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Laplacian, on the Sierpiński tetrahedron

Nizare Riane, Claire David

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Sorbonne Universités, UPMC Univ Paris 06
CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, 4, place Jussieu 75005, Paris, France

Abstract

Numerous work revolve around the Sierpiński gasket. Its three-dimensional analogue, the Sierpiński tetrahedron $\mathfrak{S}\mathfrak{T}$. In [5], the authors discuss the existence of the Laplacian on $\mathfrak{S}\mathfrak{T}$. In this work, We go further and give we give the explicit spectrum of the Laplacian, with a detailed study of the first eigenvalues. This enables us to obtain an estimate of the spectral counting function, by applying the results given in [7], [8].

Keywords: Laplacian - Sierpiński tetrahedron.

AMS Classification: 37F20- 28A80-05C63.

1 Introduction

The Laplacian plays a major role in the mathematical analysis of partial differential equations. Recently, the work of J. Kigami [1], [2], taken up by R. S. Strichartz [3], [4], allowed the construction of an operator of the same nature, defined locally, on graphs having a fractal character: the triangle of Sierpiński, the carpet of Sierpiński, the diamond fractal, the Julia sets, the fern of Barnsley.

J. Kigami starts from the definition of the Laplacian on the unit segment of the real line. For a double-differentiable function u on $[0, 1]$, the Laplacian Δu is obtained as a second derivative of u on $[0, 1]$. For any pair (u, v) belonging to the space of functions that are differentiable on $[0, 1]$, such that:

$$v(0) = v(1) = 0$$

he puts the light on the fact that, taking into account:

$$\int_0^1 (\Delta u)(x) v(x) dx = - \int_0^1 u'(x) v'(x) dx = - \lim_{n \rightarrow +\infty} \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} u'(x) v'(x) dx$$

if $\varepsilon > 0$, the continuity of u' and v' shows the existence of a natural rank n_0 such that, for any integer $n \geq n_0$, and any real number x of $\left[\frac{k-1}{n}, \frac{k}{n}\right]$, $1 \leq k \leq n$:

$$\left| u'(x) - \frac{u\left(\frac{k}{n}\right) - u\left(\frac{k-1}{n}\right)}{\frac{1}{n}} \right| \leq \varepsilon \quad , \quad \left| v'(x) - \frac{v\left(\frac{k}{n}\right) - v\left(\frac{k-1}{n}\right)}{\frac{1}{n}} \right| \leq \varepsilon$$

the relation:

$$\int_0^1 (\Delta u)(x) v(x) dx = - \lim_{n \rightarrow +\infty} n \sum_{k=1}^n \left(u\left(\frac{k}{n}\right) - u\left(\frac{k-1}{n}\right) \right) \left(v\left(\frac{k}{n}\right) - v\left(\frac{k-1}{n}\right) \right)$$

enables one to define, under a weak form, the Laplacian of u , while avoiding first derivatives. It thus opens the door to Laplacians on fractal domains.

Concretely, the weak formulation is obtained by means of Dirichlet forms, built by induction on a sequence of graphs that converges towards the considered domain. For a continuous function on this domain, its Laplacian is obtained as the renormalized limit of the sequence of graph Laplacians.

Numerous work revolve around the Sierpiński gasket. Its three-dimensional analogue, the Sierpiński tetrahedron \mathfrak{ST} , obtained by means of an iterative process which consists in repeatedly contracting a regular 3-simplex to one half of its original height, put together four copies, the frontier corners of which coincide with the initial simplex, appears as a natural extension. Yet, very few works concern \mathfrak{ST} in the existing literature. In [5], the authors discuss the existence of the Laplacian on \mathfrak{ST} . Yet, they do not give what appears to be of the higher importance, i.e. the spectrum of the Laplacian. In [6], generalizations of the Sierpiński gasket to higher dimensions are considered. Yet, despite interesting results, there are a few mistakes, and no study at all of the Dirichlet forms, whereas they are the obligatory passage to the determination of the Laplacian.

We go further and, after a detailed study, we give the explicit spectrum of the Laplacian, with a specific presentation of the first eigenvalues. This enables us to obtain an estimate of the spectral counting function (analogous of Weyl's law), by applying the results given in [7], [8].

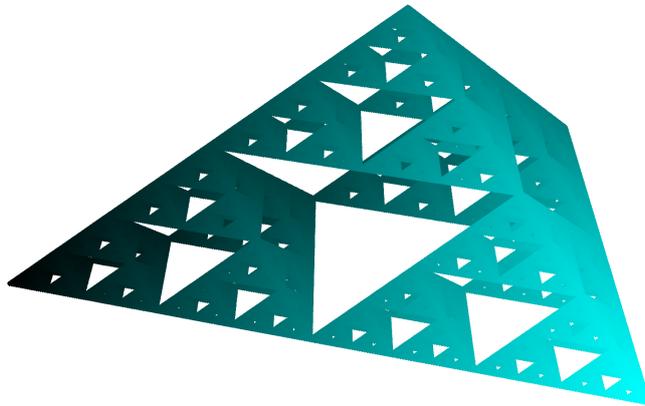


Figure 1: Sierpiński tetrahedron.

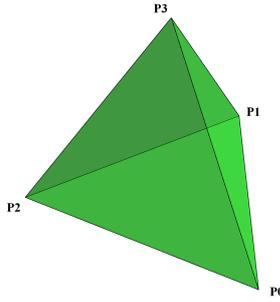


Figure 2: The initial tetrahedron.

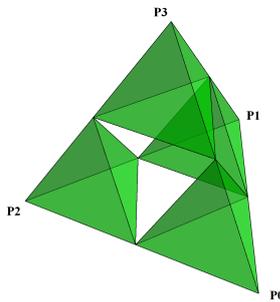


Figure 3: The tetrahedron after one iteration.

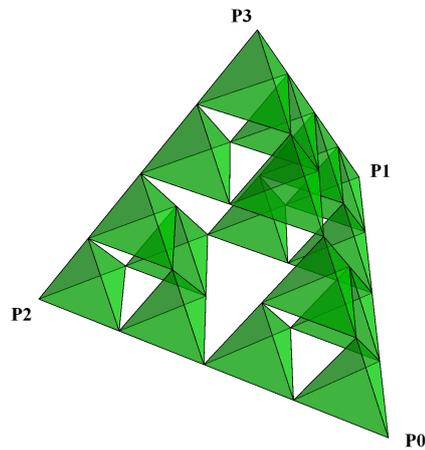


Figure 4: The tetrahedron after two iterations.

The Sierpiński tetrahedron is a self-similar set which has many beautiful properties.

2 Self-similar Sierpiński tetrahedron

We place ourselves, in the following, in the euclidian space of dimension 3, referred to a direct orthonormal frame. The usual Cartesian coordinates are (x, y, z) .

Let us denote by P_0, P_1, P_2, P_3 the points:

$$P_0 = (0, 0, 0) \quad , \quad P_1 = (6^{\frac{1}{3}}, 0, 6^{\frac{1}{3}}) \quad , \quad P_2 = (0, 6^{\frac{1}{3}}, 6^{\frac{1}{3}}) \quad , \quad P_3 = (0, 0, 6^{\frac{1}{3}})$$

Let us introduce the iterated function system of the family of maps from \mathbb{R}^2 to \mathbb{R}^2 :

$$\{f_0, \dots, f_3\}$$

where, for any integer i belonging to $\{0, \dots, 3\}$, and any $X \in \mathbb{R}^2$:

$$f_i(X) = \frac{X + P_i}{2}$$

Remark 2.1. The family $\{f_0, \dots, f_3\}$ is a family of contractions from \mathbb{R}^2 to \mathbb{R}^2 , the ratio of which is:

$$\frac{1}{2}$$

For any integer i belonging to $\{0, \dots, 3\}$, P_i is the fixed point of f_i .

Property 2.1. According to [9], there exists a unique subset $\mathfrak{S}\mathfrak{T} \subset \mathbb{R}^3$ such that:

$$\mathfrak{S}\mathfrak{T} = \bigcup_{i=0}^3 f_i(\mathfrak{S}\mathfrak{T})$$

which will be called the Sierpiński tetrahedron.

For the sake of simplicity, we set:

$$F = \bigcup_{i=0}^3 f_i$$

Definition 2.1. Hausdorff dimension of the Sierpiński tetrahedron $\mathfrak{S}\mathfrak{T}$

The Hausdorff dimension of the Sierpiński tetrahedron $\mathfrak{S}\mathfrak{T}$ is:

$$D_{\mathfrak{S}\mathfrak{T}} = \ln_2 4 = \frac{\ln 4}{\ln 2} = 2$$

Definition 2.2. We will denote by V_0 the ordered set, of the points:

$$\{P_0, \dots, P_3\}$$

The set of points V_0 , where, for any i of $\{0, 1, 2\}$, the point P_i is linked to the point P_{i+1} , constitutes an oriented graph, that we will denote by $\mathfrak{S}\mathfrak{T}_0$. V_0 is called the set of vertices of the graph $\mathfrak{S}\mathfrak{T}_0$.

For any natural integer m , we set:

$$V_m = \bigcup_{i=0}^3 f_i(V_{m-1})$$

The set of points V_m , where two consecutive points are linked, is an oriented graph, which we will denote by $\mathfrak{S}\mathfrak{T}_m$. V_m is called the set of vertices of the graph $\mathfrak{S}\mathfrak{T}_m$. We will denote, in the following, by \mathcal{N}_m the number of vertices of the graph $\mathfrak{S}\mathfrak{T}_m$.

Property 2.2. *The set of vertices $(V_m)_{m \in \mathbb{N}}$ is dense in \mathfrak{ST} .*

Proposition 2.3. *Given a natural integer m , we will denote by \mathcal{N}_m the number of vertices of the graph \mathfrak{ST}_m . One has then, for any pair of integers $(i, j) \in \{0, \dots, 3\}^2$:*

$$f_i(P_j) = f_j(P_i)$$

and

$$\mathcal{N}_m = 4 \times \mathcal{N}_{m-1} - 6$$

Proof. Let us note that, for any pair of integers $(i, j) \in \{0, 1, 2, 3\}^2$:

$$f_i(P_j) = \frac{1}{2} (P_j + P_i) = f_j(P_i)$$

Thus, six points appear twice in \mathfrak{ST}_m .

The second point results from the fact that the graph \mathfrak{ST}_m is obtained by applying four similarities to \mathfrak{ST}_{m-1} . □

Definition 2.3. Consecutive vertices of \mathfrak{ST}

Two points X and Y of \mathfrak{ST} will be called *consecutive vertices* of \mathfrak{ST} if there exists a natural integer m , and an integer j of $\{0, 1, 2, 3\}$, such that:

$$X = (f_{i_1} \circ \dots \circ f_{i_m})(P_j) \quad \text{and} \quad Y = (f_{i_1} \circ \dots \circ f_{i_m})(P_{j+1}) \quad \{i_1, \dots, i_m\} \in \{0, 1, 2, 3\}^m$$

Definition 2.4. Polyhedral domain delimited by \mathfrak{ST}_m , $m \in \mathbb{N}$

For any natural integer m , we will call **polyhedral domain delimited by \mathfrak{ST}_m** , and denote by $\mathcal{D}(\mathfrak{ST}_m)$, the reunion of the 4^m tetrahedra of \mathfrak{ST}_m .

Definition 2.5. Polyhedral domain delimited by the Sierpiński Tetrahedron \mathfrak{ST}

We will call **polyhedral domain delimited by \mathfrak{ST}** , and denote by $\mathcal{D}(\mathfrak{ST})$, the limit:

$$\mathcal{D}(\mathfrak{ST}) = \lim_{n \rightarrow +\infty} \mathcal{D}(\mathfrak{ST}_n)$$

Definition 2.6. Word, on \mathfrak{ST}

Let m be a strictly positive integer. We will call **number-letter** any integer \mathcal{W}_i of $\{0, 1, 2, 3\}$, and **word of length $|\mathcal{W}| = m$** , on the graph \mathfrak{ST} , any set of number-letters of the form:

$$\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_m)$$

We will write:

$$f_{\mathcal{W}} = f_{\mathcal{W}_1} \circ \dots \circ f_{\mathcal{W}_m}$$

Definition 2.7. Edge relation, on $\mathfrak{S}\mathfrak{T}$

Given a natural integer m , two points X and Y of $\mathfrak{S}\mathfrak{T}_m$ will be said to be **adjacent** if and only if X and Y are two consecutive vertices of $\mathfrak{S}\mathfrak{T}_m$. We will write:

$$X \underset{m}{\sim} Y$$

This edge relation ensures the existence of a word $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_m)$ of length m , such that X and Y both belong to the iterate:

$$f_{\mathcal{W}} V_0 = (f_{\mathcal{W}_1} \circ \dots \circ f_{\mathcal{W}_m}) V_0$$

Given two points X and Y of the graph $\mathfrak{S}\mathfrak{T}$, we will say that X and Y are **adjacent** if and only if there exists a natural integer m such that:

$$X \underset{m}{\sim} Y$$

Proposition 2.4. Adresses, on the the Sierpiński tetrahedron

Given a strictly positive integer m , and a word $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_m)$ of length $m \in \mathbb{N}^*$, on the graph $\mathfrak{S}\mathfrak{T}_m$, for any integer j of $\{0, \dots, 3\}$, any $X = f_{\mathcal{W}}(P_j)$ of $V_m \setminus V_0$, i.e. distinct from one of the four fixed point P_i , $0 \leq i \leq 3$, has exactly three adjacent vertices in $f_{\mathcal{W}}(V_0)$, given by:

$$f_{\mathcal{W}}(P_{j+n \pmod{4}}) \quad \text{for } n \in \{1, 2, 3\}$$

where:

$$f_{\mathcal{W}} = f_{\mathcal{W}_1} \circ \dots \circ f_{\mathcal{W}_m}$$

Proposition 2.5. Given a natural integer m , a point $X \in \mathfrak{S}\mathfrak{T}_m$ and a word \mathcal{W} of length m such that:

$$\begin{aligned} \mathcal{W} &= \mathcal{W}_1 \mathcal{W}_2 \dots \mathcal{W}_{m-1} \mathcal{W}_m \\ X &= F_{\mathcal{W}}(P_i) \in V_m \setminus V_0 \quad , \quad i \in \{0, 1, 2, 3\} \end{aligned}$$

let us write, for any integer $\mathcal{W}_m \in \{0, 1, 2, 3\}n \mathcal{W}_m := j$, so:

$$\mathcal{W} = \mathcal{W}_1 \mathcal{W}_2 \dots \mathcal{W}_{m-1} j = \tilde{\mathcal{W}} j$$

Then $\tilde{\mathcal{W}} \in \{0, 1, 2, 3\}^{m-1}$. The point X has exactly six adjacent vertices, of the form

$$f_{\tilde{\mathcal{W}} j}(P_k)$$

and

$$f_{\tilde{\mathcal{W}} i}(P_l)$$

for $k \neq i$ and $l \neq j$.

Proof. x belongs to $f_{\mathcal{W}}(V_0)$, so it has three "neighbors" or adjacent vertices of the form $f_{\tilde{\mathcal{W}}_j}(p_k)$ for $k \neq i$. And we recall that $f_{\tilde{\mathcal{W}}_j}(p_i) = f_{\tilde{\mathcal{W}}_i}(p_j)$, so x belong to $f_{\tilde{\mathcal{W}}_i}(V_0)$ too, so it has three other neighbors $f_{\tilde{\mathcal{W}}_l}(p_l)$ for $l \neq i$. □

Proposition 2.6. *Let us set:*

$$V_\star = \bigcup_{m \in \mathbb{N}} V_m$$

The set V_\star is dense in $\mathfrak{S}\mathfrak{T}$.

3 Dirichlet forms on the Sierpiński tetrahedron

Following J. Kigami's approach [2], Dirichlet forms and Laplacian on the Sierpiński tetrahedron can be respectively defined as limits of Dirichlet forms and Laplacians on $(V_m)_{m \in \mathbb{N}}$.

Definition 3.1. **Dirichlet form, on a finite set** ([12])

Let V denote a finite set V , equipped with the usual inner product which, to any pair (u, v) of functions defined on V , associates:

$$(u, v) = \sum_{P \in V} u(P) v(P)$$

A *Dirichlet form* on V is a symmetric bilinear form \mathcal{E} , such that:

1. For any real valued function u defined on V : $\mathcal{E}(u, u) \geq 0$.
2. $\mathcal{E}(u, u) = 0$ if and only if u is constant on V .
3. For any real-valued function u defined on V , if:

$$u_\star = \min(\max(u, 0), 1)$$

i.e. :

$$\forall p \in V : u_\star(p) = \begin{cases} 1 & \text{if } u(p) \geq 1 \\ u(p) & \text{si } 0 < u(p) < 1 \\ 0 & \text{if } u(p) \leq 0 \end{cases}$$

then: $\mathcal{E}(u_\star, u_\star) \leq \mathcal{E}(u, u)$ (Markov property).

Definition 3.2. **Energy, on the graph $\mathfrak{S}\mathfrak{T}_m$, $m \in \mathbb{N}$, of a pair of functions**

Let m be a natural integer, and u and v two real valued functions, defined on the set:

$$V_m = \{\mathcal{S}_0^m, \mathcal{S}_1^m, \dots, \mathcal{S}_{\mathcal{N}_m-1}^m\}$$

of the \mathcal{N}_m vertices of $\mathfrak{S}\mathfrak{T}_m$.

The energy, on the graph $\mathfrak{S}\mathfrak{T}_m$, of the pair of functions (u, v) , is:

$$E_{\mathfrak{S}\mathfrak{T}_m}(u, v) = \sum_{i=0}^{\mathcal{N}_m-2} (u(\mathcal{S}_i^m) - u(\mathcal{S}_{i+1}^m)) (v(\mathcal{S}_i^m) - v(\mathcal{S}_{i+1}^m))$$

or

$$E_{\mathfrak{S}\mathfrak{T}_m}(u, v) = \sum_{X \underset{m}{\sim} Y} (u(X) - u(Y)) (v(X) - v(Y))$$

Let us note that

$$E_{\mathfrak{S}\mathfrak{T}_m}(u, u) = 0 \quad \text{if } u \text{ is constant}$$

$E_{\mathfrak{S}\mathfrak{T}_m}$ is a Dirichlet form on $\mathfrak{S}\mathfrak{T}_m$.

Proposition 3.1. Harmonic extension of a function, on the Sierpiński Tetrahedron

For any strictly positive integer m , if u is a real-valued function defined on V_{m-1} , its **harmonic extension**, denoted by \tilde{u} , is obtained as the extension of u to V_m which minimizes the energy:

$$\mathcal{E}_{\mathfrak{S}\mathfrak{T}_m}(\tilde{u}, \tilde{u}) = \sum_{X \underset{m}{\sim} Y} (\tilde{u}(X) - \tilde{u}(Y))^2$$

The link between $\mathcal{E}_{\mathfrak{S}\mathfrak{T}_m}$ and $\mathcal{E}_{\mathfrak{S}\mathfrak{T}_{m-1}}$ is obtained through the introduction of two strictly positive constants r_m, r_{m+1} , such that:

$$r_m \sum_{X \underset{m}{\sim} Y} (\tilde{u}(X) - \tilde{u}(Y))^2 = r_{m-1} \sum_{X \underset{m-1}{\sim} Y} (u(X) - u(Y))^2$$

In particular:

$$r_1 \sum_{X \underset{1}{\sim} Y} (\tilde{u}(X) - \tilde{u}(Y))^2 = r_0 \sum_{X \underset{0}{\sim} Y} (u(X) - u(Y))^2$$

For the sake of simplicity, we will fix the value of the initial constant: $r_0 = 1$. One has then:

$$\mathcal{E}_{\mathfrak{S}\mathfrak{T}_m}(\tilde{u}, \tilde{u}) = \frac{1}{r_1} \mathcal{E}_{\mathfrak{S}\mathfrak{T}_0}(\tilde{u}, \tilde{u})$$

Let us set:

$$r = \frac{1}{r_1}$$

and:

$$\mathcal{E}_m(u) = r_m \sum_{X \underset{m}{\sim} Y} (\tilde{u}(X) - \tilde{u}(Y))^2$$

Since the determination of the harmonic extension of a function appears to be a local problem, on the graph $\mathfrak{S}\mathfrak{T}_{m-1}$, which is linked to the graph $\mathfrak{S}\mathfrak{T}_m$ by a similar process as the one that links $\mathfrak{S}\mathfrak{T}_1$ to $\mathfrak{S}\mathfrak{T}_0$, one deduces, for any strictly positive integer m :

$$\mathcal{E}_{\mathfrak{S}\mathfrak{T}_m}(\tilde{u}, \tilde{u}) = \frac{1}{r_1} \mathcal{E}_{\mathfrak{S}\mathfrak{T}_{m-1}}(\tilde{u}, \tilde{u})$$

By induction, one gets:

$$r_m = r_1^m r_0 = r^{-m}$$

If v is a real-valued function, defined on V_{m-1} , of harmonic extension \tilde{v} , we will write:

$$\mathcal{E}_m(u, v) = r^{-m} \sum_{X \sim_m Y} (\tilde{u}(X) - \tilde{u}(Y)) (\tilde{v}(X) - \tilde{v}(Y))$$

For further precision on the construction and existence of harmonic extensions, we refer to [10].

Definition 3.3. Dirichlet form, for a pair of continuous functions defined on the Sierpiński tetrahedron $\mathfrak{S}\mathfrak{T}$

We define the Dirichlet form \mathcal{E} which, to any pair of real-valued, continuous functions (u, v) defined on $\mathfrak{S}\mathfrak{T}$, associates, subject to its existence:

$$\mathcal{E}(u, v) = \lim_{m \rightarrow +\infty} \mathcal{E}_m(u|_{V_m}, v|_{V_m}) = \lim_{m \rightarrow +\infty} \sum_{X \sim_m Y} r^{-m} (u|_{V_m}(X) - u|_{V_m}(Y)) (v|_{V_m}(X) - v|_{V_m}(Y))$$

Definition 3.4. Normalized energy, for a continuous function u , defined on $\mathfrak{S}\mathfrak{T}$

Taking into account that the sequence $(\mathcal{E}_m(u|_{V_m}))_{m \in \mathbb{N}}$ is defined on

$$V_\star = \bigcup_{i \in \mathbb{N}} V_i$$

one defines the normalized energy, for a continuous function u , defined on $\mathfrak{S}\mathfrak{T}$, by:

$$\mathcal{E}(u) = \lim_{m \rightarrow +\infty} \mathcal{E}_m(u|_{V_m})$$

Property 3.2. *The Dirichlet form \mathcal{E} which, to any pair of real-valued, continuous functions defined on $\mathfrak{S}\mathfrak{T}$, associates:*

$$\mathcal{E}(u, v) = \lim_{m \rightarrow +\infty} \mathcal{E}_m(u|_{V_m}, v|_{V_m}) = \lim_{m \rightarrow +\infty} \sum_{X \sim_m Y} r^{-m} (u|_{V_m}(X) - u|_{V_m}(Y)) (v|_{V_m}(X) - v|_{V_m}(Y))$$

satisfies the self-similarity relation:

$$\mathcal{E}(u, v) = r^{-1} \sum_{i=0}^3 \mathcal{E}(u \circ f_i, v \circ f_i)$$

Proof.

$$\begin{aligned}
\sum_{i=0}^3 \mathcal{E}(u \circ f_i, v \circ f_i) &= \lim_{m \rightarrow +\infty} \sum_{i=0}^3 \mathcal{E}_m(u|_{V_m} \circ f_i, v|_{V_m} \circ f_i) \\
&= \lim_{m \rightarrow +\infty} \sum_{X \sim_m Y} r^{-m} \sum_{i=0}^3 (u|_{V_m}(f_i(X)) - u|_{V_m}(f_i(Y))) (v|_{V_m}(f_i(X)) - v|_{V_m}(f_i(Y))) \\
&= \lim_{m \rightarrow +\infty} \sum_{X \sim_{m+1} Y} r^{-m} \sum_{i=0}^3 (u|_{V_{m+1}}(X) - u|_{V_{m+1}}(Y)) (v|_{V_{m+1}}(X) - v|_{V_{m+1}}(Y)) \\
&= \lim_{m \rightarrow +\infty} r \mathcal{E}_{m+1}(u|_{V_{m+1}}, v|_{V_{m+1}}) \\
&= r \mathcal{E}(u, v)
\end{aligned}$$

□

Notation. We will denote by $\text{dom } \mathcal{E}$ the subspace of continuous functions defined on \mathfrak{ST} , such that:

$$\mathcal{E}(u) < +\infty$$

Notation. We will denote by $\text{dom}_0 \mathcal{E}$ the subspace of continuous functions defined on \mathfrak{ST} , which take the value 0 on V_0 , such that:

$$\mathcal{E}(u) < +\infty$$

Proposition 3.3. *The space $\text{dom} \mathcal{E}$, modulo the space of constant function on \mathfrak{ST} , is a Hilbert space.*

4 Explicit construction of the Dirichlet forms

Let us denote by u a real valued function defined on:

$$V_0 = \{P_0, P_1, P_2, P_3\}$$

We herafter aim at determining its harmonic extension \tilde{u} on V_1 .

For the sake of simplicity, we set:

$$u(p_0) = a \quad , \quad u(p_1) = b \quad , \quad u(p_2) = c \quad , \quad u(p_3) = d$$

One has to bear in mind that the energy on V_0 is given by:

$$E_0(u) = (a - b)^2 + (a - c)^2 + (a - d)^2 + (b - c)^2 + (b - d)^2 + (c - d)^2$$

For the sake of simplicity, we set:

$$\tilde{u}(f_0(q_1)) = x_1 \quad , \quad \tilde{u}(f_1(q_2)) = x_2 \quad , \quad \tilde{u}(f_0(q_2)) = x_3 \quad , \quad \tilde{u}(f_0(q_3)) = x_4 \quad , \quad \tilde{u}(f_1(q_3)) = x_5 \quad , \quad \tilde{u}(f_2(q_3)) = x_6$$

Thus:

$$\begin{aligned}
E_1(\tilde{u}) &= (x_1 - a)^2 + (x_1 - b)^2 + (x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_1 - x_4)^2 + (x_1 - x_5)^2 \\
&+ (x_2 - b)^2 + (x_2 - c)^2 + (x_2 - x_3)^2 + (x_2 - x_5)^2 + (x_2 - x_6)^2 \\
&+ (x_3 - a)^2 + (x_3 - c)^2 + (x_3 - x_4)^2 + (x_3 - x_6)^2 \\
&+ (x_4 - a)^2 + (x_4 - d)^2 + (x_4 - x_5)^2 + (x_4 - x_6)^2 \\
&+ (x_5 - b)^2 + (x_5 - d)^2 + (x_5 - x_6)^2 \\
&+ (x_6 - c)^2 + (x_6 - d)^2
\end{aligned}$$

The minimum of this quantity is to be obtained in the set of critical points, which leads to:

$$\begin{aligned}
6x_1 - x_2 - x_3 - x_4 - x_5 &= a + b \\
6x_2 - x_1 - x_3 - x_5 - x_6 &= b + c \\
6x_3 - x_1 - x_2 - x_4 - x_6 &= a + c \\
6x_4 - x_1 - x_3 - x_5 + x_6 &= a + d \\
6x_5 - x_1 - x_2 - x_4 - x_6 &= b + d \\
6x_6 - x_2 - x_3 - x_4 - x_5 &= c + d
\end{aligned}$$

Under matricial form:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

with

$$\mathbf{A} = \begin{pmatrix} 6 & -1 & -1 & -1 & -1 & 0 \\ -1 & 6 & -1 & 0 & -1 & -1 \\ -1 & -1 & 6 & -1 & 0 & -1 \\ -1 & 0 & -1 & 6 & -1 & -1 \\ -1 & -1 & 0 & -1 & 6 & -1 \\ 0 & -1 & -1 & -1 & -1 & 6 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} a + b \\ b + c \\ a + c \\ a + d \\ b + d \\ c + d \end{pmatrix}$$

Finally, we get:

$$\mathbf{x} = \begin{pmatrix} \frac{1}{6}(2a + 2b + c + d) \\ \frac{1}{6}(a + 2b + 2c + d) \\ \frac{1}{6}(2a + b + 2c + d) \\ \frac{1}{6}(2a + b + c + 2d) \\ \frac{1}{6}(a + 2b + c + 2d) \\ \frac{1}{6}(a + b + 2(c + d)) \end{pmatrix}$$

By substituting \mathbf{X} in the energy, one obtains:

$$\begin{aligned} E_1(\tilde{u}) &= \left\{ \frac{2}{3} (3a^2 - 2a(b+c+d) + 3b^2 - 2b(c+d) + 3c^2 - 2cd + 3d^2) \right\} \\ &= \frac{2}{3} E_0(u) \end{aligned}$$

Let us now move to the general case, and consider a natural integer m . Each point of $V_{m+1} \setminus V_m$ belongs to a m -cell of the form

$$f_{\mathcal{W}}(\mathfrak{S}\mathfrak{T})$$

where \mathcal{W} denotes a word of length m . The total energy $E_{m+1}(\tilde{u})$ is given by:

$$E_{m+1}(\tilde{u}) = \sum_{|\mathcal{W}|=m} E_1(\tilde{u} \circ f_{\mathcal{W}})$$

The global minimization problem can be reduced to 4^m local minimization problems which are of the same kind of the one we just solved.

Thus, the normalization constant is.

$$r = \frac{2}{3}$$

This enables us to define the normalized energy:

$$\mathcal{E}_m(u) = r^{-m} E_m(u)$$

and its limit:

$$\mathcal{E}(u) = \lim_{m \rightarrow +\infty} \mathcal{E}_m(u)$$

for $u \in \text{dom}(\mathcal{E})$.

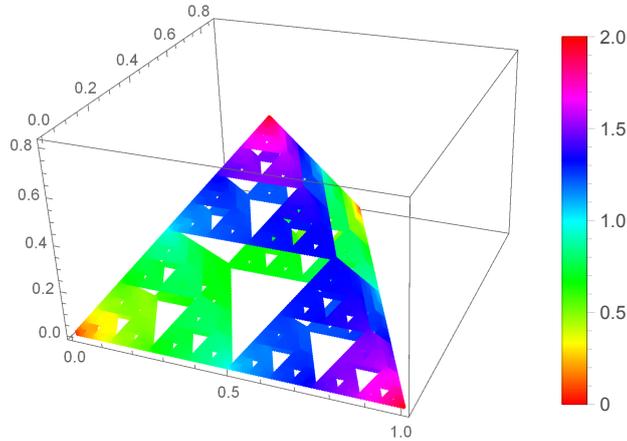


Figure 5: The harmonic extension on the Sierpiński tetrahedron of a function taking the values $a = 0$, $b = 2$, $c = 0$ and $d = 2$.

5 Laplacian, on the Sierpiński tetrahedron

Definition 5.1. Self-similar measure, on the Sierpiński tetrahedron

A measure μ on \mathbb{R}^3 will be said to be **self-similar** on the Sierpiński tetrahedron, if there exists a family of strictly positive pounds $(\mu_i)_{0 \leq i \leq 3}$ such that:

$$\mu = \sum_{i=0}^3 \mu_i \mu \circ f_i^{-1} \quad , \quad \sum_{i=0}^3 \mu_i = 1$$

For further precisions on self-similar measures, we refer to the works of J. E. Hutchinson (see [9]).

Property 5.1. *Building of a self-similar measure, for the Sierpiński tetrahedron*

The Dirichlet forms mentioned in the above require a positive Radon measure with full support. Let us set, for any integer i belonging to $\{0, \dots, 3\}$:

$$\mu_i = \frac{1}{4}$$

This enables one to define a self-similar measure μ on \mathfrak{ST} as:

$$\mu = \frac{1}{4} \sum_{i=0}^3 \mu \circ f_i$$

Definition 5.2. **Laplacian of order $m \in \mathbb{N}^*$**

For any strictly positive integer m , and any real-valued function u , defined on the set V_m of the vertices of the graph \mathfrak{ST}_m , we introduce the Laplacian of order m , $\Delta_m(u)$, by:

$$\Delta_m u(X) = \sum_{Y \in V_m, Y \sim_m X} (u(Y) - u(X)) \quad \forall X \in V_m \setminus V_0$$

Definition 5.3. **Harmonic function of order $m \in \mathbb{N}^*$**

Let m be a strictly positive integer. A real-valued function u , defined on the set V_m of the vertices of the graph \mathfrak{ST}_m , will be said to be **harmonic of order m** if its Laplacian of order m is null:

$$\Delta_m u(X) = 0 \quad \forall X \in V_m \setminus V_0$$

Definition 5.4. **Piecewise harmonic function of order $m \in \mathbb{N}^*$**

Given a strictly positive integer m , a real valued function u , defined on the set of vertices of \mathfrak{ST} , is said to be **piecewise harmonic function of order m** if, for any word \mathcal{W} of length m , $u \circ f_{\mathcal{W}}$ is harmonic of order m .

Definition 5.5. Existence domain of the Laplacian, for a continuous function on \mathfrak{ST} (see [11])

We will denote by $\text{dom } \Delta$ the existence domain of the Laplacian, on the graph \mathfrak{ST} , as the set of functions u of $\text{dom } \mathcal{E}$ such that there exists a continuous function on \mathfrak{ST} , denoted Δu , that we will call **Laplacian of u** , such that :

$$\mathcal{E}(u, v) = - \int_{\mathcal{D}(\mathfrak{ST})} v \Delta u d\mu \quad \text{for any } v \in \text{dom}_0 \mathcal{E}$$

Definition 5.6. Harmonic function

A function u belonging to $\text{dom } \Delta$ will be said to be **harmonic** if its Laplacian is equal to zero.

Notation. In the following, we will denote by $\mathcal{H}_0 \subset \text{dom } \Delta$ the space of harmonic functions, i.e. the space of functions $u \in \text{dom } \Delta$ such that:

$$\Delta u = 0$$

Given a natural integer m , we will denote by $\mathcal{S}(\mathcal{H}_0, V_m)$ the space, of dimension 4^m , of spline functions "of level m ", u , defined on \mathfrak{ST} , continuous, such that, for any word \mathcal{W} of length m , $u \circ T_{\mathcal{W}}$ is harmonic, i.e.:

$$\Delta_m (u \circ T_{\mathcal{W}}) = 0$$

Property 5.2. For any natural integer m :

$$\mathcal{S}(\mathcal{H}_0, V_m) \subset \text{dom } \mathcal{E}$$

Property 5.3. Let m be a strictly positive integer, $X \notin V_0$ a vertex of the graph \mathfrak{ST} , and $\psi_X^m \in \mathcal{S}(\mathcal{H}_0, V_m)$ a spline function such that:

$$\psi_X^m(Y) = \begin{cases} \delta_{XY} & \forall Y \in V_m \\ 0 & \forall Y \notin V_m \end{cases}, \quad \text{where } \delta_{XY} = \begin{cases} 1 & \text{if } X = Y \\ 0 & \text{else} \end{cases}$$

Then, since $X \notin V_0$: $\psi_X^m \in \text{dom}_0 \mathcal{E}$.

For any function u of $\text{dom } \mathcal{E}$, such that its Laplacian exists, definition (5.5) applied to ψ_X^m leads to:

$$\mathcal{E}(u, \psi_X^m) = \mathcal{E}_m(u, \psi_X^m) = -r^{-m} \Delta_m u(X) = - \int_{\mathcal{D}(\mathfrak{ST})} \psi_X^m \Delta u d\mu \approx -\Delta u(X) \int_{\mathcal{D}(\mathfrak{ST})} \psi_X^m d\mu$$

since Δu is continuous on \mathfrak{ST} , and the support of the spline function ψ_X^m is close to X :

$$\int_{\mathcal{D}(\mathfrak{ST})} \psi_X^m \Delta u d\mu \approx -\Delta u(X) \int_{\mathcal{D}(\mathfrak{ST})} \psi_X^m d\mu$$

By passing through the limit when the integer m tends towards infinity, one gets:

$$\lim_{m \rightarrow +\infty} \int_{\mathcal{D}(\mathfrak{ST})} \psi_X^m \Delta_m u d\mu = \Delta u(X) \lim_{m \rightarrow +\infty} \int_{\mathcal{D}(\mathfrak{ST})} \psi_X^m d\mu$$

i.e.:

$$\Delta u(X) = \lim_{m \rightarrow +\infty} r^{-m} \left(\int_{\mathcal{D}(\mathfrak{S}\mathfrak{T})} \psi_X^m d\mu \right)^{-1} \Delta_m u(X)$$

Remark 5.1. As it is explained in [13], one has just to reason by analogy with the dimension 1, more particularly, the unit interval $I = [0, 1]$, of extremities $X_0 = (0, 0)$, and $X_1 = (1, 0)$. The functions ψ_{X_1} and ψ_{X_2} such that, for any Y of \mathbb{R}^2 :

$$\psi_{X_1}(Y) = \delta_{X_1 Y} \quad , \quad \psi_{X_2}(Y) = \delta_{X_2 Y}$$

are, in the most simple way, tent functions. For the standard measure, one gets values that do not depend on X_1 , or X_2 (one could, also, choose to fix X_1 and X_2 in the interior of I) :

$$\int_I \psi_{X_1} d\mu = \int_I \psi_{X_2} d\mu = \frac{1}{2}$$

(which corresponds to the surfaces of the two tent triangles.)

Figure 6: The graphs of the spline functions ψ_{X_1} and ψ_{X_2} .

In our case, we have to build the pendant, we no longer reason on the unit interval, but on our polyhedra cells.

Given a natural integer m , and a point $X \in V_m$, the spline function ψ_X^m is supported by two m -polyhedra cells. It is such that, for every m -polyhedra cell $f_{\mathcal{W}}(\mathfrak{S}\mathfrak{T})$ the vertices of which are $X, Y \neq X, Z \neq X, T \neq X$

$$\psi_X^m + \psi_Y^m + \psi_Z^m + \psi_T^m = 1$$

Thus:

$$\int_{f_{\mathcal{W}}(\mathfrak{S}\mathfrak{T})} (\psi_X^m + \psi_Y^m + \psi_Z^m + \psi_T^m) d\mu = \mu(f_{\mathcal{W}}(\mathfrak{S}\mathfrak{T})) = \frac{1}{4^m}$$

By symmetry, all three summands have the same integral. This yields:

$$\int_{f_{\mathcal{W}}(\mathfrak{S}\mathfrak{T})} \psi_X^m d\mu = \frac{1}{4^{m+1}}$$

Taking into account the contributions of the remaining m -polyhedra cells, one has:

$$\int_{\mathfrak{S}\mathfrak{T}} \psi_X^m d\mu = \frac{2}{4^{m+1}}$$

which leads to:

$$\left(\int_{\mathfrak{S}\mathfrak{T}} \psi_X^m d\mu \right)^{-1} = \frac{4^{m+1}}{2}$$

Since:

$$r^{-m} = \left(\frac{3}{2} \right)^m$$

this enables us to obtain the point-wise formula, for $u \in \text{dom } \Delta$:

$$\forall X \in \mathfrak{S}\mathfrak{T} : \quad \Delta_\mu u(X) = 2 \lim_{m \rightarrow +\infty} 6^m \Delta_m u(X)$$

Theorem 5.4. *Let u be in $\text{dom } \Delta$. Then, the sequence of functions $(f_m)_{m \in \mathbb{N}}$ such that, for any natural integer m , and any X of $V_\star \setminus V_0$:*

$$f_m(X) = r^{-m} \left(\int_{\mathcal{D}(\mathfrak{S}\mathfrak{T})} \psi_X^m d\mu \right)^{-1} \Delta_m u(X)$$

converges uniformly towards Δu , and, reciprocally, if the sequence of functions $(f_m)_{m \in \mathbb{N}}$ converges uniformly towards a continuous function on $V_\star \setminus V_0$, then:

$$u \in \text{dom } \Delta$$

Proof. Let u be in $\text{dom } \Delta$. Then:

$$r^{-m} \left(\int_{\mathcal{D}(\mathfrak{S}\mathfrak{T})} \psi_X^m d\mu \right)^{-1} \Delta_m u(X) = \frac{\int_{\mathcal{D}(\mathfrak{S}\mathfrak{T})} \Delta u \psi_X^m d\mu}{\int_{\mathcal{D}(\mathfrak{S}\mathfrak{T})} \psi_X^m d\mu}$$

Since u belongs to $\text{dom } \Delta$, its Laplacian Δu exists, and is continuous on the graph $\mathfrak{S}\mathfrak{T}$. The uniform convergence of the sequence $(f_m)_{m \in \mathbb{N}}$ follows.

Reciprocally, if the sequence of functions $(f_m)_{m \in \mathbb{N}}$ converges uniformly towards a continuous function on $V_\star \setminus V_0$, then, for any natural integer m , and any v belonging to $\text{dom}_0 \mathcal{E}$:

$$\begin{aligned} \mathcal{E}_m(u, v) &= \sum_{(X, Y) \in V_m^2, X \sim_m Y} r^{-m} (u_{|V_m}(X) - u_{|V_m}(Y)) (v_{|V_m}(X) - v_{|V_m}(Y)) \\ &= \sum_{(X, Y) \in V_m^2, X \sim_m Y} r^{-m} (u_{|V_m}(Y) - u_{|V_m}(X)) (v_{|V_m}(Y) - v_{|V_m}(X)) \\ &= - \sum_{X \in V_m \setminus V_0} r^{-m} \sum_{Y \in V_m, Y \sim_m X} v_{|V_m}(X) (u_{|V_m}(Y) - u_{|V_m}(X)) \\ &\quad - \sum_{X \in V_0} r^{-m} \sum_{Y \in V_m, Y \sim_m X} v_{|V_m}(X) (u_{|V_m}(Y) - u_{|V_m}(X)) \\ &= - \sum_{X \in V_m \setminus V_0} r^{-m} v(X) \Delta_m u(X) \\ &= - \sum_{X \in V_m \setminus V_0} v(X) \left(\int_{\mathcal{D}(\mathfrak{S}\mathfrak{T})} \psi_X^m d\mu \right) r^{-m} \left(\int_{\mathcal{D}(\mathfrak{S}\mathfrak{T})} \psi_X^m d\mu \right)^{-1} \Delta_m u(X) \end{aligned}$$

Let us note that any X of $V_m \setminus V_0$ admits exactly three adjacent vertices which belong to $V_m \setminus V_0$, which accounts for the fact that the sum

$$\sum_{X \in V_m \setminus V_0} r^{-m} \sum_{Y \in V_m \setminus V_0, Y \sim_m X} v(X) (u_{|V_m}(Y) - u_{|V_m}(X))$$

has the same number of terms as:

$$\sum_{(X, Y) \in (V_m \setminus V_0)^2, X \sim_m Y} r^{-m} (u_{|V_m}(Y) - u_{|V_m}(X)) (v_{|V_m}(Y) - v_{|V_m}(X))$$

For any natural integer m , we introduce the sequence of functions $(f_m)_{m \in \mathbb{N}}$ such that, for any X of $V_m \setminus V_0$:

$$f_m(X) = r^{-m} \left(\int_{\mathcal{D}(\mathfrak{S}\mathfrak{T})} \psi_X^m d\mu \right)^{-1} \Delta_m u(X)$$

The sequence $(f_m)_{m \in \mathbb{N}}$ converges uniformly towards Δu . Thus:

$$\mathcal{E}_m(u, v) = - \int_{\mathcal{D}(\mathfrak{S}\mathfrak{T})} \left\{ \sum_{X \in V_m \setminus V_0} v|_{V_m}(X) \Delta u|_{V_m}(X) \psi_X^m \right\} d\mu$$

□

6 Normal derivatives

Let us go back to the case of a function u twice differentiable on $I = [0, 1]$, that does not vanish in 0 and :

$$\int_0^1 (\Delta u)(x) v(x) dx = - \int_0^1 u'(x) v'(x) dx + u'(1) v(1) - u'(0) v(0)$$

The normal derivatives:

$$\partial_n u(1) = u'(1) \quad \text{et} \quad \partial_n u(0) = u'(0)$$

appear in a natural way. This leads to:

$$\int_0^1 (\Delta u)(x) v(x) dx = - \int_0^1 u'(x) v'(x) dx + \sum_{\partial[0,1]} v \partial_n u$$

One meets thus a particular case of the Gauss-Green formula, for an open set Ω of \mathbb{R}^d , $d \in \mathbb{N}^*$:

$$\int_{\Omega} \nabla u \nabla v d\mu = - \int_{\Omega} (\Delta u) v d\mu + \int_{\partial\Omega} v \partial_n u d\sigma$$

where μ is a measure on Ω , and where $d\sigma$ denotes the elementary surface on $\partial\Omega$.

In order to obtain an equivalent formulation in the case of the graph $\mathfrak{S}\mathfrak{T}$, one should have, for a pair of functions (u, v) continuous on $\mathfrak{S}\mathfrak{T}$ such that u has a normal derivative:

$$\mathcal{E}(u, v) = - \int_{\Omega} (\Delta u) v d\mu + \sum_{V_0} v \partial_n u$$

For any natural integer m :

$$\begin{aligned} \mathcal{E}_m(u, v) &= \sum_{(X, Y) \in V_m^2, X \sim_m Y} r^{-m} (u|_{V_m}(Y) - u|_{V_m}(X)) (v|_{V_m}(Y) - v|_{V_m}(X)) \\ &= - \sum_{X \in V_m \setminus V_0} r^{-m} \sum_{Y \in V_m, Y \sim_m X} v|_{V_m}(X) (u|_{V_m}(Y) - u|_{V_m}(X)) \\ &\quad - \sum_{X \in V_0} r^{-m} \sum_{Y \in V_m, Y \sim_m X} v|_{V_m}(X) (u|_{V_m}(Y) - u|_{V_m}(X)) \\ &= - \sum_{X \in V_m \setminus V_0} v|_{V_m}(X) r^{-m} \Delta_m u|_{V_m}(X) \\ &\quad + \sum_{X \in V_0} \sum_{Y \in V_m, Y \sim_m X} r^{-m} v|_{V_m}(X) (u|_{V_m}(X) - u|_{V_m}(Y)) \end{aligned}$$

We thus come across an analogous formula of the Gauss-Green one, where the role of the normal derivative is played by:

$$\sum_{X \in V_0} r^{-m} \sum_{Y \in V_m, Y \underset{m}{\sim} X} (u|_{V_m}(X) - u|_{V_m}(Y))$$

Definition 6.1. For any X of V_0 , and any continuous function u on \mathfrak{ST} , we will say that u admits a normal derivative in X , denoted by $\partial_n u(X)$, if:

$$\lim_{m \rightarrow +\infty} r^{-m} \sum_{Y \in V_m, Y \underset{m}{\sim} X} (u|_{V_m}(X) - u|_{V_m}(Y)) < +\infty$$

We will set:

$$\partial_n u(X) = \lim_{m \rightarrow +\infty} r^{-m} \sum_{Y \in V_m, Y \underset{m}{\sim} X} (u|_{V_m}(X) - u|_{V_m}(Y)) < +\infty$$

Definition 6.2. For any natural integer m , any X of V_m , and any continuous function u on \mathfrak{ST} , we will say that u admits a normal derivative in X , denoted by $\partial_n u(X)$, if:

$$\lim_{k \rightarrow +\infty} r^{-k} \sum_{Y \in V_k, Y \underset{k}{\sim} X} (u|_{V_k}(X) - u|_{V_k}(Y)) < +\infty$$

We will set:

$$\partial_n u(X) = \lim_{k \rightarrow +\infty} r^{-k} \sum_{Y \in V_k, Y \underset{k}{\sim} X} (u|_{V_k}(X) - u|_{V_k}(Y)) < +\infty$$

Remark 6.1. One can thus extend the definition of the normal derivative of u to \mathfrak{ST} .

Theorem 6.1. *Let u be in $\text{dom } \Delta$. The, for any X of \mathfrak{ST} , $\partial_n u(X)$ exists. Moreover, for any v of $\text{dom } \mathcal{E}$, et any natural integer m , the Gauss-Green formula writes:*

$$\mathcal{E}(u, v) = - \int_{\mathcal{D}(\mathfrak{ST})} (\Delta u) v d\mu + \sum_{V_0} v \partial_n u$$

7 Spectrum of the Laplacian

In the following, let u be in $\text{dom } \Delta$. We will apply the *spectral decimation method* developed by R. S. Strichartz [13], in the spirit of the works of M. Fukushima et T. Shima [16]. In order to determine the eigenvalues of the Laplacian Δu built in the above, we concentrate first on the eigenvalues $(-\lambda_m)_{m \in \mathbb{N}}$ of the sequence of graph Laplacians $(\Delta_m u)_{m \in \mathbb{N}}$, built on the discrete sequence of graphs $(\mathfrak{ST}_m)_{m \in \mathbb{N}}$. For any natural integer m , the restrictions of the eigenfunctions of the continuous Laplacian Δu to the graph \mathfrak{ST}_m are, also, eigenfunctions of the Laplacian Δ_m , which leads to recurrence relations between the eigenvalues of order m and $m + 1$.

We thus aim at determining the solutions of the eigenvalue equation:

$$-\Delta u = \lambda u \quad \text{on } \mathfrak{S}\mathfrak{T}$$

as limits, when the integer m tends towards infinity, of the solutions of:

$$-\Delta_m u = \lambda_m u \quad \text{on } V_m \setminus V_0$$

We will call them Dirichlet eigenvalues (resp. Neumann eigenvalues) if:

$$u|_{\partial\mathfrak{S}\mathfrak{T}} = 0 \quad (\text{resp. } \partial_n u|_{\partial\mathfrak{S}\mathfrak{T}} = 0)$$

Given a strictly positive integer m , let us consider a $(m - 1)$ -polyhedron cell, with boundary vertices X_0, X_1, X_2, X_3 .

We denote by $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6$ the points of $V_m \setminus V_{m-1}$ such that:

- i.* Y_1 is between X_0 and X_1 ;
- ii.* Y_2 is between X_1 and X_2 ;
- iii.* Y_3 is between X_0 and X_2 ;
- iv.* Y_4 is between X_0 and X_3 ;
- v.* Y_5 is between X_1 and X_3 ;
- vi.* Y_6 is between X_2 and X_3 .

The discrete equation on $\mathfrak{S}\mathfrak{T}$ leads to the following system:

$$\begin{aligned} (6 - \lambda_m) u(Y_1) &= u(X_0) + u(X_1) + u(Y_2) + u(Y_3) + u(Y_4) + u(Y_5) \\ (6 - \lambda_m) u(Y_2) &= u(X_1) + u(X_2) + u(Y_1) + u(Y_3) + u(Y_5) + u(Y_6) \\ (6 - \lambda_m) u(Y_3) &= u(X_0) + u(X_2) + u(Y_1) + u(Y_2) + u(Y_4) + u(Y_6) \\ (6 - \lambda_m) u(Y_4) &= u(X_0) + u(X_3) + u(Y_1) + u(Y_3) + u(Y_5) + u(Y_6) \\ (6 - \lambda_m) u(Y_5) &= u(X_1) + u(X_3) + u(Y_1) + u(Y_2) + u(Y_4) + u(Y_6) \\ (6 - \lambda_m) u(Y_6) &= u(X_2) + u(X_3) + u(Y_2) + u(Y_3) + u(Y_4) + u(Y_5) \end{aligned}$$

Under matricial form, this writes:

$$\mathbf{x} = \mathbf{A}_m^{-1} \mathbf{b}$$

with

$$\mathbf{A}_m = \begin{pmatrix} 6 - \lambda_m & -1 & -1 & -1 & -1 & 0 \\ -1 & 6 - \lambda_m & -1 & 0 & -1 & -1 \\ -1 & -1 & 6 - \lambda_m & -1 & 0 & -1 \\ -1 & 0 & -1 & 6 - \lambda_m & -1 & -1 \\ -1 & -1 & 0 & -1 & 6 - \lambda_m & -1 \\ 0 & -1 & -1 & -1 & -1 & 6 - \lambda_m \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} u(Y_1) \\ u(Y_2) \\ u(Y_3) \\ u(Y_4) \\ u(Y_5) \\ u(Y_6) \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} u(X_0) + u(X_1) \\ u(X_1) + u(X_2) \\ u(X_0) + u(X_2) \\ u(X_0) + u(X_3) \\ u(X_1) + u(X_3) \\ u(X_2) + u(X_3) \end{pmatrix}$$

By assuming $\lambda_m \neq \{2, 6\}$, one gets:

$$\begin{aligned} u(Y_1) &= \frac{(4 - \lambda_m)(u(X_0) + u(X_1)) + 2(u(X_2) + u(X_3))}{(2 - \lambda_m)(6 - \lambda_m)} \\ u(Y_2) &= \frac{(4 - \lambda_m)(u(X_1) + u(X_2)) + 2(u(X_0) + u(X_3))}{(2 - \lambda_m)(6 - \lambda_m)} \\ u(Y_3) &= \frac{(4 - \lambda_m)(u(X_0) + u(X_2)) + 2(u(X_1) + u(X_3))}{(2 - \lambda_m)(6 - \lambda_m)} \\ u(Y_4) &= \frac{(4 - \lambda_m)(u(X_0) + u(X_3)) + 2(u(X_1) + u(X_2))}{(2 - \lambda_m)(6 - \lambda_m)} \\ u(Y_5) &= \frac{(4 - \lambda_m)(u(X_1) + u(X_3)) + 2(u(X_0) + u(X_2))}{(2 - \lambda_m)(6 - \lambda_m)} \\ u(Y_6) &= \frac{(4 - \lambda_m)(u(X_2) + u(X_3)) + 2(u(X_0) + u(X_1))}{(2 - \lambda_m)(6 - \lambda_m)} \end{aligned}$$

Let us now compare the λ_{m-1} -eigenvalues on V_{m-1} , and the λ_m -eigenvalues on V_m . To this purpose, we fix $X_0 \in V_{m-1} \setminus V_0$.

One has to bear in mind that X_0 also belongs to a $(m-1)$ -cell, with boundary points X_0, X'_1, X'_2, X'_3 and interior points $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6$.

Thus:

$$(6 - \lambda_{m-1})u(X_0) = u(X'_1) + u(X'_2) + u(X'_3) + u(X_1) + u(X_2) + u(X_3)$$

and:

$$(6 - \lambda_m)u(X_0) = u(Y_1) + u(Y_3) + u(Y_4) + u(Z_1) + u(Z_3) + u(Z_4)$$

$$\begin{aligned} u(Y_1) + u(Y_3) + u(Y_4) &= \frac{(8 - \lambda_m)(u(X_1) + u(X_2) + u(X_3)) + 3(4 - \lambda_m)u(X_0)}{(2 - \lambda_m)(6 - \lambda_m)} \\ u(Z_1) + u(Z_3) + u(Z_4) &= \frac{(8 - \lambda_m)(u(X'_1) + u(X'_2) + u(X'_3)) + 3(4 - \lambda_m)u(X_0)}{(2 - \lambda_m)(6 - \lambda_m)} \end{aligned}$$

By adding member to member, one obtains:

$$(6 - \lambda_m)u(X_0) = \frac{(8 - \lambda_m)(6 - \lambda_{m-1}) + 6(4 - \lambda_m)}{(2 - \lambda_m)(6 - \lambda_m)}u(X_0)$$

We record one more forbidden eigenvalue $\lambda_m = 8$, else λ_m is independent of λ_{m-1} . One has:

$$(6 - \lambda_m)^2 (2 - \lambda_m) - 6(4 - \lambda_m) = (8 - \lambda_m)(6 - \lambda_{m-1})$$

Finally:

$$\lambda_{m-1} = \lambda_m(6 - \lambda_m)$$

One may solve:

$$\lambda_m = 3 \pm \sqrt{9 - \lambda_{m-1}}$$

Let us introduce:

$$\lambda = 2 \lim_{m \rightarrow \infty} 6^m \lambda_m$$

One may note that the limit exists, since, when x is close to 0:

$$3 - \sqrt{9 - x} = \frac{x}{6} + \mathcal{O}(x^2)$$

Let us now look the Dirichlet eigenvalues and eigenfunctions

1. First case : $m = 1$.

The Tetrahedron with its ten vertices can be seen in the following figures:

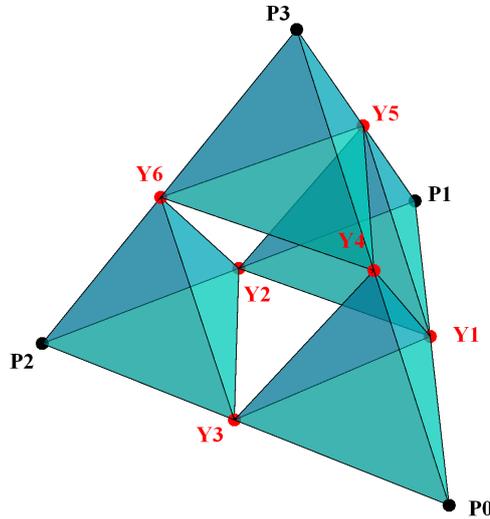


Figure 7: The tetrahedron after the first iteration.

Let us look for the kernel of the matrix \mathbf{A}_1 in the case where $\lambda_1 \in \{2, 6, 8\}$.

For $\lambda_1 = 2$, we find the one dimensional Dirichlet eigenspace

$$V_2^1 = \text{Vect} \{(1, 1, 1, 1, 1, 1)\}$$

For $\lambda_1 = 8$, we find the two dimensional Dirichlet eigenspace

$$V_8^1 = \text{Vect} \{(1, -1, 0, -1, 0, 1), (0, -1, 1, -1, 1, 0)\}$$

For $\lambda_1 = 6$, we find the three dimensional Dirichlet eigenspace

$$V_6^1 = \text{Vect} \{(-1, 0, 0, 0, 0, 1), (0, 0, -1, 0, 1, 0), (0, -1, 0, 1, 0, 0)\}$$

One can easily check that:

$$\#V_0 = 6$$

Thus, the spectrum is complete.

2. Second case : $m = 2$.

Let us now move to the $m = 2$ case.

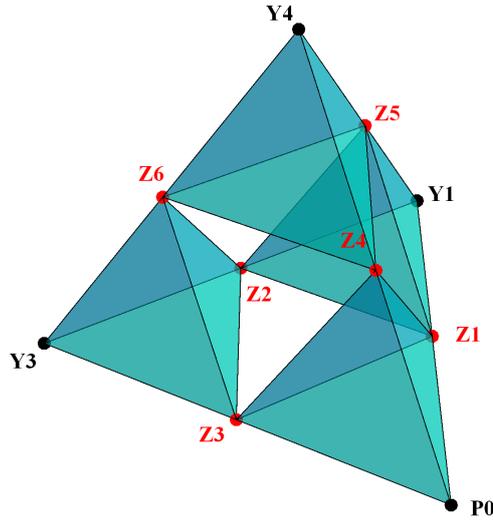


Figure 8: The cell $F_1(V_1)$

Let us denote by $Z_j^i := f_i(Y_j)$ the points of $V_2 \setminus V_1$ that belongs to the cell $f_i(V_0)$.

One has to solve the following system, taking into account the Dirichlet boundary conditions ($u(X_0) = u(X_1) = \dots$)

$$\begin{aligned} (6 - \lambda_m) u(Z_1^1) &= u(X_0) + u(Y_1) + u(Z_2^1) + u(Z_3^1) + u(Z_4^1) + u(Z_5^1) \\ (6 - \lambda_m) u(Z_2^1) &= u(Y_1) + u(Y_3) + u(Z_1^1) + u(Z_3^1) + u(Z_5^1) + u(Z_6^1) \\ (6 - \lambda_m) u(Z_3^1) &= u(X_0) + u(Y_3) + u(Z_1^1) + u(Z_2^1) + u(Z_4^1) + u(Z_6^1) \\ (6 - \lambda_m) u(Z_4^1) &= u(X_0) + u(Y_4) + u(Z_1^1) + u(Z_3^1) + u(Z_5^1) + u(Z_6^1) \\ (6 - \lambda_m) u(Z_5^1) &= u(Y_1) + u(Y_4) + u(Z_1^1) + u(Z_2^1) + u(Z_4^1) + u(Z_6^1) \\ (6 - \lambda_m) u(Z_6^1) &= u(Y_3) + u(Y_4) + u(Z_2^1) + u(Z_3^1) + u(Z_4^1) + u(Z_5^1) \end{aligned}$$

$$\begin{aligned}
(6 - \lambda_m) u(Z_1^2) &= u(X_1) + u(Y_1) + u(Z_2^2) + u(Z_3^2) + u(Z_4^2) + u(Z_5^2) \\
(6 - \lambda_m) u(Z_2^2) &= u(X_1) + u(Y_2) + u(Z_1^2) + u(Z_3^2) + u(Z_5^2) + u(Z_6^2) \\
(6 - \lambda_m) u(Z_3^2) &= u(Y_1) + u(Y_2) + u(Z_1^2) + u(Z_2^2) + u(Z_4^2) + u(Z_6^2) \\
(6 - \lambda_m) u(Z_4^2) &= u(Y_1) + u(Y_5) + u(Z_1^2) + u(Z_3^2) + u(Z_5^2) + u(Z_6^2) \\
(6 - \lambda_m) u(Z_5^2) &= u(X_1) + u(Y_5) + u(Z_1^2) + u(Z_2^2) + u(Z_4^2) + u(Z_6^2) \\
(6 - \lambda_m) u(Z_6^2) &= u(Y_2) + u(Y_5) + u(Z_2^2) + u(Z_3^2) + u(Z_4^2) + u(Z_5^2)
\end{aligned}$$

$$\begin{aligned}
(6 - \lambda_m) u(Z_1^3) &= u(Y_2) + u(Y_3) + u(Z_2^3) + u(Z_3^3) + u(Z_4^3) + u(Z_5^3) \\
(6 - \lambda_m) u(Z_2^3) &= u(X_2) + u(Y_2) + u(Z_1^3) + u(Z_3^3) + u(Z_5^3) + u(Z_6^3) \\
(6 - \lambda_m) u(Z_3^3) &= u(X_2) + u(Y_3) + u(Z_1^3) + u(Z_2^3) + u(Z_4^3) + u(Z_6^3) \\
(6 - \lambda_m) u(Z_4^3) &= u(Y_3) + u(Y_6) + u(Z_1^3) + u(Z_3^3) + u(Z_5^3) + u(Z_6^3) \\
(6 - \lambda_m) u(Z_5^3) &= u(Y_2) + u(Y_6) + u(Z_1^3) + u(Z_2^3) + u(Z_4^3) + u(Z_6^3) \\
(6 - \lambda_m) u(Z_6^3) &= u(X_2) + u(Y_6) + u(Z_2^3) + u(Z_3^3) + u(Z_4^3) + u(Z_5^3)
\end{aligned}$$

$$\begin{aligned}
(6 - \lambda_m) u(Z_1^4) &= u(Y_4) + u(Y_5) + u(Z_2^4) + u(Z_3^4) + u(Z_4^4) + u(Z_5^4) \\
(6 - \lambda_m) u(Z_2^4) &= u(Y_5) + u(Y_6) + u(Z_1^4) + u(Z_3^4) + u(Z_5^4) + u(Z_6^4) \\
(6 - \lambda_m) u(Z_3^4) &= u(Y_4) + u(Y_6) + u(Z_1^4) + u(Z_2^4) + u(Z_4^4) + u(Z_6^4) \\
(6 - \lambda_m) u(Z_4^4) &= u(X_3) + u(Y_4) + u(Z_1^4) + u(Z_3^4) + u(Z_5^4) + u(Z_6^4) \\
(6 - \lambda_m) u(Z_5^4) &= u(X_3) + u(Y_5) + u(Z_1^4) + u(Z_2^4) + u(Z_4^4) + u(Z_6^4) \\
(6 - \lambda_m) u(Z_6^4) &= u(X_3) + u(Y_6) + u(Z_2^4) + u(Z_3^4) + u(Z_4^4) + u(Z_5^4)
\end{aligned}$$

$$\begin{aligned}
(6 - \lambda_m) u(Y_1) &= u(Z_1^1) + u(Z_2^1) + u(Z_5^1) + u(Z_1^2) + u(Z_3^2) + u(Z_4^2) \\
(6 - \lambda_m) u(Y_2) &= u(Z_2^2) + u(Z_3^2) + u(Z_6^2) + u(Z_1^3) + u(Z_2^3) + u(Z_5^3) \\
(6 - \lambda_m) u(Y_3) &= u(Z_2^1) + u(Z_3^1) + u(Z_6^1) + u(Z_1^3) + u(Z_3^3) + u(Z_4^3) \\
(6 - \lambda_m) u(Y_4) &= u(Z_4^1) + u(Z_5^1) + u(Z_6^1) + u(Z_1^4) + u(Z_3^4) + u(Z_4^4) \\
(6 - \lambda_m) u(Y_5) &= u(Z_4^2) + u(Z_5^2) + u(Z_6^2) + u(Z_1^4) + u(Z_2^4) + u(Z_5^4) \\
(6 - \lambda_m) u(Y_6) &= u(Z_4^3) + u(Z_5^3) + u(Z_6^3) + u(Z_2^4) + u(Z_3^4) + u(Z_6^4)
\end{aligned}$$

The system can be written as $\mathbf{A}_2 \mathbf{x} = \mathbf{0}$ and we look for the kernel of \mathbf{A}_2 for $\lambda_2 \in \{2, 6, 8\}$. We found that there is no eigenfunction for $\lambda_2 = 2$.

For $\lambda_2 = 6$, the eigenspace is a six dimensional one, the basis vectors of which are:

$$\left(\begin{array}{c|c|c|c|c|c} 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

For $\lambda_2 = 8$, the eigenspace has dimension 14, and is generated by:

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From $\lambda_1 = 2$, the spectral decimation leads to:

$$\lambda_2 = 3 - \sqrt{7} \quad \text{and} \quad \lambda_2 = 3 + \sqrt{7}$$

Each of these eigenvalues has multiplicity 1.

From $\lambda_1 = 6$, the spectral decimation leads to:

$$\lambda_2 = 3 - \sqrt{3} \quad \text{and} \quad \lambda_2 = 3 + \sqrt{3}$$

Each of these eigenvalues has multiplicity 3.

From $\lambda_1 = 8$, the spectral decimation leads to:

$$\lambda_2 = 4$$

with multiplicity 2 (one may note that 2 is not a Dirichlet eigenvalue for $m = 2$).

One can easily check that:

$$\#V_2 \setminus V_0 = 30 = 6 + 14 + 2 \times 1 + 3 \times 2 + 1 \times 2 = 30$$

Thus, the spectrum is complete.

Let us now go back to the general case. Given a strictly positive integer m , let us introduce the respective multiplicities $M_m(6)$ and $M_m(8)$ of the eigenvalues $\lambda_m = 6$ and $\lambda_m = 8$.

One can easily check by induction that:

$$\#V_m \setminus V_0 = 2(4^m - 1)$$

and:

$$M_m(8) = 4^m - 2$$

(we here refer to [15]).

One also has:

$$\#V_{m-1} \setminus V_0 = 2(4^{m-1} - 1)$$

and:

$$M_{m-1}(8) = 4^{m-1} - 2$$

There are thus $2(4^{m-1} - 1) - (4^{m-1} - 2) = 4^{m-1}$ continued eigenvalues (the ones obtained by means of the spectral decimation), which correspond to a space of eigenfunctions, the dimension of which is:

$$(4^{m-1} - 2) + 2 \times 4^{m-1}$$

This leads to:

$$M_m(6) = 2(4^m - 1) - (4^m - 2) - ((4^{m-1} - 2) + 2 \times 4^{m-1}) = 4^m$$

8 Metric - Towards spectral asymptotics

Definition 8.1. Effective resistance metric, on $\mathfrak{S}\mathfrak{T}$

Given two points (X, Y) of $\mathfrak{S}\mathfrak{T}^2$, let us introduce the **effective resistance metric between X and Y** :

$$R_{\mathfrak{S}\mathfrak{T}}(X, Y) = \left\{ \min_{\{u \mid u(X)=0, u(Y)=1\}} \mathcal{E}(u) \right\}^{-1}$$

In an equivalent way, $R_{\mathfrak{S}\mathfrak{T}}(X, Y)$ can be defined as the minimum value of the real numbers R such that, for any function u of $\text{dom } \Delta$:

$$|u(X) - u(Y)|^2 \leq R \mathcal{E}(u)$$

Definition 8.2. Metric, on the Sierpiński Tetrahedron $\mathfrak{S}\mathfrak{T}$

Let us define, on the Sierpiński Tetrahedron $\mathfrak{S}\mathfrak{T}$, the distance $d_{\mathfrak{S}\mathfrak{T}}$ such that, for any pair of points (X, Y) of $\mathfrak{S}\mathfrak{T}^2$:

$$d_{\mathfrak{S}\mathfrak{T}}(X, Y) = \left\{ \min_{\{u \mid u(X)=0, u(Y)=1\}} \mathcal{E}(u, u) \right\}^{-1}$$

Remark 8.1. One may note that the minimum

$$\min_{\{u \mid u(X)=0, u(Y)=1\}} \mathcal{E}(u)$$

is reached for u being harmonic on the complement set, on $\mathfrak{S}\mathfrak{T}$, of the set

$$\{X\} \cup \{Y\}$$

(One might bear in mind that, due to its definition, a harmonic function u on $\mathfrak{S}\mathfrak{T}$ minimizes the sequence of energies $(\mathcal{E}_{\mathfrak{S}\mathfrak{T}_m}(u, u))_{m \in \mathbb{N}}$.

Definition 8.3. Dimension of the Sierpiński Tetrahedron $\mathfrak{S}\mathfrak{T}$, in the resistance metrics

The dimension of the Sierpiński Tetrahedron $\mathfrak{S}\mathfrak{T}$, in the resistance metrics, is the strictly positive number $d_{\mathfrak{S}\mathfrak{T}}$ such that, given a strictly positive real number r , and a point $X \in \mathfrak{S}\mathfrak{T}$, for the X -centered ball of radius r , denoted by $\mathcal{B}_r(X)$:

$$\mu(\mathcal{B}_r(X)) = r^{d_{\mathfrak{S}\mathfrak{T}}}$$

Property 8.1. *Given a natural integer m , and two points (X, Y) of $\mathfrak{S}\mathfrak{T}^2$ such that $X \sim_m Y$:*

$$\min_{\{u \mid u(X)=0, u(Y)=1\}} \mathcal{E}(u) \lesssim r^m = \left(\frac{2}{3}\right)^m$$

which also corresponds to the order of the diameter of m -polyhedra cells.

Since the Sierpiński tetrahedron $\mathfrak{S}\mathfrak{T}$ is obtained from the initial regular 3-simplex by means of four contractions, the ratio which is equal to $\frac{1}{2}$, let us look for a real number $\beta_{\mathfrak{S}\mathfrak{T}}$ such that:

$$\left(\frac{1}{2}\right)^{m \beta_{\mathfrak{S}\mathfrak{T}}} = \left(\frac{2}{3}\right)^m$$

One obtains:

$$\beta_{\mathfrak{S}\mathfrak{T}} = \frac{\ln \frac{3}{2}}{\ln 2}$$

Let us denote by μ the standard measure on $\mathfrak{S}\mathfrak{T}$ which assigns measure $\frac{1}{4^m}$ to each m -polyhedron cell. Let us now look for a real number $d_{\mathfrak{S}\mathfrak{T}}$ such that:

$$\left(\frac{2}{3}\right)^{m d_{\mathfrak{S}\mathfrak{T}}} = \frac{1}{4^m}$$

One obtains:

$$d_{\mathfrak{S}\mathfrak{T}} = \frac{\ln \frac{3}{2}}{\ln 4}$$

Given a strictly positive real number r , and a point $X \in \mathfrak{S}\mathfrak{T}$, one has then the following estimate, for the X -centered ball of radius r , denoted by $\mathcal{B}_r(X)$:

$$\mu(\mathcal{B}_r(X)) = r^{d_{\mathfrak{S}\mathfrak{T}}}$$

Definition 8.4. Eigenvalue counting function

We introduce the eigenvalue counting function $\mathcal{N}^{\mathfrak{S}\mathfrak{T}}$ such that, for any real number x :

$$\mathcal{N}^{\mathfrak{S}\mathfrak{T}}(x) = \# \{ \lambda \text{ eigenvalue of } -\Delta : \lambda \leq x \}$$

Property 8.2. *The existing results of J. Kigami [7] and R. S. Strichartz [8] lead to the modified Weyl formula, when x tends towards infinity:*

$$\mathcal{N}^{\mathfrak{S}\mathfrak{T}}(x) = G(x) x^{\alpha_{\mathfrak{S}\mathfrak{T}}} + \mathcal{O}(1)$$

where the exponent $\alpha_{\mathfrak{S}\mathfrak{T}}$ is given by:

$$\alpha_{\mathfrak{S}\mathfrak{T}} = \frac{d_{\mathfrak{S}\mathfrak{T}}}{d_{\mathfrak{S}\mathfrak{T}} + 1}$$

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