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Bi-objective matchings with the triangle inequality

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Abstract. This article deals with a bi-objective matching problem. The input is a complete graph and two values on each edge (a weight and a length) which satisfy the triangle inequality. It is unlikely that every instance admits a matching with maximum weight and maximum length at the same time. Therefore, we look for a compromise solution, i.e. a matching that simultaneously approximates the best weight and the best length. For which approximation ratio ρ can we guarantee that any instance admits a ρ-approximate matching? We propose a general method which relies on the existence of an approximate matching in any graph of small size. An algorithm for computing a 1/3-approximate matching in any instance is provided. The algorithm uses an analytical result stating that every instance on at most 6 nodes must admit a 1/2-approximate matching. We extend our analysis with a computer-aided approach for larger graphs, indicating that the general method may produce a 2/5-approximate matching. We conjecture that a 1/2-approximate matching exists in any bi-objective instance satisfying the triangle inequality.

Keywords: Bi-objective optimization, approximation algorithm, matching

1 Introduction

For many optimization problems, there is a need to deal with several aspects of a solution, at the same time. In multi-objective optimization, a problem has a set of feasible solutions S and $k > 1$ objective functions $f_i : S \to \mathbb{R}$, $i \in \{1, \ldots, k\}$. For each f_i, it is also specified if the function should be minimized or maximized. Suppose every f_i should be maximized and let opt_i denote $\max_{s \in S} f_i(s)$, for all $i \in \{1, \ldots, k\}$. For a given instance of a multi-objective problem, the *ideal point* $(opt_1, \ldots, opt_k)$ is the image of a (not necessarily feasible) solution reaching optimality on all objective functions, at the same time.

In practice, it is unlikely that the ideal point is the image of a feasible solution. We thus need to resort to approximation. A feasible solution s is said to be a ρ-approximation of the ideal point (or, in short, a ρ-approximate solution), for $\rho \in (0, 1]$, if $f_i(s) \geq \rho\, opt_i$, $i \in \{1, \ldots, k\}$.

The existence of a feasible ρ-approximate solution is not guaranteed for every possible ratio $\rho \in (0, 1]$. Therefore, it is challenging to identify, for a given problem, the best approximation factor under which the existence of a feasible ρ-approximate solution is always guaranteed. It is also relevant to have a constructive approach: what is the largest ratio $\rho \in (0, 1]$ such that a feasible ρ-approximation of the ideal point can be computed in polynomial time?

In this article, we focus on matching problems, which are central in combinatorial optimization problems. In particular, we deal with a bi-objective matching problem. The set of feasible solutions is every subset of pairwise non-adjacent edges. Each edge e has a non-negative weight $w(e)$ and a non-negative length $\ell(e)$, where both objectives satisfy the triangle inequality.

After a short review of the literature related to our problem (section 2), we give in section 3 a formal definition of the model, together with a motivation for restricting ourselves to two objectives satisfying the triangle inequality. Section 4 is devoted to the computational complexity of the bi-objective matching problem on complete graphs. In section 5 we give an approximation algorithm. The result is a general method which, combined with a technical lemma, provides an approximation algorithm with performance guarantee 1/3.

In section 6 we reuse the general method but we substitute the technical lemma with computational results, to give an insight about possible improvements of the previous performance guarantee of 1/3. Some upper bounds are given in section 7. We end this article with concluding remarks.

2   Related work

This article falls in the field of *multi-objective combinatorial optimization* (see [1] Chapter 8 in particular). A prominent solution concept in multi-objective optimization is the *Pareto set* $\mathcal{P}$ (a.k.a. Pareto curve [2]). For every feasible solution s, $\mathcal{P}$ contains a feasible solution s^* such that the image of s^* weakly dominates[1] the image of s. However, the notion of Pareto set is sometimes problematic since, for some instances, $\mathcal{P}$ must contain a number of solutions that is not polynomial in the input size. Therefore, no algorithm can compute $\mathcal{P}$ in polynomial time. Another issue comes from computational complexity because the multi-objective version of many polynomial single-objective problems is **NP**-hard (this is for example the case for the minimum cost spanning tree problem [3] and for the minimum assignment problem [4]). We can remedy to these difficulties with the help of approximation. If the maximization is retained for every f_i, then we say that s' is a ρ-approximation of s, with $\rho \in (0, 1]$, when $f_i(s') \geq \rho f_i(s)$ for all $i \in \{1, \ldots, k\}$. An *approximate Pareto set* $\mathcal{P}_\rho$ contains a ρ-approximation for every feasible solution.

Interestingly, the notion of approximate Pareto set helps to circumvent the two aforementioned difficulties. For a constant number of objectives, Papadimitriou and Yannakakis [2] have shown that for every $\rho \in (0, 1]$, there always exists an approximate Pareto set $\mathcal{P}_\rho$ such that $|\mathcal{P}_\rho|$ is polynomial in both the input size and $(1 - \rho)^{-1}$. Moreover, for the multi-objective version of some paradigmatic problems in combinatorial optimization (e.g. minimum cost spanning tree), $\mathcal{P}_\rho$ can be computed in polynomial time.

Usually, no explicit upper bound on $|\mathcal{P}_\rho|$ is given. However, we can be interested in fixing the size of an approximate Pareto set and try to provide the best performance guarantee. Because $\mathcal{P}_\rho$ may contain an overwhelming number of solutions, limiting its size is useful (in a decision process for example). When the size of an approximate Pareto set is upper bounded, we cannot guarantee that every approximation factor can be achieved. Intuitively, the larger the size of the approximation set, the more accurate it can be. In fact, it is not difficult to see that a ρ-approximation of the ideal point is just a ρ-approximate Pareto set of size 1. Therefore, we interchangeably talk about ρ-approximation of the ideal point and ρ-approximate Pareto set (with a single solution).

Approximation of the ideal point has been done for the multi-objective versions of many well-studied optimization problems. In [5–7], the authors study scheduling problems with two classical objective functions (e.g. makespan, sum of completion times, etc). They show the existence (and computation) of schedules that are approximately optimal for the objective functions, at the same time.

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[1] For k objectives functions to be maximized, s^* weakly dominates s when $f_i(s^*) \geq f_i(s)$ for all $i \in \{1, \ldots, k\}$.

A 1/3-approximation of the ideal point exists for the bi-objective maximum spanning tree problem [8]. In the same article, the authors give an approximation of two other multi-objective problems: max sat and cut-complement. In [9], the 1/3-approximation presented in [8] has been generalized to a bi-objective problem on simple matroids.

A bi-objective version of max cut is studied in [10]: there always exists a 1/2-approximation of the ideal point whereas a 0.439-approximation can be computed in polynomial time. These results were recently generalized to the maximization of two sub-modular, symmetric and non-negative functions[2] [11].

The ideal point approach has also been used on the bi-objective version of the traveling salesman problem [12–14]. For the maximization version of the problem, the best ratios for the approximation of the ideal point come from [14]: $5/12 - \varepsilon$ if the two objectives satisfy the triangle inequality, $3/8 - \varepsilon$ if only one objective satisfies the triangle inequality, and $\frac{1+2\sqrt{2}}{14} - \varepsilon$ without the triangle inequality.

Actually, the bi-objective maximum matching (where the triangle inequality is satisfied) has already been studied in [11]. This article provides a general approximation method for a class of biobjective problems. An application of the general method to the bi-objective maximum matching yields a 1/6-approximation of the ideal point. In addition, the problem is shown **NP**-hard when the instance is a collection of complete graphs in which the triangle inequality is satisfied [11]. The case of a complete graph was left open. In the present article, we improve on the approximation result since a 1/3-approximation is provided (Corollary 1). Moreover the complexity of the problem (previously left open) is shown on complete graphs (Theorem 1).

It is noteworthy that bi-objective versions of the assignment problem (i.e. matchings in bipartite graphs) have been studied previously. Exact methods for the minimization version can be found in [15] and [16].

To conclude, let us mention a popular approach in multi-objective combinatorial optimization. It consists in turning all but one objective functions in budget constraints. Given a vector $(B_2, B_3, \ldots, B_k) \in \mathbb{R}^{k-1}$, a solution $s \in S$ is said to be optimal if it optimizes $f_1(s)$ under the budget constraints $f_j(s) \geq B_j$ (or $f_j(s) \leq B_j$), $j = 2, \ldots, k$. To mention a few examples, this method is used for some multi-objective network-design problems [17] or shortest path [18]. The matching problem, and generalizations to matroids and independence systems are studied in [19, 20].

3   Model

Let $G = (V, E)$ be a complete graph. Each edge $e \in E$ has a non-negative weight $w(e)$ and a non-negative length $\ell(e)$. We assume that both w and ℓ satisfy the triangle inequality[3]. The *weight* and *length* of a set of edges $M \subset E$ are defined as $w(M) = \sum_{e \in M} w(e)$ and $\ell(M) = \sum_{e \in M} \ell(e)$, respectively. Both $w(M)$ and $\ell(M)$ should be maximized. Since it is unlikely that a matching reaches optimality for both objectives, our aim is to find a single matching which constitutes the best approximation of the ideal point.

Note that we have to restrict to a bi-objective matching where the two edge values fulfill the triangle inequality, otherwise there is not always a single matching which constitutes an ε-approximate Pareto set with any given positive ratio ε:
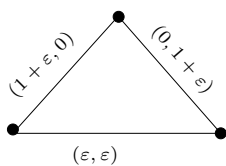
– If the triangle inequality does not hold then at least 3 solutions are required to constitute an approximate Pareto set for any given positive ratio ε. See the instance on Figure 1 which

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[2] This bi-objective problem generalizes max cut.

[3] For every triple of edges a, b and c forming a triangle, we have that $w(a) + w(b) \geq w(c)$ and $\ell(a) + \ell(b) \geq \ell(c)$.

contains 3 matchings with values $(1 + \varepsilon, 0)$, $(0, 1 + \varepsilon)$ and $(\varepsilon, \varepsilon)$, respectively. No solution is an ε-approximation of any other one, so the three solutions are required to provide an ε-approximate Pareto set.

– If only one objective fulfills the triangle inequality, then at least 2 solutions are required to constitute an ε-approximate Pareto set with any given positive ratio ε: consider a triangle with values $(0, 1)$, $(1, 0)$, and $(1, 0)$ (see Figure 2).

– If we consider 3 objectives which all fulfill the triangle inequality, then at least 3 solutions are required to constitute an ε-approximate Pareto set with any given positive ratio ε: consider a triangle with values $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$ (see Figure 3).

Fig. 1. At least 3 solutions are required if the edge valuations do not fulfill the triangle inequality.
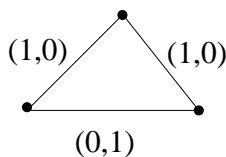
Fig. 2. At least 2 solutions are required if one edge valuation does not fulfill the triangle inequality.
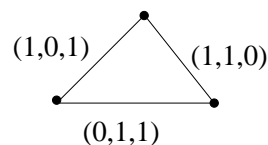
Fig. 3. At least 3 solutions are required with 3 edge valuations fulfilling the triangle inequality.

4 Complexity

We first investigate the complexity of the decision version of the biobjective maximum matching problem. This corresponds to the problem RESTRICTEDMAXMATCHING defined as follows. Let $(x_1, x_2) \in \mathbb{N}^2$, does there exist a matching M such that the weight of M is at least x_1, and the length of M is at least x_2?

Theorem 1. RESTRICTEDMAXMATCHING *is NP-complete, even in a graph in which both objectives fulfill the triangle inequality.*

Proof. We will show that PARTITION polynomially reduces to RESTRICTEDMAXMATCHING. We recall the PARTITION problem: an instance of PARTITION is a set $E = \{e_1, \ldots, e_n\}$ of n elements, and a cost function $c : E \to \mathbb{N}$ such that $\sum_{e \in E} c(e) = 2B$ and $0 < c(e) < B$. The question is: is there a subset $E' \subseteq E$ such that $\sum_{e \in E'} c(e) = \sum_{e \in E \setminus E'} c(e) = B$?
The corresponding instance of RESTRICTEDMAXMATCHING is as follows: we fix $x_1 = x_2 = (4n+1)B$, and we consider a complete graph G composed of $4n$ vertices : n vertices $\{a_1, \ldots, a_n\}$, n vertices $\{b_1, \ldots, b_n\}$, n vertices $\{c_1, \ldots, c_n\}$, and n vertices $\{d_1, \ldots, d_n\}$. For each $i \in \{1, \ldots, n\}$, the cost of edge (a_i, b_i) is equal to $(2B + c(e_i), 2B)$, and the cost of edge (b_i, c_i) is equal to $(2B, 2B + c(e_i))$. All the other edges have a cost equal to $(2B, 2B)$. This graph satisfies the triangle inequality since on each objective the cost of each edge is at least $2B$ and at most $3B$.

Let us show that there is a feasible solution of RESTRICTEDMAXMATCHING on this instance if and only if (E, c) is a yes instance of PARTITION. If (E, c) is a yes instance of PARTITION (i.e. there is a subset $E' \subseteq E$ such that $\sum_{e \in E'} c(e) = \sum_{e \in E \setminus E'} c(e) = B$), then there is a solution M of the corresponding instance of the RESTRICTEDMAXMATCHING problem: for each $i \in \{1, \ldots, n\}$, if $e_i \in E \setminus E'$ then the edges (a_i, b_i) and (c_i, d_i) are in M, and else, if $e_i \in E'$ then the edges (b_i, c_i) and (a_i, d_i) are in M (no other edge is in M). Since $\sum_{e \in E'} c(e) =$

$\sum_{e \in E \setminus E'} c(e) = B$, the weight of M is then $x_1 = 2n(2B) + B = (4n + 1)B$, and the length of M is $x_2 = 2n(2B) + B = (4n + 1)B$.

Conversely, let us now show that if there is a feasible solution M of RESTRICTEDMAX-MATCHING then (E, c) is a yes instance of PARTITION. We assume that M is a perfect matching, and is thus made of $2n$ edges; it is possible because the graph is complete, it contains an even number of nodes and any matching can be completed to become perfect without decreasing its value because all the edges have positive values for each objective. The weight of M is $|M|2B + \sum_{(a_i,b_i) \in M} c(e_i)$. Since the weight of M is at least $x_1 = (4n + 1)B$, we have $\sum_{(a_i,b_i) \in M} c(e_i) \geq B$. Let $E_1 = \{e_i \in E | (a_i, b_i) \in M\}$. We have: $\sum_{e_i \in E_1} c(e_i) \geq B$. Likewise, the length of M is $|M|2B + \sum_{(b_i,c_i) \in M} c(e_i)$. Since the length of M is at least $x_2 = (4n+1)B$, we have $\sum_{(b_i,c_i) \in M} c(e_i) \geq B$. Let $E_2 = \{e_i \in E | (b_i, c_i) \in M\}$. We have: $\sum_{e_i \in E_2} c(e_i) \geq B$. Note that for each $i \in \{1, \ldots, n\}$, at most one edge in $\{(a_i, b_i), (b_i, c_i)\}$ belongs to M, since M is a matching. Thus $E_1 \cap E_2 = \emptyset$. Since $\sum_{e \in E} c(e) = 2B$, we have $\sum_{e_i \in E_1} c(e_i) = \sum_{e_i \in E_2} c(e_i) = B$, which is a solution to the (E, c) instance of PARTITION. $\qquad \square$

5 Approximation

This section provides a general method, described in Theorem 2, which, combined with some results for small graphs, described in Lemma 2, provides an approximation algorithm with performance guarantee 1/3.

Theorem 2. *Let k be a fixed integer satisfying $k \geq 2$. Let ρ be a real satisfying $0 < \rho \leq 1$. Suppose that for every $q \leq 2k$ and every complete graph K_q of q nodes, there always exists a matching F_q satisfying $w(F_q) \geq \rho w(M_w^q)$ and $\ell(F_q) \geq \rho \ell(M_\ell^q)$, where M_w^q (resp. M_ℓ^q) denote a maximum weight (resp. length) matching of K_q. Then, there exists a single matching which constitutes a $\rho \frac{k-1}{k}$-approximate Pareto set for the biobjective Maximum Matching problem.*

Proof. Let $k \geq 2$. Build two maximum matchings of K_n with respect to w and ℓ respectively. Denote them by M_w and M_ℓ. We can suppose wlog that both M_w and M_ℓ are maximal with respect to the inclusion and then $|M_w| = |M_\ell|$. The subgraph $G' = (V, M_w \cup M_\ell)$ is made of p connected components denoted by C_i for $i \in \{1, \ldots, p\}$. One C_i may be one isolated node but every other connected component contains at least one edge. Each C_i is a cycle of even size or a path. Note that there is at most one path with at least two edges in $M_w \cup M_\ell$ (because the graph is complete and we can assume that M_w are M_ℓ are of maximum size) and this path contains an even number of edges. Each path containing a single edge must be in $M_w \cap M_\ell$ (because M_w and M_ℓ are of maximum size: at least one endpoint of each edge should be adjacent to one edge of M_ℓ – and this is the same thing for M_w).

Consider a connected component C_i with at least two edges and denote by n_i its number of vertices. Thus C_i contains n_i edges if C_i is a cycle and $n_i - 1$ if C_i is a path.

- If $n_i \leq 2k$, then by hypothesis, there exists in K_{n_i} (the subgraph induced by the vertices of C_i) a matching F_{n_i} such that:

$$w(F_{n_i}) \geq \rho w(M_w^{n_i}) \geq \rho w(C_i \cap M_w) \text{ and } \ell(F_{n_i}) \geq \rho \ell(M_\ell^{n_i}) \geq \rho \ell(C_i \cap M_\ell) \qquad (1)$$

  since $M_w^{n_i}$ (resp. $M_\ell^{n_i}$) is an optimum weight (resp. length) matching of K_{n_i}.
- Now, suppose that $n_i \geq 2k+1$. The edges of $C_i \cap M_w$ are labeled with numbers in $\{1, \ldots, k\}$ as follows. If C_i is a cycle, then it is explored from an arbitrarily chosen vertex that we denote v_0. If C_i is a path, then it is explored from an endpoint denoted by v_0. During the exploration, the first edge of $C_i \cap M_w$ is labeled 1, the second edge of $C_i \cap M_w$ is labeled 2,
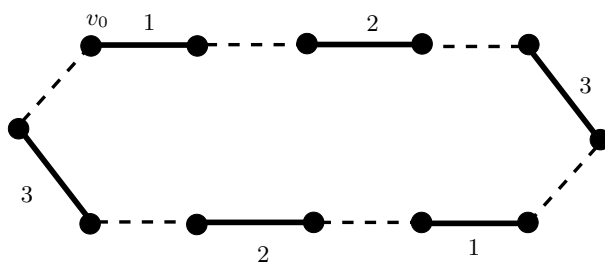
Fig. 4. The labels of the edges when $n_i = 12$ and $k = 3$. Bold edges belong to M_w and dashed edges belong to M_ℓ.

..., the $k + 1$ edge of $C_i \cap M_w$ is labeled 1, and so on. Formally, the j-th edge of $C_i \cap M_w$ is labeled $j - \lfloor \frac{j-1}{k} \rfloor k$. See Figure 4 for an illustration for a cycle of length 12 and $k = 3$. For $j \in \{1, \ldots, k\}$, $L_j(C_i \cap M_w)$ are the edges of $C_i \cap M_w$ with label j. Assume that j_w is the index such that the matching of label j_w is the lightest, i.e., $w(L_{j_w}(C_i \cap M_w)) = \min_{j=1,\ldots,k}\{w(L_j(C_i \cap M_w))\}$. Thus, we deduce that

$$w(C_i \cap M_w \setminus L_{j_w}(C_i \cap M_w)) \geq \frac{k-1}{k}w(C_i \cap M_w) \tag{2}$$

Moreover, since $L_{j_w}(C_i \cap M_\ell) = \emptyset$ (the edges which have been labelled belong to $C_i \cap M_w$ and in a connected component C_i with at least two edges $(C_i \cap M_w) \cap (C_i \cap M_\ell) = \emptyset$), we have:

$$\ell(C_i \cap M_\ell \setminus L_{j_w}(C_i \cap M_\ell)) = \ell(C_i \cap M_\ell) \tag{3}$$

Observe that if C_i is a cycle, then all the paths in $C_i \setminus L_{j_w}(C_i \cap M_w)$ except one have a length $2(k-1)+1 = 2k-1$ and the last path P_0^i (containing v_0) has a length at most equal to $2(2(k-1))+1 = 4k-3$. See Figure 5 for an illustration of the worst case with $n_i = 10$ and $k = 3$; here edge e corresponds to matching $L_3(C_i \cap M_w)$.
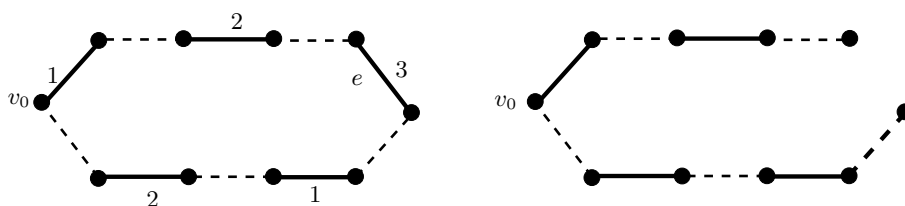
Fig. 5. On the left, the labels of the edges of C_i when $n_i = 10$ and $k = 3$ (Bold edges belong to M_w and dashed edges belong to M_ℓ). On the right, the path P_0 when the edges with label 3 for matching $L_3(C_i \cap M_w)$ have been deleted.

Let us focus on the path P_0^i when C_i is a cycle and assume that the length of this path is at least $2k$ (actually, it is at least $2k + 1$ because P_0^i is of odd length). This path contains

at most $4k - 3$ edges and by construction the two end edges are in M_ℓ. We label the edges of $P_0^i \cap M_\ell$ with numbers in $\{1, \ldots, k\}$ as previously starting from an end edge with label 1. See Figure 6 for an illustration with $n_i = 10$ and $k = 3$;
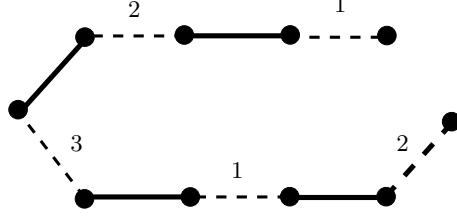
Fig. 6. The labels of the edges of $P_0^i \cap M_\ell$ when $n_i = 10$ and $k = 3$ (Bold edges belong to M_w and dashed edges belong to M_ℓ) for the example described in Figure 5.

Using a similar approach, $L_j(P_0^i \cap M_\ell)$ denotes the set of edges of $P_0^i \cap M_\ell$ with label j for $j \in \{1, \ldots, k\}$. Assume that j_ℓ is the index such that the matching of label j_ℓ is the lightest, i.e., $\ell(L_{j_\ell}(P_0^i \cap M_\ell)) = \min_{j=1,\ldots,k}\{\ell(L_j(P_0^i \cap M_\ell))\}$. Thus, we obtain:

$$\ell(P_0^i \cap M_\ell \setminus L_{j_\ell}(P_0^i \cap M_\ell)) \geq \frac{k-1}{k}\ell(P_0^i \cap M_\ell) \tag{4}$$

$$w(P_0^i \cap M_w \setminus L_{j_\ell}(P_0^i \cap M_\ell)) = w(P_0^i \cap M_w) \tag{5}$$

If C_i is a path, then all the paths in $C_i \setminus L_{j_w}(C_i \cap M_w)$ contain at most $2k - 1$ edges. In this case, we set $P_0^i = \emptyset$.

In conclusion, we have the following property: each path P of $C_i \setminus (L_{j_w}(C_i \cap M_w) \cup L_{j_\ell}(P_0^i \cap M_\ell))$ is contained in a complete subgraph $K_{q'}$ where $q' \leq 2k$ (in particular, it is the case of P_0^i when C_i is a cycle with length at most $2k - 1$). By hypothesis there is a matching $F_{q'}$ of $K_{q'}$ such that:

$$w(F_{q'}) \geq \rho w(M_w^{q'}) \geq \rho w(P \cap M_w) \text{ and } \ell(F_{q'}) \geq \rho\ell(M_\ell^{q'}) \geq \rho\ell(P \cap M_\ell) \tag{6}$$

since $M_w^{q'}$ (resp. $M_\ell^{q'}$) is optimal for w (resp. ℓ). The union of these matchings $F_{q'}$, denoted by F^i, is a matching of G induced by the vertices of C_i. Moreover, using Inequalities (2), (3), (4), (5), and (6) we get:

$$w(F^i) \geq \rho w(C_i \setminus (L_{j_w}(C_i \cap M_w) \cup L_{j_\ell}(P_0^i \cap M_\ell)) \cap M_w) \geq \rho \frac{k-1}{k} w(C_i \cap M_w) \tag{7}$$

$$\ell(F^i) \geq \rho\ell(C_i \setminus (L_{j_w}(C_i \cap M_w) \cup L_{j_\ell}(P_0^i \cap M_\ell)) \cap M_w) \geq \rho \frac{k-1}{k}\ell(C_i \cap M_\ell) \tag{8}$$

By summing up previous inequalities (1), or (7), or (8) according to the case and by adding the connected components made of a single edge, we get a matching F such that $w(F) \geq \rho\frac{k-1}{k}\sum_i w(C_i \cap M_w) = \rho\frac{k-1}{k}w(M_w)$ and $\ell(F) \geq \rho\frac{k-1}{k}\sum_i \ell(C_i \cap M_\ell) = \rho\frac{k-1}{k}\ell(M_\ell)$. □

We show now the existence of $\frac{1}{2}$-approximations for graphs of small size. For this purpose, the following simple result will be useful.

Lemma 1. *Let $G = (V, E)$ be a weighted complete graph with $|V| \geq 4$ and weight function h satisfying the triangle inequality. For any four vertices i, j, k, l, at least one of the following two inequalities holds :*

$$h(i, k) + h(j, l) \geq \frac{1}{2}(h(i, j) + h(k, l))$$

$$h(i, l) + h(j, k) \geq \frac{1}{2}(h(i, j) + h(k, l))$$

Proof. Using the triangle inequality, we have $h(i, k) + h(j, k) \geq h(i, j)$, $h(i, l) + h(j, l) \geq h(i, j)$, $h(i, k) + h(i, l) \geq h(k, l)$, and $h(j, k) + h(j, l) \geq h(k, l)$. Summing up these inequalities we get $h(i, k) + h(j, l) + h(i, l) + h(j, k) \geq h(i, j) + h(k, l)$. The result follows. □

Lemma 2. *For complete graphs with at most 6 vertices, with values on edges w and ℓ satisfying the triangle inequality, there exists a single matching M such that $w(M) \geq \frac{1}{2}w(M_w)$ and $\ell(M) \geq \frac{1}{2}\ell(M_\ell)$, where M_w and M_ℓ are optimal matchings for the weight and the length, respectively.*

Proof. Let $n \in \{3, 4, 5\}$ since the case $n = 2$ is trivial. Build two optimal matchings on K_n with respect to w and ℓ denoted respectively by M_w and M_ℓ. We assume wlog that the subgraph G' induced by $M_w \cup M_\ell$ is connected since otherwise we deal with connected components separately. Thus, the different cases are depicted in Figure 7.
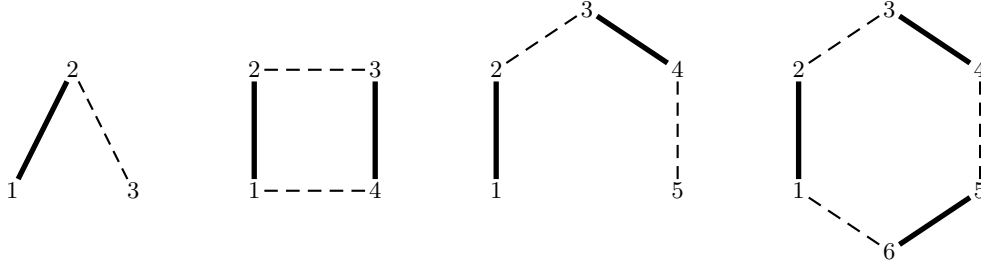
Fig. 7. The four cases $n = 3, 4, 5, 6$. Bold edges belong to M_w and dashed edges belong to M_ℓ.

Let us show now that for each of these cases, there exists a matching M satisfying $w(M) \geq \frac{1}{2}w(M_w)$ and $\ell(M) \geq \frac{1}{2}\ell(M_\ell)$.

- Case $n = 3$. G' is a path. We have $M_w = \{(1, 2)\}$ and $M_\ell = \{(2, 3)\}$. If $\ell(1, 2) \geq \ell(2, 3)/2$ then $M = \{(1, 2)\}$. If $w(2, 3) \geq w(1, 2)/2$ then $M = \{(1, 2)\}$. Otherwise, $M = \{(1, 3)\}$ since, by the triangle inequality on ℓ and w, $\ell(1, 2) < \ell(2, 3)/2$ and $w(2, 3) < w(1, 2)/2$ give $\ell(1, 3) \geq \ell(2, 3)/2$ and $w(1, 3) \geq w(1, 2)/2$, respectively.

- Case $n = 4$. G' is a cycle. We have $M_w = \{(1, 2), (3, 4)\}$ and $M_\ell = \{(1, 4), (2, 3)\}$. Using Lemma 1 for w, at least one of the following two inequalities holds :

$$w(1, 4) + w(2, 3) \geq \frac{1}{2}(w(1, 2) + w(3, 4)) \tag{9}$$

$$w(1, 3) + w(2, 4) \geq \frac{1}{2}(w(1, 2) + w(3, 4)) \tag{10}$$

Using Lemma 1 for ℓ, at least one of the following two inequalities holds :

$$\ell(1,2) + \ell(3,4) \geq \frac{1}{2}(\ell(1,4) + \ell(2,3)) \tag{11}$$

$$\ell(1,3) + \ell(2,4) \geq \frac{1}{2}(\ell(1,4) + \ell(2,3)) \tag{12}$$

If (9) holds then $M = \{(1,4),(2,3)\}$. If (11) holds then $M = \{(1,2),(3,4)\}$. Finally, if neither (9) nor (11) hold then both (10) and (12) must be true and we have $M = \{(1,3),(2,4)\}$.

- Case $n = 5$. G' is a path. We have $M_w = \{(1,2),(3,4)\}$ and $M_\ell = \{(2,3),(4,5)\}$.
  If $w(1,2) + w(4,5) \geq \frac{1}{2}(w(1,2) + w(3,4))$ and $\ell(1,2) + \ell(4,5) \geq \frac{1}{2}(\ell(2,3) + \ell(4,5))$, we have $M = \{(1,2),(4,5)\}$. Otherwise, we have:
  - either $w(1,2) + w(4,5) < \frac{1}{2}(w(1,2) + w(3,4))$. This implies $w(3,4) \geq \frac{1}{2}(w(1,2) + w(3,4))$ and $w(3,5) \geq \frac{1}{2}(w(1,2) + w(3,4))$, since otherwise the converse inequalities, summed up with the initial inequality, would lead respectively to $w(1,2) + w(4,5) + w(3,4) < w(1,2) + w(3,4)$ and $w(1,2) + w(4,5) + w(3,5) < w(1,2) + w(3,4)$ which are clearly wrong, using the triangle inequality in the second case. Moreover, using Lemma 1 for ℓ, we have either $\ell(2,5) + \ell(3,4) \geq \frac{1}{2}(\ell(2,3) + \ell(4,5))$, in which case $M = \{(2,5),(3,4)\}$, or $\ell(2,4) + \ell(3,5) \geq \frac{1}{2}(\ell(2,3) + \ell(4,5))$, in which case $M = \{(2,4),(3,5)\}$.
  - or $\ell(1,2) + \ell(4,5) < \frac{1}{2}(\ell(2,3) + \ell(4,5))$. This implies $\ell(2,3) \geq \frac{1}{2}(\ell(2,3) + \ell(4,5))$ and $\ell(1,3) \geq \frac{1}{2}(\ell(2,3) + \ell(4,5))$, since otherwise the converse inequalities, summed up with the initial inequality, would lead respectively to $\ell(1,2) + \ell(4,5) + \ell(2,3) < \ell(2,3) + \ell(4,5)$ and $\ell(1,2) + \ell(4,5) + \ell(1,3) < \ell(2,3) + \ell(4,5)$, which are clearly wrong, using the triangle inequality in the second case. Moreover, using Lemma 1 for w, we have either $w(1,4) + w(2,3) \geq \frac{1}{2}(w(1,2) + w(3,4))$, in which case $M = \{(1,4),(2,3)\}$, or $w(1,3) + w(2,4) \geq \frac{1}{2}(w(1,2) + w(3,4))$, in which case $M = \{(1,3),(2,4)\}$.

- Case $n = 6$. G' is a cycle. We have $M_w = \{(1,2),(3,4),(5,6)\}$ and $M_\ell = \{(1,6),(2,3),(4,5)\}$. Consider the three following matchings $M_1 = \{(1,2),(3,6),(4,5)\}$, $M_2 = \{(1,6),(2,5),(3,4)\}$, and $M_3 = \{(1,4),(2,3),(5,6)\}$. We first show that there exists $i \in \{1,2,3\}$ such that $\ell(M_i) \geq \frac{1}{2}\ell(M_\ell)$. By contradiction, assuming that $\ell(M_i) < \frac{1}{2}\ell(M_\ell)$ for all $i \in \{1,2,3\}$, we multiply by 2 each inequality and sum them up, which gives $2\ell(1,6) + (\ell(1,2) + \ell(2,5) + \ell(5,6)) + 2\ell(2,3) + (\ell(1,2) + \ell(1,4) + \ell(3,4)) + 2\ell(4,5) + (\ell(3,4) + \ell(3,6) + \ell(5,6)) + \ell(1,4) + \ell(2,5) + \ell(3,6) < 3\ell(1,6) + 3\ell(2,3) + 3\ell(4,5)$. Applying the triangle inequality to each term between brackets gives values respectively larger than or equal to $\ell(1,6)$, $\ell(2,3)$, and $\ell(4,5)$, contradicting the inequality.
  In the following, we assume wlog, after possibly renumbering the vertices, that $\ell(M_1) \geq \frac{1}{2}\ell(M_\ell)$. If $w(M_1) \geq \frac{1}{2}w(M_w)$ then $M = M_1$. If $w(M_1) < \frac{1}{2}w(M_w)$ then, by Lemma 1, we get $w(M_4) \geq \frac{1}{2}w(M_w)$ where $M_4 = \{(1,2),(3,5),(4,6)\}$. If $\ell(M_4) \geq \frac{1}{2}\ell(M_\ell)$ then $M = M_4$. Thus, we assume $\ell(M_4) < \frac{1}{2}\ell(M_\ell)$.

Consider now the two additional matchings $M_5 = \{(1,5),(2,6),(3,4)\}$ and $M_6 = \{(1,3),(2,4),(5,6)\}$. We show that the four following conditions hold.

$$\text{there exists } i \in \{4,5,6\} \text{ such that } \ell(M_i) \geq \frac{1}{2}\ell(M_\ell) \tag{13}$$

$$\text{there exists } i \in \{2,3,4\} \text{ such that } \ell(M_i) \geq \frac{1}{2}\ell(M_\ell) \tag{14}$$

$$\text{there exists } i \in \{2,4,6\} \text{ such that } \ell(M_i) \geq \frac{1}{2}\ell(M_\ell) \tag{15}$$

$$\text{there exists } i \in \{3,4,5\} \text{ such that } \ell(M_i) \geq \frac{1}{2}\ell(M_\ell) \tag{16}$$

Regarding (13), the real condition is: there exists $i \in \{4,5,6,w\}$ such that $\ell(M_i) \geq \frac{1}{2}\ell(M_\ell)$. Obviously, if $i = w$ then $M = M_w$, and therefore we can assume (13), provided that the real condition is true. Assume by contradiction that $\ell(M_i) < \frac{1}{2}\ell(M_\ell)$ for all $i \in \{4,5,6,w\}$. Then, adding up these four inequalities, we obtain $(\ell(1,2)+\ell(2,6))+(\ell(1,5)+\ell(5,6))+(\ell(1,2)+\ell(1,3))+(\ell(2,4)+\ell(3,4))+(\ell(3,4)+\ell(3,5))+(\ell(4,6)+\ell(5,6)) < 2\ell(1,6)+2\ell(2,3)+2\ell(4,5)$. Applying the triangle inequality to each term between brackets leads to contradict the inequality.

Regarding (14), we proceed again by contradiction, assuming that $\ell(M_i) < \frac{1}{2}\ell(M_\ell)$ for all $i \in \{2,3,4\}$. Multiplying by two the inequality for $i = 4$ and adding the two other inequalities, we obtain $\ell(1,6) + (\ell(1,4) + \ell(4,6)) + \ell(2,3) + (\ell(2,5) + \ell(3,5)) + (\ell(3,4) + \ell(3,5)) + (\ell(4,6) + \ell(5,6)) + 2\ell(1,2) < 2\ell(1,6) + 2\ell(2,3) + 2\ell(4,5)$. Applying the triangle inequality to each term between brackets leads to contradict the inequality.

Conditions (15) and (16) are established similarly (each time multiplying by 2 the inequality for $i = 4$).

Since $\ell(M_4) < \frac{1}{2}\ell(M_\ell)$, by (13) we have either $\ell(M_5) \geq \frac{1}{2}\ell(M_\ell)$ or $\ell(M_6) \geq \frac{1}{2}\ell(M_\ell)$. We conclude the proof by examining these two cases:

- Consider the case $\ell(M_5) \geq \frac{1}{2}\ell(M_\ell)$. If $w(M_5) \geq \frac{1}{2}w(M_w)$ then $M = M_5$. Otherwise, if $w(M_5) < \frac{1}{2}w(M_w)$ then by Lemma 1, we get $w(M_2) \geq \frac{1}{2}w(M_w)$. If $\ell(M_2) \geq \frac{1}{2}\ell(M_\ell)$ then $M = M_2$. If $\ell(M_2) < \frac{1}{2}\ell(M_\ell)$ then, by (14) and (15) we get respectively $\ell(M_3) \geq \frac{1}{2}\ell(M_\ell)$ and $\ell(M_6) \geq \frac{1}{2}\ell(M_\ell)$. If $w(M_3) \geq \frac{1}{2}w(M_w)$ then $M = M_3$. If $w(M_3) < \frac{1}{2}w(M_w)$ then, by Lemma 1, we have $w(M_6) \geq \frac{1}{2}w(M_w)$ and thus $M = M_6$.
- Consider finally the case $\ell(M_6) \geq \frac{1}{2}\ell(M_\ell)$. If $w(M_6) \geq \frac{1}{2}w(M_w)$ then $M = M_6$. Otherwise, if $w(M_6) < \frac{1}{2}w(M_w)$ then by Lemma 1, we get $w(M_3) \geq \frac{1}{2}w(M_w)$. If $\ell(M_3) \geq \frac{1}{2}\ell(M_\ell)$ then $M = M_3$. If $\ell(M_3) < \frac{1}{2}\ell(M_\ell)$ then, by (14) and (16) we get respectively $\ell(M_2) \geq \frac{1}{2}\ell(M_\ell)$ and $\ell(M_5) \geq \frac{1}{2}\ell(M_\ell)$. If $w(M_2) \geq \frac{1}{2}w(M_w)$ then $M = M_2$. If $w(M_2) < \frac{1}{2}w(M_w)$ then, by Lemma 1, we have $w(M_5) \geq \frac{1}{2}w(M_w)$ and thus $M = M_5$. □

Corollary 1. *We can build in polynomial time a single matching which constitutes a 1/3-approximate Pareto set for the biobjective Maximum Matching problem satisfying the triangle inequality on each objective.*

Proof. By using Theorem 2 with $\rho = 1/2$, $k = 3$ and Lemma 2 the result follows.   □

6   A computer-aided approach

Lemma 2 establishes the existence of 1/2-approximate matchings for complete graphs with at most $n = 6$ nodes. It seems difficult to establish a general result, and our proofs become more and more tedious as n grows. In order to test the existence of a single 1/2-approximate matching for larger values of n, we propose a mixed 0-1 linear programming formulation. Actually, this formulation provides, for any value of n, the best possible approximation ratio ρ^*.

To introduce this formulation, we need to consider, for a given n:

– the set $\mathcal{M}_n$ of all maximal (perfect or nearly perfect) matchings which can be defined on a complete graph of size n. Note that the number of such maximal matchings is $n!!$ when n is odd and $(n-1)!!$ when n is even[4].

– the set $\mathcal{I}_n^\Delta$ of all possible instances corresponding to a complete valued graph $G = (V, E)$ with $|V| = n$ where each edge $e \in E$ has two values $w(e)$ and $\ell(e)$ which satisfy the triangle inequality. We assume wlog that $0 \leq w(e) \leq 1$ and $0 \leq \ell(e) \leq 1$ for each edge $e \in E$. Moreover, we impose that M_w and M_ℓ, which are optimal matchings for w and ℓ respectively, are such that $w(M_w) = \ell(M_\ell) = 1$. If these conditions are not met, then compute $w(M_w)$ and $\ell(M_\ell)$, and for each edge e, replace $w(e)$ by $w(e)/w(M_w)$ and $\ell(e)$ by $\ell(e)/\ell(M_\ell)$. This transformation has no incidence on the approximation guarantee because it is a ratio. We also assume the worst case situation where the subgraph induced by $M_w \cup M_\ell$ is connected since otherwise we deal with connected components separately. Therefore, $M_w \cup M_\ell$ is either a cycle or a path which alternates edges of M_w and M_ℓ and covers all vertices (see Figure 7 for the cases $n = 3, 4, 5, 6$).

Our formulation aims at identifying an instance $I \in \mathcal{I}_n^\Delta$ minimizing a variable ρ such that for all matchings $M \in \mathcal{M}_n$ we have $w(M) \leq \rho$ or $\ell(M) \leq \rho$. Let ρ_n^* be the optimal value for variable ρ. Therefore, for any $\rho < \rho_n^*$, for any instance $I \in \mathcal{I}_n^\Delta$, there exists $M \in \mathcal{M}_n$ such that $w(M) > \rho$ and $\ell(M) > \rho$. Moreover, ρ_n^*, corresponding to a feasible solution, is such that $w(M) \leq \rho_n^*$ or $\ell(M) \leq \rho_n^*$ for all matchings $M \in \mathcal{M}_n$. It follows that ρ_n^* is the largest value such that, for any instance $I \in \mathcal{I}_n^\Delta$, there exists $M \in \mathcal{M}_n$ such that $w(M) \geq \rho_n^*$ and $\ell(M) \geq \rho_n^*$ (with at least one equality).

The mixed 0-1 linear program corresponding to the previous formulation is given by:

$$
(P_n) \begin{cases}
\min & \rho \\
\text{s.t.} \\
& w_{ij} \leq w_{ik} + w_{kj} \text{ and } \ell_{ij} \leq \ell_{ik} + \ell_{kj}, \quad \forall (i, j, k) \in V^3 & (17) \\
& \sum_{e \in M_w} w_e = 1 \text{ and } \sum_{e \in M_\ell} \ell_e = 1 & (18) \\
& \sum_{e \in M} w_e \leq 1 \text{ and } \sum_{e \in M} \ell_e \leq 1 \quad \forall M \in \mathcal{M}_n & (19) \\
& \sum_{e \in M} w_e \leq \rho + z_M, \quad \forall M \in \mathcal{M}_n & (20) \\
& \sum_{e \in M} \ell_e \leq \rho + 1 - z_M, \quad \forall M \in \mathcal{M}_n & (21) \\
& 0 \leq w_{ij} \leq 1 \text{ and } 0 \leq \ell_{ij} \leq 1, \quad \forall (i, j) \in E, \quad i < j & (22) \\
& z_M \in \{0, 1\}, \quad \forall M \in \mathcal{M}_n & (23) \\
& 0 \leq \rho \leq 1 & (24)
\end{cases}
$$

where continuous decision variables w_{ij} and ℓ_{ij}, defined in (22), represent an instance satisfying the triangle inequality through constraint (17), and 0-1 variables, defined in (23), are used to model the disjunctive constraints $w(M) \leq \rho$ or $\ell(M) \leq \rho$ imposed by (20) and (21), whereas constraints (18) - (19) define matchings M_w and M_ℓ as optimal matchings.

Owing to the quickly growing cardinality of $\mathcal{M}_n$, we were able to solve (P_n) for n up to 10, using CPLEX 12.6. The results are reported in Table 1. It is worth noting that these values coincide, up to the last (15th) computed decimal digit, with simple fractions larger than or equal to $1/2$ (see last row of Table 1).

---

[4] The *double factorial* $r!!$ is the factorial $r!$ restricted to numbers of the same parity as r, e.g. $5!! = 5 * 3 * 1$ and $8!! = 8 * 6 * 4 * 2$.

n	3	4	5	6	7	8	9	10
ρ_n^*	0.5	0.5	0.5	0.5556	0.5714	0.5	0.5714	0.6
	1/2	1/2	1/2	5/9	4/7	1/2	4/7	3/5

Table 1. The ratios obtained with the computational approach for $3 \le n \le 10$.

These computational results cannot be considered as totally valid owing to the limited precision of the computations. They suggest, however, the existence of 1/2-approximate matchings for complete graphs satisfying the triangle inequality, up to 10 nodes. Admitting this and using again Theorem 2, this improves the result of Corollary 1 about the approximation ratio from 1/3 to 2/5. More generally, these results tend to confirm that the best approximation could be 1/2.

7 Some upper bounds

In this section, we describe instances which give upper bounds on the ratio ρ such that a ρ-approximate matching is guaranteed to exist. In particular, we show that there is no $(\frac{1}{2} + \varepsilon)$-approximate algorithm, for all $\varepsilon > 0$, and that there is is no $(\frac{2}{3} + \varepsilon)$-approximate solution if we restrict ourselves to large graphs.

Let us define a family of instances containing $n = 4k$ nodes for any positive integer k (see Figure 8 for an illustration with $k = 2$), such that each instance does not admit any $(\frac{1}{2} + \varepsilon)$-approximate matching, for all $\varepsilon > 0$. The nodes are denoted by $\{1, \dots, 2k\} \cup \{1', \dots, 2k'\}$. The graph is complete and the edges are partitioned as follows :

- $E_w = \{\{i, i'\} | i \in \{1, \dots, 2k\} \text{ and } i' \in \{1', \dots, 2k'\}\}$. Each edge of E_w has value $(1/2k, 1/4k)$.
- $E_l = \{\{i, i+1\} | i \in \{1, \dots, 2k\} \text{ and } i \text{ is odd}\}$. Each edge of E_l has value $(0, 1/2k)$.
- $E_{l'} = \{\{i', i'+1\} | i' \in \{1', \dots, 2k'\} \text{ and } i' \text{ is odd}\}$. Each edge of $E_{l'}$ has value $(0, 1/2k)$.
- E_0 the set containing the remaining edges. Each edge of E_0 has value $(0, 1/4k)$.

We can take the $2k$ edges (i, i') and have a matching of maximal weight. We can take the $2k$ edges with value $(0, 1/2k)$ and have a matching with maximal length. The maximal weight and length are both equal to 1. It is not difficult to see that the triangle inequality is satisfied on both coordinates.

A Pareto optimal matching consists in picking $x \le k$ edges of value $(0, 1/2k)$ in E_l, x edges of value $(0, 1/2k)$ in $E_{l'}$ and $2(k-x)$ edges of value $(1/2k, 1/4k)$ in E_w. The resulting matching has weight $\frac{k-x}{k}$ and length $\frac{k+x}{2k}$. If $k/3$ is integral, then $x = k/3$ gives $\min\{\frac{k-x}{k}, \frac{k+x}{2k}\} = 2/3$; we have an upper bound of $2/3$. If $k/3$ is not integral, then taking $x = 0$ when $k < 3$ and $x = \lceil k/3 \rceil$ when $k \ge 3$ allows us to maximize $\min\{\frac{k-x}{k}, \frac{k+x}{2k}\}$. This gives an upper bound of $\max\{1/2, 2/3 - 1/k\}$.

We get the upper bounds (UB) provided in table 2 when the number of nodes n is equal to $4k$.

k	1	2	3	4	5	6	7
n	4	8	12	16	20	24	28
UB	1/2	1/2	2/3	1/2	1/2	2/3	11/21

Table 2. Upper bounds on the best approximation ratio for a tradeoff when $n = 4k$.
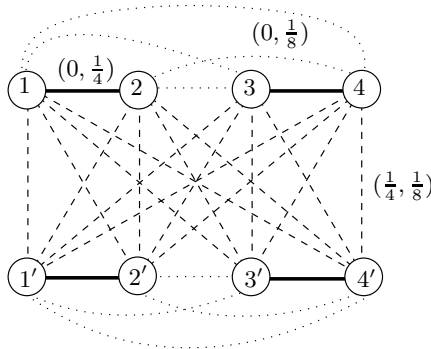
Fig. 8. An 8-node instance where no $(1/2 + \epsilon)$-approximate matching exists. Dotted edges have value $(0, \frac{1}{8})$, bold edges have value $(0, \frac{1}{4})$ and dashed edges have value $(\frac{1}{4}, \frac{1}{8})$.

8   Concluding remarks

In this article we were interested in determining, and computing, an approximate tradeoff for the bi-objective matching problem. We focused on the case where each objective satisfies the triangle inequality. We proposed a general method which, combined with results stating the existence of a good tradeoff in graphs of small size, implies the existence and computation of a good tradeoff in a graph of any size. For the existence of a good tradeoff in graphs of small size, we followed two approaches. The first one is analytical but it is limited to graphs containing at most 6 nodes. We believe that such an analysis can be conducted for slightly larger graphs. The second approach is a computer-aided determination of an approximation ratio ρ_n^*, such that any complete graph on n nodes must admit a ρ_n^*-approximate matching. The values manipulated in the computational approach have a limited precision so we cannot pretend that the given ratios hold in any case. However, they provide an insight in what ρ_n^* should be equal to.

More generally, we conjecture that a $1/2$-approximate matching exists in any bi-objective instance. We followed a constructive path but an interesting future direction is to consider this question, but only under an existential approach.

Our last observation deals with how ρ_n^* depends on the number of nodes n. From the results given in sections 6 and 7, we observe that ρ_n^* is surprisingly non-monotone with n.

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