and vice-versa. If y_j is in the interior of I then b_j must be even. If y_j is at the boundary I then b_j may be odd or even. It follows that there exists $B \in \mathcal{P}_n$, with $B > 0$ on I, such that

$$1 - f_n(x) = \prod_{j=0}^{q} |x - y_j|^{b_j} B(x),$$

for every $x \in I$. If a contact point is not isolated then f_n is identically equal to 0 or 1.

Definition 4. *Let $f_n \in \mathcal{U}_n \setminus \{0, 1\}$. The total order of contact of f_n is the integer defined by*

$$\mathrm{toc}(f_n) = \sum_{i=0}^{p} a_i + \sum_{j=0}^{q} b_j.$$

If f_n is identically equal to 0 or 1 then we set by convention $\mathrm{toc}(0) = \mathrm{toc}(1) = +\infty$ or $2n + 1$.

Actually, if $\mathrm{toc}(f_n) = 2n + 1$, then either $f_n = 0$ or $f_n = 1$. Indeed, then, f_n has at least $n + 1$ contact points (counted with their multiplicity) either with 0 or with 1, and hence either $f_n = 0$ or $f_n = 1$. It follows that $\mathrm{toc}(f_n) \leqslant 2n$ for every $f_n \in \mathcal{U}_n \setminus \{0, 1\}$, and moreover this bound is optimal. Indeed, take $f_n(x) = \frac{1}{2}(1 + T_n(2x - 1)) \in \mathcal{U}_n$, where T_n the Tchebycheff polynomial of degree n. Setting $x = \frac{1}{2}(1 + \cos\theta)$ yields $f_n(x) = \frac{1}{2}(1 + T_n(\cos\theta))$, and then it is easy to prove that $\mathrm{toc}(f_n) = \sum_{i=0}^{p} a_i + \sum_{j=1}^{q} = n + n = 2n$.

The following result can be seen as an extension of the Bojanic and DeVore theorem (see [3]).

Theorem 5. *There exists a unique maximizer $\overline{p}_n \in \mathcal{Q}_n$ of Problem (6). Moreover, $\overline{p}_n = 0$ if and only if $\mathrm{toc}(f_n) \geqslant n + 1$.*

The proof of Theorem 5 is quite lengthy and technical, and is done in Appendix A. It consists of analyzing finely and extending adequately arguments developed by Bojanic and DeVore in [3]. The obvious part of the proposition is that if $\mathrm{toc}(f_n) \geqslant n + 1$ then $\overline{p}_n = 0$: indeed, then, the graph of the function $x \mapsto \min(1 - f_n, f_n)(x)$ has at least $n + 1$ contact points (counted with their multiplicity) with the horizontal axis. In this situation polynomials in $\mathcal{Q}_n$ have $n + 1$ roots, so they must vanish because their degree is less or equal to n. The converse, which is the important and nontrivial part, will be the cornerstone to establish the uniqueness property claimed in Theorem 1, in Section 3.3.

2.3. Extremal points of $\mathcal{U}_n$. In the optimal control problem (2), since controls take their values in the convex set $\mathcal{U}_n$, it will be useful to have information on extremal points of this control constraint set. But there, also, the following result has its own interest and may be useful for other purposes, for instance in polynomial optimization (see [19]) or in polynomial optimal control (see [20]). Recall, again, that $\mathcal{U}_n$ is the set of all polynomials p_n of maximal degree n such that $0 \leqslant p_n(x) \leqslant 1$ for every $x \in I$. It is a compact convex subset (of nonempty interior) of $\mathcal{P}_n$. Note that $p_n \in \partial\mathcal{U}_n$ if and only if there exists $x^* \in I$ such that $p_n(x^*) = 0$ or 1.

The main result of this section is Theorem 8 further, which gives a characterization of the set $\mathrm{Extr}(\mathcal{U}_n)$ of extremal points of $\mathcal{U}_n$: in particular, we are going to establish that a polynomial $p_n \in \mathcal{U}_n$ is an extremal point of $\mathcal{U}_n$ if and only if $\mathrm{toc}(p_n) \geqslant n+1$. But we do not know any explicit (analytic) description of $\mathrm{Extr}(\mathcal{U}_n)$ in general. Of course, the constant polynomials 0 and 1 are extremal points of $\mathcal{U}_n$, but there are many other nontrivial ones. The geometry of the closed convex set $\mathcal{U}_n$ is not simple in general, as we are going to see.

2.3.1. Examples: $n = 1$ and $n = 2$. Before coming to the general case, we first give, hereafter, a precise and complete description $\mathcal{U}_n$ in the cases $n = 1$ and $n = 2$, where computations are easy to perform. For simplicity, we assume that $I = [0, 1]$.

Case $n = 1$. Any $p_1 \in \mathcal{U}_1$ can be written as $p_1(x) = ax + b(1-x)$ with $0 \leqslant a \leqslant 1$ and $0 \leqslant b \leqslant 1$. Therefore $\mathcal{U}_1$ has exactly 4 extremal points, that are the polynomials 0, 1, x, $1-x$.

Case $n = 2$. We start with an easy remark. Let p_2 be a polynomial of maximal degree 2. Writing $p_2(x) = ax^2 + 2bx(1-x) + c(1-x)^2$, we have $1 - p_2(x) = (1-a)x^2 + 2(1-b)x(1-x) + (1-c)(1-x)^2$.

Now, if $p_2(x) \geqslant 0$ for every $x \in [0, 1]$, then according to the Lukács theorem (see [27, 31]) it can also be written as $p_2(x) = (\alpha x + \beta(1-x))^2 + \gamma^2 x(1-x)$. Then, we must have $a = \alpha^2 \geqslant 0$, $b = \beta^2 \geqslant 0$ and $2b = 2\alpha\beta + \gamma^2$, from which it follows that $b \geqslant -\sqrt{a}\sqrt{c}$. Conversely, if $a \geqslant 0$, $c \geqslant 0$ and $b \geqslant -\sqrt{a}\sqrt{c}$ then $p_2(x) = ax^2 + 2bx(1-x) + c(1-x)^2 \geqslant (\sqrt{a}x - \sqrt{c}(1-x))^2 \geqslant 0$.

Similarly, if $p_2(x) \leqslant 1$ for every $x \in [0, 1]$, then $a \leqslant 1$, $c \leqslant 1$ and $b \leqslant 1 + \sqrt{1-a}\sqrt{1-c}$, and conversely. Therefore, we have the following lemma.

LEMMA 6. *The polynomial $p_2(x) = ax^2 + 2bx(1-x) + c(1-x)^2$ belongs to $\mathcal{U}_2$ if and only if $(a, b, c) \in \mathcal{V}_2 := \left\{(a, b, c) \in \mathbb{R}^3 \;\mid\; 0 \leqslant a \leqslant 1,\ 0 \leqslant c \leqslant 1,\ -\sqrt{a}\sqrt{c} \leqslant b \leqslant 1 + \sqrt{1-a}\sqrt{1-c}\right\}$.*

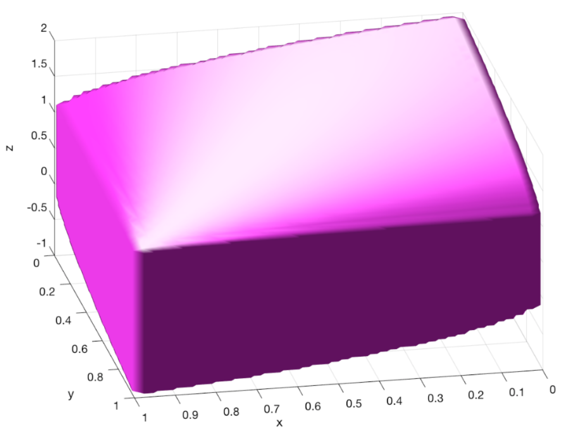

FIG. 1. *Plot of $\mathcal{V}_2$ in $\mathbb{R}^3$*

According to this lemma, $\mathcal{U}_2$ is identified with the convex set $\mathcal{V}_2$ with a linear bijective transform. Therefore $\mathrm{Extr}(\mathcal{U}_2)$ is identified with $\mathrm{Extr}(\mathcal{V}_2)$. The set $\mathcal{V}_2$ is drawn on Figure 1. On this figure, the surfaces at the sides are pieces of planes, and the bordering lower and upper surfaces, which are the graphs $(a,c)\mapsto -\sqrt{a}\sqrt{c}$ and $(a,c)\mapsto 1+\sqrt{1-a}\sqrt{1-c}$, are ruled surfaces (i.e., generated by lines). Therefore $\mathrm{Extr}(\mathcal{V}_2)$ consists of 8 points, given by the triples $(0,0,0)$, $(0,1,0)$, $(1,0,0)$, $(1,1,-1)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1)$, $(0,0,2)$ (which are the vertices that one can observe on the figure), and of curves (graphs of a square root) joining some of them, which are edges that one can be see on Figure 1. In particular, the cardinal of $\mathrm{Extr}(\mathcal{V}_2)$ is infinite.

2.3.2. General results on the geometry of $\mathcal{U}_n$. We now come to the general case, by giving precise results describing the boundary of $\mathcal{U}_n$, which is a stratified submanifold of $\mathcal{P}_n$ thanks to the following simple lemma (see, e.g., [16] for Whitney stratifications).

LEMMA 7. *The set $\mathcal{U}_n$ is semi-algebraic. As a consequence, $\mathcal{U}_n$ has a Whitney stratification, i.e., it admits a partition into smooth submanifolds of $\mathcal{P}_n$ satisfying the Whitney conditions.*

Proof. Assuming without loss of generality that $I=[0,1]$, given any $p_n\in\mathcal{P}_n^+$ (i.e., p_n is a polynomial of maximal degree n that is nonnegative on $[0,1]$, it is well known that there exist $B\in\mathcal{P}_n$ and $C\in\mathcal{P}_{n-1}$ such that

$$p_n(x)=B(x)^2+x(1-x)C(x)^2. \tag{8}$$

We write $p_n(x)=\sum_{i=0}^n a_ix^i$, $B(x)=\sum_{i=0}^n b_ix^i$ and $C(x)=\sum_{i=0}^{n-1} c_ix^i$, and we define $a=(a_0,\dots,a_n)\in\mathbb{R}^{n+1}$, $b=(b_0,\dots,b_n)\in\mathbb{R}^{n+1}$ and $c=(c_0,\dots,c_{n-1})\in\mathbb{R}^n$. By expanding (8), we get $2n+1$ relations involving the coefficients of a, b and c, are are polynomial: the set of $(a,b,c)\in\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\times\mathbb{R}^n$ satisfying these relations is algebraic. By projection, it follows that the set of all $a\in\mathbb{R}^{n+1}$ for which the corresponding $p_n\in\mathcal{P}_n$ belongs to $\mathcal{P}_n^+$ is semi-algebraic (this follows from the Tarski-Seidenberg theorem, see [2, 35]). To conclude, it suffices to note that $p_n\in\mathcal{U}_n$ if and only if $p_n\in\mathcal{P}_n^+$ and $1-p_n\in\mathcal{P}_n^+$, and that the intersection of two semi-algebraic sets is semi-algebraic. By o-minimality of semi-algebraic sets (see [35]), the Whitney stratifiability property follows. ∎

By Lemma 7, the boundary $\partial\mathcal{U}_n$ of the convex $\mathcal{U}_n$ is a stratified compact submanifold of $\mathcal{P}_n\simeq\mathbb{R}^{n+1}$ of dimension n, which is therefore the finite union of strata, each stratum being a submanifold (with boundary) of dimension $k\in\{0,\dots,n\}$. For instance, on Figure 1, strata of dimension 2 are pieces of planes at the sides and the bordering lower and upper (ruled) surfaces; strata of dimension 1 are (curved) edges, some of them consisting of extremal points; strata of diemnsion 0 are the 8 extremal points mentioned previously, which are the vertices.

Now, let us characterize points belonging to a stratum of dimension k. First of all, recall that $p_n\in\partial\mathcal{U}_n$ if and only if there exists $x^*\in I$ such that $p_n(x^*)=0$ or 1, if and only if $\mathrm{toc}(p_n)\geqslant 1$. Actually, the interior of $\mathcal{U}_n$ is

$$\overset{\circ}{\mathcal{U}}_n=\{p_n\in\mathcal{U}_n\ \mid\ \mathrm{toc}(p_n)=0\},$$

and its boundary is

$$\partial\mathcal{U}_n=\{p_n\in\mathcal{U}_n\ \mid\ \mathrm{toc}(p_n)\geqslant 1\}=\bigcup_{j=1}^{2n+1}\mathcal{V}_n^j,$$

with $\mathcal{V}_n^j=\{p_n\in\mathcal{U}_n\ \mid\ \mathrm{toc}(p_n)=j\}$. Note that, for $j=2n+1$, we have $\mathcal{V}_n^{2n+1}=\{0,1\}$. Now, each $\mathcal{V}_n^j$ is itself stratified into submanifolds, as follows. Hereafter, for simplicity we assume that $I=[0,1]$.

Case $j=1$. Any $p_n\in\mathcal{U}_n$ such that $\mathrm{toc}(p_n)=1$ must either vanish or be equal to 1 at one of the boundaries (0 or 1) of the interval I. Therefore, $\mathcal{V}_n^1$ is the union of four strata, the first of which (for instance) being $\{p_n\in\mathcal{U}_n\ \mid\ p_n(0)=0\}$. The latter is a convex subset (and thus, a flat stratum) of dimension n.

Note that such a stratum can be parametrized as follows. We write $p_n(x) = xA_{n-1}(x)$ with $A_{n-1} \in \mathcal{P}_{n-1}$ satisfying $A_{n-1}(x) > 0$ for every $x \in [0,1]$, and $p_n(x) < 1$ on $[0,1]$. Then, there exists $\varepsilon > 0$ (small enough) such that the perturbation of p_n given by

$$v_{(\delta A)_{n-1}}(x) = p_n(x) + (x - x_0)(\delta A)_{n-1}(x) = (x - x_0)\left(A_{n-1}(x) + (\delta A)_{n-1}(x)\right)$$

belongs to $\mathcal{U}_n$ (i.e., $0 \leqslant v_{(\delta A)_{n-1}} \leqslant 1$ on I), for any polynomial $(\delta A)_{n-1} \in \mathcal{P}_{n-1}$ satisfying $\|(\delta A)_{n-1}\|_{L^\infty(I)} < \varepsilon$. Such perturbations generate a flat convex subset of the boundary $\partial\mathcal{U}_n$ (because $v_{(\delta A)_{n-1}}(x_0) = 0$) of dimension n, of barycenter p_n.

Case $j = 2$. Let $p_n \in \mathcal{U}_n$ be such that $\mathrm{toc}(p_n) = 2$. Then, either p_n touches 0 or 1 at both boundaries of I, or p_n has exactly one contact point (with 0 or 1) in the interior of the interval I. Therefore, $\mathcal{V}_n^2$ is the union of six strata.

In the first case (the four first strata), one has, for instance, $p_n(0) = p_n(1) = 0$, which makes a convex subset (flat stratum) of dimension $n-1$.

In the second case (the two last strata), there exists a single contact point x_0 in the interior of I, such that (for instance) $p_n(x_0) = p_n'(x_0) = 0$ and $p_n''(x_0) > 0$. The set of such polynomials is a submanifold of dimension n, containing a convex subset of dimension $n-1$. Indeed, for x_0 fixed, the latter conditions give an open convex subset of $\mathcal{P}_n$ of codimension 2 (Hermite interpolation at x_0); then, making x_0 vary in the interior of I generates (in a nonlinear way) one more dimension.

It is interesting to make explicit a parametrization of such a stratum. Let us write $p_n(x) = (x - x_0)^2 A_{n-2}(x)$ with $A_{n-2} \in \mathcal{P}_{n-2}$ satisfying $A_{n-2}(x) > 0$ for every $x \in [0,1]$, and note that $p_n(x) < 1$ for every $x \in I$. Then, there exists $\varepsilon > 0$ (small enough) such that the perturbation of p_n given by

$$\begin{aligned} v_{(\delta A)_{n-2},\eta}(x) &= p_n(x) - \eta(2x - 2x_0 - \eta)A_{n-2}(x) + (x - x_0 - \eta)^2(\delta A)_{n-2}(x) \\ &= (x - x_0 - \eta)^2\left(A_{n-2}(x) + (\delta A)_{n-2}(x)\right) \end{aligned}$$

belongs to $\mathcal{U}_n$ (i.e., $0 \leqslant v_{(\delta A)_{n-2}} \leqslant 1$ on I), for any $\eta \in \mathbb{R}$ and any polynomial $(\delta A)_{n-2} \in \mathcal{P}_{n-2}$ satisfying $|\eta| + \|(\delta A)_{n-2}\|_{L^\infty(I)} < \varepsilon$. The second line of the above formula is obtained by noting that $(x-x_0)^2 - \eta(2x-2x_0-\eta) = (x-x_0-\eta)^2$. Making vary both parameters $(\delta A)_{n-2}$ and η gives perturbations $v_{(\delta A)_{n-2},\eta}$ that generate a subset of $\partial\mathcal{U}_n$ (because $v_{(\delta A)_{n-2},\eta}(x_0) = 0$) of dimension n, containing p_n in its interior. Among these perturbations:

- Fixing $\eta = 0$, quite similarly to the case $\mathrm{toc}(p_n) = 1$, the perturbations $(\delta A)_{n-2} \mapsto v_{(\delta A)_{n-2},0}$ generate a flat convex subset of $\partial\mathcal{U}_n$ (because $v_{(\delta A)_{n-2},0}(x_0) = 0$) of dimension $n-1$, of barycenter p_n.
- Fixing $(\delta A)_{n-2} = 0$, the perturbations $\eta \mapsto v_{0,\eta}$ generate a (1D) curve of $\partial\mathcal{U}_n$ (because $v_{0,\eta}(x_0 + \eta) = 0$), which is transverse to the previous flat convex subset. Here, we have this additional perturbation which consists of moving the contact point x_0: this was not allowed in the previous case $\mathrm{toc}(p_n) = 1$ because x_0 was at the boundary of I.

For instance for $n = 2$ we recover exactly the upper and lower surfaces of Figure 1, which are ruled but are not planes, in accordance with the above items.

Remark 2. *We stress that contact points are counted with their multiplicity. For instance if* $\mathrm{toc}(p_n) = 5$ *and* $p_n(x) = (x-x_0)(x-x_1)^4 A_{n-5}(x)$, *with* x_0 *at the left boundary of* I *and* x_1 *in the interior of* I, *then explicit perturbations are written as*

$$v_{(\delta A)_{n-5},\eta_1,\eta_2}(x) = (x - x_0)(x - x_1 - \eta_1)^2(x - x_1 - \eta_2)^2\left(A_{n-5}(x) + (\delta A)_{n-5}(x)\right),$$

and thus p_n *belongs to a flat convex subset of this stratum of dimension* $n-4$ *inside a stratum of dimension* $n-2$.

Generalization. Proceeding similarly, assuming that $\mathrm{toc}(p_n) = j$, we get that:

- If j is odd, then one of the contact points is at the boundary of I, and all other $(j-1)/2$ contact points (counted with their multiplicity) are in the interior. Reasoning as above, p_n belongs to the interior of a stratum of dimension $n-(j-1)/2$, and moreover, p_n belongs to a flat convex subset of this stratum of dimension $n-j+1$.
- If j is even, then:
  - Either the two boundaries of I are contact points, and all other $j/2-1$ contact points (counted with their multiplicity) are in the interior. Then p_n belongs to the interior of a stratum of dimension $n-j/2$, and moreover, p_n belongs to a flat convex subset of this stratum of dimension $n-j+1$.
  - Or all $j/2$ contact points are in the interior. Then p_n belongs to the interior of a stratum of dimension $n-j/2+1$, and moreover, p_n belongs to a flat convex subset of this stratum of dimension $n-j+1$.

The computation of codimensions is done as follows. Assume for instance that p_n has exactly two contact points x_0 and y_0 in the interior of I, the first one with 0 and the second one with 1 (then $\mathrm{toc}(p_n) = 4$). The set

$$\{p_n \in \mathcal{U}_n \mid \exists x_0 \in (0,1),\ p_n(x_0) = p_n'(x_0) = 0,\ p_n''(x_0) > 0, \\ \exists y_0 \in (0,1),\ p_n(y_0) = 1,\ p_n'(y_0) = 0,\ p_n''(y_0) < 0\}$$

is a submanifold of $\mathcal{U}_n$ of dimension $n-1$, containing an open convex subset (flat stratum) of dimension $n-3$. Indeed, for x_0 and y_0 fixed, the above conditions give an open convex subset of $\mathcal{P}_n$ of codimension 4 (Hermite interpolation at x_0 and y_0); then, making vary x_0 and y_0 in the interior of I generates (in a nonlinear way) two more dimensions.

Note however that, in contrast to what has been written above, it seems difficult here to write explicit perturbations. But an implicit characterization is enough for our needs.

We have therefore obtained the following result.

Theorem 8. *Let $p_n \in \partial\mathcal{U}_n$, and let $j = \mathrm{toc}(p_n) \in \{1,\dots,2n\}$.*

- *If j is odd then p_n belongs to the interior of a stratum of dimension $n-(j-1)/2$.*
- *If j is even then p_n belongs to the interior of a stratum of dimension either equal to $n-j/2+1$ if all contact points are in the interior of I, or $n-j/2$ if the two boundaries of I are contact points.*

In all cases, if moreover $j \leqslant n$ then p_n belongs to a flat convex subset contained in this stratum, of dimension $n-j+1$ (and thus p_n is not extremal).

In particular:

- *Strata of dimension greater than $n/2$ contain no extremal point in their interior.*
- *Flat strata of $\partial\mathcal{U}_n$ of dimension n exactly consist of all $p_n \in \partial\mathcal{U}_n$ such that $\mathrm{toc}(p_n) = 1$.*
- *A polynomial $p_n \in \mathcal{U}_n$ is an extremal point of $\mathcal{U}_n$ if and only if $\mathrm{toc}(p_n) \geqslant n+1$.*

Remark 3. *If $p_n \in \mathrm{Extr}(\mathcal{U}_n) \setminus \{0,1\}$ then p_n must have contact points with 0 and with 1. Indeed, otherwise, since $\mathrm{toc}(p_n) \geqslant n+1$ by the last item of Theorem 8, either p_n or $1-p_n$ would have $n+1$ zeros, but then p_n would be identically equal to 0 or 1 because its maximal degree is n.*

2.4. A useful remark on a polynomial optimization problem. Let $r_n \in \mathcal{P}_n$ be fixed. We consider the polynomial maximization problem

$$\max_{p_n \in \mathcal{U}_n} \int_0^1 r_n(x) p_n(x)\, dx. \tag{9}$$

Since $\mathcal{U}_n$ is compact and convex in $\mathcal{P}_n$, this problem is well posed, and since the maximization functional is linear, any maximizer $\overline{p}_n$ must either be an extremal point of $\mathcal{U}_n$, or belong to a segment joining two distinct extremal points of $\mathcal{U}_n$.

If $r_n \in \mathcal{P}_n^+$, i.e., if $r_n \geqslant 0$ on $[0,1]$, then Problem (9) has a unique maximizer, which is $\overline{p}_n = 1$. This is in accordance with Theorem 5, because in Problem 6 we can take $w = r_n \geqslant 0$ and in (5) we take $f_n = 1/2$. Similarly, if $r_n \in \mathcal{P}_n^-$, i.e., if $r_n \leqslant 0$ on $[0,1]$, then Problem (9) has a unique maximizer, which is $\overline{p}_n = 0$.

But if r_n does not keep a constant sign over $[0,1]$, then it may happen that Problem (9) have several distinct maximizers. Indeed, (9) corresponds to maximizing the scalar product $\langle r_n, p_n \rangle_{L^2(0,1)}$ over all $p_n \in \mathcal{U}_n$. If r_n is orthogonal (in $L^2(0,1)$) to a supporting hyperplane of $\mathcal{U}_n$ containing at least two distinct extremal points of $\mathcal{U}_n$, then Problem (9) has an infinite number of maximizers.

We have seen in the previous section that this may happen: for $n = 2$ the upper (or the lower) surface of $\mathcal{V}_2$ is ruled, and for instance it contains the segment joining $(1,1,1)$ to $(0,0,2)$. Then it suffices to consider the hyperplane containing this segment and tangent to the surface, and to choose p_2 corresponding to a normal to this hyperplane: this gives an example where there is no uniqueness of the maximizer in (9).

However, nonuniqueness is in some sense exceptional with respect to r_n. Indeed, nonuniqueness may only happen whenever r_n is orthogonal to a supporting hyperplane of $\mathcal{U}_n$ that contains (at least) a nontrivial segment of points of $\mathcal{U}_n$. According to Theorem 8, this happens if and only if r_n is orthogonal to a flat convex subset of some stratum, which implies some nontrivial orthogonality conditions on r_n thus encoding a nongenericity condition.

We conclude that there exists an open dense subset $\mathcal{O}_n$ of $\mathcal{P}_n$ such that, if $r_n \in \mathcal{O}_n$ (meaning that r_n is in a "generic" position with respect to the convex set $\mathcal{U}_n$), then the maximizer of (9) is unique.

3. Proof of Theorem 1.

In the proof of Theorem 1, the delicate point is to establish uniqueness of the optimal solution $\overline{u}_n$. This strongly relies on Theorem 5, which has been stated previously (and separately because the result has its own interest).

3.1. Preliminaries.

It is convenient to reformulate Problem (1) using a time integration by parts. We define $z_n(t) = y_n(t) - \frac{q_n}{T}t$, where we recall that $y_n(t) = \int_0^t u_n(\tau)\,d\tau$. We have $\partial_t z_n(t) = u_n(t) - \frac{q_n}{T}$, $z_n(0) = 0$, and we have $z_n(T) = 0$ if and only if $y_n(T) = q_n$. Now, we define the sets

$$
\begin{aligned}
\mathcal{R}_n(T) &= \left\{ z_n \in W^{1,\infty}((0,+\infty),\mathcal{P}_n) \;\mid\; z_n(0) = 0 \text{ and } z_n(t) = 0 \text{ for } t > T \right\}, \\
\mathcal{S}_n(T,q_n) &= \left\{ z_n \in \mathcal{R}_n(T) \;\mid\; 0 \leqslant \frac{q_n(x)}{T} + \partial_t z_n(t,x) \leqslant 1 \text{ for } (t,x) \in [0,T]\times I \right\},
\end{aligned}
\tag{10}
$$

and we define the functional

$$
S(z_n) = \int_I \int_0^{+\infty} z_n(t,x) s''(t)\,dt\,w(x)\,dx, \qquad z_n \in \mathcal{R}_n(T).
$$

Since $T \geqslant \|q_n\|_{L^\infty(I)}$, by Remark 1, the set $\mathcal{S}_n(T,q_n)$ contains the zero function (and thus is nonempty).

Lemma 9. *The function $\overline{u}_n \in \mathcal{K}_n(T,q_n)$ is a minimizer of J if and only if the function $\overline{z}_n \in \mathcal{S}_n(T,q_n)$ is a maximizer of S.*

Proof. Integrating by parts, we have

$$\begin{aligned} J(u_n) &= \int_I \int_0^T \left(\partial_t z_n(t,x) + \frac{q_n}{T}\right) s'(t)\,dt\,dx \\ &= -\int_I \int_0^T s''(t) z_n(t,x)\,dt\,w(x)\,dx + \frac{s(T)-s(0)}{T}\int_I q_n(x)w(x)\,dx \\ &= -S(z_n) + \frac{s(T)-s(0)}{T}\int_I q_n(x)w(x)\,dx. \end{aligned}$$

Since the last term is a constant independent of u_n and of z_n, the lemma follows. ∎

3.2. Existence of an optimal solution.

LEMMA 10. *There exists a maximizer $\overline{z}_n \in \mathcal{S}_n(T,q_n)$ of S, and there exists a minimizer $\overline{u}_n \in \mathcal{K}_n(T,q_n)$ of J.*

Proof. With the formulation of Problem 1 as an optimal control problem, existence of optimal solutions follows immediately from standard existence results in optimal control (see, e.g., [8, 33]), using the fact (also easy to establish) that $\mathcal{U}_n$ is convex and compact in $\mathcal{P}_n$ endowed with its L^2_w norm. Let us however provide a direct argument of proof.

Let us prove that there exists $C_n > 0$ such that $\|z_n\|_{W^{1,\infty}((0,T)\times I)} \leqslant C_n$ for every $z_n \in \mathcal{S}_n(T,q_n)$. Indeed: a) the derivative of z_n with respect t is bounded by definition (see (10)); b) the function z_n is bounded by integration of the time derivative from 0 to $t \in [0,T]$; c) the derivative with respect to x is bounded by the Markov theorem (see [13, page 97]), which implies that $\|\partial_x \mathcal{P}_n\|_{L^\infty(I)} \leqslant n^2 \|\mathcal{P}_n\|_{L^\infty(I)}$ for polynomials of degree less than or equal to n. Therefore $\mathcal{S}_n(T,q_n)$ is embedded in a ball in $W^{1,\infty}((0,T)\times I)$ and is thus compactly embedded in $L^1((0,T)\times I)$.

Functions in $\mathcal{S}_n(T,q_n)$ are uniformly bounded in L^∞, so the cost function I is also bounded by construction over this set. The compact embedding of $\mathcal{S}_n(T,q_n)$ in $L^1((0,T)\times I)$ shows that maximizing sequences of I converge to optimal solutions. Extracting a subsequence if necessary, we obtain the existence of a maximizer z_n for S, and thus by Lemma 9 a minimizer of J in $\mathcal{K}_n(T,q_n)$. ∎

3.3. Uniqueness of the optimal solution. Let $\overline{z}_n \in \mathcal{S}_n(T,q_n)$ be a maximizer of S. We are going to define a new function $z_n^{\mu,\varepsilon}$, which is equal to $\overline{z}_n$ except along the interval $[\mu-\varepsilon,\mu+\varepsilon] \subset [0,T]$ (with $\mu \in [\varepsilon, T-\varepsilon]$ for $0 < \varepsilon < T/2$) where it is equal to a test function which is maximal in a sense explained below. Comparison inequalities between $\overline{z}_n$ and the test function will give a characterization of the maximizer, which in turn, combined with Theorem 5, will prove that the maximizer is unique.

We first define $\widehat{z}_n^{\mu,\varepsilon}$, the linear interpolation of z_n between $\mu-\varepsilon$ and $\mu+\varepsilon$, given by

$$\widehat{z}_n^{\mu,\varepsilon}(t,x) = \overline{z}_n(\mu-\varepsilon,x) + \frac{t-\mu+\varepsilon}{2\varepsilon}\left(\overline{z}_n(\mu+\varepsilon,x) - \overline{z}_n(\mu-\varepsilon,x)\right), \quad \mu-\varepsilon \leqslant t \leqslant \mu+\varepsilon.$$

Now, let $z_n^{\mu,\varepsilon} \in \mathcal{R}_n(T)$ be defined by

$$\begin{array}{ll} \text{for } 0 \leqslant t < \mu-\varepsilon, & z_n^{\mu,\varepsilon}(t,x) = \overline{z}_n(t,x), \\ \text{for } \mu-\varepsilon \leqslant t \leqslant \mu+\varepsilon, & z_n^{\mu,\varepsilon}(t,x) = \widehat{z}_n^{\mu,\varepsilon}(t,x) + \varepsilon\varphi(\frac{t-\mu}{\varepsilon}) r_n^{\mu,\varepsilon}(x), \\ \text{for } \mu+\varepsilon < t \leqslant T, & z_n^{\mu,\varepsilon}(t,x) = \overline{z}_n(t,x), \end{array} \tag{11}$$

for some $r_n^{\mu,\varepsilon} \in \mathcal{P}_n$, where $\varphi : [-1,1] \to \mathbb{R}$ is the "hat" function $\varphi(t) = \min(1+t, 1-t)$. The function $z_n^{\mu,\varepsilon}$ is continuous by construction at $t = \mu \pm \varepsilon$. In order to determine an "optimal" $r_n^{\mu,\varepsilon}$, we define the set

$$\mathcal{Q}_n^{\mu,\varepsilon} = \{s_n \in \mathcal{P}_n \ \mid\ -g_n^{\mu,\varepsilon} \leqslant s_n \leqslant g_n^{\mu,\varepsilon} \text{ on } I\}$$

with $g_n^{\mu,\varepsilon} = \min(1 - f_n^{\mu,\varepsilon}, f_n^{\mu,\varepsilon})$ and $f_n^{\mu,\varepsilon} = \frac{1}{T} q_n + \frac{z_n(\mu+\varepsilon) - z_n(\mu-\varepsilon)}{2\varepsilon}$. Since $\overline{z}_n \in \mathcal{S}_n(T, q_n)$, we have $f_n^{\mu,\varepsilon} = \frac{1}{2\varepsilon} \int_{\mu-\varepsilon}^{\mu+\varepsilon} \left(\frac{1}{T} q_n + \partial_t \overline{z}_n(s)\right) ds \in \mathcal{U}_n$. Hence $g_n^{\mu,\varepsilon} \geqslant 0$ and the definition of $\mathcal{Q}_n^{\mu,\varepsilon}$ makes sense. The next result provides a condition on $r_n^{\mu,\varepsilon}$ such that $z_n^{\mu,\varepsilon}$ is admissible.

Lemma 11. *We have $z_n^{\mu,\varepsilon} \in \mathcal{S}_n(T, q_n)$ if and only if $r_n^{\mu,\varepsilon} \in \mathcal{Q}_n^{\mu,\varepsilon}$.*

Proof. By construction $z_n^{\mu,\varepsilon}(0,x) = z_n^{\mu,\varepsilon}(T,x) = 0$ for every $x \in I$. The condition (10) on the derivative is satisfied by construction for $0 < t < \mu - \varepsilon$ and $\mu + \varepsilon < t < T$. For $\mu - \varepsilon < t < \mu + \varepsilon$, the time derivative is computed accordingly to (11). Then, the condition (10) is equivalent to

$$0 \leqslant \frac{1}{T} q_n(x) + \frac{\overline{z}_n(\mu+\varepsilon, x) - \overline{z}_n(\mu-\varepsilon, x)}{2\varepsilon} \pm r_n^{\mu,\varepsilon}(x) \leqslant 1, \quad x \in I$$

which is equivalent to the claim. The lemma is proved. ∎

Since $\mathcal{Q}_n^{\mu,\varepsilon}$ has the structure described in (5), it defines a problem similar to (6). The maximizer, which exists and is unique thanks to Theorem 5, is written as

$$r_n^{\mu,\varepsilon} = \underset{s_n \in \mathcal{Q}_n^{\mu,\varepsilon}}{\operatorname{argmax}} \int_I s_n(x) w(x) dx. \tag{12}$$

Now that $z_n^{\mu,\varepsilon}$ and $r_n^{\mu,\varepsilon}$ are constructed in such a way that are good candidates to locally "test" the maximizer $\overline{z}_n$, let us establish some of their properties.

Lemma 12. *We have $0 \leqslant \int_I r_n^{\mu,\varepsilon}(x) w(x) dx$.*

Proof. It suffices to note that $0 \in \mathcal{Q}_n^{\mu,\varepsilon}$. ∎

Lemma 13. *We have*

$$\begin{aligned} \varepsilon^2 s''_- \int_I r_n^{\mu,\varepsilon}(x) w(x)\, dx \leqslant & \int_I \int_{\mu-\varepsilon}^{\mu+\varepsilon} \frac{\mu+\varepsilon-t}{2\varepsilon} \left(\overline{z}_n(t,x) - \overline{z}_n(\mu-\varepsilon,x)\right) s''(t) w(x)\, dt\, dx \\ & - \int_I \int_{\mu-\varepsilon}^{\mu+\varepsilon} \frac{t-\mu+\varepsilon}{2\varepsilon} \left(\overline{z}_n(\mu+\varepsilon,x) - \overline{z}_n(t,x)\right) s''(t) w(x)\, dt\, dx. \end{aligned} \tag{13}$$

Proof. Since $\overline{z}_n$ is a maximizer, we have $S(z_n^{\mu,\varepsilon}) \leqslant S(\overline{z}_n)$ for all possible μ, ε. Discarding the integrals for $t \leqslant \mu - \varepsilon$ and $t \geqslant \mu + \varepsilon$ where $z_n^{\mu,\varepsilon}$ and $\overline{z}_n$ are equal by definition (see (11)), this yields the inequality

$$\int_I \int_{\mu-\varepsilon}^{\mu+\varepsilon} z_n^{\mu,\varepsilon}(t,x) s''(t) w(x)\, dt\, dx \leqslant \int_I \int_{\mu-\varepsilon}^{\mu+\varepsilon} \overline{z}_n(t,x) s''(t) w(x)\, dt\, dx.$$

Elimination of $z_n^{\mu,\varepsilon}$ in function of $\overline{z}_n$ and $r_n^{\mu,\varepsilon}$ yields the inequality

$$\begin{aligned} & \int_I \int_{\mu-\varepsilon}^{\mu+\varepsilon} \left(\overline{z}_n(\mu-\varepsilon,x) + \frac{t-\mu+\varepsilon}{2\varepsilon}\left(\overline{z}_n(\mu+\varepsilon,x) - \overline{z}_n(\mu-\varepsilon,x)\right)\right) s''(t) w(x)\, dt\, dx \\ & \quad + \varepsilon \int_I r_n^{\mu,\varepsilon}(x) w(x)\, dx \int_{\mu-\varepsilon}^{\mu+\varepsilon} \varphi\left(\frac{t-\mu+\varepsilon}{\varepsilon}\right) s''(t)\, dt \leqslant \int_I \int_{\mu-\varepsilon}^{\mu+\varepsilon} \overline{z}_n(t,x) s''(t) w(x)\, dt\, dx. \end{aligned}$$

Since $\varepsilon^2 s''_- = \varepsilon \int_{\mu-\varepsilon}^{\mu+\varepsilon} s''_- \varphi\left(\frac{t-\mu+\varepsilon}{\varepsilon}\right) dt \leqslant \varepsilon \int_{\mu-\varepsilon}^{\mu+\varepsilon} s''(t) \varphi\left(\frac{t-\mu+\varepsilon}{\varepsilon}\right) dt$, it follows that

$$\begin{aligned} \varepsilon^2 s''_- \int_I r_n^{\mu,\varepsilon}(x) w(x)\, dx \leqslant & \int_I \int_{\mu-\varepsilon}^{\mu+\varepsilon} \overline{z}_n(t,x) s''(t) w(x)\, dt\, dx \\ & - \int_I \int_{\mu-\varepsilon}^{\mu+\varepsilon} \left(\frac{\mu+\varepsilon-t}{2\varepsilon} \overline{z}_n(\mu-\varepsilon,x) + \frac{t-\mu+\varepsilon}{2\varepsilon} \overline{z}_n(\mu+\varepsilon,x)\right) s''(t) w(x)\, dt\, dx. \end{aligned}$$

Using that $1 = \frac{\mu+\varepsilon-t}{2\varepsilon} + \frac{t-\mu+\varepsilon}{2\varepsilon}$, the lemma follows. ∎

LEMMA 14. *We have*

$$\lim_{\varepsilon\to 0^+}\int_\varepsilon^{T-\varepsilon}\int_I r_n^{\mu,\varepsilon}(x)w(x)\,dx\,d\mu = 0. \tag{14}$$

Proof. Integrating (13) over $\mu\in[\varepsilon, T-\varepsilon]$, we get an inequality $A^\varepsilon \leqslant B^\varepsilon - C^\varepsilon$, with

$$0\leqslant A^\varepsilon = \varepsilon^2 s''_- \int_\varepsilon^{T-\varepsilon}\int_I r_n^{\mu,\varepsilon}(x)w(x)\,dx\,d\mu \tag{15}$$

where the lower bound follows from Lemma 12,

$$B^\varepsilon = \int_e^{T-\varepsilon}\int_I\int_{\mu-\varepsilon}^{\mu+\varepsilon}\frac{\mu+\varepsilon-t}{2\varepsilon}\left(\overline{z}_n(t,x)-\overline{z}_n(\mu-\varepsilon,x)\right)s''(t)w(x)\,dt\,dx\,d\mu,$$

$$C^\varepsilon = \int_\varepsilon^{T-\varepsilon}\int_I\int_{\mu-\varepsilon}^{\mu+\varepsilon}\frac{t-\mu+\varepsilon}{2\varepsilon}\left(\overline{z}_n(\mu+\varepsilon,x)-\overline{z}_n(t,x)\right)s''(t)w(x)\,dt\,dx\,d\mu.$$

Performing in C^ε the change of variables $t\mapsto t-\varepsilon$, $\mu\mapsto\mu-\varepsilon$, we get that $C^\varepsilon = C_1^\varepsilon + C_2^\varepsilon - C_3^\varepsilon$ with

$$C_1^\varepsilon = \int_\varepsilon^{T-\varepsilon}\int_I\int_{t-\varepsilon}^{t+\varepsilon}\frac{\mu-t+\varepsilon}{2\varepsilon}\left(\overline{z}_n(t,x)-\overline{z}_n(\mu-\varepsilon,x)\right)s''(\mu-\varepsilon)w(x)\,dt\,dx\,d\mu,$$
$$C_2^\varepsilon = \int_{T-\varepsilon}^{T}\int_I\int_{t-\varepsilon}^{t+\varepsilon}\frac{\mu-t+\varepsilon}{2\varepsilon}\left(\overline{z}_n(t,x)-\overline{z}_n(\mu-\varepsilon,x)\right)s''(\mu-\varepsilon)w(x)\,dt\,dx\,d\mu,$$
$$C_3^\varepsilon = \int_\varepsilon^{2\varepsilon}\int_I\int_{t-\varepsilon}^{t+\varepsilon}\frac{\mu-t+\varepsilon}{2\varepsilon}\left(\overline{z}_n(t,x)-\overline{z}_n(\mu-\varepsilon,x)\right)s''(\mu-\varepsilon)w(x)\,dt\,dx\,d\mu.$$

We obtain the inequality

$$|A^\varepsilon| \leqslant |B^\varepsilon - C_1^\varepsilon| + |C_2^\varepsilon| + |C_3^\varepsilon|. \tag{16}$$

Note that B^ε and C_1^ε are integrals over $\mathcal{D} = \{\varepsilon<\mu<T-\varepsilon\}\cap\{\mu-\varepsilon<t<\mu+\varepsilon\}$ which is rectangular, but they have a slightly different integrand because $s''(t)$ in B^ε becomes $s"(\mu-\varepsilon)$ in C_1^ε. We have

$$B^\varepsilon - C_1^\varepsilon = \int_{\mathcal{D}}\int_I a(x,t,\mu,\varepsilon)\left(s''(t)-s''(\mu-\varepsilon)\right)\,dt\,d\mu\,w(x)\,dx$$

with $a(x,t,\mu,\varepsilon) = \frac{\mu+\varepsilon-t}{2\varepsilon}\left(\overline{z}_n(t,x)-\overline{z}_n(\mu-\varepsilon,x)\right)$. The definition of $\mathcal{D}$ yields $0<\mu+\varepsilon-t<2\varepsilon$, hence $|a(x,t,\mu,\varepsilon)|\leqslant|\overline{z}_n(t,x)-\overline{z}_n(\mu-\varepsilon,x)|$ for $(t,\mu)\in\mathcal{D}$. Moreover $|\partial_t\overline{z}_n|\leqslant 1$ as a consequence of the (10). Since $\mu-\varepsilon-t\leqslant 2\varepsilon$, we infer that

$$|B^\varepsilon - C_1^\varepsilon| \leqslant 2\varepsilon\int_I w(x)\,dx\int_{\mathcal{D}}|s''(t)-s''(\mu-\varepsilon)|\,dt\,d\mu.$$

For a given ε, performing another change of variable adapted to the rectangular structure of $\mathcal{D}$, namely $\tau = t-\mu+\varepsilon$, $\mu' = \mu-\varepsilon$, we get

$$\int_{\mathcal{D}}|s''(t)-s''(\mu-\varepsilon)|\,dt\,d\mu = \int_0^{2\varepsilon}\int_0^{T-2\varepsilon}|s''(\mu'+\tau)-s''(\mu')|\,d\mu'\,d\tau \leqslant 2\varepsilon\,\mathrm{mod}_1(s'',2\varepsilon),$$

where $\mathrm{mod}_1(s'',2\varepsilon)$ is the L^1-modulus of continuity of s''. Therefore

$$|B^\varepsilon - C_1^\varepsilon| \leqslant 4\varepsilon^2\mathrm{mod}_1(s'',2\varepsilon)\int_I w(x)\,dx.$$

The other terms are simpler to treat. We have $|C_2^\varepsilon| \leqslant \left(2\int_I w(x)\,dx\right)\varepsilon^2\int_{T-2\varepsilon}^{T} s''(\tau)\,d\tau$ and $|C_3^\varepsilon| \leqslant \left(2\int_I w(x)\,dx\right)\varepsilon^2\int_0^{2\varepsilon} s''(\tau)\,d\tau$. Plugging in (16) and rescaling by $\frac{1}{\varepsilon^2}$, we obtain $\frac{1}{\varepsilon^2}A^\varepsilon \to 0$ as $\varepsilon \to 0^+$. Using (15), the result follows. ∎

In (14), it is tempting to pass to the limit and to define $r_n^{\mu,0}$, and then to use Theorem 5 to establish that $r_n^{\mu,0}$ vanishes. However, using another path, it is possible to avoid the technical problem of proving that $r_n^{\mu,\varepsilon}$ has a limit as $\varepsilon \to 0^+$. It consists in reformulating (14) for another function to which Theorem 5 applies immediately. This is performed with the help of a comparison argument, as follows. Given any $t \in [0,T]$, we define

$$\mathcal{Q}_n(t) = \{s_n \in \mathcal{P}_n \ \mid\ -g_n(t) \leqslant s_n \leqslant g_n(t) \text{ on } I\},$$

with $g_n(t,x) = \min(1 - f_n(t,x), f_n(t,x))$ and $f_n(t) = \frac{1}{T}q_n + \partial_t \overline{z}_n(t)$. Since $\overline{z}_n$ is a maximizer, one has that $f_n \in \mathcal{U}_n$ for almost every $t \in [0,T]$. With the help of Theorem 5, we define the polynomial

$$r_n(t) = \underset{s_n \in \mathcal{Q}_n(t)}{\operatorname{argmax}} \int_I s_n(x) w(x)\,dx.$$

Lemma 15. *We have $0 \leqslant \int_I r_n(t,x)w(x)\,dx$ and*

$$\frac{1}{2\varepsilon}\int_{\mu-\varepsilon}^{\mu+\varepsilon}\int_I r_n(t,x)w(x)\,dx \leqslant \int_I r_n^{\mu,\varepsilon}(x)w(x)\,dx.$$

Proof. For the first inequality, the lower bound is immediate since $0 \in \mathcal{Q}_n(t)$. By definition of $r_n \in \mathcal{Q}_n(t)$, we have $-f_n(t,x) \leqslant r_n(t,x) \leqslant f_n(t,x)$ and $-1 + f_n(t,x) \leqslant r_n(t,x) \leqslant 1 - f_n(t,x)$, for $x \in I$. Noting that $\frac{1}{2\varepsilon}\int_{\mu-\varepsilon}^{\mu+\varepsilon} f_n(t)\,dt = f_n^{\mu,\varepsilon}$, integrating in time yields

$$-\min\left(1 - f_n^{\mu,\varepsilon}, f_n^{\mu,\varepsilon}\right) \leqslant \frac{1}{2\varepsilon}\int_{\mu-\varepsilon}^{\mu+\varepsilon} r_n(t,x)\,dt \leqslant \min\left(1 - f_n^{\mu,\varepsilon}, f_n^{\mu,\varepsilon}\right),$$

and thus $\frac{1}{2\varepsilon}\int_{\mu-\varepsilon}^{\mu+\varepsilon} r_n(t,\cdot)\,dt \in \mathcal{Q}_n^{\mu,\varepsilon}$. By definition the polynomial $r_n^{\mu,\varepsilon} \in \mathcal{Q}_n^{\mu,\varepsilon}$ is the maximizer of the integral in (12), hence the second inequality follows. ∎

Lemma 16. *We have $\int_0^T \int_I r_n(t,x)w(x)\,dx\,d\mu = 0$.*

Proof. It suffices to combine Lemma 15 with the estimate (14). ∎

Lemma 17. *We have* $\operatorname{toc}\left(\frac{1}{T}q_n + \partial_t\overline{z}_n(t)\right) \geqslant n+1$ *for almost every $t \in [0,T]$.*

Proof. If $\operatorname{toc}\left(\frac{1}{T}q_n + \partial_t z_n(t)\right) \leqslant n$, then $r_n(t) \neq 0$ and $0 < \int_I r_n(t,x)w(x)\,dx$ for almost every $t \in [0,T]$. This yields a contradiction with Lemma 16. ∎

We are now in a position to finish the proof of Theorem 1.

Proof (End of the proof of Theorem 1.). The proof goes by contradiction (as in [3]). Assume that there are two distinct maximizers $\overline{z}_n$ and $\widehat{z}_n$. Then $\frac{1}{T}q_n(\cdot) + \partial_t\overline{z}_n(t,\cdot)$ and $\frac{1}{T}q_n(\cdot) + \partial_t\widehat{z}_n(t,\cdot)$ have both $n+1$ contact points for almost every $t \in [0,T]$. Since the cost is linear, $z_n = \frac{1}{2}\left(\overline{z}_n + \widehat{z}_n\right)$ is another maximizer. Therefore $\frac{1}{T}q_n(\cdot) + \partial_t z_n(t,\cdot) = \frac{1}{2}\left(\frac{1}{T}q_n(\cdot) + \partial_t\overline{z}_n(t,\cdot)\right) + \frac{1}{2}\left(\frac{1}{T}q_n(\cdot) + \partial_t\widehat{z}_n(t,\cdot)\right)$ has also $n+1$ contact points for almost every $t \in [0,T]$. Since the half-sum of two distinct functions that have $n+1$ contact points for almost every t cannot have $n+1$ contact points for almost every t, this raises a contradiction. Therefore the maximizer $\overline{z}_n$ is unique. Thanks to Lemma 9, Theorem 1 is proved. ∎

3.4. Properties of the optimal solution for large T. The properties of the optimal solution for $T \geqslant \|q_n\|_{L^\infty(I)}$ large are established with the notations of Problem (1) for which $\overline{u}_n = \frac{1}{T}q_n + \partial_t \overline{z}_n \in \mathcal{K}_n(T,q_n)$ is the minimizer of J, associated with the maximizer $\overline{z}_n \in \mathcal{S}_n(T,q_n)$ of S. Since $\overline{u}_n$ is a minimizer and the weight function $t \mapsto s'(t)$ increases, we expect that $\overline{u}_n(t)$ vanishes if t is large enough. This is what we are going to prove.

LEMMA 18. *The function $t \mapsto \int_I \overline{u}_n(t,x)w(x)\,dx = \langle \overline{u}_n(t), 1\rangle_{L^2_w(I)}$ is nonincreasing.*

Proof. First of all, integrating by parts and noting that $\int_0^T u_n(t)\,dt = q_n$ for every $u_n \in \mathcal{K}_n(T,q_n)$, we have

$$J(u_n) = \int_0^T s'(t)\langle u_n(t), 1\rangle_{L^2_w(I)}\,dt = s'(T)\langle q_n, 1\rangle_{L^2_w(I)} - \int_0^T s''(t)\int_0^t \langle u_n(\tau), 1\rangle_{L^2_w(I)}\,d\tau\,dt. \tag{17}$$

The nonincreasing rearrangement (see, e.g., [28]) of the function $f(t) = \langle u_n(t), 1\rangle_{L^2_w(I)}$ on $[0,T]$ is the nonincreasing function f^* on $[0,T]$, and, by the first Hardy-Littlewood inequality, we have

$$\int_0^t f(\tau)\,d\tau \leqslant \int_0^t f^*(\tau)\,d\tau \qquad \forall t \in [0,T]. \tag{18}$$

We are now going to prove that there exists $\tilde{u}_n \in \mathcal{K}_n(T,q_n)$ such that $\langle \tilde{u}_n(t), 1\rangle_{L^2_w(I)} = f^*(t)$ for almost every $t \in [0,T]$, by using a theorem due to Ryff (see [30] and comment in [5]).

There exists [30] a measurable mapping $\sigma : [0,T] \to [0,T]$, preserving the Lebesgue measure, such that $f^* \circ \sigma = f$, i.e., $T_\sigma f^* = f$, where $T_\sigma : L^2(0,T) \to L^2(0,T)$ is the (doubly stochastic) operator defined by $T_\sigma g = g \circ \sigma$ for every $g \in L^2(0,T)$. Note that T_σ is an isometry which is not necessarily surjective and that its adjoint T_σ^* is not necessarily induced by a measure-preserving function (see once again [30]). However we have $f^* = T_\sigma^* f$. Indeed, since $f = T_\sigma f^*$ and since T_σ is an isometry, we have $\langle T_\sigma^* f, g\rangle_{L^2(0,T)} = \langle f, T_\sigma g\rangle_{L^2(0,T)} = \langle T_\sigma f^*, T_\sigma g\rangle_{L^2(0,T)} = \langle f, g\rangle_{L^2(0,T)}$ for any $g \in L^2(0,T)$, whence the claim.

Given any $u_n \in \mathcal{K}_n(T,q_n)$, we write $u_n(t,x) = a_0(t) + a_1(t)x + \cdots + a_n(t)x^n$. This inversion formula for the rearrangement leads us to define $\tilde{a}_i = T_\sigma^* a_i$, for $i = 0,\dots,n$, and we define $\tilde{u}_n = T_\sigma^* u_n \in L^2(0,T;\mathcal{P}_n)$ by $\tilde{u}_n(t,x) = (T_\sigma^* u_n)(t,x) = \tilde{a}_0(t) + \tilde{a}_1(t)x + \cdots + \tilde{a}_n(t)x^n$.

First of all, by linearity, we have $\langle \tilde{u}_n(t), 1\rangle_{L^2_w(I)} = (T_\sigma^* f)(t) = f(t)$ for almost every $t \in [0,T]$.

Let us prove that $\tilde{u}_n \in L^\infty(0,T;\mathcal{U}_n)$. Let $x \in I$ be arbitrary. Let $t \in [0,T]$ be an arbitrary Lebesgue point of the function $t \mapsto \tilde{u}_n(t,x)$. Then, denoting by $\chi_{[t-\varepsilon,t+\varepsilon]}$ the characteristic function of the interval $[t-\varepsilon,t+\varepsilon]$, and noting that $T_\sigma \chi_{[t-\varepsilon,t+\varepsilon]} = \chi_{[t-\varepsilon,t+\varepsilon]} \circ \sigma = \chi_{\sigma^{-1}([t-\varepsilon,t+\varepsilon])}$, we have

$$\begin{aligned}\tilde{u}_n(t,x) &= \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} \tilde{u}_n(s,x)\,ds = \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_0^T (T_\sigma^* u_n)(s,x)\ \chi_{[t-\varepsilon,t+\varepsilon]}(s)\,ds \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_0^T u_n(s,x)\ T_\sigma\chi_{[t-\varepsilon,t+\varepsilon]}(s)\,ds = \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_{\sigma^{-1}([t-\varepsilon,t+\varepsilon])} u_n(s,x)\,ds,\end{aligned}$$

from which we infer that $\tilde{u}_n(t,x) \in [0,1]$, because $u_n(s,x) \in [0,1]$ and the Lebesgue measure of $\sigma^{-1}([t-\varepsilon,t+\varepsilon])$ is equal to 2ε since σ is measure-preserving. The claim is proved.

Finally, we claim that $\tilde{u}_n \in \mathcal{K}_n(T,q_n)$. Indeed we have the expansion $\int_0^T \tilde{u}_n(t,x)\,dt = \int_0^T \tilde{a}_0(t)\,dt + \cdots + \int_0^T \tilde{a}_n(t)\,dt\ x^n$. Since $T_\sigma 1 = 1$, we have $\int_0^T \tilde{a}_i(t)\,dt = \langle T_\sigma^* a_i, 1\rangle_{L^2(0,T)} = \langle a_i, T_\sigma 1\rangle_{L^2(0,T)} = \int_0^T a_i(t)\,dt$, for $i = 0,\dots,n$. It follows that $\int_0^T \tilde{u}_n(t,x)\,dt = \int_0^T u_n(t,x)\,dt = q_n(x)$, for every $x \in I$. We have thus proved that $\tilde{u}_n \in \mathcal{K}_n(T,q_n)$ is such that $\langle \tilde{u}_n(t), 1\rangle_{L^2_w(I)} = f^*(t)$ for almost every $t \in [0,T]$.

Now, it follows from (17) and (18) that $J(\tilde{u}_n) \leqslant J(\overline{u}_n)$ (this is also a consequence of Lemma 4 in [30]). So by uniqueness of the optimal solution $\overline{u}_n = \tilde{u}_n$ and $f = f^*$ is nonincreasing. The lemma is proved. ∎

Let $C_n > 0$ the smallest constant such that $\|p_n\|_{L^\infty(I)} \leqslant C_n \|p_n\|_{L^1_w(I)} = C_n \int_I p_n(x) w(x)\, dx$ for every $p_n \in \mathcal{P}_n$. Note that $C_n \int_I w(x)\, dx \geqslant 1$ because the inequality must be satisfied for the constant polynomial $p_n = 1$.

Hereafter, for simplicity we make the additional assumption that $s(0) = 0$.

Lemma 19. *We have* $\|\overline{u}_n(t)\|_{L^\infty(I)} \leqslant \frac{s(\|q_n\|_{L^\infty(I)})}{s(t)} C_n \int_I w(x)\, dx$, *for almost every* $t \in (0, T]$.

Proof. Using Lemma 18 to obtain the first inequality below, we have

$$
\begin{aligned}
s(t) \int_I \overline{u}_n(t,x) w(x)\, dx &= \int_0^t \int_I \overline{u}_n(t,x) s'(\tau) w(x)\, dx\, d\tau \leqslant \int_0^t \int_I \overline{u}_n(\tau,x) s'(\tau) w(x)\, dx\, d\tau \\
&\leqslant \int_0^T \int_I \overline{u}_n(\tau,x) s'(\tau) w(x)\, dx\, d\tau.
\end{aligned} \tag{19}
$$

Defining the function $g_n(\tau, x) = \frac{q_n(x)}{\|q_n\|_{L^\infty(I)}}$ if $0 < \tau < \|q_n\|_{L^\infty(I)}$ and $g_n(\tau, x) = 0$ if $\tau > \|q_n\|_{L^\infty(I)}$, we have $0 \leqslant g_n \leqslant 1$ and $\int_0^T g_n(t)\, dt = q_n$, and thus $g_n \in \mathcal{K}_n(T, q_n)$. Since $\overline{u}_n$ is the minimizer of J, we infer that the right-hand side member of (19) is estimated by

$$
\begin{aligned}
\int_0^T \int_I \overline{u}_n(\tau,x) s'(\tau) w(x)\, dx\, d\tau \quad &\leqslant \int_0^T \int_I g_n(\tau,x) s'(\tau) w(x)\, dx\, d\tau \\
&= \int_0^{\|q_n\|_{L^\infty(I)}} s'(\tau)\, d\tau \int_I \frac{q_n(x) w(x)}{\|q_n\|_{L^\infty(I)}}\, dx \\
&= s\left(\|q_n\|_{L^\infty(I)}\right) \frac{\int_I q_n(x) w(x)\, dx}{\|q_n\|_{L^\infty(I)}} \leqslant s\left(\|q_n\|_{L^\infty(I)}\right) \int_I w(x)\, dx.
\end{aligned}
$$

Using (19), we get $\|\overline{u}_n(t)\|_{L^1_w(I)} \leqslant \frac{s(\|q_n\|_{L^\infty(I)})}{s(t)} \int_I w(x)\, dx$. The lemma follows. ∎

We define what we call the *critical time* $T_* = T_*(n, q_n)$ by

$$
s(T_*) = s\left(\|q_n\|_{L^\infty(I)}\right) C_n \int_I w(x)\, dx. \tag{20}
$$

It depends on q_n and n but not on T.

Proposition 20. *If $T > T_*$ then $\overline{u}_n(t) = 0$ for $t \in (T_*, T)$.*

Proof. By using the inequality given in Lemma 19, and by definition of the critical time, we have $\|\overline{u}_n(t)\|_{L^\infty(I)} < 1$ for $T_* < t$. Besides, by Proposition 17, we have $\mathrm{toc}(\overline{u}_n(t)) \geqslant n + 1$. Therefore all contact points of $\overline{u}_n(t)$ occur at the lower value 0, hence $\overline{u}_n(t) \in \mathcal{P}_n$ has a number of roots that is greater or equal to $n + 1$ (counted with their multiplicity). Therefore $\overline{u}_n(t)$ vanishes identically if $t > T_*$. ∎

The next result establishes that the optimal solution $\overline{u}_n$ is actually independent of T if $T_* < T$. To state it, we consider the minimizer $\tilde{u}_n$ of J over the set $\mathcal{K}_n(T_*, q_n)$: it is well defined because $\|q_n\|_{L^\infty(I)} \leqslant T_*$.

Corollary 21. *If $T > T_*$ then $\overline{u}_n = \tilde{u}_n$.*

Proof. Since $\overline{u}_n(t) = 0$ for $T_* < t < T$, with a slight abuse of notation we have $\overline{u}_n \in \mathcal{K}_n(T_*, q_n)$, and hence $J(\tilde{u}_n) \leqslant J(\overline{u}_n)$. Similarly, we extend $\tilde{u}_n$ by 0 for $T_* < t < T$, and with a slight abuse of notation we have $\tilde{u}_n \in \mathcal{K}_n(T, q_n)$, and thus $J(\overline{u}_n) \leqslant J(\tilde{u}_n)$. Therefore $J(\tilde{u}_n) = J(\overline{u}_n)$, whence the conclusion by uniqueness. ∎

4. Proof of Theorem 2. Recalling that Problem (1) can be equivalently formulated as the optimal control problem (2), in this section we analyze it by means of optimal control theory, and in particular we study the first-order optimality system resulting from the application of the Pontryagin maximum principle, which characterizes the unique optimal solution in terms of Lagrange multipliers.

For simplicity, throughout the section we take $I=[0,1]$, $w(x)=1$ and $s'(t)=t$, but all our results readily extend to more general entropy functions and weights, as stated in Theorem 2.

4.1. Application of the Pontryagin maximum principle. We denote by $\overline{y}_n$ the optimal trajectory, solution of Problem (2), associated with the optimal control $\overline{u}_n$. We have seen that $\overline{u}_n \in L^\infty(0,T;\mathcal{U}_n)$ is the unique optimal solution. In the sequel, we extend it by 0 for $t>T$, so that we have $\overline{u}_n \in \mathcal{K}_n(T,q_n)$.

The Hamiltonian $H : \mathbb{R}\times\mathcal{P}_n\times\mathcal{P}_n\times\mathbb{R}\times\mathcal{P}_n \to \mathbb{R}$ of the optimal control problem (2) is

$$\begin{aligned} H(t,y_n,\lambda_n,\lambda^0,p_n) &= \langle \lambda_n, p_n\rangle_{L^2(I)} + t\lambda^0\langle p_n, 1\rangle_{L^2(I)} \\ &= \langle \lambda_n + \lambda^0 t, p_n\rangle_{L^2(I)} = \int_0^1 (\lambda_n(x)+\lambda^0 t)p_n(x)\,dx. \end{aligned}$$

According to the Pontryagin maximum principle (see [21, 26, 33]) which is a first-order necessary condition for optimality, since $\overline{u}_n$ is optimal, there must exist $\lambda^0\in\{0,-1\}$ and $\lambda_n\in\mathcal{P}_n$ (note indeed that the adjoint equation gives that λ_n does not depend on t), with $(\lambda_n,\lambda^0)\neq(0,0)$, such that

$$\overline{u}_n(t) = \underset{p_n\in\mathcal{U}_n}{\operatorname{argmax}} \int_0^1 \left(\lambda_n(x)+\lambda^0 t\right) p_n(x)\,dx = \underset{p_n\in\mathcal{U}_n}{\operatorname{argmax}} \left\langle \lambda_n+\lambda^0 t, p_n\right\rangle_{L^2(I)}, \tag{21}$$

for almost every $t\in[0,T]$. The quadruple $(\overline{y}_n,\lambda_n,\lambda^0,\overline{u}_n)$ is called an extremal lift of the optimal trajectory. The couple (λ_n,λ^0) is a Lagrange multiplier. If $\lambda^0=-1$ then the extremal lift (and the Lagrange multiplier) is said to be *normal*, and if $\lambda^0=0$ it is said to be *abnormal*.

By the way, here, it is interesting to note that, by convexity of the optimal control problem, the first-order optimality condition given by the Pontryagin maximum principle is also sufficient: If there exists a multiplier (λ_n,λ^0) such that $u_n(t)=\operatorname{argmax}_{p_n\in\mathcal{U}_n}\int_0^1\left(\lambda_n(x)+\lambda^0 t\right)p_n(x)\,dx$ for almost every $t\in[0,T]$, and if $\int_0^T u_n(t)\,dt = q_n$, then $u_n=\overline{u}_n$ is the (unique) optimal solution of Problem 2.

REMARK 4. *It is interesting to note that, for almost every $t\in[0,T]$, the maximization problem (21) is of the form of Problem (9) studied in Section 2.4, with $r_n(x)=\lambda_n(x)+\lambda^0 t$ (here, t is fixed), for which we have seen that uniqueness of the maximizer is not ensured.*

Anyway, thanks to Theorem 1, we know that the optimal control $\overline{u}_n\in L^\infty(0,T;\mathcal{U}_n)$ is unique, which implies that the solution $\overline{u}_n(t)$ at time t of the maximization problem (21) must be unique for almost every $t\in[0,T]$. This is not contradictory with the fact that, for exceptional values of t, the maximization problem (21) may have several distinct maximizers. This is well known in optimal control, and such times may typically correspond to switching times.

Now, let us infer the expression of the optimal control $\overline{u}_n(t)$ at time t from the maximization condition (21). Following what has been said in Section 2.4, since the function $t\mapsto\lambda_n(x)+\lambda^0 t$ is nonincreasing on $[0,T]$ for any fixed x, at this stage what we can say is that there exist $(t_1,t_2)\in[0,T]^2$ such that, for almost every t,

$$\begin{array}{ll} \overline{u}_n(t)=1 & \text{if } 0\leqslant t\leqslant t_1, \\ \overline{u}_n(t)\in \operatorname{Extr}(\mathcal{U}_n) & \text{if } t_1\leqslant t\leqslant t_2, \\ \overline{u}_n(t)=0 & \text{if } t_2\leqslant t\leqslant T. \end{array} \tag{22}$$

For instance if $\lambda_n(\cdot)>0$ on $[0,1]$ then $t_1>0$, and if $\lambda_n(\cdot)+\lambda^0 T<0$ on $[0,1]$ then $t_2<T$. What happens inbetween t_1 and t_2 depends on the geometric position of $\lambda_n(x)+\lambda^0 t$ with respect to the convex set $\mathcal{U}_n$.

Remark 5. *It may happen that the optimal control $\overline{u}_n$ have several extremal lifts.*

Actually, we are going to prove that $\overline{u}_n$ always admits a normal extremal lift, without any assumption on $q_n \in \mathcal{P}_n^+$. But in some situations it may also have an abnormal extremal lift. Before studying normal extremals, let us comment on the abnormal case.

Abnormal case. Let us assume that $\overline{u}_n$ has an abnormal extremal lift, i.e., a Lagrange multiplier (λ_n, λ^0) with $\lambda^0 = 0$. Then

$$\int_0^1 \lambda_n(x)\overline{u}_n(t,x)\,dx = \max_{p_n\in\mathcal{U}_n} \int_0^1 \lambda_n(x)p_n(x)\,dx. \tag{23}$$

According to the remarks done Section 2.4, if the Lagrange multiplier λ_n is in a "generic" position with respect to $\mathcal{U}_n$ then there is a unique maximizer, and thus $\overline{u}_n(t,x) = \overline{u}_n(x)$ does not depend on t. Since $\partial_t\overline{y}_n = \overline{u}_n$, it follows that $\overline{y}_n(t,x) = t\overline{u}_n(x)$, and since $\overline{y}_n(T) = q_n$, we get that $\overline{u}_n = \frac{1}{T}q_n$ (note that this is the control used in Remark 1 to show that $\mathcal{K}_n(T,q_n)$ is nonempty if and only if $T \geqslant \|q_n\|_{L^\infty(0,1)}$). But then, since $\overline{u}_n \in \mathrm{Extr}(\mathcal{U}_n)$, we must have $\frac{1}{T}q_n \in \mathrm{Extr}(\mathcal{U}_n)$. Using Remark 3, it follows that $\frac{1}{T}q_n$ must have contact points with 0 and with 1: taking into account that $T \geqslant \|q_n\|_{L^\infty(I)}$, this may occur only if $T = \|q_n\|_{L^\infty(I)}$. This argument shows that, if $T > \|q_n\|_{L^\infty(I)}$ and if we are in the abnormal case, then λ_n is not in a "generic" position with respect to $\mathcal{U}_n$, in the sense of Section 2.4, i.e., λ_n is orthogonal to a nontrivial flat stratum F of $\partial\mathcal{U}_n$; and then we must have $\overline{u}_n(t) = \frac{1}{T}q_n + r_n(t) \in F$ with $\int_0^T r_n(t)\,dt = 0$.[1]

Actually, abnormal extremals are easy to construct if q_n vanishes at some point $x_\star \in I = [0,1]$. Indeed, then, we first note that, given any admissible solution, satisfying $\partial_t y_n(t,x) = u_n(t,x) \in [0,1]$, with $y_n(0,x) = 0$ and $y_n(T,x) = q_n(x)$, we must have $u_n(t,x_\star) = 0$ for almost every $t \in [0,T]$. Now, we consider the polynomial $\lambda_n \in \mathcal{P}_n$ enjoying the property[2] $\int_0^1 \lambda_n(x)p_n(x)\,dx = -p_n(x_\star)$ for every $p_n \in \mathcal{P}_n$. Since $\overline{u}_n(t,x_\star) = 0$ for any time t, it is clear that λ_n satisfies (23). Indeed, $\int_0^1 \lambda_n(x)\overline{u}_n(t,x)\,dx = -\overline{u}_n(t,x_\star) = 0$, and $\max_{p_n\in\mathcal{U}_n}\int_0^1 \lambda_n(x)p_n(x)\,dx = \max_{p_n\in\mathcal{U}_n}(-p_n(x_\star)) = 0$. We conclude that, if q_n vanishes at some point $x_\star \in I$, then the optimal control $\overline{u}_n$ has (at least) an abnormal extremal lift.

By the way, this argument shows that, if q_n has several distinct zeros on I then $\overline{u}_n$ has several (independent) abnormal extremal lifts.

We have thus obtained the following result.

Lemma 22. *If q_n vanishes k times on I then the optimal control $\overline{u}_n$ has at least k linearly independent abnormal extremal lifts.*

This does not exclude that $\overline{u}_n$ have also a normal extremal lift, and this is indeed the case as we are going to see next.

4.2. Existence of a normal extremal lift. Let us introduce several notations. Given any control $u_n \in L^\infty(0,T;\mathcal{U}_n)$, we define the end-point mapping E_T (this notion is classical in optimal control, see [33, 34]) by $E_T(u_n) = y_n(T) = \int_0^T u_n(t)\,dt$, where y_n is the solution of the Cauchy problem $\partial_t y_n = u_n$, $y_n(0) = 0$. Given a target $q_n \in \mathcal{P}_n^+$, we define the value function at q_n by

$$V_T(q_n) = \min\{J_T(u_n) \mid u_n \in L^\infty(0,T;\mathcal{U}_n),\ E_T(u_n) = q_n\} = J_T(\overline{u}_n),$$

[1] Indeed, searching $\overline{u}_n(t)$ in the form $\overline{u}_n(t) = v_n(x) + w_n(t,x) \in F$ with $v_n \in F$, we must have $Tv_n(x) + \int_0^T w_n(t,x)\,dt = q_n(x)$, whence $\overline{u}_n(t) = \frac{1}{T}q_n(x) + w_n(t,x) - \frac{1}{T}\int_0^T w_n(t,x)\,dt$.

[2] We consider the closed subspace $\mathcal{E}_n = \{p_n \in \mathcal{P}_n \mid p_n(x_\star) = 0\}$ of $\mathcal{P}_n$. Its orthogonal $\mathcal{E}_n^\perp$ in $\mathcal{P}_n$ (for the scalar product of $L^2(I)$) is of dimension 1 and is spanned by some $\lambda_n \in \mathcal{P}_n \setminus \{0\}$ (satisfying necessarily $\lambda_n(x_\star) \neq 0$). Multiplying by an appropriate scalar, we assume that $\lambda_n(x_\star) = -\|\lambda_n\|^2_{L^2(I)}$. Now, any $p_n \in \mathcal{P}_n$ can be written in a unique way as $p_n = q_n + s\lambda_n$ for some $q_n \in \mathcal{E}_n$ and $s \in \mathbb{R}$. By definition, $q_n(x_\star) = 0$, hence $p_n(x_\star) = s\lambda_n(x_\star) = -s\|\lambda_n\|^2_{L^2(I)}$. It follows that $\int_I \lambda_n(x)p_n(x)\,dx = \langle\lambda_n, q_n + s\lambda_n\rangle = s\|\lambda_n\|^2_{L^2(I)} = -p_n(x_\star)$.

where $\overline{u}_n \in L^\infty(0,T;\mathcal{U}_n)$ is an optimal control minimizing J over all controls $u_n \in L^\infty(0,T;\mathcal{U}_n)$ such that $E_T(u_n) = q_n$. We have $V_T(q_n) < +\infty$ if $\|q_n\|_{L^\infty(I)} \leqslant T$, and by convention we set $V_T(q_n) = +\infty$ when q_n is not reachable in time T. Accordingly, we also extend V_T to the whole space $\mathcal{P}_n$ by setting $V_T(r_n) = +\infty$ for every $r_n \in \mathcal{P}_n \setminus \mathcal{P}_n^+$.

Lemma 23. *The value function V_T is convex.*

Proof. The proof is obvious. Let q_n^1 and q_n^2 in $\mathcal{P}_n^+$ be such that $\|q_n^i\|_{L^\infty(I)} \leqslant T$ for $i = 1, 2$, and let $\tau \in [0,1]$. Let $\overline{u}_n^i \in L^\infty(0,T;\mathcal{U}_n)$ be the unique optimal control such that $V_T(q_n^i) = J_T(\overline{u}_n^i)$, for $i = 1, 2$. Setting $u_n^\tau = \tau\overline{u}_n^1 + (1-\tau)\overline{u}_n^2 \in L^\infty(0,T;\mathcal{U}_n)$, we have $E_T(u_n^\tau) = \tau q_n^1 + (1-\tau)q_n^2$. Hence u_n^τ is an admissible control steering the control system from 0 to $\tau q_n^1 + (1-\tau)q_n^2$ in time T. Therefore we must have $V_T(\tau q_n^1 + (1-\tau)q_n^2) \leqslant J_T(u_n^\tau)$. Since $J_T(u_n^\tau) = \tau J_T(\overline{u}_n^1) + (1-\tau)J_T(\overline{u}_n^2) = \tau V_T(q_n^1) + (1-\tau)V_T(q_n^2)$, the convexity property follows. ∎

Since V_T is convex, its subdifferential $\partial V_T(q_n)$ at any $q_n \in \mathcal{P}_n^+$ such that $V_T(q_n) < +\infty$ is a convex nonempty subset.

Proposition 24. *Let $q_n \in \mathcal{P}_n^+$ and let $T \geqslant \|q_n\|_{L^\infty(I)}$. For every $\lambda_n \in \partial V_T(q_n)$, the quadruple $(\overline{y}_n, \lambda_n, -1, \overline{u}_n)$ is a normal extremal lift of the (unique) optimal control $\overline{u}_n$ solution of Problem 2.*

Proof. We follow an argument developed in [29, Proposition 2]. Let $\lambda_n \in \partial V_T(q_n)$. Since $\overline{u}_n \in L^\infty(0,T;\mathcal{U}_n)$ minimizes the functional J_T over all $u_n \in L^\infty(0,T;\mathcal{U}_n)$ satisfying $E_T(u_n) = q_n$, we have $E_T(\overline{u}_n) = q_n$ and $V_T(q_n) = J_T(\overline{u}_n)$. Besides, by definition of the subdifferential, for every $q_n \in \mathcal{P}_n^+$, there exists a smooth function $\phi : \mathcal{P}_n \to \mathbb{R}$ such that $V_T - \phi$ reaches a (global) minimum at q_n

$$V_T(q_n) - \phi(q_n) \leqslant V_T(r_n) - \phi(r_n) \qquad \forall r_n \in \mathcal{P}_n,$$

Taking $r_n = E_T(u_n)$, we have $V_T(q_n) - \phi(q_n) \leqslant V_T(E_T(u_n)) - \phi(E_T(u_n))$ for every $u_n \in L^\infty(0,T;\mathcal{U}_n)$. Since $V_T(E_T(u_n)) \leqslant J_T(u_n)$ by definition, we infer that

$$J_T(\overline{u}_n) - \phi(E_T(\overline{u}_n)) + \phi(q_n) = J_T(\overline{u}_n) = V_T(q_n) \leqslant J_T(u_n) - \phi(E_T(u_n)) + \phi(q_n),$$

and hence $\overline{u}_n \in L^\infty(0,T;\mathcal{U}_n)$ minimizes the functional $J_T - \phi \circ E_T + \phi(q_n)$ over $L^\infty(0,T;\mathcal{U}_n)$.

Therefore, $\overline{u}_n$ is also an optimal solution of the auxiliary optimal control problem:

$$\begin{aligned}
&\partial_t y_n(t) = u_n(t), \qquad \text{for a.e. } t \in [0,T],\\
&y_n(0) = 0,\\
&u_n(t) \in \mathcal{U}_n, \qquad \text{for a.e. } t \in [0,T],\\
&\min\left(J_T(u_n) - \phi(E_T(u_n)) + \phi(q_n)\right) = \int_0^T t\int_0^1 u_n(t,x)\,dx\,dt - \phi(y_n(T)) + \phi(q_n).
\end{aligned}$$

Compared with Problem 2, this optimal control problem (which has a different minimization functional) is now with a free final point: there is no constraint on $E_T(u_n) = y_n(T)$. Applying the Pontryagin maximum principle, the transversality condition on the adjoint vector implies immediately that the extremal lift must be normal (the argument is classical in optimal control, see [33, 34]), and moreover the normal Lagrange multiplier $(\lambda_n, -1)$ is given by $\lambda_n = \nabla\phi(y_n(T)) = \nabla\phi(q_n)$. The proposition is proved. ∎

Remark 6. *We claim that V_T is continuous on the set of $q_n \in \mathcal{P}_n^+$ such that $\|q_n\|_{L^\infty(I)} \leqslant T$.*

Indeed, it suffices to follow the proof of [32, Theorem 4.6]. We sketch the proof. Considering a sequence of $q_n^k \in \mathcal{P}_n^+$ converging to q_n as $k \to +\infty$, we assume that $\overline{u}_n^k \in L^\infty(0,T;\mathcal{U}_n)$ is an optimal control reaching q_n^k, i.e., $V_T(q_n^k) = J_T(u_n^k)$ and $E_T(u_n^k) = q_n^k$. Up to some subsequence, and since $\mathcal{U}_n$ is convex, we have $u_n^k \rightharpoonup u$ for some $u_n \in L^\infty(0,T;\mathcal{U}_n)$ in weak-star topology. Since J_T and E_T are linear and continuous, we have $J_T(u_n^k) \to J_T(u_n)$ and $E_T(u_n^k) \to E_T(u_n)$, and thus $E_T(u_n) = q_n$ and u_n is optimal, i.e., $V_T(q_n) = J_T(u_n)$.

REMARK 7. *According to Proposition 24, the dimension of the cone generated by normal Lagrange multipliers of the optimal control $\overline{u}_n$ is equal to the dimension of the convex subset $\partial V_T(q_n)$ of $\mathcal{P}_n$. In particular if V_T is Fréchet differentiable at q_n then $\overline{u}_n$ has a unique normal extremal lift. However, due to the nonsmoothness of the boundary of $\mathcal{U}_n$, even in the interior of $\mathcal{P}_n^+$, V_T may happen to be not differentiable.*

To see this, it suffices to take $q_n = c$ with $c > 0$ and $T = c$. We claim that $\overline{u}_n(t) = 1$ on I for every $t \in [0,T]$. Indeed, this fact is obviously true for every fixed x, without taking into account the polynomial constraint (absolute optimal solution); now this $\overline{u}_n$ is indeed polynomial in x for every t. Obviously, any $\lambda_n \in \mathcal{P}_n$ satisfying $\lambda_n(x) \geqslant c$ for every $x \in I$ is a normal Lagrange multiplier, for which (21) holds true with $\lambda^0 = -1$. Hence, in this case, $\overline{u}_n(t,x)$ has an infinite number of normal Lagrange multipliers, and V_T is not Fréchet differentiable at q_n.

At this step, Theorem 2 is proved. Hereafter and in the next subsection, we provide more results on normal extremals and we comment on the shooting method.

4.3. Study of normal extremals. According to Proposition 24, given any $q_n \in \mathcal{P}_n^+$ and any $T \geqslant \|q_n\|_{L^\infty(I)}$, the unique optimal control $\overline{u}_n$ solution of Problem 2 has at least one normal extremal lift, with a normal Lagrange multplier $(\lambda_n, -1)$ (which may not be unique).

In this section, we study *all* normal extremals, by letting λ_n vary in $\mathcal{P}_n$. It is then useful, here, to change a bit notations: in what follows, given any $\lambda_n \in \mathcal{P}_n$, we denote by$(y_{\lambda_n}, \lambda_n, -1, u_{\lambda_n})$ the normal extremal associated on the time interval $[0,+\infty)$ with the Lagrange multiplier λ_n. It is well defined according to the next result.

LEMMA 25. *Let $\lambda_n \in \mathcal{P}_n$ be arbitrary.*

- *There exists a unique control $u_{\lambda_n} \in L^\infty(0,+\infty;\mathcal{U}_n)$ such that*

$$\int_0^1 (\lambda_n(x) - t) u_{\lambda_n}(t,x)\, dx = \max_{p_n \in \mathcal{U}_n} \int_0^1 (\lambda_n(x) - t) p_n(x)\, dx, \tag{24}$$

  for almost every $t \in [0,+\infty)$.
- *There exist $(t_{1,\lambda_n}, t_{2,\lambda_n}) \in [0,+\infty)^2$, satisfying*

$$t_{1,\lambda_n} \geqslant \max\left(0, \min_{x\in[0,1]} \lambda_n(x)\right) \qquad \text{and} \qquad t_{2,\lambda_n} \leqslant \max_{x\in[0,1]} \lambda_n(x),$$

  such that $u_{\lambda_n}(t) = 1$ if $0 \leqslant t \leqslant t_{1,\lambda_n}$ and $u_{\lambda_n}(t) = 0$ if $t \geqslant t_{2,\lambda_n}$.
- *Denoting by y_{λ_n} the trajectory generated on $[0,+\infty)$ by the control u_{λ_n}, i.e., satisfying $\partial_t y_{\lambda_n} = u_{\lambda_n}$ and $y_{\lambda_n}(0) = 0$, for every $T \geqslant t_{2,\lambda_n}$ the control $u_{\lambda_n} \in L^\infty(0,T;\mathcal{U}_n)$ is the unique optimal control steering the control system to the target $y_{\lambda_n}(T) = y_{\lambda_n}(t_{2,\lambda_n}) = y_{\lambda_n}(+\infty)$.*

Note that, with respect to the statement of Theorem 2, with a slight ambiguity of notation, we have $\overline{u}_n = u_{\overline{\lambda}_n}$ and $\overline{y}_n = y_{\overline{\lambda}_n}$ (the optimal solution of Problem (2)) if $T \geqslant t_{2,\overline{\lambda}_n}$. For $\|q_n\|_{L^\infty(I)} \leqslant T \leqslant t_{2,\overline{\lambda}_n}$, the equality $\overline{u}_n = u_{\overline{\lambda}_n}$ may fail.

Of course, making λ_n vary, the corresponding final point $y_{\lambda_n}(+\infty) = y_{\lambda_n}(t_{2,\lambda_n})$ varies as well (we have a parametrization of all normal extremals), and in the shooting method we will want to tune λ_n such that $y_{\lambda_n}(T) = q_n$ for a given $q_n \in \mathcal{P}_n^+$. This equation can always be solved when $T \geqslant t_{2,\overline{\lambda}_n}$.

Proof (Proof of Lemma 25.). A key ingredient is the uniqueness result stated in Theorem 1. Let $u_n(t)$ be a maximizer of (24) for almost every $t \in [0,T]$. Integrating by parts, we have

$$\int_0^T \int_0^1 (\lambda_n(x) - t) u_n(t,x)\, dx\, dt = \int_0^T \int_0^1 y_n(t,x)\, dx\, dt + \int_0^1 (\lambda_n(x) - T) y_n(T,x)\, dx.$$

Since $u_n(t)$ is a maximizer of (24), the function y_n maximizes the functional

$$Q(y_n) = \int_0^T \int_0^1 y_n(t,x)\,dx\,dt + \int_0^1 (\lambda_n(x) - T) y_n(T,x)\,dx$$

under the constraint $\partial_t y_n \in \mathcal{U}_n$. In particular it maximizes Q over a subset of all such states, which is defined conveniently as the set of $\widehat{y}_n$ such that $\partial_t \widehat{y}_n = \widehat{u}_n \in \mathcal{U}_n$ and $\widehat{y}_n(T) = y_n(T)$. By Theorem 1, y_n is unique, u_n is unique, and by Lemma 17, u_n is such that $\mathrm{toc}(u_n(t)) \geqslant n+1$ for almost every $t \in [0,T]$. Now, let us consider the set A of all times $t \in [0,T]$ at which (21) has two distinct solutions with order of contact $\geqslant n+1$. If the Lebesgue measure of A is positive, then there exist two different solutions $u_n^1(t)$ and $u_n^2(t)$ with total order of contact $n+1$ or more. The half-sum $u_n^3 = \frac{1}{2}(u_n^1 + u_n^2)$ has a total order of contact which is less than $n+1$ over a set of times of positive measure. But its primitive y_n^3 is also a maximizer of $Q(y_n^3)$. This contradicts the previous point, and therefore the measure of A is zero. The first item of the lemma follows.

To prove the second item, it suffices to make the same reasoning as the one done to obtain (22).

This third item follows by uniqueness, and by using the fact that the first-order condition given by the Pontryagin maximum principle is sufficient since the optimal control problem is convex. ∎

Remark 8. *Note that the uniqueness of u_{λ_n} proved in Lemma 25 is quite subtle. Not only it follows from a complicated result (the proof of uniqueness in Theorem 1 is the difficult part), but also, we do not necessarily have uniqueness of the maximizer for the following* static *maximization problem (9) studied in Section 2.4. Here, the difference in (24) is that we deal with a dynamic maximization problem, where the term r_n of (9) is replaced with $\lambda_n(x)-t$. Then, for a particular given time $t \in [0,T]$, the maximizer $u_{\lambda_n}(t)$ may not be unique, but the maximizer $u_{\lambda_n} \in L^\infty(0,+\infty;\mathcal{U}_n)$ such that (24) is satisfied for almost every $t \in [0,+\infty)$ is unique.*

Lemma 26. *Given any $\lambda_n \in \mathcal{P}_n$, the function $t \mapsto u_{\lambda_n}(t) \in \mathcal{U}_n$ is piecewise continuous, with a finite number of discontinuities along $[0,+\infty)$.*

Proof. Recall that $u_{\lambda_n}(t) \in \mathrm{Extr}(\mathcal{U}_n)$ for almost every t. By Remark 8, the set of times t for which the maximizer $u_{\lambda_n}(t)$ is not unique is of Lebesgue measure zero (otherwise uniqueness of the optimal control would fail). This set corresponds to switching times, i.e., to possible discontinuities of $t \mapsto u_{\lambda_n}(t)$. Now, noting that the curve $t \mapsto \lambda_n(x) - t$ is algebraic and that the set $\mathcal{U}_n$ is semi-algebraic by Lemma 7, the finiteness property follows. ∎

Lemma 27. *Given any $\lambda_n \in \mathcal{P}_n$, the maximizer u_{λ_n} of (24) is also the unique maximizer of the functional $v_n \mapsto \int_0^{+\infty}\int_0^1 (\lambda_n(x)-t) v_n(t,x)\,dx\,dt$ over all possible $v_n \in L^\infty(0,+\infty;\mathcal{U}_n)$, that is,*

$$\int_0^{+\infty}\int_0^1 (\lambda_n(x)-t) u_{\lambda_n}(t,x)\,dx\,dt = \max_{v_n \in L^\infty(0,+\infty;\mathcal{U}_n)} \int_0^{+\infty}\int_0^1 (\lambda_n(x)-t) v_n(t,x)\,dx\,dt.$$

Proof. It suffices to notice that

$$\max_{v_n \in L^\infty(0,+\infty;\mathcal{U}_n)} \int_0^{+\infty}\int_0^1 (\lambda_n(x)-t) v_n(t,x)\,dx\,dt = \int_0^{+\infty} \max_{p_n \in \mathcal{U}_n} \int_0^1 (\lambda_n(x)-t) p_n(t,x)\,dx\,dt,$$

where this equality follows by applying a measurable selection lemma (see, e.g., [21, Lemmas 2A and 3A page 161]). ∎

4.4. Case $T=+\infty$. We have seen that $u_{\lambda_n}(t) = 0$ if $t \geqslant t_{2,\lambda_n}$. In other words, given some $\lambda_n \in \mathcal{P}_n$, the corresponding normal control is zero when t is large enough, but t_{2,λ_n} depends on λ_n. In order to establish results that do not depend on λ_n, in this section we take $T=+\infty$.

We have the following homogeneity property for the controls u_{λ_n}.

Lemma 28. *Given any $\alpha > 0$, we have*

$$u_{\alpha\lambda_n}(\alpha t) = u_{\lambda_n}(t) \ \ \text{for a.e. } t \geqslant 0, \qquad y_{\alpha\lambda_n}(\alpha t) = \alpha y_{\lambda_n}(t) \ \ \forall t \geqslant 0.$$

Proof. By uniqueness of the maximizer in Lemma 25, given any $\alpha > 0$, we know that $u_{\alpha\lambda_n}$ is the unique control in $L^\infty(0,+\infty;\mathcal{U}_n)$ such that $\int_0^1 (\alpha\lambda_n(x) - t) u_{\alpha\lambda_n}(t,x)\,dx = \max_{p_n\in\mathcal{U}_n} \int_0^1 (\alpha\lambda_n(x) - t)p_n(x)\,dx$ for almost every $t > 0$. Replacing t with αt and by uniqueness, the result follows. ∎

We consider the maximized Hamiltonian defined on $[0,+\infty) \times \mathcal{P}_n$ by

$$H_r(t,\lambda_n) = \max_{p_n\in\mathcal{U}_n} \int_0^1 (\lambda_n(x) - t)p_n(x)\,dx = \int_0^1 (\lambda_n(x) - t)u_{\lambda_n}(t,x)\,dx, \tag{25}$$

where $u_{\lambda_n} \in L^\infty(0,+\infty;\mathcal{U}_n)$ is the unique maximizer given by Lemma 25. We also set

$$K(\lambda_n) = \int_0^{+\infty} H_r(t,\lambda_n)\,dt = \int_0^{+\infty}\int_0^1 (\lambda_n(x) - t)u_{\lambda_n}(t,x)\,dx\,dt, \tag{26}$$

for every $\lambda_n \in \mathcal{P}_n$. The time integral is well defined because $u_{\lambda_n}(t) = 0$ if $t > \max_{x\in[0,1]}\lambda_n(x)$ by Lemma 25. Note also that $K(\lambda_n) \geqslant 0$ (take $p_n = 0$ in (25)).

Lemma 29. *The functional K defined by (26) has the following properties:*

- *K is convex on $\mathcal{P}_n$.*
- *K is Fréchet differentiable at any $\lambda_n \in \mathcal{P}_n$, and $dK(\lambda_n).h_n = \int_0^{+\infty}\int_0^1 h_n(x)u_{\lambda_n}(t,x)\,dt$. Identifying $dK(\lambda_n)$ with the gradient $\nabla K(\lambda_n)$ at λ_n with the scalar product of $L^2(I)$, we have*

$$\nabla K(\lambda_n) = \int_0^{+\infty} u_{\lambda_n}(t)\,dt = y_{\lambda_n}(+\infty) \in \mathcal{P}_n^+, \tag{27}$$

  i.e., $\nabla K(\lambda_n)$ is the final point $y_{\lambda_n}(+\infty) \in \mathcal{P}_n^+$ reached (in large enough time) by the trajectory y_{λ_n} generated by the control u_{λ_n} (solution of the Cauchy problem $\partial_t y_{\lambda_n} = u_{\lambda_n}$, $y_{\lambda_n}(0) = 0$).
- *K is positively 2-homogeneous, i.e., $K(\alpha\lambda_n) = \alpha^2 K(\lambda_n)$, for every $\alpha > 0$ and for every $\lambda_n \in \mathcal{P}_n$. Additionally ∇K is positively 1-homogeneous, i.e., $\nabla K(\alpha\lambda_n) = \alpha\nabla K(\lambda_n)$.*
- *for every $\lambda_n \in \mathcal{P}_n$, we have*

$$K(\lambda_n) = \frac{1}{2}\int_0^{+\infty}\int_0^1 \lambda_n(x)u_{\lambda_n}(t,x)\,dx\,dt = \int_0^{+\infty} t\int_0^1 u_{\lambda_n}(t,x)\,dx\,dt = J(u_{\lambda_n}). \tag{28}$$

Proof. It suffices to prove that H_r is convex with respect to λ. Given any $(\lambda_n^1,\lambda_n^2) \in \mathcal{P}_n^2$ and any $\alpha \in [0,1]$, we have, for almost every $t \in [0,T]$,

$$\begin{aligned}
H_r(t,\alpha\lambda_n^1 + (1-\alpha)\lambda_n^2) &= \int_0^1 (\alpha\lambda_n^1(x) + (1-\alpha)\lambda_n^2(x) - t)u_{\alpha\lambda_n^1+(1-\alpha)\lambda_n^2}(t,x)\,dx \\
&= \alpha\int_0^1 (\lambda_n^1(x) - t)u_{\alpha\lambda_n^1+(1-\alpha)\lambda_n^2}(t,x)\,dx + (1-\alpha)\int_0^1 (\lambda_n^2(x) - t)u_{\alpha\lambda_n^1+(1-\alpha)\lambda_n^2}(t,x)\,dx \\
&\leqslant \alpha\int_0^1 (\lambda_n^1(x) - t)u_{\lambda_n^1}(t,x)\,dx + (1-\alpha)\int_0^1 (\lambda_n^2(x) - t)u_{\lambda_n^2}(t,x)\,dx \\
&= \alpha H_r(t,\lambda_n^1) + (1-\alpha)H_r(t,\lambda_n^2),
\end{aligned}$$

where we have used (24) to obtain the above inequality. The convexity property of K (first item) follows.

Let us establish the differentiability property (second item). Applying the Danskin theorem (see [1, 10]) to the function H_r defined by (25), it follows that, for almost every $t \in [0,T]$, the function $\lambda_n \mapsto H_r(t,\lambda_n)$ is directionally differentiable at any $\lambda_n \in \mathcal{P}_n$, and

$$\frac{\partial H_r}{\partial \lambda_n}(t,\lambda_n).h_n = \max\left\{ \int_0^1 h_n(x)\widehat{p}_n(x)\,dx \ \mid \ \widehat{p}_n \in \widehat{\mathcal{U}}_n(t,\lambda_n) \right\},$$

where $\widehat{\mathcal{U}}_n(t,\lambda_n)$ is the set of all $\widehat{p}_n \in \mathcal{P}_n$ maximizing the functional $p_n \mapsto \int_0^1 (\lambda_n(x)-t)p_n(x)\,dx$ (with t fixed). Note that $\widehat{\mathcal{U}}_n(t,\lambda_n)$ contains the element $u_{\lambda_n(t)}$ for almost every $t \in [0,T]$, but may not be a singleton (see Remark 8). But since the maximizer u_{λ_n} is unique in $L^\infty(0,T;\mathcal{U}_n)$, it follows that the functional K is directionally differentiable, and $dK(\lambda_n).h_n = \int_0^{+\infty}\int_0^1 h_n(x)u_{\lambda_n}(t,x)\,dx\,dt = \int_0^{+\infty}\langle u_{\lambda_n}(t),h_n\rangle_{L^2(I)}\,dt$. Since K is convex and thus locally Lipschitz, and since $\mathcal{P}_n$ is finite-dimensional, Gâteaux differentiability implies Fréchet differentiability (see [9]).

The 2-homogeneity property of K (third item) obviously follows by making the change of variable $t \mapsto t/\alpha$ in the integral and by using Lemma 28. The 1-homogeneity property of ∇K is then obtained by differentiating $K(\alpha\lambda_n) = \alpha^2 K(\lambda_n)$ with respect to λ_n.

It remains to prove the fourth item. Since $K(\alpha\lambda_n) = \alpha^2 K(\lambda_n)$, derivating in α and taking $\alpha = 1$ gives $\langle \nabla K(\lambda_n),\lambda_n\rangle_{L^2(I)} = 2K(\lambda_n)$ (Euler equation), which yields (28) by using (27). ☐

Remark 9. *We can notice that, given any $\lambda_n \in \mathcal{P}_n^-$, i.e., $\lambda_n \in \mathcal{P}_n$ such that $\lambda_n(x) \leqslant 0$ for every $x \in [0,1]$, we have $u_{\lambda_n} = 0$ (this follows from Lemma 25), and thus, by Lemma 29, $K(\lambda_n) = J(u_{\lambda_n}) = 0$ and $\nabla K(\lambda_n) = 0$. In other words, the convex functional K is identically zero on the convex subset $\mathcal{P}_n^-$ of $\mathcal{P}_n$, as well as its gradient.*

Moreover, as an easy consequence of Lemma 29, we have:

$$K(\lambda_n) = 0 \ \Leftrightarrow\ \nabla K(\lambda_n) = 0 \ \Leftrightarrow\ u_{\lambda_n} = 0.$$

Another remark is that, if $\lambda_n(x) > 0$ for every $x \in I = [0,1]$, then $\nabla K(\lambda_n) = \int_0^{+\infty} u_{\lambda_n}(t)\,dt > 0$ on I, and if $\int_0^1 \lambda_n(x)\,dx > 0$ (or more generally, if there exists $p_n \in \mathcal{U}_n$ such that $\int_0^1 \lambda_n(x)p_n(x)\,dx > 0$), then $K(\lambda_n) > 0$ and $\nabla K(\lambda_n) \in \mathcal{P}_n^+ \setminus \{0\}$.

Remark 10. *Using (28), it is interesting to note that, for the optimal control $\overline{u}_n$ solution of Problem 2, if $T \geqslant t_{2,\overline{\lambda}_n}$ where $(\overline{\lambda}_n,-1)$ is a normal Lagrange multiplier (whose existence follows from Proposition 24), then $\overline{u}_n = u_{\overline{\lambda}_n}$ and*

$$V_T(q_n) = J_T(\overline{u}_n) = J(\overline{u}_n) = K(\overline{\lambda}_n),$$

$$q_n = \nabla K(\overline{\lambda}_n) = \int_0^{+\infty} \overline{u}_n(t)\,dt = E_T(\overline{u}_n) = \overline{y}_n(T) = \overline{y}_n(+\infty).$$

In particular we have obtained the relation $V_T(\nabla K(\lambda_n)) = K(\lambda_n)$, for every $\lambda_n \in \mathcal{P}_n$ and every $T \geqslant t_{2,\lambda_n}$, and thus,

$$V_{+\infty} \circ \nabla K = K,$$

where $V_{+\infty}(\lambda_n)$ is defined as the limit of $V_T(\lambda_n)$ as $T \to +\infty$ (the limit is reached in finite time, for $T \geqslant t_{2,\lambda_n}$). We have also obtained that, given any $q_n \in \mathcal{P}_n^+$, for every $\lambda_n \in \partial V_T(q_n)$ and every $T \geqslant t_{2,\lambda_n}$ we have $q_n = \nabla K(\lambda_n)$, which we can write in the form

$$\nabla K \circ \partial V_{+\infty} = \mathrm{id}_{\mathcal{P}_n}.$$

It can also be noted that V_T is positively 2-homogeneous.

Shooting method. We have seen how to parametrize the "normal extremal flow", i.e., all possible normal extremals, by the $(n+1)$-parameter $\lambda_n \in \mathcal{P}_n$. The shooting method consists of finding $\lambda_n \in \mathcal{P}_n$ such that $\nabla K(\lambda_n) = q_n$. Note that, as planned by the theory, the shooting problem consists of $n+1$ unknowns for $n+1$ equations (see [34] for a survey on well-posedness and implementation issues of the shooting method). Proposition 24 shows that this is always possible: it suffices to pick some $\lambda_n \in \partial V_{+\infty}(q_n)$; and thus that the mapping $\nabla K : \mathcal{P}_n \to \mathcal{P}_n^+$ is surjective. Given that $\nabla K = 0$ on $\mathcal{P}_n^-$ (see Remark 9), and given that ∇K is positively 1-homonegenous (see Lemma 29), we even have a more precise result: the mapping $\nabla K : \mathcal{P}_n \setminus \mathcal{P}_n^- \to S\mathcal{P}_n^+$ is surjective (where $S\mathcal{P}_n^+$ is the quotient of $\mathcal{P}_n^+$ by positive homotheties).

However, a given $q_n \in \mathcal{P}_n^+$ may have several preimages $\lambda_n \in \partial V_{+\infty}(q_n)$ under ∇K, i.e., several normal Lagrange multipliers. This is so as soon as the subdifferential $\partial V_{+\infty}(q_n)$ is not reduced to a singleton, which is equivalent to say that $V_{+\infty}$ is not Fréchet differentiable at q_n.

In particular, we have seen in Lemma 22 that, if $q_n \in \mathcal{P}_n^+$ vanishes at some point of I, then the optimal control u_n such that $E_{+\infty}(u_n) = q_n$ has an abnormal Lagrange multiplier $(\mu_n, 0)$ (actually, as many as q_n has zeros on I). Since u_n has also, according to Proposition 24, a normal Lagrange multiplier $(\lambda_n, -1)$, it follows that u_n has an infinite number of normal Lagrange multipliers $(\lambda_n + \beta\mu_n, -1)$, for any $\beta \geqslant 0$. In particular, $V_{+\infty}$ is not Fréchet differentiable at q_n.

However, we think (but we do not know how to prove) that $V_{+\infty}$ is Fréchet differentiable at any $q_n \in \mathcal{P}_n$ such that $q_n > 0$ on I. This is not contradictory with what is stated in Remark 7.

5. Convergence with respect to n. In this section, we investigate some convergence properties as $n \to +\infty$. The question is natural in the context of polynomial approximation (see [13, 31]). As in Section 4, to simplify we assume that $I = [0,1]$, that $w = 1$ and that $s(t) = t^2/2$.

We assume that $q_n \in \mathcal{P}_n^+$ converges uniformly on I to a nonnegative continuous function q as $n \to +\infty$. We first address the convergence of $J(\overline{u}_n)$. The indicatrix function (4) gives the lower bound

$$J(\overline{u}_n) \geqslant \int_0^{q_n(x)} t \int_I dx\, dt = \frac{1}{2}\int_I q_n(x)^2\, dx \underset{n\to+\infty}{\longrightarrow} \frac{1}{2}\int_I q(x)^2\, dx. \tag{29}$$

The main point is then to obtain an upper bound for $J(\overline{u}_n)$. We make use of a polynomial convolution kernel $F_n\star : C^0([0,1]) \to \mathcal{P}_n$ such that $\|q - F_n \star q\|_{L^\infty(I)} \leqslant \varepsilon_n(q)$ with $\lim_{n\to+\infty} \varepsilon_n(q) = 0$ (see [12, 13]), such as the Fejer kernel or the Jackson kernel (for the latter, one has $\varepsilon_n(q) \leqslant C\|q''\|_{L^\infty(I)}/n^2$). These kernels are nonnegative with mass 1 on $I = [0,1]$; more precisely, we have $F_n \star 1 = 1$ and $F_n \star p \geqslant 0$ for $p \in C^0([0,1])$ such that $p \geqslant 0$ (see [13]).

LEMMA 30. *We assume that $q \in C^0(I)$ and that $q > 0$ on I. Then:*
- $\lim_{n\to+\infty} J(\overline{u}_n) = \frac{1}{2}\int_I q(x)^2\, dx$.
- *The optimal control $\overline{u}_n$ converges to the indicatrix function $\mathbf{1}_{\{0<t<q(x)\}}$ for the weak-star topology of $L^\infty(I)$.*

Proof. Writing $q_n = (q_n - F_n \star q + \mu_n) + F_n \star (q - \mu_n)$ with $\mu_n = \|q - q_n\|_{L^\infty(I)} + \varepsilon_n(q)$ and $\|q - F_n \star q\|_{L^\infty(I)} \leqslant \varepsilon_n(q)$, we have $q_n - F_n\star q + \mu_n = (q - q_n) + (q - F_n\star q) + \|q - q_n\|_{L^\infty(I)} + \varepsilon_n(q)$, and thus $0 \leqslant q_n - F_n \star q + \mu_n \leqslant 2\mu_n$ on I. Defining

$$v_n(t,x) = \frac{1}{2\mu_n}\Big(\underbrace{q_n - F_n \star q + \mu_n}_{\in\mathcal{P}_n}\Big)(x)\mathbf{1}_{\{0<t<2\mu_n\}} + \underbrace{F_n \star \mathbf{1}_{\{2\mu_n<t<q(x)+\mu_n\}}}_{\in\mathcal{P}_n}(x),$$

by construction we have $v_n \in L^\infty(0,+\infty;\mathcal{U}_n)$ for $T > \|q\|_{L^\infty(I)} + \sup_n \mu_n$ and

$$\begin{aligned}\int_0^{+\infty} v_n(t,x)\, dt &= (q_n - F_n \star q + \mu_n)(x) + F_n \star \left(\int_0^{+\infty} \mathbf{1}_{\{2\mu_n<t<q(x)+\mu_n\}} dt\right)(x)\\ &= (q_n - F_n \star q + \mu_n)(x) + F_n \star (q - \mu_n)(x) = q_n(x).\end{aligned}$$

Therefore $v_n \in \mathcal{K}_n(T, q_n)$ is admissible and

$$
\begin{aligned}
J(u_n) \leqslant J(v_n) &= \int_0^{+\infty} t \int_I v_n(t,x)\,dx\,dt \\
&= \mu_n \int_I (q_n - F_n \star q + \mu_n)(x)\,dx + \int_I F_n \star \left(\int_0^\infty \mathbf{1}_{\{2\mu_n < t < q(x)+\mu_n\}} t\,dt \right) dx \\
&\leqslant 2\mu_n^2 + \frac{1}{2} \int_I F_n \star \left((q(x)+\mu_n)^2 - 4\mu_n^2 \right) dx \leqslant \frac{1}{2} \int_I F_n \star (q(x)+\mu_n)^2\,dx.
\end{aligned}
$$

Using that $\lim_{n\to+\infty} \frac{1}{2}\int_I F_n \star (q(x)+\mu_n)^2 dx = \frac{1}{2}\int_I (q(x)^2)\,dx$ (see [13]) and the lower bound (29), the first item follows.

Since $0 \leqslant \overline{u}_n \leqslant 1$, up to some subsequence u_n converges in weak-star topology to some $u \in L^\infty(\mathbb{R}^+ \times I)$ satisfying $0 \leqslant u \leqslant 1$. Defining $w_n(t,x) = \frac{q_n(x)}{\|q_n\|_{L^\infty(I)}} \mathbf{1}_{\{0<t<\|q_n\|_{L^\infty(I)}\}}$, we have $w_n \in \mathcal{K}_n(T, q_n)$, and thus $J(\overline{u}_n) \leqslant J(w_n) = \frac{1}{2}\|q_n\|_{L^\infty(I)}\|q_n\|_{L^1(I)} \leqslant C$ uniformly with respect to n since q_n converges to q pointwise. Hence $J(\overline{u}_n) \to J(u)$. Using the first item, we get that $J(u) = \frac{1}{2}\int_I q(x)^2\,dx$. Since $0 \leqslant u \leqslant 1$ and $\int_0^{+\infty} u(t,x)\,dt = q(x)$ for almost every x, the Brenier principle (see [4]) yields that $u(t,x) = \mathbf{1}_{\{0<t<q(x)\}}$. Now since the limit is unique, the whole sequence converges to u in weak-star topology. The second item is proved. ◻

6. Numerical illustration. To implement the minimization problem 1 (equivalent to Problem 2), we use the modeling language `AMPL` language (see [15]) combined with the optimization routine `IpOpt` (see [36]). An example is provided where the time-space domain $[0,5] \times [0,1]$ is discretized with a finite differences. The number of cells in t (resp., in x) is N_t (resp. N_x). We take $q_3(x) = 1 + x + x^2 + x^3$.

We see on Figures 2 and 3 that $u_n(t) = 0$ t approximately greater that $\max q_n = 4$. Taking into account the errors brought by the discretization, the cuts of u_n show that $\mathrm{toc}(u_n) \geqslant n+1$ for almost any time. Indeed, on Figure 2 (right), one sees 4 contact points (counted with multiplicity), in accordance with the theory.

The parameters are doubled in Figure 3, but the target $q_6(x) = q_3(x)$ is the same. Convergence results are given in Table 1. The asymptotic value of the cost is $K_\infty = \frac{1}{2}\int_0^1 (1+x+x^2+x^3)^2\,dx \approx 2.54$, and we indeed observe convergence to this optimal value, in accordance with Lemma 30.

We have seen in Lemma 26 that the number of switching times is finite, and indeed we observe that the function $t \mapsto u_n(t)$ has discontinuities (jumps) for a finite number of times (see Figures 2 and 3). This is a reminiscence of the bang-bang principle in optimal control theory. Numerical simulations suggest that this number is less than $n+1$, but establishing such a property is open.

Note that implementation of polynomials in $\mathcal{U}_n$ with a new technique developed in [11] is under study.

$n =$	3	6	12	∞
$N_x =$	40	80	60	∞
$N_t =$	100	200	400	∞
$K(u_n) =$	2.73	2.61	2.57	2.54

TABLE 1

Convergence table of the cost function with respect to the polynomial degree: the target is the same: $q_3(x) = q_6(x) = q_{12}(x) = 1 + x + x^2 + x^3$. The parameters n, N_x and N_t are doubled from one test to the other. The last column is the exact solution.

7. Conclusion, further comments and perspectives. The combination on the one hand of L^1 optimization techniques à la Bojanic and DeVore extended to space-time formulations and on

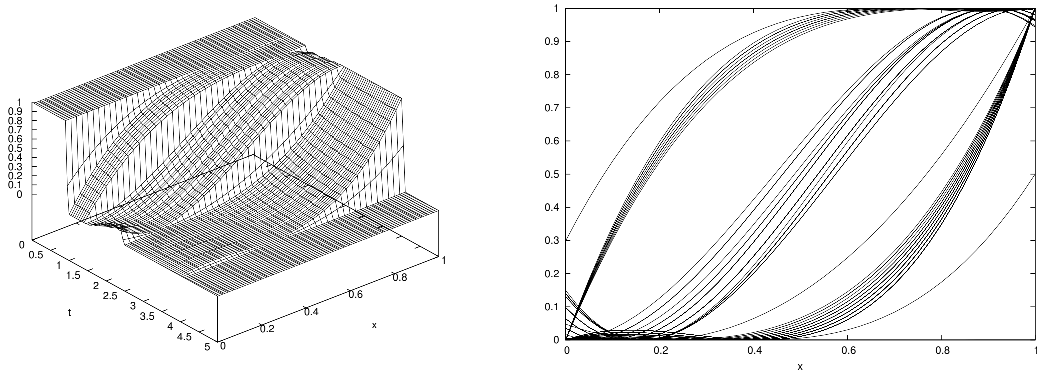

FIG. 2. *Parameters: $N_x = 40$, $N_t = 100$, $n = 3$, $q_3 = 1 + x + x^2 + x^3$. The function $(x, t) \mapsto u_n(x, t)$ is represented on the left as a surface, and is represented on the right as many curves $x \mapsto u(x; t)$ parametrized by t.*

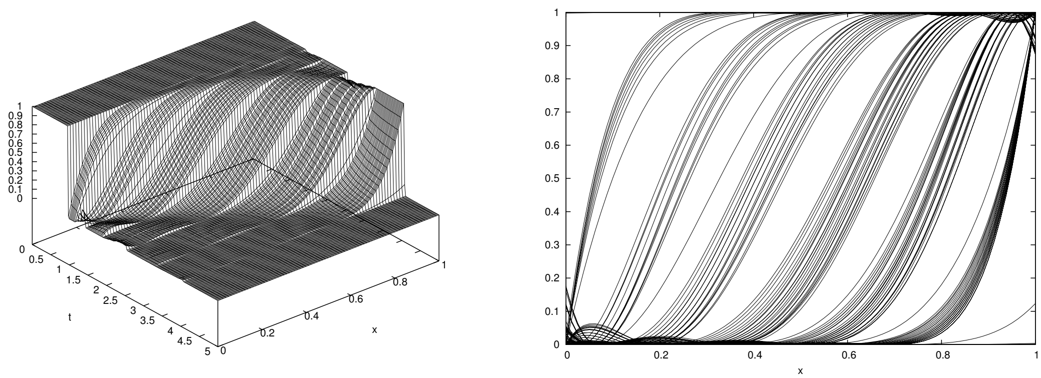

FIG. 3. *Parameters: $N_x = 80$, $N_t = 200$, $n = 6$, $q_6 = 1 + x + x^2 + x^3$. The function $(x, t) \mapsto u_n(x, t)$ is represented on the left as a surface, and is represented on the right as many curves $x \mapsto u(x; t)$ parametrized by t.*

the other hand of optimal control techniques yields powerful tools for space-time L^1 optimization of polynomial problems. The reason is that the rigid algebraic polynomial structure is compatible with the Pontryagin maximum principle. Based on these tools, it has been possible to derive in Lemma 16 a characterization of optimal solutions which is a key for the proof of uniqueness. The compatibility is evidenced with the notion of total order of contact of polynomial which is the number of points (counted with multiplicity) where the graph of a polynomial f_n between 0 and 1 touches the bounds 0 and 1. A key result is that a polynomial f_n is an extremal point in the convex set $\mathcal{U}_n$ if and only if its total order of contact is greater or equal to $\deg(f_n) + 1$. Probably more research is needed to fully understand the Bojanic and DeVore theorem in relation with the geometry of $\mathcal{U}_n$. A still open problem is to explain why the optimal solution has a finite number of switching times that is uniformly bounded for a given n, as evidenced by the numerical illustrations.

Extending our results to multivariate polynomials (see [19]) is a completely open issue which is probably quite difficult. We expect huge obstructions because the structure of positive polynomials is delicate to establish. We also notice that we are not aware of any multivariate extension of Bojanic and DeVore results.

Among natural perspectives, firstly, we mention the existence and uniqueness result of Theorem 1 which justifies the theory of kinetic polynomials (see [12]) by providing the missing piece that was the uniqueness of $\overline{u}_n$. Secondly, we mention the possibility to use the new parametrization of $\mathcal{U}_n$ proposed in [11] in order to accelerate computations.

Appendix A. Proof of Theorem 5. Theorem 5 is the technical cornerstone in order to establish uniqueness in Theorem 1. To prove it, we derive interesting features of polynomial L^1 minimization. We will use the notations of Section 2.1. The starting point is the following seminal result, due to Bojanic and DeVore.

THEOREM 31 ([3]). *Let f be a function on I, continuous and differentiable in the interior of I. We assume that $\int_I w(x)\,dx > 0$. Then there exists a unique maximizer*

$$\overline{p}_n = \operatorname{argmax}\left\{ \int_I p_n(x)w(x)\,dx \ \mid \ p_n \in \mathcal{P}_n,\ p_n \leqslant f \ \text{on } I \right\}.$$

The polynomial $\overline{p}_n$ is the solution of a one-sided polynomial optimization problem. We recall hereafter the main ideas of the proof of this result. Existence of a maximizer $\overline{p}_n$ is straightforward, but we sketch the main arguments to establish uniqueness since we are going to extend them to our context (where the optimization problem is two-sided). By the way, it is interesting to note that uniqueness fails in general if f is only continuous.

Proof. (Sketch of proof) First, [3, Lemma 3 page 145] is proved by showing that there exists at least $\lfloor \frac{n}{2} \rfloor + 1$ contact points in I of $\overline{p}_n$ with f. The argument goes by contradiction. Assume that there exist $k < \lfloor \frac{n}{2} \rfloor + 1$ contact points $x_1 < \cdots < x_k$ in I, i.e., $\overline{p}_n(x_i) = f(x_i)$, and consider the polynomial $Q_\varepsilon \in \mathcal{P}_n$ defined by

$$Q_\varepsilon(x) = (x-(x_1-\varepsilon))(x-(x_1+\varepsilon))\dots(x-(x_k-\varepsilon))(x-(x_k+\varepsilon)), \tag{30}$$

where $\varepsilon > 0$ is a small enough so that $\int_I Q_\varepsilon(x)w(x)\,dx > 0$. Setting $\mathcal{I} = \cup_{i=1}^k (x_i-\varepsilon, x_i+\varepsilon) \cap G \subset G$, we have the following properties:

$$\begin{array}{ll} f(x) - \overline{p}_n(x) \geqslant d > 0 & \forall x \in I - \mathcal{I}, \\ \eta Q_\varepsilon(x) \leqslant d \leqslant f(x) - \overline{p}_n(x) & \forall x \in I - \mathcal{I}, \quad \eta = d/\max_G Q_\varepsilon(x) > 0, \\ \eta Q_\varepsilon(x) \leqslant 0 \leqslant f(x) - \overline{p}_n(x) & \forall x \in \mathcal{I}. \end{array} \tag{31}$$

Hence $\overline{p}_n + \eta Q_\varepsilon \in \mathcal{Q}_n(f)$ and $\int_I \overline{p}_n(x)w(x)\,dx < \int_I (\overline{p}_n(x) + \eta Q_\varepsilon(x))w(x)\,dx$. This is a contradiction.

Second, [3, Lemma 4 page 147] consists of proving that, if n is odd, then all $\lfloor \frac{n}{2} \rfloor + 1$ contact points belong to the interior I, and if n is even, then at least $\lfloor \frac{n}{2} \rfloor$ contact points belong to the interior of I. The proof is similar to the one above.

Finally, [3, Theorem 3 page 149] consists of counting the number of contact points: assume for example that n is odd and assume that there are two maximizers $\overline{p}_n^1$ and $\overline{p}_n^2$. Both must have $\lfloor \frac{n}{2} \rfloor + 1$ contact points in common inside I. Indeed, otherwise, consider $\overline{p}_n^3 = \frac{1}{2}(\overline{p}_n^1 + \overline{p}_n^2)$. The maximizers coincide at these points as well as and their derivatives. If n is odd, then we have $2(\lfloor \frac{n}{2} \rfloor + 1) = n+1$ independent equality constraints and thus $\overline{p}_n^1 = \overline{p}_n^2$. If n is even, the maximizers coincide at $\lfloor \frac{n}{2} \rfloor + 1$ contact points and their derivatives coincide at least at $\lfloor \frac{n}{2} \rfloor$ interior contact points, hence we have $\lfloor \frac{n}{2} \rfloor + 1 + \lfloor \frac{n}{2} \rfloor = n+1$ equality constraints and thus $\overline{p}_n^1 = \overline{p}_n^2$. ∎

We use for our purposes a reformulation with two bounds. Uniqueness for one-sided L^1 approximations of differentiable functions is treated in generality in [25] and our results are similar. However we deal hereafter with two-sided problems, which in general may not be C^1 (see (5)). This is why we need to make a complete proof.

PROPOSITION 32. *Let f and g be two differentiable functions on I, satisfying $g < 0 < f$. We assume that $\int_I w(x)\,dx > 0$. Then there exists a unique maximizer*

$$\overline{p}_n = \operatorname{argmax}\left\{ \int_I p_n(x)w(x)\,dx \ \mid \ p_n \in P^n,\ g \leqslant p_n \leqslant f \right\}.$$

Proof. We follow the arguments of the previous proof. Existence is obvious.

We claim that, if $\overline{p}_n$ is a maximizer, then there exist at least $\lfloor \frac{n}{2} \rfloor + 1$ contact points in I of $\overline{p}_n$ with f. The proof goes by contradiction. We denote by $x_1 < \cdots < x_k$ the contact points with f. Consider the polynomial $Q_\varepsilon \in \mathcal{P}_n$ defined by (30). Using (31), we have $\overline{p}_n + \eta Q_\varepsilon \leqslant f$. It remains to show that $g \leqslant \overline{p}_n + \eta Q_\varepsilon$. We have $Q_\varepsilon(x) \geqslant 0$ for $x \in I - \mathcal{I}$, and thus $g(x) \leqslant \overline{p}_n(x) \leqslant \overline{p}_n(x) + \eta Q_\varepsilon(x)$ for $x \in I - \mathcal{I}$.

If $x \in \mathcal{I}$, then $Q_\varepsilon(x) \leqslant 0$ by construction. But at the same time we have $\overline{p}_n(x_i) = f(x_i)$. Since $f > 0 > g$ on I, it follows that $\overline{p}_n(x) \geqslant \alpha > 0$ for $x \in \mathcal{I}$. Hence if $\varepsilon, d, \eta > 0$ are taken sufficiently small, then $g(x) < 0 < \overline{p}_n(x) + \eta Q_\varepsilon(x)$ for $x \in \mathcal{I}$. Therefore $g \leqslant \overline{p}_n + \eta Q_\varepsilon$ in I. This shows that the lower bound $g \leqslant \overline{p}_n$ does not change the structure of the proofs in the one-sided case provided $g < 0 < f$ in I. Therefore [3, Lemma 3] is still valid in this two-sided case.

It is straightforward to extend [3, Lemma 4 and Theorem 3]. The result follows. ◻

We are now in a position to prove Theorem 5. We set $f = \min(1 - f_n, f_n)$ and $g = -\min(1 - f_n, f_n)$. If $0 < f_n < 1$ on I, Theorem 5 follows from Proposition 32. But since $f_n \in \mathcal{U}_n$, the function f_n may touch the bounds 0 or 1: for example assume that there exists $x_* \in I$ such that $f_n(x_*) = 0$. In this case, we have $g(x_*) = -\min(1 - f_n, f_n)(x_*) = 0 = \min(1 - f_n, f_n)(x_*) = f(x_*)$. Hereafter, we address this case with a convenient polynomial factorization at the contact points.

Lemma 33. *If $f_n = 0$ or $f_n = 1$ then the unique solution of Theorem 5 is $\overline{p}_n = 0$. Otherwise, we have* $\mathrm{toc}(f_n) \leqslant 2n$.

Proof. The first statement is obvious. Let us prove the second one. With the notations of Section 2.1, we define $W(x) = \prod_{i=0}^p |x - x_i|^{a_i} \times \prod_{j=0}^q |x - y_j|^{b_j}$. Since $f_n \neq 0$ and $f_n \neq 1$, W is well defined and we have to prove that $\deg(W) \leqslant 2n$. By contradiction, if $\deg(W) \geqslant 2n + 1$, then either $\sum_{i=0}^p a_i \geqslant n + 1$ or $\sum_{j=0}^q b_j \geqslant n + 1$. In the first case, the factorization (7) with $\prod_{i=0}^p |x - x_i|^{a_i}$ of degree $\geqslant n + 1$ shows that $a = 0$, and thus $f_n = 0$. The second case yields similarly that $f_n = 1$. In both cases we have a contradiction. ◻

We define the continuous function

$$Z(x) = \min\left(\frac{1 - f_n(x)}{W(x)}, \frac{f_n(x)}{W(x)}\right) = \min\left(\frac{b(x)}{\prod_{i=0}^p |x - x_i|^{a_i}}, \frac{a(x)}{\prod_{j=0}^q |x - y_j|^{b_j}}\right). \tag{32}$$

By construction, there exists $\alpha > 0$ such that $Z(x) \geqslant \alpha > 0$ for every $x \in I$. The polynomial $\overline{p}_n \in \mathcal{Q}_n$ considered in Theorem 5 satisfies $-W(x)Z(x) = g \leqslant \overline{p}_n(x) \leqslant f = W(x)Z(x)$ for every $x \in I$. This shows that the polynomial W factorizes $\overline{p}_n$.

Lemma 34. *Assume that $f_n \neq 0$ and $f_n \neq 1$, and that $\mathrm{toc}(f_n) \geqslant n + 1$. Then $\overline{p}_n = 0$.*

Proof. All roots of W are also roots of f_n or of $1 - f_n$, and thus are also roots of $\overline{p}_n$. Therefore $\overline{p}_n$ is a polynomial with $n + 1$ roots, and thus $\overline{p}_n = 0$. More precisely, $\mathcal{Q}_n = \{0\}$. ◻

The last case is when $\mathrm{toc}(f_n) \leqslant n$. The idea is to factorize $\overline{p}_n = W r_m$, with $r_m \in \mathcal{P}_m$ such that $\deg(W) + m = n$. With these notations, the maximization problem (6) consider in Proposition 5 is equivalent to:

Find $r_m \in Q_m = \{s_m \in \mathcal{P}_m \mid -Z \leqslant s_m \leqslant Z$ on $I\}$ such that

$$r_m = \underset{s_m \in Q_m}{\operatorname{argmax}} \int_I s_m(x) W(x) w(x)\, dx. \tag{33}$$

The proof of Theorem 5 is done with the formulation (33), thanks to Proposition 32 applied with $g = -Z$ and $f = Z$. We have $g < 0 < f$ on I. The point is that f and g are differentiable everywhere except maybe at some points in the interior of I, which we denote generically by

z_*, where the two rational fractions in (32) have the same value, i.e., $f_n(z_*) = 1 - f_n(z_*)$, thus $f_n(z_*) = \frac{1}{2}$. Two cases occur.

If $f_n'(z_*) = 0$, then Z is differentiable at z_* with $Z'(z_*) = 0$.

If $f_n'(z_*) \neq 0$, then $Z(x) = f_n(z_*) - |f_n'(z_*)|\,|x - z_*| + \mathrm{O}\left(|x - z_*|^2\right)$ for $z_* - \varepsilon < x < z_* + \varepsilon$. Since z_* cannot be a contact point between a polynomial s_m and Z, this case never occurs.

Hence the contact points between r_m and $\pm Z$ are points where $\pm Z$ is differentiable. A regularization of Z near non-differentiable points suffices to apply Proposition 32 and prove the claim. Proposition 5 follows.

Note that if the function Z defined by (32) is defined with a maximum instead of a minimum, then the result fails because $Y(x) = f_n(z_*) + |f_n'(z_*)|\,|x - z_*| + \mathrm{O}\left(|x - z_*|^2\right)$ may have the contact point z_* between a polynomial and Y, even if $f_n'(z_*) \neq 0$.

Appendix B. Construction of exact solutions for $n = 2$. We use the Pontryagin maximum principle to determine an explicit analytical solution of Problem 2 for $n = 2$. We take $I = [0, 1]$, $s'(t) = t$ and $w(x) = 1$. We define the set

$$S = \left\{(a, b, r) \ \mid \ a = \cos\theta,\ b = \cos\psi,\ r^2 \leqslant 2(1 - \cos(\theta - \psi)),\ \theta, \psi \in \mathbb{R}\right\}.$$

Lemma 35. *Let $v_2 \in \mathcal{P}_2$. Then $v_2 \in \mathcal{U}_2$ if and only it can be written either as $v_2(x) = (ax + b(1-x))^2 + r^2 x(1-x)$ for some $(a, b, r) \in S$, or as $v_2(x) = 1 - (\overline{a}x + \overline{b}(1-x))^2 - \overline{r}^2 x(1-x)$ for some $(\overline{a}, \overline{b}, \overline{r}) \in S$.*

Proof. From the Lukács theorem (see [31]), nonnegative polynomials of degree 2 can be written as $w(x) = (ax + b(1-x))^2 + r^2 x(1-x)$. Since $v_2 \geqslant 0$ and $1 - v_2 \geqslant 0$, we get $v_2(x) = (ax + b(1-x))^2 + r^2 x(1-x)$ and $1 - v_2(x) = (\overline{a}x + \overline{b}(1-x))^2 + \overline{r}^2 x(1-x)$. Therefore $1 = (ax + b(1-x))^2 + r^2 x(1-x) + (\overline{a}x + \overline{b}(1-x))^2 + \overline{r}^2 x(1-x)$, i.e., $x^2 + 2x(1-x) + (1-x)^2 = (a^2 + \overline{a}^2)x^2 + (2ab + 2\overline{a}\overline{b} + r^2 + \overline{r}^2)x(1-x)(b^2 + \overline{b}^2)(1-x)^2$. Identifying the coefficients yields $1 = a^2 + \overline{a}^2$, $2 = 2ab + 2\overline{a}\overline{b} + r^2 + \overline{r}^2$ and $1 = b^2 + \overline{b}^2$. The result follows. ∎

According to our results, we must have $\overline{u}_2(t) = \operatorname{argmax}\{\int_0^1 (\lambda_2(x) - t)\, p_2(x)\, dx \ \mid \ p_2 \in \mathcal{U}_2\}$ for almost every t, for some normal Lagrange multiplier $\lambda_2 \in \mathcal{P}_2$ which can be defined by its moments $k_1 = 3\int_0^1 \lambda_2(x) x^2\, dx$, $k_2 = 6\int_0^1 \lambda_2(x) x(1-x)\, dx$ and $k_3 = 3\int_0^1 \lambda_2(x)(1-x)^2\, dx$. We propose hereafter a method to construct the solution represented by either (a, b, r) or $(\overline{a}, \overline{b}, \overline{r})$. We distinguish between several cases:

a) Assume that

$$t < k_2, \tag{34}$$

and write $u_2(t)$ with the second representation of Lemma 35 (if $k_2 < t$ we take the other representation). We define

$$M(a, b, r) \equiv \int_0^1 (\lambda_2(x) - t)\,(v_2(x) - 1)\, dx = \frac{1}{3}(t - k_1)\overline{a}^2 + \frac{1}{6}(t - k_2)(2\overline{a}\overline{b} + \overline{r}^2) + \frac{1}{3}(t - k_3)\overline{b}^2. \tag{35}$$

Since $M(\overline{a}, \overline{b}, \overline{r})$ must reach its maximum, we have $\overline{r} = 0$. We set $A(\overline{a}, \overline{b}) = M(\overline{a}, \overline{b}, 0) = \frac{1}{3}(t - k_1)\overline{a}^2 + \frac{1}{6}(t - k_2)2\overline{a}\overline{b} + \frac{1}{3}(t - k_3)\overline{b}^2$. Since $t - k_2 \geqslant 0$, the optimal solution is such that $\overline{a}\overline{b} \leqslant 0$. Moreover $A(\overline{a}, \overline{b}) \geqslant A(\overline{b}, \overline{a})$, i.e., $(k_1 - k_3)(\overline{b}^2 - \overline{a}^2) \geqslant 0$.

b) Assume that $k_1 < k_3$, then $\overline{a} = \lambda\overline{b}$ with $-1 \leqslant \lambda \leqslant 0$ and $A(\overline{a}, \overline{b}) = \frac{1}{3}\overline{b}^2 h(\lambda)$ with

$$h(\lambda) = A(\lambda, 1) = (t - k_1)\lambda^2 + (t - k_2)\lambda + (t - k_3). \tag{36}$$

The function h is quadratic and $h'(0) = t - k_2 < 0$. The discussion is now made according to the sign of $h'(-1) = 2k_1 - k_2 - t$.

c) If $h'(-1) \leqslant 0$, then $\max_{-1\leqslant\mu\leqslant 0} h(\mu) = h(-1) = -k_1+k_2-k_3+t$. Noting that $-k_1+k_2-k_3+t < -2k_1+k_2+t \leqslant 0$, we get $(\overline{a},\overline{b}) = (-1,1)$, and $u_2(t,x) = 1-(-x+(1-x))^2 = 1-(1-2x)^2 = 4x(1-x)$. Since $t < k_2$, the condition $2k_1 - k_2 - t \leqslant 0$ makes sense only if $2k_1 - k_2 < k_2$ i.e., $k_1 < k_2$ and if $2k_1 - k_2 \leqslant t < k_2$.
d) If $h'(-1) > 0$, then h reaches its maximum at $h'(\lambda) = 0$, that is $\lambda = -\frac{t-k_2}{2(t-k_1)}$. Then $\max_{-1\leqslant\mu\leqslant 0} h(\mu) = h(\lambda) = (t-k_3)-\frac{(t-k_2)^2}{4(t-k_1)}$. If this number is negative then $\overline{b} = 0$ and $\max A(\overline{a},\overline{b}) = 0$. If it is nonnegative then $b = 1$. Summing up, we have $\max A(\overline{a},\overline{b}) = \frac{1}{3}\max((t-k_3) - \frac{(t-k_2)^2}{4(t-k_1)}, 0)$ and $(\overline{a},\overline{b}) = (\lambda(t),1)$. It yields $u_2(t,x) = 1-(\lambda(t)x+1-x)^2$ if $(t-k_3) - \frac{(t-k_2)^2}{4(t-k_1)} \geqslant 0$ and then $u_2(t,x) = 1$ if $(t-k_3) - \frac{(t-k_2)^2}{4(t-k_1)} < 0$. Let us define two thresholds $T^{\pm}_{\text{th.}}$ as the solutions of

$$(37) \qquad (T^{\pm}_{\text{th.}} - k_3) - \frac{(T^{\pm}_{\text{th.}} - k_2)^2}{4(T_{\text{th.}} - k_1)} = 0 \Longleftrightarrow 3(T^{\pm}_{\text{th.}})^2 + (2k_2 - 4k_1 - 4k_3)T^{\pm}_{\text{th.}} + 4k_1k_3 - k_2^2 = 0.$$

The product is $3T^{-}_{\text{th.}}T^{+}_{\text{th.}} = 4k_1k_3 - k_2^2$.
e)All other cases can be deduced from the above discussion.

Example of an exact solution. We take $(k_1,k_2,k_3) = (2.5, 3, 5/3)$. Then $T^{-}_{\text{th.}} = 1$ is solution of (37), and $T^{+}_{\text{th.}} = 50/9$. Since $T^{+}_{\text{th.}} > k_2 = 3$ does not satisfies the bound (34), it does not enter in the discussion. It yields $u_2(t,x) = 4x(1-x)$ for $2 < t < 3$, $u_2(t,x) = 1-(\lambda(t)x+1-x)^2$ for $1 < t < 2$ and $u_2(t,x) = 1$ for $t < 1$. One checks with (35) also that $u_2(t,x) = 0$ for $3 < t$. We have $q_2(x) = \int_0^{+\infty} u_2(t,x)\,dt = 1 + \left(7 - \frac{1}{2}\ln 3\right)x + \left(-\frac{19}{3} + \frac{3}{4}\ln 3\right)x^2$.

LEMMA 36. *For the above example, the last switching time is different from the maximum of q_n, i.e., $T_{\text{last switch}} > \max_{x\in I} q_2(x)$.*

Proof. The maximum of the convex polynomial q_2 is reached at $q_2'(x_*) = 0$, and we have $q_2(x_*) = 1 + \left(7 - \frac{1}{2}\ln 3\right)^2/4\left(\frac{19}{3} - \frac{3}{4}\ln 3\right) \approx 2.88$. This numerical value is different from the last switching time $T_{\text{last switch}} = 3$ where $u_n(t)$ switches from a non zero polynomial to 0. ∎

Another property also observed in numerical simulations is the dependence of the optimal solution on the entropy function (see Proposition 3). Let us finally prove Proposition 3.

Proof. (of Proposition 3.) Take $w = 1$ and a general entropy function $s(t)$ (not necessarily equal to $t^2/2$). Using that $\int_I(\lambda_n(x)-s'(t))u_n(t,x)\,dx \geqslant \int_I(\lambda_n(x)-s'(t))v_n(x)\,dx$ for every $v_n \in \mathcal{U}_n$, then (36) becomes $h(\lambda) = (s'(t)-k_1)\lambda^2 + (s'(t)-k_2)\lambda + (s'(t)-k_3)$ and $\lambda(t) = -\frac{s'(t)-k_2}{2(s'(t)-k_3)}$. Therefore the optimal solution $\overline{u}_2(t,x) = \left(\lambda(t)x + (1-x)\right)^2$ depends on the entropy. ∎

Another possibility to understand the influence of the entropy s on the solution is to consider the critical time T_*, defined in (20), above which the optimal solution $u_n(t)$ vanishes identically. Let us decide of an entropy $s(t) = t^m$ where $m \geqslant 2$. One obtains a critical time function of m

$$T_*(m) = \|q_n\|_{L^\infty(I)}\left(C_n\int_I w(x)dx\right)^{\frac{1}{m}}.$$

Let $m \to 0$, then $T_*(m) \to \|q_n\|_{L^\infty(I)} = \max_I q_n(x)$ which is asymptotically different of course from the threshold enlighted in Lemma 36 for $m = 2$.

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