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Sébastien Lefort*, Marie-Jeanne Lesot*, Elisabetta Zibetti†, Charles Tijus† and Marcin Detyniecki*‡
*Sorbonne Universités, UPMC Univ Paris 06,
CNRS, LIP6 UMR 7606, 4 place Jussieu 75005 Paris, France
Email: {sebastien.lefort, marie-jeanne.lesot, marcin.detyniecki}@lip6.fr
†Laboratoire CHArt-LUTIN, EA 4004, Université Paris 8,
2 rue de la liberté, 93526, Saint-Denis - Cedex 02, France
Email: {ezibetti,tijus}@univ-paris8.fr
‡Polish Academy of Sciences, IBS PAN, Warsaw, Poland

***Abstract*—Imprecisions related to numerical expressions are pervasive in human communication. The way they are propagated in calculations is still an issue. Fuzzy logic is an attempt to account for human imprecise reasoning. In this paper, a comparison between human imprecise calculation and fuzzy arithmetic is experimentally performed. An empirical study has been conducted to collect real intervals resulting from imprecise products and additions from participants. Fuzzy intervals are elicited from these data and fuzzy arithmetic is applied to the collected imprecise operands. Comparisons of the fuzzy intervals show that the fuzzy product and addition do not fit well they way human beings perform these operations on imprecise operands. Moreover, they show that the participants, rather than taking into account the imprecisions in the calculation, realise exact calculation and then approximate the exact result.**

I. Introduction

Imprecise numerical knowledge is pervasive in human interaction and reasoning. Estimations, implying imprecisions, are made about various things, from the cost of an object to the duration of a project and the length of a trip.

In the scientific and technical domains the imprecisions in any measure can be controlled through the use of appropriate measurement tools. Moreover, imprecision propagation through the measuring chain can be precisely computed, according to several formal frameworks (e.g., fuzzy logic [1], differential calculus [2]).

However, in daily life estimations are performed by human agents who do not possess such measurement and error analysis tools. The literature from the cognitive psychology domain shows that there is an imprecision inherent in any quantity representation, even if it is exactly known [3], [4]. To the best of our knowledge, no empirical study has been conducted to examine how human beings deal with imprecisions in reasoning tasks and more specifically in calculations.

Fuzzy logic tries to represent how human beings reason with imprecise knowledge [5], [6]. Among other fuzzy generalisations of the classical logic, the extension principle and the fuzzy arithmetic [7] allow to perform calculations with fuzzy operands. However, their fitting with the way the human cognitive system performs calculations with imprecise operands remains to be demonstrated.

The aim of this paper is an attempt towards the understanding of imprecision propagation in human calculation. More specifically, its goal is twofold: (i) to assess whether fuzzy arithmetic fits human imprecise calculation and (ii) to assess whether the human cognitive system takes into account the imprecision of the operands to estimate the resulting imprecision. To achieve this aim, an empirical study is conducted to collect intervals corresponding to imprecise calculations. These intervals have been used to build membership functions of fuzzy intervals representing the imprecision around the operands and the results of the calculations.

Finally, an experimental study is designed to perform three comparisons. Firstly, to assess the fitting with fuzzy arithmetics, the fuzzy intervals of the results are compared to the ones obtained by applying fuzzy arithmetic to the fuzzy operands. Secondly, to assess whether the imprecision of the operands are taken into account in the calculation, the fuzzy intervals of the calculation results are compared to two other ones corresponding to the value of the calculation result, without performing the calculation (i.e., 600 for the calculation $20{\cdot}30$): (i) the ones elicited from the collected intervals and (ii) the ones predicted by a model of interpretation of approximate numerical expressions [8].

This paper is structured as follow: Section II is dedicated to the material and methods used to collect and pre-process the data related to human imprecise calculations. Section III presents the methods used to build of the fuzzy intervals which are compared in an experimental study described in Section IV. Section V exposes the results. Finally, conclusions and future works are discussed in Section VI.

II. Data collection

An empirical study has been conducted to collect the results of imprecise calculations, performed by human beings, in the form of intervals. This section presents the material and methods used to collect and pre-process the data.

A. Material

21 calculations, 5 products and 16 additions, referenced in Table I, are considered. They are selected in order to minimise the required arithmetical skills of the participants. The imprecise operands are of the form "*about* x", where $x \in \mathbb{N}^+$. The calculations are not semantically contextualised, no cues are given as to what is measured or counted.

An online questionnaire has been designed to collect the data. It consists of two parts. The first one is dedicated to the 21 calculations. Each numerical value is preceded by the approximator "*about*" to avoid any ambiguity with regards to the imprecise nature of the operands. Instructions, given in French can be translated as: "*In your opinion, what are the MINIMUM and MAXIMUM values associated with the result of about $x \odot$ about y?*", where x and y are the operands of the calculation and $\odot$ the considered operation, i.e., + or $\cdot$. Each participant fills the 21 items corresponding to the 21 calculations. The order in which they are presented is randomly set for each participant.

The second part of the questionnaire consists in asking the participant to give the intervals corresponding to the approximations of the 22 operands and exact results of the calculations (e.g., for $\simeq 20 \cdot \simeq 30$, the intervals corresponding to "*about 20*", "*about 30*" and "*about 600*"). The instructions of this part can be translated as "*In your opinion, what are the MINIMUM and MAXIMUM values associated with about x?*", where x is the considered value. As for the first part, the order in which the 22 items are presented is randomly set for each participant. Moreover, the order in which the two parts of the questionnaire are filled is also randomly set.

For each calculation performed by a participant p, four intervals are collected. The first one, $I_C^p(z)$, corresponds to the interval given in the first part of the questionnaire, i.e., the result of the calculation performed on the imprecise operands, $I_C^p(z) = \simeq x \odot \simeq y$. The three other intervals, $I_P^p(x)$, $I_P^p(y)$ and $I_P^p(z)$ are the ones given in the second part of the questionnaire and correspond to the range of values denoted by the operands $\simeq x$ and $\simeq y$ and by the exact result $\simeq z$. Collecting $I_P^p(x)$ and $I_P^p(y)$ allow us to elicit fuzzy intervals (see Section III) later used in the fuzzy arithmetics, while collecting $I_P^p(z)$, the approximation of the exact resulting value, allow us to compare it with the result of the calculations $I_C^p(z)$ to see whether the participants really take into account the imprecision of the operands during the calculation.

B. Population

Participants were recruited through an announcement diffused on mailing-lists. 146 adults, all native French speakers, volunteered to freely take part in the study, 102 women and 44 men, aged 20 to 70 (mean= 38.6; standard deviation= 14.2).

C. Data cleaning

In order to exclude outliers intervals from the set, data are preprocessed according to the following procedure (please note that $I(x) = [u, v]$ is used as the representative interval, the procedure is performed for $I_P^p(x)$, $I_P^p(y)$, $I_P^p(z)$ and $I_C^p(z)$):

1) An interval is considered as outliers because either:
   - it is inadequate (e.g., $[0, \infty]$)
   - the right endpoint is below the reference value or the left endpoint is above the reference value, formally: $v < x$ or $u > x$ (e.g., $I(600) = [500, 550]$ or $I(600) = [610, 650]$)
   - an endpoint is greater than ten times or lesser than one tenth the reference value, formally $v > 10 \cdot x$ or $u < \frac{x}{10}$ (e.g., $I(600) = [59, 6001]$)
2) For both endpoints of each operand, exact result or calculation result, the mean and standard deviation of the data remaining after step 1 are computed. Any endpoint value beyond three standard deviations of the mean is considered as outliers.
3) Participants are considered as untrustworthy and are excluded because more than 70% of their interval endpoints are missing values or outliers.
4) All calculations for which at least one over the four collected intervals is considered as an outliers is excluded.

From 3066 calculations in the original corpus, 2304 (75%) are used in the experimental study. The next section describes the methods used to elicit fuzzy intervals from these data.

TABLE I
CALCULATIONS USED IN THE QUESTIONNAIRE (SEE SECTION II) AND THEIR EXACT RESULT.

Calculation			Exact result
about 20	$\cdot$	about 30	600
about 20	$\cdot$	about 400	8000
about 20	$\cdot$	about 50	1000
about 30	$\cdot$	about 50	1500
about 40	$\cdot$	about 50	2000
about 50	+	about 100	150
about 30	+	about 4700	4730
about 150	+	about 8000	8150
about 20	+	about 80	100
about 200	+	about 800	1000
about 400	+	about 600	1000
about 500	+	about 1000	1500
about 2000	+	about 6000	8000
about 100	+	about 500	1500
about 200	+	about 400	600
about 20	+	about 30	50
about 440	+	about 560	1000
about 40	+	about 400	440
about 40	+	about 110	150
about 400	+	about 1100	1500
about 500	+	about 1500	2000

III. METHODS

In the experimental study, four membership functions, listed in Table II, are compared: μC_z, corresponding to the result of the imprecise calculation, based on the $I_C^p(z)$ intervals, μP_z, corresponding to the approximation of the exact result, based on the $I_P^p(z)$ intervals, μA_z, obtained by applying the fuzzy arithmetics to the fuzzy intervals μP_x and μP_y, based on the $I_P^p(x)$ and $I_P^p(y)$ intervals respectively, and μM_z, the fuzzy

TABLE II

MEMBERSHIP FUNCTIONS COMPARED IN THE EXPERIMENTAL STUDY (SEE SECTION IV), THEIR DESCRIPTION AND THEIR COMPUTATION METHOD.

μ	Description
μC_z	Elicited from the result of the imprecise **C**alculation
μP_z	Elicited from the approximation of the exact result given by **P**articipants
μA_z	Result of the fuzzy **A**rithmetic applied to the imprecise operands
μM_z	Prediction made by the interpretation **M**odel of approximate numerical expressions

interval built according to our proposed model of interpretation of numerical expressions [8].

This section presents how these membership functions are built. The first subsection describe the methods used to elicit membership functions from the collected intervals. The fuzzy product and addition are presented in the second subsection. Finally, a brief description of our interpretation model is provided in the third subsection.

A. Computation of μP and μC: Elicitation from intervals

Five semantic interpretations of membership functions has been proposed by [9]: likelihood view, random set view, similarity view, utility view and measurement view. Among them, the random set perspective is the most relevant according to the way our data are collected. Indeed, in this view, the membership degree of a candidate number (e.g., 595 for $\simeq 20 \cdot \simeq 30$) is the cumulative frequency of participants thinking that it belongs to the interval resulting from the calculation. Thus, if half of the population think that 595 is included in $\simeq 20 \cdot \simeq 30$, the truth value of 595 is 0.5.

Figure 1 illustrates three membership functions elicited for the calculation $\simeq 200 + \simeq 400$: μP_{200} and μP_{400}, corresponding to the operands, and μC_{600}, corresponding to the result of the imprecise calculation.

B. Computation of μA: Fuzzy product and addition

Two approaches can be considered to apply product and addition to fuzzy intervals. The first one is based on Zadeh's extension principle [7] while the second one uses the $\alpha-$cut representation of membership functions [10]. In this work, the μA_z fuzzy intervals are computed according to the second approach. Computing the resulting fuzzy interval is performed through a three step procedure.

In a first time, the membership functions μP_x and μP_y, of the operands x and y respectively, are decomposed into 100 α-cuts $[\mu P_x]_\alpha = [s, t]_\alpha$ and $[\mu P_y]_\alpha = [u, v]_\alpha$.

Secondly, $[\mu P_x]_\alpha$ and $[\mu P_y]_\alpha$ are combined to obtain $[\mu A_z]_\alpha$, the α-cuts corresponding to the desired resulting fuzzy interval μA_z, according to the interval calculus [11]. In the case of addition, $[\mu A_z]_\alpha = [s + u, t + v]$. In the case of products, because s, t, u and v are all positive natural numbers, i.e. $\in \mathbb{N}^+$, the computation is simpler than the general case discussed by Moore [11]: $[\mu A_z]_\alpha = [s \cdot u, t \cdot v]$.

Finally, the resulting fuzzy interval μA_z is obtained by linearly interpolating the endpoints of its α-cuts, $[\mu A_z]_\alpha$.

C. Computation of μM: Interpretation model of approximate numerical expressions

In a previous work [8], we proposed and validated a computational model to interpret approximate numerical expressions, i.e., expressions of the form "*about* x" where $x \in \mathbb{N}^+$, as fuzzy numbers. The aim of the model is to provide three α-cuts of their membership functions: the support, the 0.5-cut and the kernel. Two steps are considered to generate the α-cuts.

Firstly, a pool of candidate values are selected among the integers to be the endpoints values. These good candidates are the ones that best realises a compromise between two properties characterising them: (i) their distance to the reference value x; (ii) their cognitive salience, capturing how easily they can be evoked by the human cognitive system, and measured by their complexity [8].

Secondly, the good candidates form a list which is sorted from the closest to the farthest from the reference value x. The endpoints values are then selected among them, according to their rank in the list and taking into account arithmetical dimensions of the reference value.

Figure 2 illustrates three membership functions predicted by the interpretation model, related to the calculation $\simeq 200 + \simeq 400$: μM_{200}, μM_{400} and μM_{600}.

IV. EXPERIMENTAL STUDY

This Section describes the experimental study designed to assess (i) whether human imprecise calculation fits the fuzzy arithmetic and (ii) whether the participants of the empirical study take into account the imprecision related to the operands to estimate the result.

To do so, μC_z, the fuzzy interval corresponding to the result of the calculations, is compared to three other ones: (i) μA_z, the fuzzy interval resulting from the use of fuzzy arithmetic on the fuzzy operands to achieve the first assessment, (ii) μP_z, the fuzzy interval elicited from the collected data, corresponding to the approximation of the result of the calculation and (iii) μM_z, the fuzzy interval generated by the interpretation model. The two latter comparisons are considered to achieve the second assessment.

Beyond a qualitative visual assessment of the membership functions, we propose a quantitative criterion to compare the fitting of two fuzzy intervals. It is designed as follow.

Firstly, rather than producing a single score for the whole membership functions, we propose to assess several α-cuts. Indeed, as illustrated in Figure 3, the difference between two membership functions is not constant across all α-cuts. Each α-cut receive its own score and, in the end, a graph plotting the fitting score as a function of α is produced.

Secondly, the score is computed as the sum of the absolute differences between the endpoints of the two compared α-cuts. However, because the fitting score is not comparable from one calculation to another (e.g., a difference of 10 units should be considered as much more important when considering $\simeq 20 + \simeq 30$ than when considering $\simeq 500 + \simeq 1000$), it should be relative to the reference value of the exact result z. Considering z in itself may lead to another bias. Indeed,

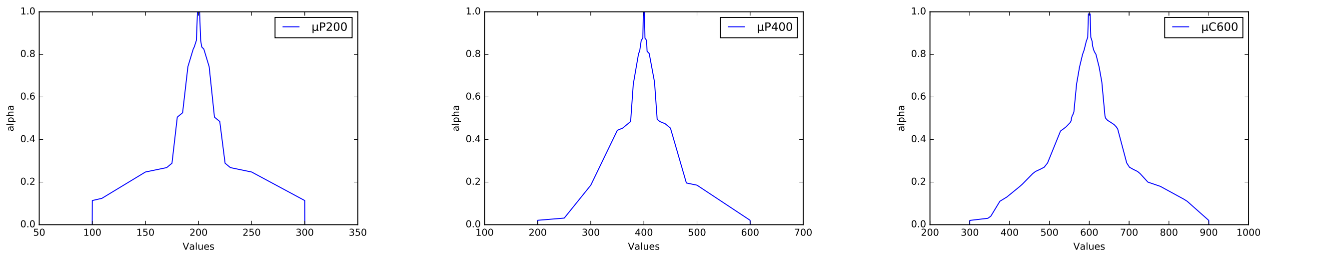

Fig. 1. Membership functions elicited from the collected intervals of the calculation $\simeq 200+ \simeq 400$, as described in Section III-A: μP_{200} (left), μP_{400} (center) of the operands and μC_{600} (right) of the result.

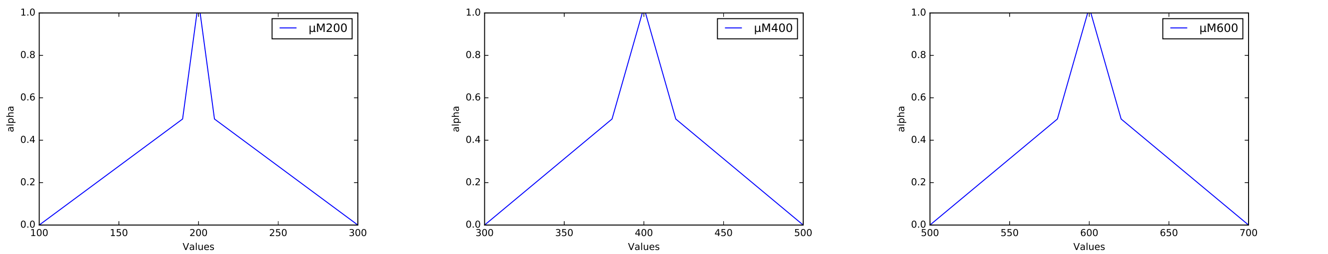

Fig. 2. Membership functions predicted by the interpretation model [8], as described in Section III-C: μM_{200} (left), μM_{400} (center) and μM_{600} (right).

the same difference should be less important, for instance, for 4730 than for 4700. We therefore propose to use the precision of the reference value as the relative factor.

The precision of a number x, noted $Prec(x)$ is defined as its last significant digit followed by its ending zeros. Noting x in the decimal $x = \sum_{i=0}^{q} a_i \cdot 10^i$, where $a_i \in [\![0, 9]\!]$, $Prec(x)$ is computed as $Prec(x) = a_{i^*} \cdot 10^{i^*}$, where $i^* = \min\{i | a_i \neq 0\}$. For instance, $Prec(4730) = 30$, $Prec(4700) = 700$ and $Prec(4000) = 4000$.

The fitting score, noted α_{Fit} is computed as:

$$\alpha_{Fit}(\mu 1_z, \mu 2_z, \alpha) = \frac{|u_1 - u_2| + |v_1 - v_2|}{Prec(z)} \quad (1)$$

where $\mu 1_z$ and $\mu 2_z$ are the fuzzy intervals to be compared and $[\mu 1_z]_\alpha = [u_1, v_1]$, $[\mu 2_z]_\alpha = [u_2, v_2]$ their considered α-cuts.

A low α_{Fit} score indicates a good fitting. The comparison of two fuzzy intervals is made on the basis of 100 α-cuts ($\alpha = \{0.01; 0.02; \ldots; 1\}$).

V. Results

Figure 3 illustrates four fuzzy intervals: μC_z, elicited from the intervals provided as results of the calculations, μA_z, obtained by applying fuzzy arithmetic, μP_z, elicited from the intervals provided as the approximation of the exact result and μM_z, generated by the interpretation model. The illustrated fuzzy intervals correspond to four calculations: $\simeq 20 \cdot \simeq 400$, $\simeq 30 \cdot \simeq 50$, $\simeq 50+ \simeq 100$ and $\simeq 150+ \simeq 8000$.

Figure 4 illustrates the average fitting scores over all products and all additions, as a function of α and Table III present the average fitting scores over α, in products and additions.

TABLE III
Average fitting scores resulting from the comparison of μA_z, μP_z and μM_z to μC_z, in products and in additions.

	Fitting scores	
Fuzzy interval compared to μC_z	Products	Additions
μA_z	$\alpha_{Fit} = 0.564$	$\alpha_{Fit} = 2.057$
μP_z	$\alpha_{Fit} = 0.170$	$\alpha_{Fit} = 0.378$
μM_z	$\alpha_{Fit} = 0.209$	$\alpha_{Fit} = 0.771$

A. Qualitative comparison of μC_z and μA_z

From the visual assessment of the membership functions, one can observe that the fuzzy intervals obtained by fuzzy arithmetic, μA_z, tend to be wider and to overestimate the membership degree of values, compared to the one elicited from participants' calculations, μC_z.

Three cases can be distinguished. Firstly, the overestimation of the imprecision is especially observed in products. This observation is consistent with the fact that the imprecision is quadratically propagated in interval and fuzzy products while it is linearly propagated in additions. Secondly, μA_z fits well μC_z in additions whose precisions of the operands are comparable in magnitude. This case is illustrated by $\simeq 50+ \simeq 100$ (see Figure 3), where $Prec(50) = 50$ and $Prec(100) = 100$. Additions whose precisions of the operands are different in magnitude belong to the third case. It is illustrated by $\simeq 150+ \simeq 8000$ (see Figure 3), where $Prec(150) = 50$ and $Prec(8000) = 8000$. In these cases,

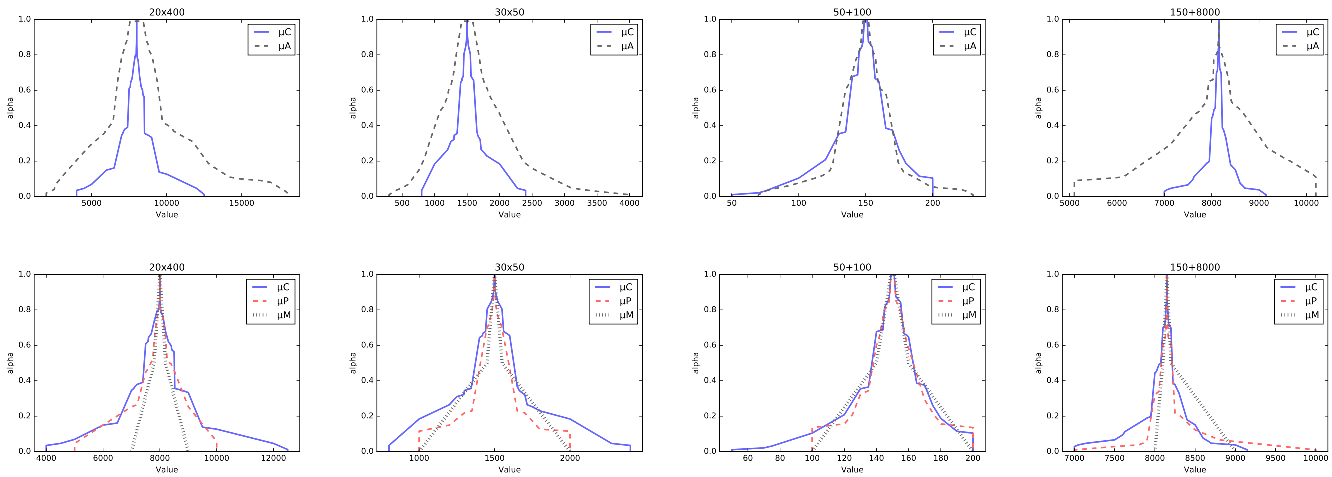

Fig. 3. Fuzzy intervals corresponding to four calculations: $\simeq 20 \cdot \simeq 400$, $\simeq 30 \cdot \simeq 50$, $\simeq 50 + \simeq 100$ and $\simeq 150 + \simeq 8000$. On the first row, comparisons of the fuzzy intervals μC_z (in plain line), elicited from the calculation results provided by the participants, with μA_z (in dashed line), the fuzzy intervals obtained by applying fuzzy arithmetic. On the second row, comparisons of μC_z (in plain line) with (i) μP_z (in dashed line), the fuzzy intervals elicited from the intervals provided as approximations of the exact result (8000, 1500, 150 and 8150 respectively) and (ii) μM_z (in dotted line), the fuzzy intervals predicted by the interpretation model.

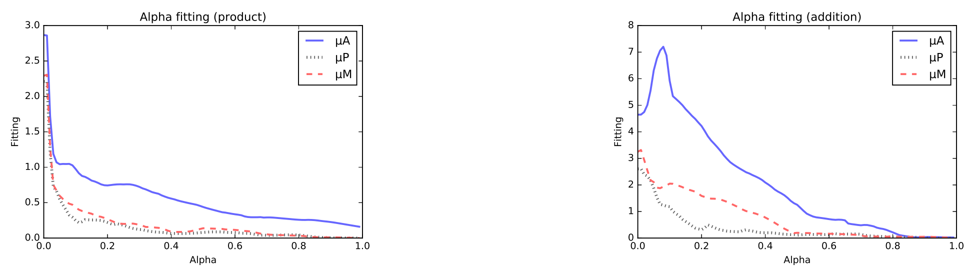

Fig. 4. Averages of the fitting scores of μA_z (plain line), μP_z (dotted line) and μM_z (dashed line) with μC_z, as a function of α, for products (left) and additions (right).

on can observe that fuzzy arithmetic lead to overestimated imprecisions, much more than in products.

On can conclude from these observations that fuzzy arithmetic does not fit well the way human beings propagate imprecisions in imprecise products and addtions.

B. *Qualitative comparison of μC_z, μP_z and μM_z*

From the second row of Figure 3, it can be observed that μC_z tend to be very close to μP_z, the fuzzy interval elicited from the intervals provided as approximation of the exact result (e.g., "*about 8000*" for $\simeq 20 \cdot \simeq 400$). Consequently, it seems that the participants of the study do not take into account the imprecision they attribute to the operands.

It is especially obvious in additions such as $\simeq 30 + \simeq 4700$ and $\simeq 150 + \simeq 8000$ (see Figure 3), in which the imprecision related to the result of the calculation (i.e., 4730 and 8150) is much more lower than the one related to the large operand (i.e., 4700 and 8000). This observation indicates that the imprecision of the large operand is not taken into account during the calculations.

Finally, the bottom row of Figure 3 also shows that the fuzzy intervals generated by the model of interpretation of approximate numerical expressions, μM_z, visually fit well the ones elicited from the results of the calculations, μC_z.

C. *Quantitative comparisons of μC_z with μA_z, μP_z and μM_z*

The fitting scores (see Table III) support the visual analysis. Indeed, they show that μP_z and μM_z better fit μC_z than μA_z.

They also show that the fitting scores tend to be lower for products than for additions. This observation can be explained by the fact that, contrary to the additions, the products provided to the participants do not include any calculation whose operands are very different in the magnitude of their precision (e.g., $\simeq 150 \cdot \simeq 8000$).

The fuzzy intervals obtained by applying fuzzy arithmetic, μA_z are the one that lesser fit μC_z. On the contrary, both the fuzzy intervals predicted by the interpretation model, μM_z and the ones elicited from the approximation of the exact result, μP_z, fit well μC_z, showing that these intervals better account for the observed results of calculations.

Taken together, the results tend to show that participants of the study seem not to take into account the imprecision related to the operands to estimate the one related to the result of the calculation. The comparisons of μC_z and μP_z suggest that the imprecision attributed to the result correspond to the one attributed to the approximation of the exact result (e.g., to "*about 8150*" for $\simeq 150+ \simeq 8000$). Consequently, our main hypothesis to explain these observations is that participant tend to perform the exact calculation and then to approximate the exact result (e.g., "*about* $150 + 8000 = 8150$" for $\simeq 150+ \simeq 8000$). Moreover, as observed, the interpretation model predicting μM_z is able to give better estimations of the results of human additions and products than fuzzy arithmetics.

VI. Conclusion and Future Works

Imprecisions in numerical expressions are pervasive in daily human communication. Fuzzy logic, and more specifically fuzzy arithmetic, is an attempt to capture the way human beings deal with these imprecisions [5], [6]. While these imprecisions have been studied and estimated [12], [8], the issue of their propagation in human calculation have not, to the best of our knowledge, been addressed in the literature.

By means of an empirical study, this paper aims at assessing the fitting of fuzzy arithmetic with human imprecise calculation. Beyond fuzzy arithmetic, and because cognitive studies of imprecision are sparse, this paper is also an attempt to better understand how human beings take into account the imprecision of the calculation operands to estimate the result.

To do so, intervals corresponding to imprecise products and additions have been collected from participants. An experimental study has been conducted to compare three fuzzy intervals to the reference one, elicited from the intervals corresponding to the result of the calculation: (i) one obtained by applying fuzzy arithmetic to the fuzzy operands, (ii) one corresponding to the approximation of the exact result and (iii) one generated by a model of interpretation of approximate numerical expressions [8].

The qualitative and quantitative comparisons show that fuzzy arithmetic tends to produce membership functions that overestimate the resulting imprecision and that do not fit well the observed resulting fuzzy interval. The comparisons also show that the fuzzy intervals corresponding to the approximation of the exact result and the one generated by the interpretation model better fit the observed ones.

Taken together, these results tend to show that fuzzy arithmetic do not account for the human imprecise calculations. Conversely, human beings do not seem to propagate imprecision in a mathematically correct manner, especially in products. More specifically, human beings seem not take into account at all the imprecision related to the operands during calculation. Indeed, the fuzzy intervals elicited from the calculations results tend to be very close to the ones corresponding to the approximation of the result (e.g., $\mu(\simeq 150+ \simeq 8000)$ $\simeq$ $\mu($ "*about 8150*" $)$). Consequently our main hypothesis to explain these results is that, when confronted with an imprecise calculation, human beings tend to perform the exact calculation, and then to provide an approximation of the result of this exact calculation.

Future work will include more detailed study of imprecision propagation in human imprecise calculations.

Firstly, using contextualised calculations, such as a sum of estimated costs. Indeed, it has been shown that human beings better deal with arithmetical facts when they are concrete, such as in money manipulation [13]. It may be that contextualised operands lead to calculation results closer to the fuzzy arithmetic. Secondly, other arithmetic operations will be studied, particularly the subtraction.

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