Laplacian, on the graph of the Weierstrass-Hadamard function

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1 Introduction

The Laplacian plays a major role in the mathematical analysis of partial differential equations. Recently, the work of J. Kigami [1], [2], taken up by R. S. Strichartz [3], [4], allowed the construction of an operator of the same nature, defined locally, on graphs having a fractal character: the triangle of Sierpiński, the carpet of Sierpiński, the diamond fractal, the Julia sets, the fern of Barnsley.

J. Kigami starts from the definition of the Laplacian on the unit segment of the real line. For a double-differentiable function u on $[0,1]$, the Laplacian $\Delta\, u$ is obtained as a second derivative of u on $[0,1]$. For any pair (u,v) belonging to the space of functions that are differentiable on $[0,1]$, such that:

$$v(0)=v(1)=0$$

he puts the light on the fact that, taking into account:

$$\int_0^1 (\Delta u)\,(x)\,v(x)\,dx = -\int_0^1 u'(x)\,v'(x)\,dx = -\lim_{n\to+\infty}\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} u'(x)\,v'(x)\,dx$$

if $\varepsilon>0$, the continuity of u' and v' shows the existence of a natural rank n_0 such that, for any integer $n\geqslant n_0$, and any real number x of $\left[\dfrac{k-1}{n},\dfrac{k}{n}\right]$, $1\leqslant k\leqslant n$:

$$\left| u'(x) - \frac{u\left(\frac{k}{n}\right)-u\left(\frac{k-1}{n}\right)}{\frac{1}{n}} \right| \leqslant \varepsilon \quad , \quad \left| v'(x) - \frac{u\left(\frac{k}{n}\right)-v\left(\frac{k-1}{n}\right)}{\frac{1}{n}} \right| \leqslant \varepsilon$$

the relation:

$$\int_0^1 (\Delta u)\,(x)\,v(x)\,dx = -\lim_{n\to+\infty} n\sum_{k=1}^{n}\left(u\left(\frac{k}{n}\right)-u\left(\frac{k-1}{n}\right)\right)\left(v\left(\frac{k}{n}\right)-v\left(\frac{k-1}{n}\right)\right)$$

enables one to define, under a weak form, the Laplacian of u, while avoiding first derivatives. It thus opens the door to Laplacians on fractal domains.

Concretely, the weak formulation is obtained by means of Dirichlet forms, built by induction on a sequence of graphs that converges towards the considered domain. For a continuous function on this domain, its Laplacian is obtained as the renormalized limit of the sequence of graph Laplacians.

If the work of J. Kigami is, in means of analysis on fractals, seminal, it is to **Robert S. Strichartz** that one owes its rise. Robert S. Strichartz goes further than J. Kigami : on the ground of the Sierpiński gasket, he deepens, develops, exploits, generalizes, and reconstructs the classical functional spaces.

Strangely, the case of the **graph of the Weierstrass function**, introduced in 1872 by K. Weierstrass [7], which presents self similarity properties, does not seem to have been considered anywhere, **neither by Robert S. Strichartz, neither by others**. It is yet **an obligatory passage**, in the perspective of studying **diffusion phenomena in irregular structures.**
Let us recall that, being given $\lambda \in]0,1[$, and b such that $\lambda\, b > 1 + \dfrac{3\,\pi}{2}$, the Weierstrass function

$$x \in \mathbb{R} \mapsto \sum_{n=0}^{+\infty} \lambda^n \, \cos\left(\pi\, b^n\, x\right)$$

is continuous everywhere, while nowhere differentiable. The original proof, by K. Weierstrass [7], can also be found in [8]. It has been completed by the one, now a classical one, in the case where $\lambda\, b > 1$, by G. Hardy [9].

After the works of A. S. Besicovitch and H. D. Ursell [10], it is Benoît Mandelbrot [11] who particularly highlighted the fractal properties of the graph of the Weierstrass function. He also conjectured that the Hausdorff dimension of the graph is $D_{\mathcal{W}} = 2 + \dfrac{\ln \lambda}{\ln b}$. Interesting discussions in relation to this question have been given in the book of K. Falconer [12]. A proof was given by B. Hunt [13] in 1998 in the case where arbitrary phases are included in each cosinusoidal term of the summation. Recently, K. Barańsky, B. Bárány and J. Romanowska [14] proved that, for any value of the real number b, there exists a threshold value λ_b belonging to the interval $\left]\dfrac{1}{b}, 1\right[$ such that the aforementioned dimension is equal to $D_{\mathcal{W}}$ for every b in $]\lambda_b, 1[$. In [15], G. Keller proposes what appears as a much simpler proof.

In [5], [6], we have asked ourselves the following question: **given a continuous function on the graph of the Weierstrass function u, under which conditions is it possible to associate to u a function $\Delta\, u$ which is, in the weak sense, its Laplacian, so that this new function $\Delta\, u$ is also defined and continuous on the graph of the Weierstrass function ?**

Following our results, it was natural to consider, then, the case of the Weierstrass-Hadamard function $\mathcal{W}^{\mathcal{H}}$, i.e. the lacunary complex series, such that, for any complex number z, the modulus of which is less or equal to 1:

$$\mathcal{W}^{\mathcal{H}}(z) = \sum_{n=0}^{+\infty} \lambda^n \, z^{N_b^n} \quad , \quad \lambda \in]0,1[\quad , \quad b \in \mathbb{R} \,|\, \lambda\, b > 1 + \frac{3\,\pi}{2}$$

The novelty consists in working into an entirely complex space, $\overline{\mathbb{D}} \times \mathbb{C}$.

We present thus, in the following, the results obtained by following the approach of J. Kigami and R. S. Strichartz. Ours is made **in a completely renewed framework**, as regards, the one, affine, of the Sierpiński gasket. First, we concentrate on Dirichlet forms, on the graph of the Weierstrass function, which enable us the, subject to its existence, to define the Laplacian of a continuous function on this graph. This Laplacian appears as the renormalized limit of a sequence of discrete Laplacians on a sequence of **graphs** which converge to the one of the Weierstrass function. The normalization constants related to each graph Laplacian are obtained thanks Dirichlet forms.

In addition to the Dirichlet forms, we have come across several delicate points: the building of a self-similar measure related to the graph of the function, as well as the one of spline functions on the vertices of the graph.

The **spectrum of the Laplacian** thus built is obtained through spectral decimation. Beautifully, as regards to the method developed by Robert S. Strichartz in the case of the de Sierpiński gasket, our results come as the most natural illustration of the iterative process that gives birth to the discrete sequence of graphs.

2 Dirichlet forms, on the graph of the Weierstrass-Hadamard function

Notation. In the following, λ and b are two real numbers such that:

$$0 < \lambda < 1 \quad , \quad b = N_b \in \mathbb{N} \quad \text{and} \quad \lambda\, N_b > 1$$

We consider, in the following, the function what we choose to call the Weierstrass Weierstrass-Hadamard (or lacunary Hadamard series) function $\mathcal{W}^{\mathcal{H}}$, such that, for any complex number z, the modulus of which is less or equal to 1:

$$\mathcal{W}^{\mathcal{H}}(z) = \sum_{n=0}^{+\infty} \lambda^n\, z^{N_b^n} \quad , \quad \lambda \in]0,1[\quad , \quad b \in \mathbb{R} \,\big|\, \lambda\, b > 1 + \frac{3\,\pi}{2}$$

which can also be written as:

$$\mathcal{W}^{\mathcal{H}}\left(\rho\, e^{i\,\theta}\right) = \sum_{n=0}^{+\infty} \lambda^n\, \rho^{N_b^n}\, e^{i\,N_b^n\,\theta} = \sum_{n=0}^{+\infty} \lambda^n\, \rho^{N_b^n}\, \left\{\cos\left(N_b^n\,\theta\right) + i\,\sin\left(N_b^n\,\theta\right)\right\}$$

i.e., by identifying $\mathbb{R}^2$ and the complex plane $\mathbb{C}$:

$$\mathcal{W}^{\mathcal{H}}\left(\rho\, e^{i\,\theta}\right) = \left(\sum_{n=0}^{+\infty} \lambda^n\, \rho^{N_b^n}\, \cos\left(N_b^n\,\theta\right), \sum_{n=0}^{+\infty} \lambda^n\, \rho^{N_b^n}\, \sin\left(N_b^n\,\theta\right)\right)$$

2.1 Theoretical point of view

We place ourselves, in the following, in the euclidian plane of dimension 3, referred to a direct orthonormal frame. The usual Cartesian coordinates are (x, y, z).

Property 2.1. ***Periodic properties of the Weierstrass-Hadamard function***
For any real number θ:

$$\mathcal{W}^{\mathcal{H}}\left(\rho\, e^{i\,(\theta+2\,\pi)}\right) = \mathcal{W}^{\mathcal{H}}\left(\rho\, e^{i\,\theta}\right)$$

The study of the restriction of the Weierstrass-Hadamard function can be restricted to the complex closed unit disk, that we will denote by $\overline{\mathbb{D}}$. We will identify $\overline{\mathbb{D}}$ with $[0,1] \times [0, 2\,\pi]$.

By following the method developed by J. Kigami, and developed by Cl. David [5], [6], we approximate the restriction of $\Gamma_{\mathcal{W}^{\mathcal{H}}}$ to $\overline{\mathbb{D}}$, of the graph of the Weierstrass-Hadamard function, by a sequence of graphs, built through an iterative process. To this purpose, we introduce the iterated function system of the family of C^∞ functions from $\overline{\mathbb{D}}\times\mathbb{C}$ to $\overline{\mathbb{D}}\times\mathbb{C}$:

$$\{T_0,...,T_{N_b-1}\}$$

where, for any integer k belonging to $\{0,...,N_b-1\}$, and any (ρ,θ,z) of $[0,1]\times[0,2\,\pi]\times\mathbb{C}$:

$$T_k\,(\rho,\theta,z)=\left(\rho^{\frac{1}{N_b}},\frac{\theta+2\,k\,\pi}{N_b},\lambda\,z+e^{\,i\,\frac{\theta+2\,k\,\pi}{N_b}}\right)$$

Lemme 2.2. *For any integer k belonging to $\{0,...,N_b-1\}$, the map T_k is a bijection of $\Gamma_{\mathcal{W}^{\mathcal{H}}}$.*

Proof. Let $k\in\{0,...,N_b-1\}$.
Consider a point $\left(\rho'\,e^{i\,\theta'},\mathcal{W}(\rho'\,e^{i\,\theta'})\right)$ of $\Gamma_{\mathcal{W}^{\mathcal{H}}}$, and let us look for two real numbers $\rho\in[0,1]$ and θ in $[0,2\,\pi]$ such that:

$$T_k\left(\rho\,\theta,\mathcal{W}(\rho\,e^{i\,\theta})\right)=\left(\rho',\theta',\mathcal{W}(\rho'\,e^{i\,\theta'}\right)$$

One has then:

$$\rho'=\rho^{\frac{1}{N_b}}$$

and:

$$\theta'=\frac{\theta+2\,k\,\pi}{N_b}$$

It follows that:

$$\rho=\left(\rho'\right)^{N_b}\quad,\quad\theta=N_b\,\theta'-2\,k\,\pi$$

This enables one to obtain:

$$\begin{array}{rcl}\mathcal{W}^{\mathcal{H}}\left(\rho\,e^{i\,\theta}\right)&=&\mathcal{W}^{\mathcal{H}}\left((\rho')^{N_b}\;e^{i\,(N_b\,\theta'-2\,k\,\pi)}\right)\\&=&\displaystyle\sum_{n=0}^{+\infty}\lambda^n\,\left(\rho'\right)^{N_b^{n+1}}\;e^{i\,N_b^{n+1}\,\theta'}\end{array}$$

and:

$$\begin{array}{rcl}T_k\left(\rho,\theta,\mathcal{W}^{\mathcal{H}}\left(\rho\,e^{i\,\theta}\right)\right)&=&\left(\rho^{\frac{1}{N_b}}\,e^{\,i\,\frac{\theta+2\,k\,\pi}{N_b}},\lambda\,\mathcal{W}^{\mathcal{H}}\left(\rho\,e^{i\,\theta}\right)+e^{\,i\,\frac{\theta+2\,k\,\pi}{N_b}}\right)\\&=&\left(\rho'\,e^{i\,\theta'},\lambda\displaystyle\sum_{n=0}^{+\infty}\lambda^n\,\left(\rho'\right)^{N_b^{n+1}}\;e^{i\,N_b^{n+1}\,\theta'}+e^{i\,\theta'}\right)\\&=&\left(\rho'\,e^{i\,\theta'},\displaystyle\sum_{n=0}^{+\infty}\lambda^{n+1}\,\left(\rho'\right)^{N_b^{n+1}}\;e^{i\,N_b^{n+1}\,\theta'}+e^{i\,\theta'}\right)\\&=&\left(\rho'\,e^{i\,\theta'},\displaystyle\sum_{n=0}^{+\infty}\lambda^{n}\,\left(\rho'\right)^{N_b^{n}}\;e^{i\,N_b^{n}\,\theta'}\right)\\&=&\left(\rho',\theta',\mathcal{W}(\rho'\,e^{i\,\theta'}\right)\end{array}$$

There exists thus a unique pair of real numbers (ρ, θ) belonging to $[0,1] \times [0, 2\pi]$ such that:

$$T_k \left(\rho, \theta, \mathcal{W}(\rho\, e^{i\,\theta}) \right) = \left(\rho', \theta', \mathcal{W}(\rho'\, e^{i\,\theta'} \right)$$

□

Property 2.3.

$$\Gamma_{\mathcal{W}\mathcal{H}} = \bigcup_{k=0}^{N_b-1} T_k\left(\Gamma_{\mathcal{W}\mathcal{H}}\right)$$

Remark 2.1. The family $\{T_0, ..., T_{N_b-1}\}$ is a family of contractions from $\overline{\mathbb{D}} \times \mathbb{C}$ into $\overline{\mathbb{D}} \times \mathbb{C}$.

Proof. Let us equip $\overline{\mathbb{D}} \times \mathbb{C}$ of the distance $d_{\overline{\mathbb{D}}\times\mathbb{C}}$ such that, for any $((\rho, \theta, z), (\rho', \theta', z'))$ of $(]0,1] \times]0, 2\pi] \times \mathbb{C}^\star)^2$:

$$\begin{cases} d_{\overline{\mathbb{D}}\times\mathbb{C}}\left((\rho,\theta,z),(\rho',\theta',z')\right) & = \left|\ln \dfrac{\rho\,\theta\,|z|}{\rho'\,\theta'\,|z'|}\right| \\ d_{\overline{\mathbb{D}}\times\mathbb{C}}\left((\rho,\theta,z),(0,\theta,z)\right) & = \left|\ln \dfrac{\rho\,\theta\,|z|}{\theta'\,|z'|}\right| \\ d_{\overline{\mathbb{D}}\times\mathbb{C}}\left((\rho,\theta,z),(\rho',0,z')\right) & = \left|\ln \dfrac{\rho\,\theta\,|z|}{\rho'\,|z'|}\right| \\ d_{\overline{\mathbb{D}}\times\mathbb{C}}\left((\rho,\theta,z),(0,\theta',z')\right) & = \left|\ln \dfrac{\rho\,\theta\,|z|}{\theta'\,|z'|}\right| \\ d_{\overline{\mathbb{D}}\times\mathbb{C}}\left((\rho,\theta,z),(0,0,0)\right) & = \left|\ln \rho\,\theta\,|z|\right| \end{cases}$$

One can easily check the triangular inequality ; for each $((\rho, \theta, z), (\rho', \theta', z'), (\rho, \theta, z))$ belonging to $(]0,1] \times]0, 2\pi] \times \mathbb{C}^\star)^3$, one has:

$$\begin{aligned} d_{\overline{\mathbb{D}}\times\mathbb{C}}\left((\rho,\theta,z),(\rho',\theta',z')\right) &= \left|\ln \frac{\rho\,\theta\,|z|}{\rho'\,\theta'\,|z'|}\right| \\ &= \left|\ln \frac{\rho\,\theta\,|z|}{\rho''\,\theta''\,|z''|}\,\frac{\rho''\,\theta''\,|z''|}{\rho'\,\theta'\,|z'|}\right| \\ &\leqslant \left|\ln \frac{\rho\,\theta\,|z|}{\rho''\,\theta''\,|z''|}\right| + \left|\ln \frac{\rho''\,\theta''\,|z''|}{\rho'\,\theta'\,|z'|}\right| \\ &= d_{\overline{\mathbb{D}}\times\mathbb{C}}\left((\rho,\theta,z),(\rho'',\theta'',z'')\right) + d_{\overline{\mathbb{D}}\times\mathbb{C}}\left((\rho',\theta',z'),(\rho'',\theta'',z'')\right) \end{aligned}$$

One has then, for any $((\rho,\theta,z),(\rho',\theta',z'))$ belonging to $(]0,1]\times]0,2\pi]\times\mathbb{C})^2$:

$$d_{\overline{\mathbb{D}}\times\mathbb{C}}\left(T_k(\rho,\theta,z),T_k(\rho',\theta',z')\right) = \left|\ln\frac{\rho^{\frac{1}{N_b}}\,(\theta+2\,k\,\pi)\,|\lambda\,z+e^{i\,\frac{\theta+2\,k\,\pi}{N_b}}|}{\rho'^{\frac{1}{N_b}}\,(\theta'+2\,k\,\pi)\,|\lambda\,z'+e^{i\,\frac{\theta'+2\,k\,\pi}{N_b}}|}\right|$$

Since (ρ,θ,z), et (ρ',θ',z') play symmetric parts, it is natural to consider the case when:

$$\rho'\leqslant\rho \quad,\quad \theta'\leqslant\theta \quad,\quad |z'|\leqslant|z|$$

One has then:

$$\frac{\theta+2\,k\,\pi}{\theta'+2\,k\,\pi}\,\frac{|\lambda\,z+e^{i\,\frac{\theta+2\,k\,\pi}{N_b}}|}{|\lambda\,z'+e^{i\,\frac{\theta'+2\,k\,\pi}{N_b}}|}\leqslant\frac{\theta}{\theta'}\,\frac{|\lambda|\,|z|+\left|e^{i\,\frac{\theta+2\,k\,\pi}{N_b}}\right|}{|\lambda|\,|z'|+\left|e^{i\,\frac{\theta'+2\,k\,\pi}{N_b}}\right|}=\frac{\theta}{\theta'}\,\frac{|\lambda|\,|z|+1}{|\lambda|\,|z'|+1}\leqslant\frac{\theta}{\theta'}\,\frac{|\lambda|\,|z|}{|\lambda|\,|z'|}=\frac{\theta}{\theta'}\,\frac{|z|}{|z'|}$$

Since the logarithm is non increasing, it yields:

$$\begin{aligned}
d_{\overline{\mathbb{D}}\times\mathbb{C}}\left(T_k(\rho,\theta,z),T_k(\rho',\theta',z')\right) &\leqslant \left|\ln\frac{\rho^{\frac{1}{N_b}}\,\theta\,|z|}{\rho'^{\frac{1}{N_b}}\,\theta'|z'|}\right| \\
&= \frac{1}{N_b}\,\ln\frac{\rho}{\rho'}+\ln\frac{\theta\,|z|}{\theta'|z'|} \\
&\leqslant K\,\ln\frac{\rho\,\theta\,|z|}{\rho'\,\theta'\,|z'|} \quad,\quad 0<K<1 \\
&= K\,\frac{1}{N_b}\,d_{\overline{\mathbb{D}}\times\mathbb{C}}\left((\rho,\theta,z),(\rho',\theta',z')\right)
\end{aligned}$$

Enfin, comme, pour tout $((0,\theta,z),(\rho',\theta',z'))$ de $(\{0\}\times]0,2\pi]\times\mathbb{C})\times(]0,1]\times]0,2\pi]\times\mathbb{C})$:

$$d_{\overline{\mathbb{D}}\times\mathbb{C}}\left(T_k(0,\theta,z),T_k(\rho',\theta',z')\right) = \left|\ln\frac{(\theta+2\,k\,\pi)\,|\lambda\,z+e^{i\,\frac{\theta+2\,k\,\pi}{N_b}}|}{\rho'^{\frac{1}{N_b}}\,(\theta'+2\,k\,\pi)\,|\lambda\,z'+e^{i\,\frac{\theta'+2\,k\,\pi}{N_b}}|}\right|$$

In the same way, one shows that:

$$d_{\overline{\mathbb{D}}\times\mathbb{C}}\left(T_k(0,\theta,z),T_k(\rho',\theta',z')\right) \leqslant K\,d_{\overline{\mathbb{D}}\times\mathbb{C}}\left((0,\theta,z),(\rho',\theta',z')\right)$$

$\square$

Definition 2.1. For any integer k belonging to $\{0,...,N_b-1\}$, let us denote by:

$$P_k=\left(\rho_k\,e^{i\,\theta_k},z_k\right)=\left(e^{i\,\frac{2\,k\,\pi}{N_b-1}},\frac{1}{1-\lambda}\,e^{i\,\frac{2\,k\,\pi}{N_b-1}}\right)$$

and:

$$Q_k=(0,z_k)=\left(0,\frac{1}{1-\lambda}\,e^{i\,\frac{2\,k\,\pi}{N_b-1}}\right)$$

the two fixed points of the contraction map T_k.

One may note that the sequence of points $(P_k)_{0\leqslant k\leqslant N_b-1}$ belong to the plane $\rho = 1$, while the one $(Q_k)_{0\leqslant k\leqslant N_b-1}$ belong to the plane $\rho = 0$.

Property 2.4. *For any integer k belonging to $\{0, ..., N_b - 1\}$, the plane $\rho = 0$ is invariant under the contraction T_k. In the same way, the plane $\rho = 1$ is invariant under the contraction T_k.*

Definition 2.2. Projection of the graph $\Gamma_{\mathcal{W}^{\mathcal{H}}}$ on a plane, the equation of which is of the form $\rho = \rho_0$, $\rho_0 \in [0,1]$

Given a real number ρ_0 of $[0,1]$, we will call **projection, on the plane** $\rho = \rho_0$, of the graph $\Gamma_{\mathcal{W}^{\mathcal{H}}}$, that we will denote by:

$$\Gamma_{\mathcal{W}^{\mathcal{H}}_{|\rho=\rho_0}}$$

the graph of the restriction of the function $\mathcal{W}^{\mathcal{H}}$ to the plane $\rho = \rho_0$; for any $\theta \in [0, 2\pi]$:

$$\mathcal{W}^{\mathcal{H}}_{|\rho=\rho_0}\left(\rho\, e^{i\,\theta}\right) = \mathcal{W}^{\mathcal{H}}\left(\rho_0\, e^{i\,\theta}\right) = \sum_{n=0}^{+\infty} \lambda^n\, \rho_0^{N_b^n}\, e^{i\,N_b^n\,\theta}$$

Definition 2.3. Frontier set of vertices of order m, $m \in \mathbb{N}$

Let us denote by $V_{0,\rho=0}$ the ordered set (according to increasing complex arguments), of the points:

$$\{Q_0, ..., Q_{N_b-1}\}$$

and by $V_{0,\rho=1}$ the ordered set (according to increasing complex arguments), of the points:

$$\{P_0, ..., P_{N_b-1}\}$$

since, for any integer k belonging to $\{0, ..., N_b - 2\}$:

$$\theta_k \leqslant \theta_{k+1}$$

We set:

$$V_0 = V_{0,\rho=0} \cup V_{0,\rho=1}$$

The set of points $V_{0,\rho=0}$, where, for any integer k belonging to $\{0, ..., N_b - 2\}$, the point Q_k is linked to the point Q_{k+1}, is a connected graph (according to increasing complex arguments), that we will denote by $\Gamma_{\mathcal{W}^{\mathcal{H}}_0{}_{|\rho=0}}$.

In the same way, the set of points $V_{0,\rho=1}$, where, for any integer k belonging to $\{0, ..., N_b - 2\}$, the point P_k is connected to the point P_{k+1}, is a connected graph (according to increasing complex arguments), that we will denote by $\Gamma_{\mathcal{W}^{\mathcal{H}}_0{}_{|\rho=1}}$.

The set of points $V_0 = V_{0,\rho=0} \cup V_{0,\rho=0}$, where, for any integer k belonging to $\{0, ..., N_b - 2\}$, the point Q_k is connected to the point Q_{k+1}, the point P_k is connected to the point P_{k+1}, and where the point P_k is connected to the point Q_k, is a connected graph (according to increasing complex arguments), that we will denote by $\Gamma_{\mathcal{W}_0^{\mathcal{H}}}$. V_0 will be called the set of vertices of the graph $\Gamma_{\mathcal{W}_0^{\mathcal{H}}}$.

For any natural integer m, we set:

$$V_{m,\rho=0} = \bigcup_{k=0}^{N_b-1} T_k\left(V_{m-1,\rho=0}\right)$$

$$V_{m,\rho=1} = \bigcup_{k=0}^{N_b-1} T_k\left(V_{m-1,\rho=1}\right)$$

$$V_{m-1} = V_{m-1,\rho=0} \cup V_{m-1,\rho=1}$$

$$V_m = \bigcup_{k=0}^{N_b-1} T_k\left(V_{m-1}\right)$$

The set of points V_m, where three consecutive points are connected, is a connected graph (according to increasing arguments), that we will denote by $\Gamma_{\mathcal{W}_m}$. V_m will be called **frontier set of vertices of order** m, of the graph $\Gamma_{\mathcal{W}_m^{\mathcal{H}}}$.
In the following, we will denote by:

$$\mathcal{N}^{\mathcal{S}}_{m,\rho=i} \quad , \quad i=0,1$$

the number of frontier vertices of the oriented graph $\Gamma_{\mathcal{W}^{\mathcal{H}}_{m\,|\rho=i}}$ obtained by projection of $\Gamma_{\mathcal{W}_m^{\mathcal{H}}}$ on the plane $\rho=i$, $i=0,1$. One may note that:

$$\mathcal{N}^{\mathcal{S}}_{m,\rho=0} = \mathcal{N}^{\mathcal{S}}_{m,\rho=1}$$

For the sake of simplicity, we will set:

$$\mathcal{N}^{\mathcal{S}}_{m,\rho=0} = \mathcal{N}^{\mathcal{S}}_{m,\rho=1} = \mathcal{N}^{\mathcal{S}}_{m}$$

Thus, the number of frontier vertices of the graph $\Gamma_{\mathcal{W}_m^{\mathcal{H}}}$ is:

$$2\,\mathcal{N}^{\mathcal{S}}_{m} = \mathcal{N}^{\mathcal{S}}_{m,\rho=0} + \mathcal{N}^{\mathcal{S}}_{m,\rho=1}$$

We will write:

$$V_m = \left\{\mathcal{S}_0^{m,\rho=0}, \mathcal{S}_1^{m,\rho=0}, \ldots, \mathcal{S}^{m,\rho=0}_{\mathcal{N}^{\mathcal{S}}_m-1}, \mathcal{S}_0^{m,\rho=1}, \mathcal{S}_1^{m,\rho=1}, \ldots, \mathcal{S}^{m,\rho=1}_{\mathcal{N}^{\mathcal{S}}_m-1}\right\}$$

where, for any integer k belonging to $\left\{0, ..., \mathcal{N}^{\mathcal{S}}_m-1\right\}$:

$$\mathcal{S}_k^{m,\rho=i} \in \Gamma_{\mathcal{W}^{\mathcal{H}}_{m\,|\rho=i}} \quad , \quad i=0,1$$

Property 2.5. *For any natural integer m, $V_{m,\rho=0}$ is the projection of $V_{m,\rho=1}$ on the plane $\rho=0$.*

Property 2.6. *For any natural integer m:*

$$V_m \subset V_{m+1}$$

Property 2.7. *For any k of $\{0, ..., N_b - 2\}$:*

$$T_k\left(P_{N_b-1}\right) = T_{k+1}\left(P_0\right) \quad , \quad T_k\left(Q_{N_b-1}\right) = T_{k+1}\left(Q_0\right)$$

Proof. It is obvious, since:

$$\rho_0 = \rho_{N_b-1} = 1 \quad \text{or} \quad \rho_0 = \rho_{N_b-1} = 0 \quad , \quad \theta_0 = 0 \quad , \quad \theta_{N_b-1} = 2\,\pi$$

and:

$$z_0 = z_{N_b-1} = \frac{1}{1-\lambda}$$

One gets then:

$$T_k\left(P_{N_b-1}\right) = T_{k+1}\left(P_0\right) = \left(e^{i\,\frac{2\,k\,\pi}{N_b}}, \frac{\lambda}{1-\lambda} + e^{i\,\frac{2\,k\,\pi}{N_b}}\right)$$

$$T_k\left(Q_{N_b-1}\right) = T_{k+1}\left(Q_0\right) = \left(0, \frac{\lambda}{1-\lambda} + e^{i\,\frac{2\,k\,\pi}{N_b}}\right)$$

$\square$

Definition 2.4. Mesh of order m, $m \in \mathbb{N}$, on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$

Given a natural integer m, we will call **mesh of order m, on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$**, the sequence of graphs

$$\left(\Gamma_{\mathcal{W}_m^{\mathcal{H}}|\rho=\frac{i}{N_b^m}}\right)_{0\leqslant i\leqslant N_b^m}$$

obtained as projections of the graphs $\Gamma_{\mathcal{W}_m^{\mathcal{H}}}$ on the planes, the equation of which is:

$$\rho = \frac{i}{N_b^m} \quad , \quad 0 \leqslant i \leqslant N_b^m$$

We will denote by:

$$\mathcal{V}_m = \left(M_{m,j,|\rho=\frac{i}{N_b^m}}\right)_{0\leqslant i\leqslant N_b^m, 0\leqslant j\leqslant \mathcal{N}_m^{\mathcal{S}}}$$

the family of points of the mesh $\left(\Gamma_{\mathcal{W}_m^{\mathcal{H}}|\rho=\frac{i}{N_b^m}}\right)_{0\leqslant i\leqslant N_b^m}$, that we will call **vertices of the graph $\Gamma_{\mathcal{W}_m^{\mathcal{H}}}$**.

Property 2.8. *For any natural integer m, the number of vertices of the graph $\Gamma_{\mathcal{W}_m^{\mathcal{H}}}$ is:*

$$N_b^m\,\mathcal{N}_m^{\mathcal{S}}$$

Property 2.9. *The sequence $\left(\mathcal{N}_m^{\mathcal{S}}\right)_{m\in\mathbb{N}}$ is an arithmetico-geometric one, with $\mathcal{N}_0^{\mathcal{S}} = N_b$ as first term:*

$$\forall\, m \in \mathbb{N} : \quad \mathcal{N}_{m+1}^{\mathcal{S}} = N_b\,\mathcal{N}_m^{\mathcal{S}} - (N_b - 2)$$

which leads to:

$$\forall\, m \in \mathbb{N} : \quad \mathcal{N}^{\mathcal{S}}_{m+1} = N_b^m \left(\mathcal{N}_0 - (N_b - 2) \right) + (N_b - 2) = 2\, N_b^m + N_b - 2$$

Proof. This results comes from the fact that each graph $\Gamma^{\mathcal{H}}_{\mathcal{W}_m}$, $m \in \mathbb{N}^\star$, is built from its predecessor $\Gamma^{\mathcal{H}}_{\mathcal{W}_{m-1}}$ by applying the N_b contractions T_k, $0 \leqslant k \leqslant N_b - 1$, to the vertices of $\Gamma_{\mathcal{W}_{m-1}}$. Since, for any i of $\{0, ..., N_b - 2\}$:

$$T_k \left(P_{N_b - 1} \right) = T_{k+1} \left(P_0 \right)$$

the, $N_b - 2$ points appear twice if one takes into account the images of the $\mathcal{N}_{m-1}$ vertices of $\Gamma_{\mathcal{W}_{m-1}}$ by the whole set of contractions T_k, $0 \leqslant k \leqslant N_b - 1$.

□

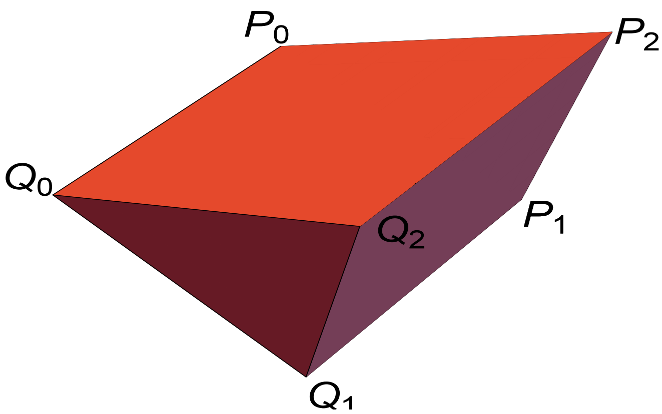

Figure 1: Cylindrical view, in the space $\overline{\mathbb{D}} \times \mathbb{R}$, of the fixed points P_0, P_1, P_2, Q_0, Q_1, Q_2, in the case where $\lambda = \dfrac{1}{2}$, and $N_b = 3$.

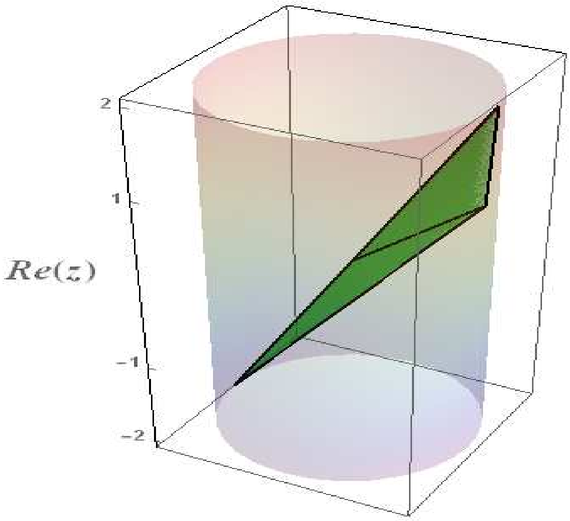

Figure 2: View, in the space (ρ, θ, z), of the initial polyhedron $P_0P_1P_2Q_0 = P_0P_1P_2Q_1 = P_0P_1P_2Q_2$, in the case where $\lambda = \dfrac{1}{2}$, and $N_b = 3$.

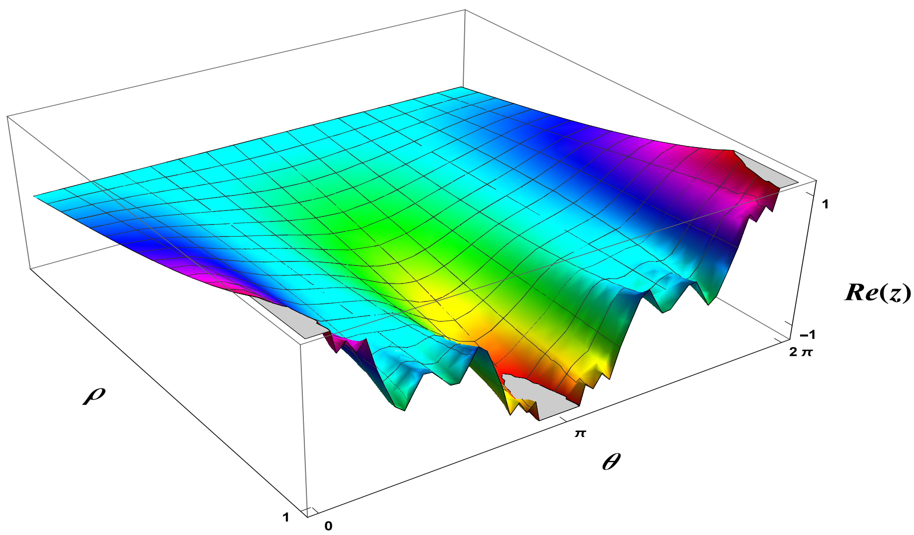

Figure 3: View, in the space (ρ, θ, z), of the graph of the real part of $\Gamma_{\mathcal{W}^{\mathcal{H}}}$, in the case where $\lambda = \dfrac{1}{2}$, and $N_b = 3$.

Definition 2.5. Word, on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$

Let m be a strictly positive integer. We will call **number-letter** any integer $\mathcal{M}_i$ of $\{0, \ldots, N_b - 1\}$, and **word of length** $|\mathcal{M}| = m$, on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$, any set of number-letters of the form:

$$\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_m)$$

We will write:

$$T_{\mathcal{M}} = T_{\mathcal{M}_1} \circ \ldots \circ T_{\mathcal{M}_m}$$

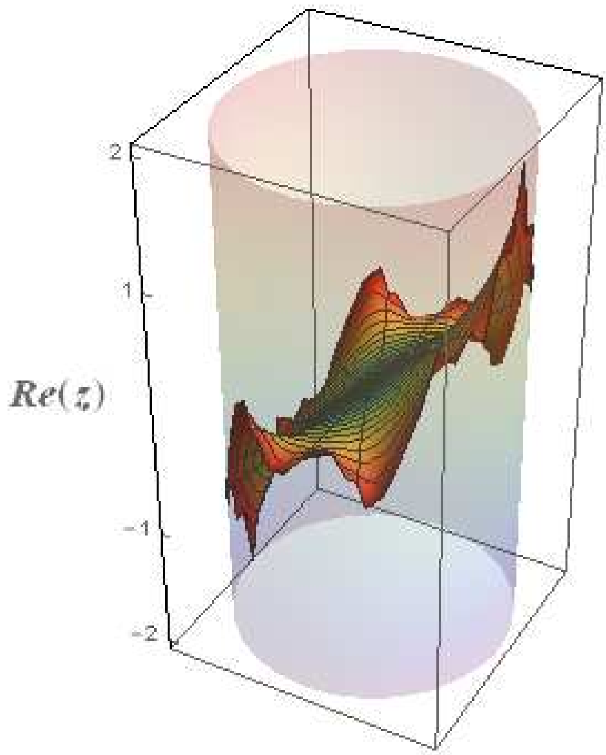

Figure 4: Cylindrical view, in the space $\overline{\mathbb{D}} \times \mathbb{R}$, of the graph of the real part of $\Gamma_{\mathcal{W}^{\mathcal{H}}}$, in the case where $\lambda = \frac{1}{2}$, and $N_b = 3$.

Property 2.10. *For any natural integer m :*

$$\Gamma_{\mathcal{W}}^{\mathcal{H}} = \overline{\bigcup_{|\mathcal{M}|=k\geqslant m} T_{\mathcal{M}}\left(\Gamma_{\mathcal{W}}^{\mathcal{H}}\right)}$$

Definition 2.6. Projection of a word, on a plane $\rho = \rho_0$, $\rho_0 \in [0,1]$

Given a real number ρ_0 belonging to the interval $[0,1]$, and a strictly positive integer m, we will call **projection, on the plane** $\rho = \rho_0$, of the word of length $|\mathcal{M}| = m$, on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$, the restriction, to the plane, the equation of which is $\rho = \rho_0$,of :

$$\mathcal{M}_{|\rho=\rho_0} = (\mathcal{M}_1, \ldots, \mathcal{M}_m)_{|\rho=\rho_0}$$

Definition 2.7. Consecutive vertices on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$

Two points X and Y of $\Gamma_{\mathcal{W}}$ will be called ***consecutive vertices*** of the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$ if there exists a natural integer m, a natural integer j of $\{0, ..., N_b - 2\}$, and a natural integer i of $\{0, ..., N_b^m\}$, such that:

$$X = (T_{i_1} \circ \ldots \circ T_{i_m})_{|\rho=\frac{i}{N_b^m}} (P_j) \quad \text{et} \quad Y = (T_{i_1} \circ \ldots \circ T_{i_m})_{|\rho=\frac{i}{N_b^m}} (P_{j+1}) \qquad \{i_1, \ldots, i_m\} \in \{0, ..., N_b - 1\}^m$$

or:

$$X = (T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_m}) \, (P_{N_b-1})_{|\rho=\frac{i}{N_b^m}} \quad \text{et} \quad Y = (T_{i_1+1} \circ T_{i_2} \ldots \circ T_{i_m})_{|\rho=\frac{i}{N_b^m}} \, (P_0)$$

Remark 2.2. It is important to note that X and Y cannot be in the same time the images of P_j and P_{j+1}, $0 \leqslant j \leqslant N_{b-2}$, by $T_{i_1} \circ \ldots \circ T_{i_m}$, $(i_1, \ldots, i_m) \in \{0, ..., N_b - 2\}$, and of P_k and P_{k+1}, $0 \leqslant k \leqslant N_{b-2}$, by $T_{p_1} \circ \ldots \circ T_{p_m}$, $(p_1, \ldots, p_m) \in \{0, ..., N_b - 2\}$. This result can be proved by induction, since, for any pair of integers (j, k) of $\{0, ..., N_b - 2\}^2$, for any i_m of $\{0, ..., N_b - 2\}$, and any p_m of $\{0, ..., N_b - 2\}$:

$$(i_m \neq p_m \quad \text{and} \quad j \neq k) \Longrightarrow (T_{i_m}(P_j) \neq T_{j_m}(P_k) \quad \text{and} \quad T_{i_m}(P_j) \neq T_{j_m}(P_k))$$

Each contraction T_k, $0 \leqslant k \leqslant N_b - 1$ is indeed injective.
Since the vertices of the initial graph $\Gamma_{\mathcal{W}_0}$ are distincts, one gets the expected result.

Definition 2.8. Opposed and connected vertices, on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$

Two points X and Y of $\Gamma_{\mathcal{W}^{\mathcal{H}}}$, with the same angular coordinates $\theta_X = \theta_Y \in [0, 2\pi]$, will be called ***opposed and connected vertices*** on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$ if there exists a natural integer m a natural integer i, $0 \leqslant i \leqslant N_b^m - 2$, such that:

$$X \in \Gamma_{\mathcal{W}_m^{\mathcal{H}}}{}_{|\rho=\frac{i}{N_b^m}} \quad \text{and} \quad Y \in \Gamma_{\mathcal{W}_m^{\mathcal{H}}}{}_{|\rho=\frac{i+1}{N_b^m}}$$

or:

$$X \in \Gamma_{\mathcal{W}_m^{\mathcal{H}}}{}_{|\rho=\frac{i+1}{N_b^m}} \quad \text{and} \quad Y \in \Gamma_{\mathcal{W}_m^{\mathcal{H}}}{}_{|\rho=\frac{i}{N_b^m}}$$

Definition 2.9. Edge relation, on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$

Given a natural integer m, two points X and Y of $\Gamma_{\mathcal{W}_m}^{\mathcal{H}}$ will be called ***adjacents*** if and only if X and Y are connected vertices, or connected and opposed vertices of $\Gamma_{\mathcal{W}_m}$. We will write then:

$$X \underset{m}{\sim} Y$$

Given two points X and Y of the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$, we will say that X et Y are ***adjacents*** if and only if there exists a natural integer m such that:

$$X \underset{m}{\sim} Y$$

Definition 2.10. For any natural integer m, the points $\left(M_{m,j,|\rho=\frac{i}{N_b^m}}\right)_{0\leqslant i\leqslant N_b^m, 0\leqslant j\leqslant \mathcal{N}_m^{\mathcal{S}}}$ also appear to be the vertices of N_b^{2m} polyhedra $\mathcal{P}_{m,i,j}$, $(i,j) \in \{0,\ldots,N_b^m-1\}^2$, each polyhedron having N_b+2 faces and $2\,N_b$ vertices. For any natural integer m, and any pair of integers (i,j) of $\{0,\ldots,N_b^m-1\}^2$, each polyhedron is obtained by connecting the point number j of the plane $\rho = \dfrac{i}{N_b^m}$, i.e. the point $M_{m,j,|\rho=\frac{i}{N_b^m}}$ to the point $j+1$ of the same plane, i.e. the point $M_{m,j+1,|\rho=\frac{i}{N_b^m}}$ if $j = i \bmod N_b$, $0 \leqslant i \leqslant N_b-2$, the point number j of the plane $\rho = \dfrac{i}{N_b^m}$ to the point number $j-N_b+1$ of the same plane if $j = -1 \bmod N_b$, the point number j of the plane $\rho = \dfrac{i}{N_b^m}$, i.e. the point $M_{m,j,|\rho=\frac{i+1}{N_b^m}}$ to the point to the point number j of the plane $\rho = \dfrac{i+1}{N_b^m}$, i.e. the point $M_{m,j,|\rho=\frac{i+1}{N_b^m}}$, the point number j of the plane $\rho = \dfrac{i}{N_b^m}$ to the point number $j-N_b+1$ of the same plane if $j = -1 \bmod N_b$. These polyhedra generate a Borel set of $\overline{\mathbb{D}} \times \mathbb{C}$.

Definition 2.11. Polyhedral domain delimited by the graph $\Gamma_{\mathcal{W}_m^{\mathcal{H}}}$, $m \in \mathbb{N}$

For any natural integer m, we will call **polyhedral domain delimited by the graph** $\Gamma_{\mathcal{W}_m^{\mathcal{H}}}$, that we will denote by $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}^{\mathcal{H}}\right)$, the reunion of the N_b^{2m} polyhedra $\mathcal{P}_{m,i,j}$, $(i,j) \in \{0,\ldots,N_b^m-1\}^2$ with N_b+2 faces.

Proposition 2.11. ***Adresses, on the graph of the Weierstrass-Hadamard function***

Given a strictly positive integer m, and a word $\mathcal{M} = (\mathcal{M}_1,\ldots,\mathcal{M}_m)$ of length $m \in \mathbb{N}^\star$, on the graph $\Gamma_{\mathcal{W}_m}^{\mathcal{H}}$, for any integer j of $\{1,...,N_b-1\}$, each point

$$X = T_{\mathcal{M}_{|\rho=\frac{i}{N_b^m}}}(P_j) \quad , \quad 1 \leqslant i \leqslant N_b^m - 2$$

has exactly four adjacent vertices, given by:

$$T_{\mathcal{M}_{|\rho=\frac{i}{N_b^m}}}(P_{j+1}) \quad , \quad T_{\mathcal{M}_{|\rho=\frac{i}{N_b^m}}}(P_{j-1}) \quad , \quad T_{\mathcal{M}_{|\rho=\frac{i+1}{N_b^m}}}(P_j) \quad et \quad T_{\mathcal{M}_{|\rho=\frac{i-1}{N_b^m}}}(P_j)$$

where:

$$T_{\mathcal{M}} = T_{\mathcal{M}_1} \circ \ldots \circ T_{\mathcal{M}_m}$$

Each point

$$X = T_{\mathcal{M}_{|\rho=0}}(P_j) \quad , \quad 1 \leqslant j \leqslant N_b - 2$$

has exactly three adjacent vertices, given by:

$$T_{\mathcal{M}_{|\rho=0}}(P_{j+1}) \quad , \quad T_{\mathcal{M}_{|\rho=0}}(P_{j-1}) \quad , \quad T_{\mathcal{M}_{|\rho=\frac{1}{N_b^m}}}(P_j)$$

Each point

$$X = T_{\mathcal{M}_{|\rho=1}}(P_j) \quad , \quad 1 \leqslant j \leqslant N_b - 2$$

has exactly three adjacent vertices, given by:

$$T_{\mathcal{M}_{|\rho=1}}(P_{j+1}) \quad , \quad T_{\mathcal{M}_{|\rho=1}}(P_{j-1}) \quad , \quad T_{\mathcal{M}_{|\rho=\frac{N_b^m-1}{N_b^m}}}(P_j)$$

By convention, the adjacent vertices of $T_{\mathcal{M}}(P_0)$ are $T_{\mathcal{M}}(P_1)$ and $T_{\mathcal{M}}(P_{N_b-1})$, and those of $T_{\mathcal{M}}(P_{N_b-1})$, $T_{\mathcal{M}}(P_{N_b-2})$ and $T_{\mathcal{M}}(P_0)$.
In the same way, the adjacent vertices of $T_{\mathcal{M}}(Q_0)$ are $T_{\mathcal{M}}(Q_1)$ and $T_{\mathcal{M}}(Q_{N_b-1})$, and those of $T_{\mathcal{M}}(Q_{N_b-1})$, $T_{\mathcal{M}}(Q_{N_b-2})$ and $T_{\mathcal{M}}(Q_0)$.

Property 2.12. *The set of vertices $(\mathcal{V}_m)_{m\in\mathbb{N}}$ is dense in $\Gamma_{\mathcal{W}}^{\mathcal{H}}$.*

Definition 2.12. Measure, on the domain delimited by the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$

We will call **domain delimited by the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$**, that we will denote by $\mathcal{D}(\Gamma_{\mathcal{W}})$, the limit:

$$\mathcal{D}\left(\Gamma_{\mathcal{W}}^{\mathcal{H}}\right) = \lim_{n\to+\infty} \mathcal{D}\left(\Gamma_{\mathcal{W}_m}^{\mathcal{H}}\right)$$

which is to be understood in the following sense: given a continuous function u on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$, and a full support measure μ on $\overline{\mathbb{D}} \times \mathbb{C}$:

$$\int_{\mathcal{D}(\Gamma_{\mathcal{W}})} u \, d\mu = \lim_{m\to+\infty} \sum_{j=0}^{N_b^m-1} \sum_{X \text{ vertex of } \mathcal{P}_{m,j}} u(X)\, \mu(\mathcal{P}_{m,j})$$

We will say that μ is a **measure, on the domain delimited by the graph** $\Gamma_{\mathcal{W}}^{\mathcal{H}}$.

Definition 2.13. Dirichlet form (we refer to the paper [19], or the book [20])

Given a measured space (E, μ), a ***Dirichlet form*** on E is a bilinear symmetric form, that we will denote by $\mathcal{E}$, defined on a vectorial subspace D dense in $L^2(E, \mu)$, such that:

1. For any real-valued function u defined on D : $\mathcal{E}(u, u) \geqslant 0$.

2. D, equipped with the inner product which, to any pair (u,v) of $D \times D$, associates:

$$(u,v)_{\mathcal{E}} = (u,v)_{L^2(E,\mu)} + \mathcal{E}(u,v)$$

is a Hilbert space.

3. For any real-valued function u defined on D, if:

$$u_\star = \min\left(\max(u,0),1\right) \in D$$

then : $\mathcal{E}(u_\star,u_\star) \leqslant \mathcal{E}(u,u)$ (Markov property, or lack of memory property).

Definition 2.14. Dirichlet form, on a finite set ([21])

Let V denote a finite set V, equipped with the usual inner product which, to any pair (u,v) of functions defined on V, associates:

$$(u,v) = \sum_{p \in V} u(p)\, v(p)$$

A ***Dirichlet form***on V is a symmetric bilinear form $\mathcal{E}$, such that:

1. For any real valued function u defined on V: $\mathcal{E}(u,u) \geqslant 0$.
2. $\mathcal{E}(u,u) = 0$ if and only if u is constant on V.
3. For any real-valued function u defined on V, if:

$$u_\star = \min\left(\max(u,0),1\right)$$

i.e. :

$$\forall\, p \in V \; : \quad u_\star(p) = \begin{cases} 1 & \text{if} \quad u(p) \geqslant 1 \\ u(p) & \text{si} \quad 0 < u(p) < 1 \\ 0 & \text{if} \quad u(p) \leqslant 0 \end{cases}$$

then: $\mathcal{E}(u_\star,u_\star) \leqslant \mathcal{E}(u,u)$ (Markov property).

Remark 2.3. In order to understand the underlying theory of Dirichlet forms, one can only refer to the work of A. Beurling and J. Deny [19]. The Dirichlet space $\mathcal{D}$ of fonctions u, complex valued functions, infinitely differentiable, the support of which belongs to a domain $\omega \subset \mathbb{R}^p$, $p \in \mathbb{N}^\star$, is equipped with the hilbertian norm:

$$u \mapsto \|u\|_{\mathcal{D}} = \int_\omega |\mathrm{grad}\, u(x)|^2\, dx$$

If the complement set of ω is not "too small", the space $\mathcal{D}$ can be completed by adding functions defined almost everywhere in ω. The space thus obtained $\mathcal{D}_\omega$, equipped with the Lebesgue measure ξ, satisfies the following properties:

i. For any compact $K \subset \omega$, there exists a positive constant C_K such that, for any u of $\mathcal{D}_\omega$:

$$\int_K |u(x)|\, d\xi(x) \leqslant C_K \, \|u\|_{\mathcal{D}_\omega}$$

ii. If one denotes by $\mathcal{C}$ the space of complex-valued, continuous functions with compact support, then: $\mathcal{C} \cap \mathcal{D}_\omega$ is dense in $\mathcal{C}$ and in $\mathcal{D}_\omega$.

iii. For any contraction of the complex plane, and any u of $\mathcal{D}_\omega$:

$$T\, u \in \mathcal{D}_\omega \quad \text{et} \quad \|T\, u\|_{\mathcal{D}_\omega} \leqslant \|u\|_{\mathcal{D}_\omega}$$

The Dirichlet space $\mathcal{D}_\omega$ is generated by the Green potentials of finite energy, which are defined in a direct way, as the functions u of $\mathcal{D}_\omega$ such that there exists a Radon measure μ such that:

$$\forall \varphi \in \mathcal{C} \cap \mathcal{D}_\omega : \quad (u, \varphi) = \int_\omega \bar{\varphi}\, d\mu$$

Such a map u will be called **potential generated by** μ.
The linear map Δ which, to any potential u of $\mathcal{D}_\omega$, associates the measure μ that generates this potential, is called **generalized Laplacian for the space** $\mathcal{D}$.

It is interesting to note that the original theory of Dirichlet spaces concerned functions defined on a Hausdorff space (separated espace), with a positive Radon measure of full support (every non-empty open set has a strictly positive measure).

Remark 2.4. One may wonder why the Markov property is of such importance in our building of a Laplacian ? Very simply, the lack of memory - or the fact that the future state which corresponds, for any natural integer m, to the values of the considered function on the graph $\Gamma_{\mathcal{W}_{m+1}}^{\mathcal{H}}$, depends only of the present state, i.e. the values of the function on the graph $\Gamma_{\mathcal{W}_m}^{\mathcal{H}}$, accounts for the building of the Laplacian step by step.

Definition 2.15. Energy, on the graph $\Gamma_{\mathcal{W}_m^{\mathcal{H}}}$, $m \in \mathbb{N}$, of a pair of functions

Let m be a natural integer, and u and v two real-valued, continuous functions, on the mesh of order m

$$\left(\Gamma_{\mathcal{W}_m^{\mathcal{H}}{}_{|\rho=\frac{i}{N_b^m}}} \right)_{0 \leqslant i \leqslant N_b^m}$$

of $\Gamma_{\mathcal{W}_m}^{\mathcal{H}}$.

The energy, on the graph $\Gamma_{\mathcal{W}_m}^{\mathcal{H}}$, of the pair of functions (u, v), is:

$$\begin{aligned}
\mathcal{E}_{\Gamma_{\mathcal{W}_m}^{\mathcal{H}}}(u, v) \;=\; & \sum_{i=0}^{N_b^m} \sum_{j=0}^{\mathcal{N}_m^{\mathcal{S}}-2} \left(u\left(M_{m,j,\rho=\frac{i}{N_b^m}}\right) - u\left(M_{m,j+1,\rho=\frac{i}{N_b^m}}\right)\right) \left(v\left(M_{m,j,\rho=\frac{i}{N_b^m}}\right) - v\left(M_{m,j+1,\rho=\frac{i}{N_b^m}}\right)\right) \\
& + \sum_{i=0}^{N_b^m-1} \sum_{j=0}^{\mathcal{N}_m^{\mathcal{S}}-1} \left(u\left(M_{m,j,\rho=\frac{i}{N_b^m}}\right) - u\left(M_{m,j,\rho=\frac{i+1}{N_b^m}}\right)\right) \left(v\left(M_{m,j,\rho=\frac{i}{N_b^m}}\right) - v\left(M_{m,j,\rho=\frac{i+1}{N_b^m}}\right)\right)
\end{aligned}$$

For the sake of simplicity, we will write it under the form:

$$\mathcal{E}_{\Gamma_{\mathcal{W}_m}^{\mathcal{H}}}(u,v) = \sum_{X \underset{m}{\sim} Y} \left(u(X) - u(Y)\right)\,\left(v(X) - v(Y)\right)$$

Property 2.13. *Given a natural integer m, and a real-valued function u, defined on the set of vertices of $\Gamma_{\mathcal{W}_m}^{\mathcal{H}}$, the map, which, to any pair of real-valued, continuous functions (u,v) defined on the set $\mathcal{V}_m$ of vertices of $\Gamma_{\mathcal{W}_m}^{\mathcal{H}}$, associates:*

$$\mathcal{E}_{\Gamma_{\mathcal{W}_m}^{\mathcal{H}}}(u,v) = \sum_{X \underset{m}{\sim} Y} \left(u(X) - u(Y)\right)\,\left(v(X) - v(Y)\right)$$

is a Dirichlet form on $\Gamma_{\mathcal{W}_m}^{\mathcal{H}}$.
Moreover:

$$\mathcal{E}_{\Gamma_{\mathcal{W}_m}^{\mathcal{H}}}(u,u) = 0 \Leftrightarrow u \text{ is constant}$$

Proposition 2.14. ***Harmonic extension of a function, on the graph of the Weierstrass function***

*For any strictly positive integer m, if u is a real-valued function defined on $\mathcal{V}_{m-1}$, its **harmonic extension**, denoted by $\tilde{u}$, is obtained as the extension of u to $\mathcal{V}_m$ which minimizes the energy:*

$$\mathcal{E}_{\Gamma_{\mathcal{W}_m}^{\mathcal{H}}}(\tilde{u},\tilde{u}) = \sum_{X \underset{m}{\sim} Y} \left(\tilde{u}(X) - \tilde{u}(Y)\right)^2$$

The link between $\mathcal{E}_{\Gamma_{\mathcal{W}_m}^{\mathcal{H}}}$ and $\mathcal{E}_{\Gamma_{\mathcal{W}_{m-1}}^{\mathcal{H}}}$ is obtained through the introduction of two strictly positive constants r_m and r_{m+1} such that:

$$r_m \sum_{X \underset{m}{\sim} Y} \left(\tilde{u}(X) - \tilde{u}(Y)\right)^2 = r_{m-1} \sum_{X \underset{m-1}{\sim} Y} \left(u(X) - u(Y)\right)^2$$

In particular:

$$r_1 \sum_{X \underset{1}{\sim} Y} \left(\tilde{u}(X) - \tilde{u}(Y)\right)^2 = r_0 \sum_{X \underset{0}{\sim} Y} \left(u(X) - u(Y)\right)^2$$

For the sake of simplicity, we will fix the value of the initial constant: $r_0 = 1$. One has then:

$$\mathcal{E}_{\Gamma_{\mathcal{W}_m}^{\mathcal{H}}}(\tilde{u},\tilde{u}) = \frac{1}{r_1}\,\mathcal{E}_{\Gamma_{\mathcal{W}_0}^{\mathcal{H}}}(\tilde{u},\tilde{u})$$

Let us set:

$$r = \frac{1}{r_1}$$

and:

$$\mathcal{E}_m(u) = r_m \sum_{X \underset{m}{\sim} Y} \left(\tilde{u}(X) - \tilde{u}(Y)\right)^2$$

Since the determination of the harmonic extension of a function appears to be a local problem, on the graph $\Gamma_{\mathcal{W}_{m-1}}^{\mathcal{H}}$, which is linked to the graph $\Gamma_{\mathcal{W}_m}^{\mathcal{H}}$ by a similar process as the one that links $\Gamma_{\mathcal{W}_1}^{\mathcal{H}}$ to $\Gamma_{\mathcal{W}_0}^{\mathcal{H}}$, one deduces, for any strictly positive integer m:

$$\mathcal{E}_{\Gamma_{\mathcal{W}_m}^{\mathcal{H}}}(\tilde{u},\tilde{u}) = \frac{1}{r_1}\,\mathcal{E}_{\Gamma_{\mathcal{W}_{m-1}}^{\mathcal{H}}}(\tilde{u},\tilde{u})$$

By induction, one gets:

$$r_m = r_1^m\, r_0 = r^{-m}$$

If v is a real-valued function, defined on V_{m-1}, of harmonic extension $\tilde{v}$, we will write:

$$\mathcal{E}_m(u,v) = r^{-m} \sum_{X \underset{m}{\sim} Y} \left(\tilde{u}(X) - \tilde{u}(Y)\right)\left(\tilde{v}(X) - \tilde{v}(Y)\right)$$

For further precision on the construction and existence of harmonic extensions, we refer to [18].

Definition 2.16. Dirichlet form, for a pair of continuous functions defined on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$

We define the Dirichlet form $\mathcal{E}$ which, to any pair of real-valued, continuous functions (u,v) defined on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$, associates, subject to its existence:

$$\mathcal{E}(u,v) = \lim_{m\to+\infty} \mathcal{E}_m\left(u_{|\mathcal{V}_m}, v_{|\mathcal{V}_m}\right) = \lim_{m\to+\infty} \sum_{X \underset{m}{\sim} Y} r^{-m}\,\left(u_{|\mathcal{V}_m}(X) - u_{|\mathcal{V}_m}(Y)\right)\,\left(v_{|\mathcal{V}_m}(X) - v_{|\mathcal{V}_m}(Y)\right)$$

Definition 2.17. Normalized energy, for a continuous function u, defined on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$
Taking into account that the sequence $\left(\mathcal{E}_m\left(u_{|\mathcal{V}_m}\right)\right)_{m\in\mathbb{N}}$ is defined on

$$\mathcal{V}_\star = \bigcup_{i\in\mathbb{N}} \mathcal{V}_i$$

one defines the normalized energy, for a continuous function u, defined on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$, by:

$$\mathcal{E}(u) = \lim_{m\to+\infty} \mathcal{E}_m\left(u_{|\mathcal{V}_m}\right)$$

Property 2.15. *The Dirichlet form $\mathcal{E}$ which, to any pair of real-valued, continuous functions defined on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$, associates:*

$$\mathcal{E}(u,v) = \lim_{m\to+\infty} \mathcal{E}_m\left(u_{|\mathcal{V}_m}, v_{|\mathcal{V}_m}\right) = \lim_{m\to+\infty} \sum_{X \underset{m}{\sim} Y} r^{-m}\,\left(u_{|\mathcal{V}_m}(X) - u_{|\mathcal{V}_m}(Y)\right)\,\left(v_{|\mathcal{V}_m}(X) - v_{|\mathcal{V}_m}(Y)\right)$$

satisfies the self-similarity relation:

$$\mathcal{E}(u,v) = r^{-1} \sum_{k=0}^{N_b-1} \mathcal{E}\left(u \circ T_k, v \circ T_k\right)$$

Proof.

$$\begin{aligned}
\sum_{k=0}^{N_b-1} \mathcal{E}\left(u \circ T_k, v \circ T_k\right) &= \lim_{m\to+\infty} \sum_{k=0}^{N_b-1} \mathcal{E}_m \left(u_{|\mathcal{V}_m} \circ T_k, v_{|\mathcal{V}_m} \circ T_k\right) \\
&= \lim_{m\to+\infty} \sum_{X \underset{m}{\sim} Y} r^{-m} \sum_{i=0}^{N_b-1} \left(u_{|\mathcal{V}_m}\left(T_k(X)\right) - u_{|\mathcal{V}_m}\left(T_k(Y)\right)\right) \left(v\left(T_k(X)\right) - v\left(T_k(Y)\right)\right) \\
&= \lim_{m\to+\infty} \sum_{X \underset{m+1}{\sim} Y} r^{-m} \sum_{i=0}^{N_b-1} \left(u_{|\mathcal{V}_m}(X) - u_{|\mathcal{V}_m}(Y)\right) \left(v(X) - v(Y)\right) \\
&= \lim_{m\to+\infty} r\, \mathcal{E}_{m+1}\left(u_{|\mathcal{V}_{m+1}}, v_{|\mathcal{V}_{m+1}}\right) \\
&= r\mathcal{E}(u,v)
\end{aligned}$$

□

Notation. We will denote by $\operatorname{dom} \mathcal{E}$ the subspace of continuous functions defined on $\Gamma_{\mathcal{W}}^{\mathcal{H}}$, such that:

$$\mathcal{E}(u) < +\infty$$

Notation. We will denote by $\operatorname{dom}_0 \mathcal{E}$ the subspace of continuous functions defined on $\Gamma_{\mathcal{W}}^{\mathcal{H}}$, which take the value zero on V_0, such that:

$$\mathcal{E}(u) < +\infty$$

3 Laplacian of a continuous function, on the graph of the Weierstrass-Hadamard function

3.1 Theoretical aspect

Definition 3.1. Self-similar measure, for the graph of the Weierstrass-Hadamard function

A mesure μ on $\overline{D} \times \mathbb{C}$ is said **self-similar** for the domain delimited by the graph of the Weierstrass-Hadamard function if there exists a family of strictly positive pounds $(\mu_k)_{0 \leqslant k \leqslant N_b-1}$, such that:

$$\mu = \sum_{k=0}^{N_b-1} \mu_k\, \mu \circ T_k^{-1} \quad , \quad \sum_{k=0}^{N_b-1} \mu_k = 1$$

For further precisions on self-similar measures, we refer to the works of J. E. Hutchinson (see [25]).

Property 3.1. ***Building of a self-similar measure, for the domain delimited by the graph of the Weierstrass-Hadamard function***

The Dirichlet forms mentioned in the above require a positive Radon measure with full support. The choice of a self-similar measure, which is, mots of the time, built with regards to a reference set, of measure 1, appears, first, as very natural. R. S. Strichartz (cite [3], [26]) showed that one can simply consider auto-replicant measures $\tilde{\mu}$, i.e. measures $\tilde{\mu}$ such that:

$$\tilde{\mu} = \sum_{k=0}^{N_b-1} \tilde{\mu}_k \, \tilde{\mu} \circ T_k^{-1} \qquad (\star)$$

where $(\tilde{\mu}_i)_{0\leqslant k\leqslant N_b-1}$ denotes a family of strictly positive pounds.

This latter approach appears as the best suited in our study, since, in the case of the graph $\Gamma_{\mathcal{W}}$, the initial set consists of the polygon $\mathcal{P}_0$, the measure of which, equal to its surface, is not necessarily equal to 1.

Let us assume that there exists a measure $\tilde{\mu}$ satisfying $(\star)$. Relation $(\star)$ yields, for any set of polyhedra $\mathcal{P}_{m,i,j}$, $m \in \mathbb{N}$, $0 \leqslant i,j \leqslant N_b^m - 1$ with $2\,N_b$ vertices and N_b+2 faces :

$$\tilde{\mu}\left(\bigcup_{0\leqslant i,j\leqslant N_b^m-1} \mathcal{P}_{m,i,j}\right) = \sum_{k=0}^{N_b-1} \tilde{\mu}_k \, \tilde{\mu}\left(T_k^{-1}\left(\bigcup_{0\leqslant i,j\leqslant N_b^m-1} \mathcal{P}_{m,i,j}\right)\right)$$

and, in particular:

$$\tilde{\mu}\left(T_0\left(\mathcal{P}_0\right) \cup T_1\left(\mathcal{P}_0\right) \cup T_2\left(\mathcal{P}_0\right) \cup \ldots \cup T_{N_b-1}\left(\mathcal{P}_0\right)\right) = \sum_{k=0}^{N_b-1} \tilde{\mu}_k \, \tilde{\mu}\left(\mathcal{P}_0\right)$$

i.e.:

$$\sum_{k=0}^{N_b-1} \tilde{\mu}\left(T_k\left(\mathcal{P}_0\right)\right) = \sum_{k=0}^{N_b-1} \tilde{\mu}_k \, \tilde{\mu}\left(\mathcal{P}_0\right)$$

The convenient choice, for any k of $\{0,\ldots,N_b-1\}$, is :

$$\tilde{\mu}_k = \frac{\tilde{\mu}\left(T_k\left(\mathcal{P}_0\right)\right)}{\tilde{\mu}\left(\mathcal{P}_0\right)}$$

If $\mu_{\mathcal{L}}$ is the Lebesgue measure on $\overline{\mathbb{D}} \times \mathbb{C}$, the choice $\tilde{\mu} = \mu_{\mathcal{L}}$ yields the expected result.

One can, from the measure $\tilde{\mu}$, build the self-similar measure μ, such that:

$$\mu = \sum_{k=0}^{N_b-1} \mu_k \, \mu \circ T_k^{-1}$$

where $(\mu_k)_{0\leqslant k\leqslant N_b-1}$ is a family of strictly positive pounds, the sum of which is equal to 1.

One has simply to set, for any k de $\{0,\ldots,N_b-1\}$:

$$\mu_k = \frac{\tilde{\mu}\left(T_k\left(\mathcal{P}_0\right)\right)}{\displaystyle\sum_{j=0}^{N_b-1} \tilde{\mu}\left(T_j\left(\mathcal{P}_0\right)\right)}$$

The measure μ is such that:

$$\mu\left(\mathcal{P}_0\right) = 1$$

The choice $\mu = \dfrac{\mu_{\mathcal{L}}}{\mu_{\mathcal{L}}\left(\mathcal{P}_0\right)} = \dfrac{\tilde{\mu}}{\tilde{\mu}\left(\mathcal{P}_0\right)}$ *yields the expected result.*

The measure μ *is self-similar, for the domain delimited by the graph of the Weierstrass-Hadamard function.*

Definition 3.2. Laplacian of order $m \in \mathbb{N}^\star$

For any strictly positive integer m, and any real-valued function u, defined on the set $\mathcal{V}_m$ of the vertices of the graph $\Gamma_{\mathcal{W}_m}^{\mathcal{H}}$, we introduce the Laplacian of order m, $\Delta_m(u)$, by:

$$\Delta_m u(X) = \sum_{Y \in \mathcal{V}_m,\, Y \underset{m}{\sim} X} \left(u(Y) - u(X)\right) \quad \forall\, X \in \mathcal{V}_m \setminus V_0$$

Definition 3.3. Harmonic function of order $m \in \mathbb{N}^\star$

Let m be a strictly positive integer. A real-valued function u,defined on the set $\mathcal{V}_m$ of the vertices of the graph $\Gamma_{\mathcal{W}_m}^{\mathcal{H}}$, will be said to be **harmonic of order** m if its Laplacian of order m is null:

$$\Delta_m u(X) = 0 \quad \forall\, X \in \mathcal{V}_m \setminus V_0$$

Definition 3.4. Piecewise harmonic function of order $m \in \mathbb{N}^\star$

Given a strictly positive integer m, a real valued function u, defined on the set of vertices of $\Gamma_{\mathcal{W}}^{\mathcal{H}}$, is said to be **piecewise harmonic function of order** m if, for any word $\mathcal{M}$ of length m, $u \circ T_{\mathcal{M}}$ is harmonic of order m.

Definition 3.5. Existence domain of the Laplacian, for a continuous function on the graph $\Gamma_{\mathcal{W}}\mathcal{H}$ (see[19])

We will denote by $\operatorname{dom}\Delta$ the existence domain of the Laplacian, on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$, as the set of functions u of $\operatorname{dom}\mathcal{E}$such that there exists a continuous function on $\Gamma_{\mathcal{W}}\mathcal{H}$, denoted $\Delta\, u$, that we will call **Laplacian of** u, such that :

$$\mathcal{E}(u,v) = -\int_{\mathcal{D}(\Gamma_{\mathcal{W}}\mathcal{H})} v\, \Delta u\, d\mu \quad \text{for any } v \in \operatorname{dom}_0 \mathcal{E}$$

Definition 3.6. Harmonic function

A function u belonging to $\operatorname{dom}\Delta$ will be said to be **harmonic** if its Laplacian is equal to zero.

Notation. In the following, we will denote by $\mathcal{H}_0 \subset \operatorname{dom}\Delta$ the space of harmonic functions, i.e. the space of functions $u \in \operatorname{dom}\Delta$ such that:

$$\Delta\, u = 0$$

Given a natural integer m, we will denote by $\mathcal{S}\left(\mathcal{H}_0, \mathcal{V}_m\right)$ the space, of dimension N_b^m, of spline functions " of level m", u, defined on $\Gamma_{\mathcal{W}}\mathcal{H}$, continuous, such that, for any word $\mathcal{M}$ of length m, $u \circ T_{\mathcal{M}}$ is harmonic, i.e.:

$$\Delta_m\,\left(u \circ T_{\mathcal{M}}\right) = 0$$

Property 3.2. *For any natural integer m:*

$$\mathcal{S}\left(\mathcal{H}_0, \mathcal{V}_m\right) \subset dom\; \mathcal{E}$$

Property 3.3. *Let m be a strictly positive integer, $X \notin V_0$ a vertex of the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$, and $\psi_X^m \in \mathcal{S}\left(\mathcal{H}_0, \mathcal{V}_m\right)$ a spline function such that:*

$$\psi_X^m(Y) = \left\{ \begin{array}{ccc} \delta_{XY} & \forall & Y \in \mathcal{V}_m \\ 0 & \forall & Y \notin \mathcal{V}_m \end{array} \right. \quad , \quad where \quad \delta_{XY} = \left\{ \begin{array}{ccc} 1 & if & X = Y \\ 0 & else & \end{array} \right.$$

Then, since $X \notin V_0$: $\psi_X^m \in dom_0\, \mathcal{E}$.
For any function u of $dom\,\mathcal{E}$, such that its Laplacian exists, definition (3.5) applied to ψ_X^m leads to:

$$\mathcal{E}(u, \psi_X^m) = \mathcal{E}_m(u, \psi_X^m) = -r^{-m}\,\Delta_m u(X) = -\int_{\mathcal{D}(\Gamma_{\mathcal{W}})} \psi_X^m\,\Delta u\,d\mu \approx -\Delta u(X)\int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})} \psi_X^m\,d\mu$$

since Δu is continuous on $\Gamma_{\mathcal{W}}^{\mathcal{H}}$, and the support of the spline function ψ_X^m is close to X:

$$\int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})} \psi_X^m\,\Delta u\,d\mu \approx -\Delta u(X)\int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})} \psi_X^m\,d\mu$$

By passing through the limit when the integer m tends towards infinity, one gets:

$$\lim_{m \to +\infty} \int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})} \psi_X^m\,\Delta_m u\,d\mu = \Delta u(X)\lim_{m \to +\infty} \int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})} \psi_X^m\,d\mu$$

i.e.:

$$\Delta u(X) = \lim_{m \to +\infty} r^{-m}\left(\int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})} \psi_X^m\,d\mu\right)^{-1}\Delta_m u(X)$$

3.2 Explicit determination of the Laplacian of a function u of dom Δ

The explicit determination of the Laplacian of a function u de $\operatorname{dom}\Delta$ requires to know:

$$\int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})} \psi_X^m\,d\mu$$

As it is explained in [24], one has just to reason by analogy with the dimension 1, more particulary, the unit interval $I = [0,1]$, of extremities $X_0 = (0,0)$, and $X_1 = (1,0)$. The functions ψ_{X_1} and ψ_{X_2} such that, for any Y of $\mathbb{R}^2$:

$$\psi_{X_1}(Y) = \delta_{X_1 Y} \quad , \quad \psi_{X_2}(Y) = \delta_{X_2 Y}$$

are, in the most simple way, tent functions. For the standard measure, one gets values that do not depend on X_1, or X_2 (one could, also, choose to fix X_1 and X_2 in the interior of I) :

$$\int_I \psi_{X_1}\, d\mu = \int_I \psi_{X_2}\, d\mu = \frac{1}{2}$$

(which corresponds to the surfaces of the two tent triangles.)

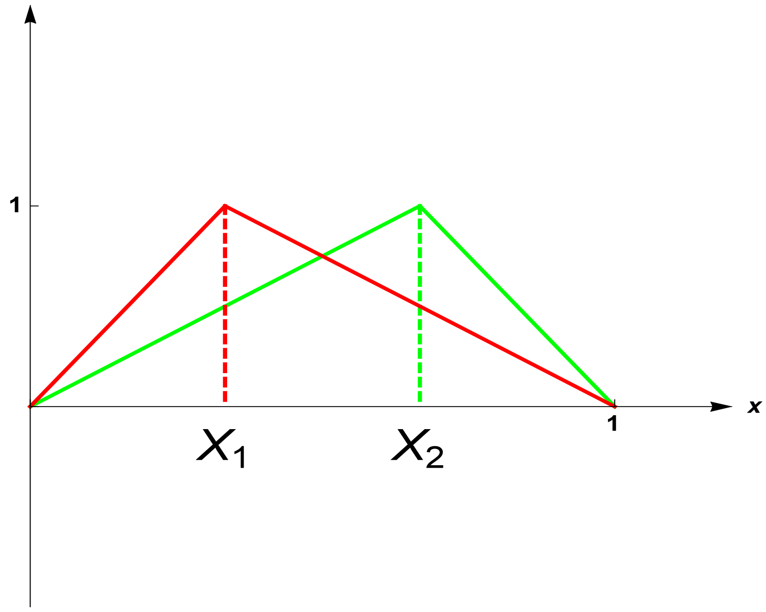

Figure 5: The graphs of the spline functions ψ_{X_1} and ψ_{X_2}.

In our case, we have to build the pendant, we no longer reason on the unit interval, but on our polyhedra with $2\, N_b$ vertices and $N_b + 2$ faces.

Given a strictly positive integer m, and a vertex X of the graph $\Gamma_{\mathcal{W}_m}^{\mathcal{H}}$, three configurations can occur:

i. the vertex X belongs to one and only one polyhedron with $2\,N_b$ vertices and N_b+2 faces, $\mathcal{P}_{m,i,j}$, $0 \leqslant i, j \leqslant N_b^m - 1$.

In this case, if one considers the spline functions ψ_Z^m which correspond to the $2\,N_b - 1$ distinct vertices X of this polyhedron:

$$\sum_{Z \text{ vertex of } \mathcal{P}_{m,i,j}} \int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})} \psi_Z^m \, d\mu = \mu\left(\mathcal{P}_{m,i,j}\right)$$

i.e., by symmetry:

$$2\,N_b \int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})} \psi_X^m \, d\mu = \mu\left(\mathcal{P}_{m,j}\right)$$

Thus:

$$\int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})} \psi_X^m \, d\mu = \frac{1}{N_b}\, \mu\left(\mathcal{P}_{m,j}\right)$$

ii. the vertex X is the intersection point of two polyhedra with $2\,N_b$ vertices and N_b+2 faces, $\mathcal{P}_{m,i,j}$ and $\mathcal{P}_{m,i+1,j}$, $0 \leqslant i, j \leqslant N_b^m - 2$.

On has then to take into account the contributions of both polyhedra, which leads to:

$$\int_{\mathcal{D}(\Gamma_{\mathcal{W}})} \psi_X^m \, d\mu = \frac{1}{4\,N_b} \left\{\mu\left(\mathcal{P}_{m,i,j}\right) + \mu\left(\mathcal{P}_{m,i+1,j}\right)\right\}$$

iii. the vertex X is the intersection point of four polyhedra with $2\,N_b$ vertices and N_b+2 faces, $\mathcal{P}_{m,i,j}$, $\mathcal{P}_{m,i,j+1}$, $\mathcal{P}_{m,i+1,j}$, $\mathcal{P}_{m,i+1,j+1}$, $0 \leqslant i, j \leqslant N_b^m - 2$.

On has then to take into account the contributions of the four polyhedra, which leads to:

$$\int_{\mathcal{D}(\Gamma_{\mathcal{W}})} \psi_X^m \, d\mu = \frac{1}{8\,N_b} \left\{\mu\left(\mathcal{P}_{m,i,j}\right) + \mu\left(\mathcal{P}_{m,i,j+1}\right) + \mu\left(\mathcal{P}_{m,i+1,j}\right) + \mu\left(\mathcal{P}_{m,i+1,j+1}\right)\right\}$$

Theorem 3.4. *Let u be in $dom\,\Delta$. Then, the sequence of functions $(f_m)_{m\in\mathbb{N}}$ such that, for any natural integer m, and any X of $\mathcal{V}_\star \setminus V_0$:*

$$f_m(X) = r^{-m} \left(\int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})} \psi_X^m \, d\mu\right)^{-1} \Delta_m\, u(X)$$

converges uniformly towards $\Delta\, u$, and, reciprocally, if the sequence of functions $(f_m)_{m\in\mathbb{N}}$ converges uniformly towards a continuous function on $\mathcal{V}_\star \setminus V_0$, then:

$$u \in dom\,\Delta$$

Proof. Let u be in $\operatorname{dom}\Delta$. Then:

$$r^{-m}\left(\int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})}\psi_X^m\,d\mu\right)^{-1}\Delta_m\,u(X)=\frac{\displaystyle\int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})}\Delta\,u\,\psi_X^m\,d\mu}{\displaystyle\int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})}\psi_X^m\,d\mu}$$

Since u belongs to $\operatorname{dom}\Delta$, its Laplacian $\Delta\,u$ exists, and is continuous on the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$. The uniform convergence of the sequence $(f_m)_{m\in\mathbb{N}}$ follows.

Reciprocally, if the sequence of functions $(f_m)_{m\in\mathbb{N}}$ converges uniformly towards a continuous function on $\mathcal{V}_\star\setminus V_0$, then, for any natural integer m, and any v belonging to $\operatorname{dom}_0\mathcal{E}$:

$$\begin{array}{rcl}
\mathcal{E}_m(u,v) & = & \displaystyle\sum_{(X,Y)\in\mathcal{V}_m^2,\,X\underset{m}{\sim}Y} r^{-m}\left(u_{|\mathcal{V}_m}(X)-u_{|\mathcal{V}_m}(Y)\right)\left(v_{|\mathcal{V}_m}(X)-v_{|V_m}(Y)\right)\\
& = & \displaystyle\sum_{(X,Y)\in\mathcal{V}_m^2,\,X\underset{m}{\sim}Y} r^{-m}\left(u_{|\mathcal{V}_m}(Y)-u_{|\mathcal{V}_m}(X)\right)\left(v_{|\mathcal{V}_m}(Y)-v_{|V_m}(X)\right)\\
& = & \displaystyle -\sum_{X\in\mathcal{V}_m\setminus V_0} r^{-m}\sum_{Y\in V_m,\,Y\underset{m}{\sim}X} v_{|\mathcal{V}_m}(X)\left(u_{|\mathcal{V}_m}(Y)-u_{|V_m}(X)\right)\\
& & \displaystyle -\sum_{X\in V_0} r^{-m}\sum_{Y\in\mathcal{V}_m,\,Y\underset{m}{\sim}X} v_{|\mathcal{V}_m}(X)\left(u_{|V_m}(Y)-u_{|\mathcal{V}_m}(X)\right)\\
& = & \displaystyle -\sum_{X\in\mathcal{V}_m\setminus V_0} r^{-m}\,v(X)\,\Delta_m\,u(X)\\
& = & \displaystyle -\sum_{X\in\mathcal{V}_m\setminus V_0} v(X)\left(\int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})}\psi_X^m\,d\mu\right) r^{-m}\left(\int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})}\psi_X^m\,d\mu\right)^{-1}\Delta_m\,u(X)
\end{array}$$

Let us note that any X of $\mathcal{V}_m\setminus V_0$ admits exactly two adjacent vertices which belong to $V_m\setminus V_0$, which accounts for the fact that the sum

$$\sum_{X\in\mathcal{V}_m\setminus V_0} r^{-m}\sum_{Y\in V_m\setminus V_0,\,Y\underset{m}{\sim}X} v(X)\left(u_{|\mathcal{V}_m}(Y)-u_{|V_m}(X)\right)$$

has the same number of terms as:

$$\sum_{(X,Y)\in(\mathcal{V}_m\setminus V_0)^2,\,X\underset{m}{\sim}Y} r^{-m}\left(u_{|\mathcal{V}_m}(Y)-u_{|\mathcal{V}_m}(X)\right)\left(v_{|\mathcal{V}_m}(Y)-v_{|\mathcal{V}_m}(X)\right)$$

For any natural integer m, we introduce the sequence of functions $(f_m)_{m\in\mathbb{N}}$ such that, for any X of $\mathcal{V}_m\setminus V_0$:

$$f_m(X)=r^{-m}\left(\int_{\mathcal{D}(\Gamma_{\mathcal{W}}^{\mathcal{H}})}\psi_X^m\,d\mu\right)^{-1}\Delta_m\,u(X)$$

The sequence $(f_m)_{m\in\mathbb{N}}$ converges uniformly towards $\Delta\,u$. Thus:

$$\mathcal{E}_m(u,v)\;=\;-\int_{\mathcal{D}(\Gamma_{\mathcal{W}})}\left\{\sum_{X\in\mathcal{V}_m\setminus V_0} v_{|\mathcal{V}_m}(X)\,\Delta\,u_{|\mathcal{V}_m}(X)\,\psi_X^m\right\}d\mu$$

$\square$

4 Normal derivatives

Let us go back to the case of a function u twice differentiable on $I=[0,1]$, that does not vanish in 0 and :

$$\int_0^1 (\Delta u)\,(x)\,v(x)\,dx = -\int_0^1 u'(x)\,v'(x)\,dx + u'(1)\,v(1) - u'(0)\,v(0)$$

The normal derivatives:

$$\partial_n u(1) = u'(1) \quad \text{et} \quad \partial_n u(0) = u'(0)$$

appear in a natural way. This leads to:

$$\int_0^1 (\Delta u)\,(x)\,v(x)\,dx = -\int_0^1 u'(x)\,v'(x)\,dx + \sum_{\partial\,[0,1]} v\,\partial_n\,u$$

One meets thus a particular case of the Gauss-Green formula, for an open set Ω of $\mathbb{R}^d$, $d \in \mathbb{N}^\star$:

$$\int_\Omega \nabla\,u\,\nabla\,v\,d\mu = -\int_\Omega (\Delta u)\,\,v\,d\mu + \int_{\partial\,\Omega} v\,\partial_n\,u\,d\sigma$$

where μ is a measure on Ω, and where $d\sigma$ denotes the elementary surface on $\partial\,\Omega$.

In order to obtain an equivalent formulation in the case of the graph $\Gamma_{\mathcal{W}}^{\mathcal{H}}$, one should have, for a pair of functions (u,v) continuous on $\Gamma_{\mathcal{W}}^{\mathcal{H}}$ such that u has a normal derivative:

$$\mathcal{E}(u,v) = -\int_\Omega (\Delta u)\,\,v\,d\mu + \sum_{V_0} v\,\partial_n\,u$$

For any natural integer m :

$$\begin{aligned}
&\mathcal{E}_m(u,v) \\
&= \sum_{(X,Y)\in\mathcal{V}_m^2,\,X\underset{m}{\sim}Y} r^{-m}\,\left(u_{|\mathcal{V}_m}(Y)-u_{|V_m}(X)\right)\,\left(v_{|\mathcal{V}_m}(Y)-v_{|\mathcal{V}_m}(X)\right) \\
&= -\sum_{X\in\mathcal{V}_m\setminus V_0} r^{-m} \sum_{Y\in\mathcal{V}_m,\,Y\underset{m}{\sim}X} v_{|\mathcal{V}_m}(X)\,\left(u_{|\mathcal{V}_m}(Y)-u_{|\mathcal{V}_m}(X)\right) \\
&\quad -\sum_{X\in V_0} r^{-m} \sum_{Y\in\mathcal{V}_m,\,Y\underset{m}{\sim}X} v_{|V_m}(X)\,\left(u_{|\mathcal{V}_m}(Y)-u_{|\mathcal{V}_m}(X)\right) \\
&= -\sum_{X\in\mathcal{V}_m\setminus V_0} v_{|\mathcal{V}_m}(X)\,r^{-m}\,\Delta_m\,u_{|\mathcal{V}_m}(X) \\
&\quad +\sum_{X\in V_0}\sum_{Y\in\mathcal{V}_m,\,Y\underset{m}{\sim}X} r^{-m}\,v_{|\mathcal{V}_m}(X)\,\left(u_{|\mathcal{V}_m}(X)-u_{|\mathcal{V}_m}(Y)\right)
\end{aligned}$$

We thus come across an analogous formula of the Gauss-Green one, where the role of the normal derivative is played by:

$$\sum_{X\in V_0} r^{-m} \sum_{Y\in\mathcal{V}_m,\,Y\underset{m}{\sim}X} \left(u_{|\mathcal{V}_m}(X)-u_{|\mathcal{V}_m}(Y)\right)$$

Definition 4.1. For any X of V_0, and any continuous function u on $\Gamma_{\mathcal{W}}$, we will say that u admits a normal derivative in X, denoted by $\partial_n\,u(X)$, if:

$$\lim_{m\to+\infty} r^{-m} \sum_{Y\in\mathcal{V}_m,\,Y\underset{m}{\sim}X} \left(u_{|\mathcal{V}_m}(X)-u_{|\mathcal{V}_m}(Y)\right) < +\infty$$

We will set:

$$\partial_n u(X) = \lim_{m \to +\infty} r^{-m} \sum_{Y \in \mathcal{V}_m,\, Y \underset{m}{\sim} X} \left(u_{|\mathcal{V}_m}(X) - u_{|\mathcal{V}_m}(Y) \right) < +\infty$$

Definition 4.2. For any natural integer m, any X of $\mathcal{V}_m$, and any continuous function u on $\Gamma_{\mathcal{W}}^{\mathcal{H}}$, we will say that u admits a normal derivative in X, denoted by $\partial_n u(X)$, if:

$$\lim_{k \to +\infty} r^{-k} \sum_{Y \in V_k,\, Y \underset{k}{\sim} X} \left(u_{|\mathcal{V}_k}(X) - u_{|\mathcal{V}_k}(Y) \right) < +\infty$$

We will set:

$$\partial_n u(X) = \lim_{k \to +\infty} r^{-k} \sum_{Y \in \mathcal{V}_k,\, Y \underset{k}{\sim} X} \left(u_{|\mathcal{V}_k}(X) - u_{|\mathcal{V}_k}(Y) \right) < +\infty$$

Remark 4.1. One can thus extend the definition of the normal derivative of u to $\Gamma_{\mathcal{W}}^{\mathcal{H}}$.

Theorem 4.1. *Let u be in $\operatorname{dom}\Delta$. The, for any X of $\Gamma_{\mathcal{W}}^{\mathcal{H}}$, $\partial_n u(X)$ exists. Moreover, for any v of $\operatorname{dom}\mathcal{E}$, et any natural integer m, the Gauss-Green formula writes:*

$$\mathcal{E}(u,v) = -\int_{\Gamma_{\mathcal{W}}} (\Delta u)\, v\, d\mu + \sum_{V_0} v\, \partial_n u$$

5 Spectrum of the Laplacian

In the following, let u be in $\operatorname{dom}\Delta$. We will apply the ***spectral decimation method*** déveloped by R. S. Strichartz [24], in the spirit of the works of M. Fukushima et T. Shima [27]. In order to determine the eigenvalues of the Laplacian Δu built in the above, we concentrate first on the eigenvalues $(-\Lambda_m)_{m\in\mathbb{N}}$ of the sequence of graph Laplacians $(\Delta_m u)_{m\in\mathbb{N}}$, built on the discrete sequence of graphs $\left(\Gamma_{\mathcal{W}_m}^{\mathcal{H}}\right)_{m\in\mathbb{N}}$. For any natural integer m, the restrictions of the eigenfunctions of the continuous Laplacian Δu to the graph $\Gamma_{\mathcal{W}_m}^{\mathcal{H}}$ are, also, eigenfunctions of the Laplacian Δ_m, which leads to recurrence relations between the eigenvalues of order m and $m+1$.

We thus aim at determining the solutions of the eigenvalue equation:

$$-\Delta u = \Lambda u \quad \text{on } \Gamma_{\mathcal{W}}^{\mathcal{H}}$$

as limits, when the integer m tends towards infinity, of the solutions of:

$$-\Delta_m u = \Lambda_m u \quad \text{on } \mathcal{V}_m \setminus V_0$$

Let $m \geqslant 1$. We consider an eigenfunction u_{m-1} on $\mathcal{V}_{m-1} \setminus V_0$, for the eigenvalue Λ_{m-1}. The aim is to extend u_{m-1} on $\mathcal{V}_m \setminus V_0$ in a function u_m, which will itself be an eigenfunction of Δ_m, for the eigenvalue Λ_m, and, thus, to obtain a recurrence relation between the eigenvalues Λ_m and Λ_{m-1}.

Given five vertices of $\Gamma^{\mathcal{H}}_{\mathcal{W}_{m-1}}$, denoted respectively by $X_{k,i}$, $X_{k+1,i+1}$, $X_{k,i+1}$ $X_{k+2,i+1}$ $X_{k+1,i+2}$ where k denotes a generic natural integer, we will denote by:

i. $Y_{k+1,i+1}$, $\ldots$, $Y_{k+1,i+N_b-1}$, the points of $\mathcal{V}_m \setminus \mathcal{V}_{m-1}$ such that: $Y_{k+1,i+1}$, $\ldots$, $Y_{k+1,i+N_b-1}$ are between $X_{k+1,i}$ and $X_{k+1,i+1}$;

ii. $Y_{k+1,i+N_b+1}$, $\ldots$, $Y_{k+1,i+2\,N_b-1}$, the points of $\mathcal{V}_m \setminus \mathcal{V}_{m-1}$ such that: $Y_{k+1,i+N_b 1}$, $\ldots$, $Y_{k+1,i+2\,N_b-1}$ are between $X_{k+1,i+1}$ and $X_{k+1,i+2}$;

iii. $Y_{k+1,i+N_b}$, $\ldots$, $Y_{k+N_b-1,i+N_b}$, the points of $\mathcal{V}_m \setminus \mathcal{V}_{m-1}$ such that: $Y_{k+1,i+N_b}$, $\ldots$, $Y_{k+N_b-1,i+N_b}$ are between $X_{k,i+1}$ and $X_{k+1,i+1}$;

iv. $Y_{k+N_b+1,i+N_b}, \ldots, Y_{k+2\,N_b-1,i+N_b}$, the points of $\mathcal{V}_m \setminus \mathcal{V}_{m-1}$ such that: $Y_{k+N_b+1,i+N_b}, \ldots, Y_{k+2\,N_b-1,i+N_b}$, are between $X_{k+1,i+1}$ and $X_{k+2,i+1}$.

For consistency, let us set:

$$Y_{k+1,i} = X_{k+1,i} \quad , \quad Y_{k+N_b,i+1} = X_{k+1,i+1} \quad , \quad Y_{k+N_b,i+N_b} = X_{k+1,i+2}$$

$$Y_{k,i+N_b} = X_{k,i+1} \quad \text{and} \quad Y_{k+2\,N_b,i+N_b} = X_{k+1,i+1}$$

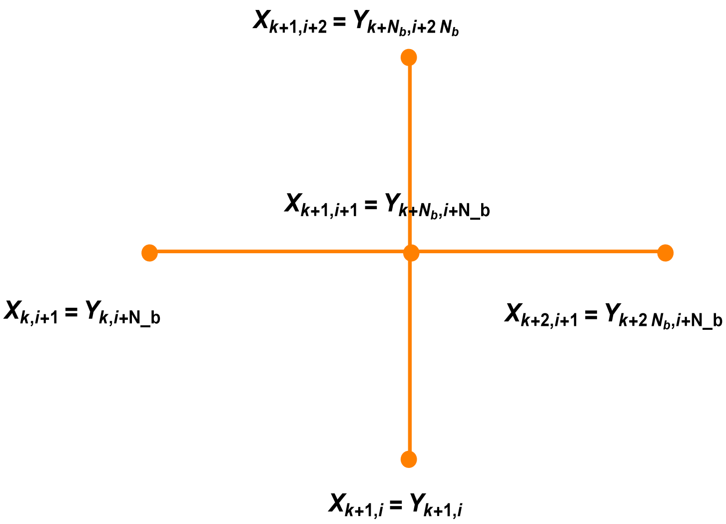

Figure 6: The points $Y_{k+1,i} = X_{k+1,i}$, $Y_{k+N_b,i+1} = X_{k+1,i+1}$, $Y_{k+N_b,i+N_b} = X_{k+1,i+2}$, $Y_{k,i+N_b} = X_{k,i+1}$, $Y_{k+2\,N_b,i+N_b} = X_{k+1,i+1}$.

The eigenvalue equation in Λ_m leads to the following systems, for any integer j, $1 \leqslant j \leqslant N_b - 3$:

$$\left\{ \begin{array}{ccc} \{\Lambda_m - 2\}\ u_m\left(Y_{k+1,i}\right) & = & -u_{m-1}\left(X_{k,i}\right) - u_m\left(Y_{k+2,i}\right) - u_m\left(Y_{k+1,i+1}\right) - u_m\left(Y_{k+1,i-1}\right) \\ \{\Lambda_m - 2\}\ u_m\left(Y_{k+j,i}\right) & = & -u_m\left(Y_{k+j-1,i}\right) - u_m\left(Y_{k+j+1,i}\right) - u_m\left(Y_{k+j,i+1}\right) - u_m\left(Y_{k+j,i-1}\right) \\ \{\Lambda_m - 2\}\ u_m\left(Y_{k+N_b-1,i}\right) & = & -u_{m-1}\left(X_{k+1,i}\right) - u_m\left(Y_{k+N_b-2,i}\right) - u_m\left(Y_{k+N_b-1,i+1}\right) \end{array} \right.$$

Let us concentrate on the relation:

$$\{\Lambda_m - 2\}\ u_m\left(Y_{k,i}\right) = -u_m\left(Y_{k+1,i}\right) - u_m\left(Y_{k-1,i}\right) - u_m\left(Y_{k,i+1}\right) - u_m\left(Y_{k,i-1}\right)$$

By analogy with the one-dimensional case (we hereby refer to [5], [6]), we first look for the $u_m\left(Y_{k+1,i}\right)$ under the form:

$$u_m\left(Y_{k,i}\right) = r_{1m}^k\, r_{2m}^i$$

where r_{1m} are r_{2m} are scalars. One has then:

$$\{\Lambda_m - 2\}\ r_{1m}^k\, r_{2m}^i = -r_{1m}^{k+1}\, r_{2m}^i - r_{1m}^{k-1}\, r_{2m}^i - r_{1m}^k\, r_{2m}^{i+1} - r_{1m}^k\, r_{2m}^{i-1}$$

which yields:

$$\{\Lambda_m - 2\}\ r_{1m}\, r_{2m} = -r_{1m}^2\, r_{2m} - r_{2m} - r_{1m}\, r_{2m}^2 - r_{1m}$$

Let us denote by I_5 the 5×5 identity matrix. The vectors

$$\begin{pmatrix} r_{1m}^{k-1}\, r_{2m}^i \\ r_{1m}^k\, r_{2m}^{i-1} \\ r_{1m}^k\, r_{2m}^i \\ r_{1m}^k\, r_{2m}^{i-1} \\ r_{1m}^k\, r_{2m}^{i+1} \end{pmatrix}$$

belong to the kernel of the matrix:

$$\{\Lambda_m - 2\}\ I_5 - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

the spectra of which is:

$$\{5 - \Lambda_m, 5 - \Lambda_m, 5 - \Lambda_m, 5 - \Lambda_m, -\Lambda_m\}$$

The eigenspaces are generated by the vectors:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Thus, $\begin{pmatrix} r_{2m} \\ r_{1m} \\ r_{1m}^k\, r_{2m}^i \\ r_{1m}^k\, r_{2m}^{i1} \\ r_{1m}^k\, r_{2m}^{i+1} \end{pmatrix}$ is a linear combination of:

$$\{-2-\Lambda_m\}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \{3-\Lambda_m\}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \{3-\Lambda_m\}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \{3-\Lambda_m\}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \{3-\Lambda_m\}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The roots r_{1m}, r_{2m} may thus take the following values:

$$r_{1m} = -2-\Lambda_m \quad \text{or} \quad r_{1m} = 3-\Lambda_m$$

$$r_{2m} = -2-\Lambda_m \quad \text{or} \quad r_{2m} = 3-\Lambda_m$$

From this point, the following compatibility conditions have to be satisfied:

$$u_m\left(Y_{k,i}\right) = u_{m-1}\left(X_{k,i}\right) = r_{1,m-1}^k\, r_{2,m-1}^i$$

$$u_m\left(Y_{k+N_b,i}\right) = r_{1,m}^{k+N_b}\, r_{2,m}^i = u_{m-1}\left(X_{k+1,i}\right) = r_{1,m-1}^{k+1}\, r_{2,m}^i$$

$$u_m\left(Y_{k,i+N_b}\right) = r_{1,m}^k\, r_{2,m}^{i+N_b} = u_{m-1}\left(X_{k,i+1}\right) = r_{1,m-1}^k\, r_{2,m}^{i+1}$$

For the specific values $i = k = 0$, one obtains:

$$r_{1,m-1} = r_{1m}^{N_b} \quad , \quad r_{2,m-1} = r_{2m}^{N_b}$$

This leads to:

$$\{3-\Lambda_m\}^{N_b} = 3-\Lambda_{m-1}$$

$$\{-2-\Lambda_m\}^{N_b} = -2-\Lambda_{m-1}$$

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