# Resistance metric, and spectral asymptotics, on the graph of the Weierstrass function

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## 1 Introduction

Following our work on the graph of the Weierstrass function [Dav17], in the spirit of those of J. Kigami [Kig89], [Kig93], and [Str99], [Str06], which enabled us to build a Laplacian on the aforementioned graph, it was natural to go further and give the related explicit resistance metric.

The aim of this work is twofold. We had a special interest in the study of the spectral properties of the Laplacian. In [Dav17], we have given the explicit the spectrum on the graph of the Weierstrass function. In the case of Laplacians on post-critically finite fractals, previous works, by J. Kigami and M. Lapidus [KL03], and R. S. Strichartz [Str06], make the link between resistance metric, and asymptotic properties of the spectrum of the Laplacian, by means of an analoguous of Weyl's formula.

So we asked ourselves wether those results were still valid, for the graph of the Weierstrass function.

## 2 Framework of the study

In this section, we recall results that are developed in [Dav17].

**Notation.** In the following, $\lambda$ and $N_b$ are two real numbers such that:

$$0<\lambda<1 \quad , \quad N_b \in \mathbb{N} \quad \text{and} \quad \lambda\, N_b>1$$

We will consider the $(1-$periodic) Weierstrass function $\mathcal{W}$, defined, for any real number $x$, by:

$$\mathcal{W}(x)=\sum_{n=0}^{+\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right)$$

We place ourselves, in the sequel, in the Euclidean plane of dimension 2, referred to a direct orthonormal frame. The usual Cartesian coordinates are $(x,y)$.

The restriction $\Gamma_{\mathcal{W}}$ to $[0,1[\times\mathbb{R}$, of the graph of the Weierstrass function, is approximated by means of a sequence of graphs, built through an iterative process. To this purpose, we introduce the iterated function system of the family of $C^\infty$ contractions from $\mathbb{R}^2$ to $\mathbb{R}^2$:

$$\{T_0, ..., T_{N_b-1}\}$$

where, for any integer $i$ belonging to $\{0, ..., N_b - 1\}$, and any $(x, y)$ of $\mathbb{R}^2$:

$$T_i(x, y) = \left(\frac{x+i}{N_b}, \lambda\, y + \cos\left(2\,\pi\left(\frac{x+i}{N_b}\right)\right)\right)$$

**Property 2.1.**

$$\Gamma_{\mathcal{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}})$$

**Definition 2.1.** For any integer $i$ belonging to $\{0, ..., N_b - 1\}$, let us denote by:

$$P_i = (x_i, y_i) = \left(\frac{i}{N_b - 1}, \frac{1}{1-\lambda}\,\cos\left(\frac{2\,\pi\, i}{N_b - 1}\right)\right)$$

the fixed point of the contraction $T_i$.

We will denote by $V_0$ the ordered set (according to increasing abscissa), of the points:

$$\{P_0, ..., P_{N_b-1}\}$$

The set of points $V_0$, where, for any $i$ of $\{0, ..., N_b - 2\}$, the point $P_i$ is linked to the point $P_{i+1}$, constitutes an oriented graph (according to increasing abscissa)), that we will denote by $\Gamma_{\mathcal{W}_0}$. $V_0$ is called the set of vertices of the graph $\Gamma_{\mathcal{W}_0}$.

For any natural integer $m$, we set:

$$V_m = \bigcup_{i=0}^{N_b-1} T_i\left(V_{m-1}\right)$$

The set of points $V_m$, where two consecutive points are linked, is an oriented graph (according to increasing abscissa), which we will denote by $\Gamma_{\mathcal{W}_m}$. $V_m$ is called the set of vertices of the graph $\Gamma_{\mathcal{W}_m}$. We will denote, in the sequel, by

$$\mathcal{N}_m^{\mathcal{S}} = 2\, N_b^m + N_b - 2$$

the number of vertices of the graph $\Gamma_{\mathcal{W}_m}$, and we will write:

$$V_m = \left\{\mathcal{S}_0^m, \mathcal{S}_1^m, \ldots, \mathcal{S}_{\mathcal{N}_m-1}^m\right\}$$

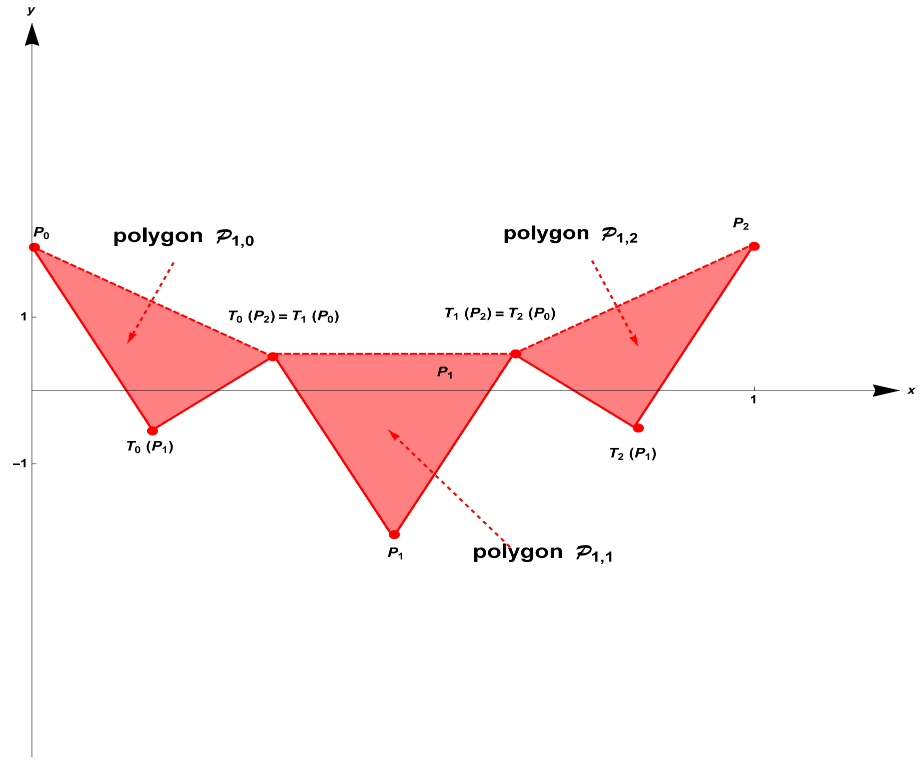


Figure 1: The polygons $\mathcal{P}_{1,0}$, $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, in the case where $\lambda = \frac{1}{2}$, and $N_b = 3$.

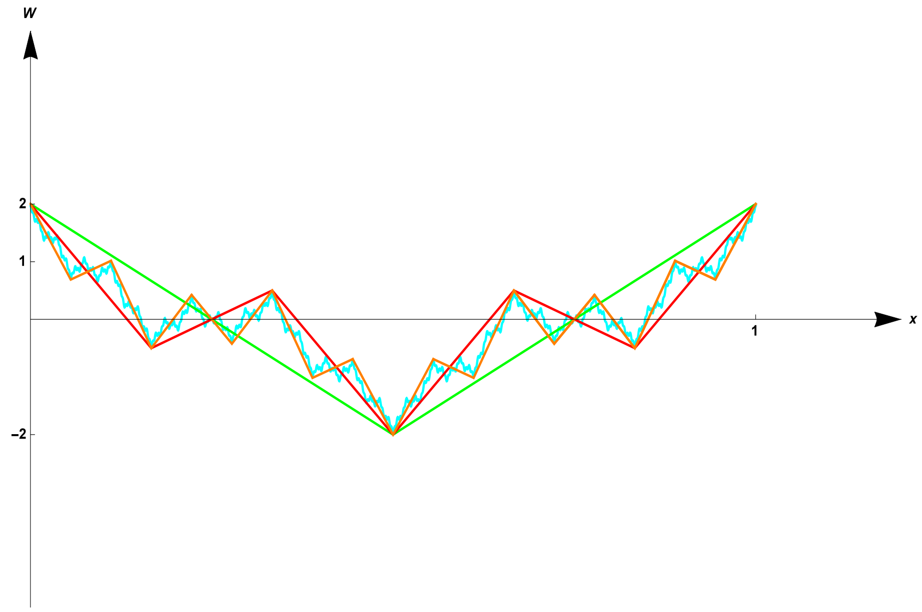


Figure 2: The graphs $\Gamma_{\mathcal{W}_0}$ (in green), $\Gamma_{\mathcal{W}_1}$ (in red), $\Gamma_{\mathcal{W}_2}$ (in orange), $\Gamma_{\mathcal{W}}$ (in cyan), in the case where $\lambda = \frac{1}{2}$, and $N_b = 3$.

**Definition 2.2. Consecutive vertices on the graph $\Gamma_{\mathcal{W}}$**

Two points $X$ et $Y$ de $\Gamma_{\mathcal{W}}$ will be called ***consecutive vertices*** of the graph $\Gamma_{\mathcal{W}}$ if there exists a natural integer $m$, and an integer $j$ of $\{0, ..., N_b - 2\}$, such that:

$$X = (T_{i_1} \circ \ldots \circ T_{i_m})(P_j) \quad \text{et} \quad Y = (T_{i_1} \circ \ldots \circ T_{i_m})(P_{j+1}) \qquad \{i_1, \ldots, i_m\} \in \{0, ..., N_b - 1\}^m$$

or:

$$X = \left(T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_m}\right)\left(P_{N_b-1}\right) \quad \text{et} \quad Y = \left(T_{i_1+1} \circ T_{i_2} \ldots \circ T_{i_m}\right)\left(P_0\right)$$

**Definition 2.3.** For any natural integer $m$, the $\mathcal{N}_m^{\mathcal{S}}$ consecutive vertices of the graph $\Gamma_{\mathcal{W}_m}$ are, also, the vertices of $N_b^m$ simple polygons $\mathcal{P}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 1$, with $N_b$ sides. For any integer $j$ such that $0 \leqslant j \leqslant N_b^m - 1$, one obtains each polygon by linking the point number $j$ to the point number $j+1$ if $j = i \bmod N_b$, $0 \leqslant i \leqslant N_b - 2$, and the point number $j$ to the point number $j - N_b + 1$ if $j = -1 \bmod N_b$. These polygons generate a Borel set of $\mathbb{R}^2$.

**Definition 2.4. Polygonal domain delimited by the graph $\Gamma_{\mathcal{W}_m}$, $m \in \mathbb{N}$**

For any natural integer $m$, well call **polygonal domain delimited by the graph** $\Gamma_{\mathcal{W}_m}$, and denote by $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$, the reunion of the $N_b^m$ polygons $\mathcal{P}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 1$, with $N_b$ sides.

**Definition 2.5. Polygonal domain delimited by the graph $\Gamma_{\mathcal{W}}$**

We will call **polygonal domain delimited by the graph** $\Gamma_{\mathcal{W}}$, and denote by $\mathcal{D}\left(\Gamma_{\mathcal{W}}\right)$, the limit:

$$\mathcal{D}\left(\Gamma_{\mathcal{W}}\right) = \lim_{n \to +\infty} \mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$$

**Definition 2.6. Word, on the graph $\Gamma_{\mathcal{W}}$**

Let $m$ be a strictly positive integer. We will call **number-letter** any integer $\mathcal{M}_i$ of $\{0, \ldots, N_b - 1\}$, and **word of length** $|\mathcal{M}| = m$, on the graph $\Gamma_{\mathcal{W}}$, any set of number-letters of the form:

$$\mathcal{M} = \left(\mathcal{M}_1, \ldots, \mathcal{M}_m\right)$$

We will write:

$$T_{\mathcal{M}} = T_{\mathcal{M}_1} \circ \ldots \circ T_{\mathcal{M}_m}$$

**Definition 2.7. Edge relation, on the graph $\Gamma_{\mathcal{W}}$**

Given a natural integer $m$, two points $X$ and $Y$ of $\Gamma_{\mathcal{W}_m}$ will be called ***adjacent*** if and only if $X$ and $Y$ are two consecutive vertices of $\Gamma_{\mathcal{W}_m}$. We will write:

$$X \underset{m}{\sim} Y$$

This edge relation ensures the existence of a word $\mathcal{M} = \left(\mathcal{M}_1, \ldots, \mathcal{M}_m\right)$ of length $m$, such that $X$ and $Y$ both belong to the iterate:

$$T_{\mathcal{M}}\, V_0 = \left(T_{\mathcal{M}_1} \circ \ldots \circ T_{\mathcal{M}_m}\right)\, V_0$$

Given two points $X$ and $Y$ of the graph $\Gamma_{\mathcal{W}}$, we will say that $X$ and $Y$ are ***adjacent*** if and only if there exists a natural integer $m$ such that:

$$X \underset{m}{\sim} Y$$

**Proposition 2.2.** ***Adresses, on the graph of the Weierstrass function***

*Given a strictly positive integer $m$, and a word $\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_m)$ of lenghth $m \in \mathbb{N}^\star$, on the graph $\Gamma_{\mathcal{W}_m}$, for any integer $j$ of $\{1, ..., N_b - 2\}$, any $X = T_{\mathcal{M}}(P_j)$ de $V_m \setminus V_0$, i.e. distinct from one of the $N_b$ fixed point $P_i$, $0 \leqslant i \leqslant N_b - 1$, has exactly two adjacent vertices, given by:*

$$T_{\mathcal{M}}(P_{j+1}) \quad et \quad T_{\mathcal{M}}(P_{j-1})$$

*where:*

$$T_{\mathcal{M}} = T_{\mathcal{M}_1} \circ \ldots \circ T_{\mathcal{M}_m}$$

*By convention, the adjacent vertices of $T_{\mathcal{M}}(P_0)$ are $T_{\mathcal{M}}(P_1)$ and $T_{\mathcal{M}}(P_{N_b-1})$, those of $T_{\mathcal{M}}(P_{N_b-1})$, $T_{\mathcal{M}}(P_{N_b-2})$ and $T_{\mathcal{M}}(P_0)$ .*

**Definition 2.8. Measure, on the domain delimited by the graph $\Gamma_{\mathcal{W}}$**

We will call **domain delimited by the graph** $\Gamma_{\mathcal{W}}$, and denote by $\mathcal{D}\left(\Gamma_{\mathcal{W}}\right)$, the limit:

$$\mathcal{D}\left(\Gamma_{\mathcal{W}}\right) = \lim_{n \to +\infty} \mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$$

which has to be understood in the following way: given a continuous function $u$ on the graph $\Gamma_{\mathcal{W}}$, and a measure with full support $\mu$ on $\mathbb{R}^2$, then:

$$\int_{\mathcal{D}(\Gamma_{\mathcal{W}})} u \, d\mu = \lim_{m \to +\infty} \sum_{j=0}^{N_b^m - 1} \sum_{X \text{ vertice of } \mathcal{P}_{m,j}} u\left(X\right) \mu\left(\mathcal{P}_{m,j}\right)$$

We will say thar $\mu$ is a **measure, on the domain delimited by the graph** $\Gamma_{\mathcal{W}}$.

**Proposition 2.3.** ***Harmonic extension of a function, on the graph of the Weierstrass function***

*For any strictly positive integer $m$, if $u$ is a real-valued function defined on $V_{m-1}$, its **harmonic extension**, denoted by $\tilde{u}$, is obtained as the extension of $u$ to $V_m$ which minimizes the energy:*

$$\mathcal{E}_{\Gamma_{\mathcal{W}_m}}(\tilde{u}, \tilde{u}) = \sum_{X \underset{m}{\sim} Y} (\tilde{u}(X) - \tilde{u}(Y))^2$$

*The link between $\mathcal{E}_{\Gamma_{\mathcal{W}_m}}$ and $\mathcal{E}_{\Gamma_{\mathcal{W}_{m-1}}}$ is obtained through the introduction of two strictly positive constants $r_m$ and $r_{m+1}$ such that:*

$$r_m \sum_{X \underset{m}{\sim} Y} (\tilde{u}(X) - \tilde{u}(Y))^2 = r_{m-1} \sum_{X \underset{m-1}{\sim} Y} (u(X) - u(Y))^2$$

*In particular:*

$$r_1 \sum_{X \underset{1}{\sim} Y} (\tilde{u}(X) - \tilde{u}(Y))^2 = r_0 \sum_{X \underset{0}{\sim} Y} (u(X) - u(Y))^2$$

*For the sake of simplicity, we will fix the value of the initial constant:* $r_0 = 1$. *One has then:*

$$\mathcal{E}_{\Gamma_{\mathcal{W}_m}}(\tilde{u}, \tilde{u}) = \frac{1}{r_1} \mathcal{E}_{\Gamma_{\mathcal{W}_0}}(\tilde{u}, \tilde{u})$$

*Let us set:*

$$r = \frac{1}{r_1}$$

*and:*

$$\mathcal{E}_m(u) = r_m \sum_{X \underset{m}{\sim} Y} (\tilde{u}(X) - \tilde{u}(Y))^2$$

*Since the determination of the harmonic extension of a function appears to be a local problem, on the graph* $\Gamma_{\mathcal{W}_{m-1}}$, *which is linked to the graph* $\Gamma_{\mathcal{W}_m}$ *by a similar process as the one that links* $\Gamma_{\mathcal{W}_1}$ *to* $\Gamma_{\mathcal{W}_0}$, *one deduces, for any strictly positive integer* $m$:

$$\mathcal{E}_{\Gamma_{\mathcal{W}_m}}(\tilde{u}, \tilde{u}) = \frac{1}{r_1} \mathcal{E}_{\Gamma_{\mathcal{W}_{m-1}}}(\tilde{u}, \tilde{u})$$

*By induction, one gets:*

$$r_m = r_1^m \, r_0 = r^{-m}$$

*If* $v$ *is a real-valued function, defined on* $V_{m-1}$, *of harmonic extension* $\tilde{v}$, *we will write:*

$$\mathcal{E}_m(u, v) = r^{-m} \sum_{X \underset{m}{\sim} Y} (\tilde{u}(X) - \tilde{u}(Y)) \, (\tilde{v}(X) - \tilde{v}(Y))$$

**Property 2.4.** ***Self-similar measure, for the domain delimited by the graph of the Weierstrass function***

*Let us denote by* $\mu_{\mathcal{L}}$ *the Lebesgue measure on* $\mathbb{R}^2$. *We set, for any* $i$ *of* $\{0, \ldots, N_b - 1\}$ :

$$\mu_i = \frac{\mu_{\mathcal{L}}\left(T_i\left(\mathcal{P}_0\right)\right)}{\mu_{\mathcal{L}}\left(\mathcal{P}_0\right)}$$

*The measure* $\mu$, *such that:*

$$\mu = \sum_{i=0}^{N_b - 1} \mu_i \, \mu \circ T_i^{-1}$$

*is self-similar, for the domain delimited by the graph of the Weierstrass function. We refer to [Dav17] for further details.*

**Definition 2.9. Laplacian of order** $m \in \mathbb{N}^\star$

For any strictly positive integer $m$, and any real-valued function $u$, defined on the set $V_m$ of the vertices of the graph $\Gamma_{\mathcal{W}_m}$, we introduce the Laplacian of order $m$, $\Delta_m(u)$, by:

$$\Delta_m u(X) = \sum_{Y \in V_m,\, Y \underset{m}{\sim} X} (u(Y) - u(X)) \quad \forall\, X \in V_m \setminus V_0$$

**Definition 2.10. Existence domain of the Laplacian, for a continuous function on the graph** $\Gamma_{\mathcal{W}}$ (see [BD85b])

We will denote by $\operatorname{dom}\Delta$ the existence domain of the Laplacian, on the graph $\Gamma_{\mathcal{W}}$, as the set of functions $u$ of $\operatorname{dom}\mathcal{E}$such that there exists a continuous function on $\Gamma_{\mathcal{W}}$, denoted $\Delta\, u$, that we will call **Laplacian of** $u$, such that :

$$\mathcal{E}(u,v) = -\int_{\mathcal{D}(\Gamma_{\mathcal{W}})} v\, \Delta u\, d\mu \quad \text{for any } v \in \operatorname{dom}_0 \mathcal{E}$$

**Notation.** In the following, we will denote by $\mathcal{H}_0 \subset \operatorname{dom}\Delta$ the space of harmonic functions, i.e. the space of functions $u \in \operatorname{dom}\Delta$ such that:

$$\Delta\, u = 0$$

Given a natural integer $m$, we will denote by $\mathcal{S}(\mathcal{H}_0, V_m)$ the space, of dimension $N_b^m$, of spline functions " of level $m$", $u$, defined on $\Gamma_{\mathcal{W}}$, continuous, such that, for any word $\mathcal{M}$ of length $m$, $u \circ T_{\mathcal{M}}$ is harmonic, i.e.:

$$\Delta_m\, (u \circ T_{\mathcal{M}}) = 0$$

**Property 2.5.** *Let $m$ be a strictly positive integer, $X \notin V_0$ a vertex of the graph $\Gamma_{\mathcal{W}}$, and $\psi_X^m \in \mathcal{S}(\mathcal{H}_0, V_m)$ a spline function such that:*

$$\psi_X^m(Y) = \begin{cases} \delta_{XY} & \forall\ \ Y \in V_m \\ 0 & \forall\ \ Y \notin V_m \end{cases} \quad , \quad \textit{where} \quad \delta_{XY} = \begin{cases} 1 & \textit{if} \quad X = Y \\ 0 & \textit{else} \end{cases}$$

*For any function $u$ of $\operatorname{dom}\mathcal{E}$, such that its Laplacian exists:*

$$\Delta u(X) = \lim_{m \to +\infty} r^{-m} \left( \int_{\mathcal{D}(\Gamma_{\mathcal{W}})} \psi_X^m\, d\mu \right)^{-1} \Delta_m u(X)$$

**Notation.** We will denote by $\operatorname{dom}\mathcal{E}$ the subspace of continuous functions defined on $\Gamma_{\mathcal{W}}$, such that:

$$\mathcal{E}(u) < +\infty$$

**Property 2.6.** ***Spectrum of the Laplacian****(We refer to our work [Dav17])*

*Let us consider the eigenvalues $(-\Lambda_m)_{m\in\mathbb{N}}$ of the sequence of graph Laplacians $(\Delta_m)_{m\in\mathbb{N}}$, built on the discrete sequence of graphs $(\Gamma_{\mathcal{W}_m})_{m\in\mathbb{N}}$.*

*The spectral decimation method leads to the following recurrence relations between the eigenvalues of order $m$ and $m+1$:*

$$\Lambda_m = \frac{\left\{\left(\dfrac{-2+\Lambda_{m-1}-\varepsilon\left(\{\Lambda_{m-1}-2\}^2-4\right)^{\frac{1}{2}}}{2}\right)^{\frac{1}{N_b}}+1\right\}^2}{\left(\dfrac{-2+\Lambda_{m-1}-\varepsilon\left(\{\Lambda_{m-1}-2\}^2-4\right)^{\frac{1}{2}}}{2}\right)^{\frac{1}{N_b}}}$$

*where $\varepsilon \in \{-1,1\}$.*

# 3 Effective resistance metric, on the graph of the Weierstrass function

## 3.1 Subcells

**Definition 3.1.** **$m^{th}$−order subcell, $m \in \mathbb{N}^\star$, related to a pair of points of the graph $\Gamma_{\mathcal{W}}$**

Given a strictly positive integer $m$, and two points $X$ and $Y$ of $V_m$ such that $X \underset{m}{\sim} Y$, we will call **$m^{th}$−order subcell, related to the pair of points** $(X,Y)$, the polygon, the vertices of which are $X$, $Y$, and the intersection points of the edge between the vertices at the extremities of the polygon, i.e. the respective intersection points of polygons of the type $\mathcal{P}_{m,j-1}$ and $\mathcal{P}_{m,j}$, $1 \leqslant j \leqslant N_b^m - 1$, on the one hand, and of the type $\mathcal{P}_{m,j}$ and $\mathcal{P}_{m,j+1}$, $0 \leqslant j \leqslant N_b^m - 2$, on the other hand.

**Notation.** For any integer $j$ belonging to $\{0, ..., N_b - 1\}$, any natural integer $m$, and any word $\mathcal{M}$ of length $m$, we set:

$$T_{\mathcal{M}}(P_j) = (x(T_{\mathcal{M}}(P_j)), y(T_{\mathcal{M}}(P_j))) \quad , \quad T_{\mathcal{M}}(P_{j+1}) = (x(T_{\mathcal{M}}(P_{j+1})), y(T_{\mathcal{M}}(P_{j+1})))$$

$$L_{j,m} = x(T_{\mathcal{M}}(P_{j+1})) - x(T_{\mathcal{M}}(P_j)) = \frac{1}{(N_b-1)\,N_b^m}$$

and:

$$\delta_m = \max\left\{\frac{1}{(N_b-1)\,N_b^m}, \eta^m\right\}$$

**Proposition 3.1.** *Let us denote by:*

$$D_{\mathcal{W}} = 2 + \frac{\ln\lambda}{\ln N_b}$$

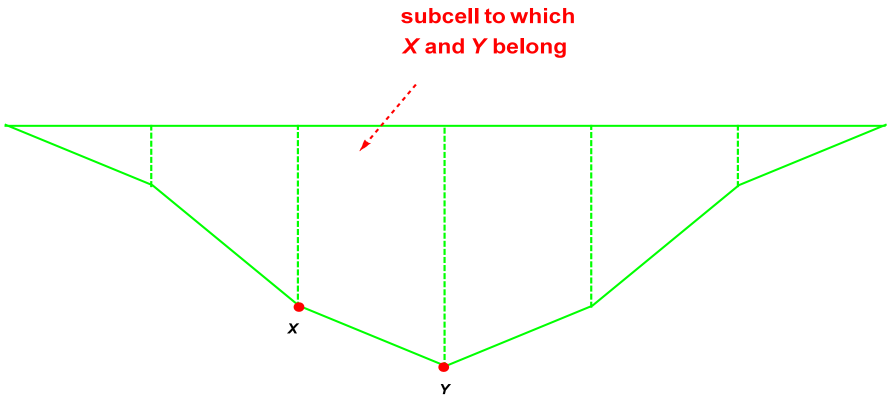


Figure 3: A $m^{th}-$order subcell, in the case where $\lambda = \frac{1}{2}$, and $N_b = 7$.

*the box-dimension (equal to the Hausdorff dimension), of the graph* $\Gamma_{\mathcal{W}}$*.*
*Given a strictly positive integer* $m$*, and two points* $X$ *and* $Y$ *belonging to* $V_m$*, such that* $X \underset{m}{\sim} Y$*, the* $m^{th}-$*order subcell, related to the pair of points* $(X, Y)$*, is included in the rectangle, whose* $X$ *and* $Y$ *are two vertices, of width:*

$$L_{j,m} = \frac{1}{(N_b - 1)\, N_b^m}$$

*and height:*

$$\eta^m \quad = \quad \eta_{2-D_{\mathcal{W}}}\, L_{j,m}^{2-D_{\mathcal{W}}} + \eta_1\, L_{j,m} + \eta_2\, L_{j,m}^2$$

*where the real constants* $\eta_{2-D_{\mathcal{W}}}$*,* $\eta_{3-D_{\mathcal{W}}}$*,* $\eta_{4-D_{\mathcal{W}}}$ *are given by :*

$$\left\{ \begin{array}{ccc} \eta_{2-D_{\mathcal{W}}} & = & (N_b-1)^{2-D_{\mathcal{W}}} \left\{ \dfrac{2}{1-N_b^{(D_{\mathcal{W}}-2)}} \dfrac{\pi^2\,(2\,N_b-1)}{(N_b-1)^2} + \dfrac{2\,\pi^2\,(2\,N_b-1)}{(N_b-1)^2} \dfrac{1}{N_b^{D_{\mathcal{W}}}-1} + \dfrac{4\,\pi^2}{(N_b^{D_{\mathcal{W}}}-1)} \right\} \\ \eta_1 & = & 8\,\pi^2 \\ \eta_2 & = & 2\,\pi^2\,(2\,N_b-1) \end{array} \right.$$

**Proposition 3.2.** ***An upper bound, for the box-dimension of the graph*** $\Gamma_{\mathcal{W}}$

*For any integer* $j$ *belonging to* $\{0, 1, \ldots, N_b - 1\}$*, and each natural integer* $m$*, let us consider the rectangle, the width of which is:*

$$L_{j,m} = x\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right) - x\left(T_{\mathcal{M}}\left(P_j\right)\right) = \frac{1}{(N_b-1)\, N_b^m}$$

*and the length of which, is:*

$$h_{j,m} = |y\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right) - y\left(T_{\mathcal{M}}\left(P_j\right)\right)| \leqslant \eta^m$$

*such that the points $T_{\mathcal{M}}\left(P_{j+1}\right)$ and $T_{\mathcal{M}}\left(P_{j+1}\right)$ are two vertices of .*

*Let us consider a natural integer $N_{j,m} > L_{j,m}$, and divide $L_{j,m}$ in $N_{j,m}$ intervals of the same length $\dfrac{L_{j,m}}{N_{j,m}}$.*

*Then:*

i. *In the case where $\dfrac{1}{\lambda} < N_b$, the values of $h_{j,m}$ on each of those interval vary at most of:*

$$\eta_{2-D_{\mathcal{W}}} \left(\frac{L_{j,m}}{N_{j,m}}\right)^{2-D_{\mathcal{W}}} + \eta_1 \frac{L_{j,m}}{N_{j,m}} + \eta_2 \left(\frac{L_{j,m}}{N_{j,m}}\right)^2 \leqslant C \left(\frac{L_{j,m}}{N_{j,m}}\right)^{2-D_{\mathcal{W}}}$$

*where $C$ denotes a positive constant which does not depend on $N_{j,m}$.*

*The graph $\Gamma_{\mathcal{W}}$ on $L_{j,m}$ can be covered by at most:*

$$N_{j,m} \left\{ C \left(\frac{L_{j,m}}{N_{j,m}}\right)^{1-D_{\mathcal{W}}} + 1 \right\} = C\, L_{j,m}^{1-D_{\mathcal{W}}}\, N_{j,m}^{D_{\mathcal{W}}} + N_{j,m}$$

*squares, the side length of which is $\dfrac{L_{j,m}}{N_{j,m}}$.*

ii. *In the case where $N_b < \dfrac{1}{\lambda}$, the values of $h_{j,m}$ on each of those intervall vary at most of;*

$$\eta_{2-D_{\mathcal{W}}} \left(\frac{L_{j,m}}{N_{j,m}}\right)^{2-D_{\mathcal{W}}} + \eta_1 \frac{L_{j,m}}{N_{j,m}} + \eta_2 \left(\frac{L_{j,m}}{N_{j,m}}\right)^2 \leqslant C \frac{L_{j,m}}{N_{j,m}}$$

*where $C$ denotes a positive constant which does not depend on $N_{j,m}$.*

*The graph $\Gamma_{\mathcal{W}}$ on $L_{j,m}$ can be covered by at most:*

$$N_{j,m} + 1$$

*squares, the side length of which is $\dfrac{L_{j,m}}{N_{j,m}}$.*

*Proof.* For any pair of integers $(i_m, j)$ of $\{0, ..., N_b - 2\}^2$:

$$T_{i_m}\left(P_j\right) = \left(\frac{x_j + i_m}{N_b}, \lambda\, y_j + \cos\left(2\,\pi\left(\frac{x_j + i_m}{N_b}\right)\right)\right)$$

For any pair of integers $(i_m, i_{m-1}, j)$ of $\{0, ..., N_b - 2\}^3$:

$$\begin{aligned} T_{i_{m-1}}\left(T_{i_m}\left(P_j\right)\right) &= \left(\frac{\frac{x_j + i_m}{N_b} + i_{m-1}}{N_b}, \lambda^2\, y_j + \lambda\, \cos\left(2\,\pi\left(\frac{x_j + i_m}{N_b}\right)\right) + \cos\left(2\,\pi\left(\frac{\frac{x_j + i_m}{N_b} + i_{m-1}}{N_b}\right)\right)\right) \\ &= \left(\frac{x_j + i_m}{N_b^2} + \frac{i_{m-1}}{N_b}, \lambda^2\, y_j + \lambda\, \cos\left(2\,\pi\left(\frac{x_j + i_m}{N_b}\right)\right) + \cos\left(2\,\pi\left(\frac{x_j + i_m}{N_b^2} + \frac{i_{m-1}}{N_b}\right)\right)\right) \end{aligned}$$

For any pair of integers $(i_m, i_{m-1}, i_{m-2}, j)$ of $\{0, ..., N_b-2\}^4$:

$$\begin{array}{rcl} T_{i_{m-2}}\left(T_{i_{m-1}}\left(T_{i_m}\left(P_j\right)\right)\right) & = & \left(\dfrac{x_j+i_m}{N_b^3}+\dfrac{i_{m-1}}{N_b^2}+\dfrac{i_{m-2}}{N_b},\right. \\ & & \lambda^3\,y_j+\lambda^2\,\cos\left(2\,\pi\,\left(\frac{x_j+i_m}{N_b}\right)\right) \\ & & \left.+\lambda\,\cos\left(2\,\pi\,\left(\dfrac{x_j+i_m}{N_b^2}+\dfrac{i_{m-1}}{N_b}\right)\right)+\cos\left(2\,\pi\,\left(\dfrac{x_j+i_m}{N_b^3}+\dfrac{i_{m-1}}{N_b^2}+\dfrac{i_{m-2}}{N_b}\right)\right)\right) \end{array}$$

Given a strictly positive integer $m$, and two points $X$ and $Y$ of $V_m$ such that:

$$X \underset{m}{\sim} Y$$

there exists a word $\mathcal{M}$ of length $|\mathcal{M}|=m$, on the graph $\Gamma_{\mathcal{W}}$, and an integer $j$ of $\{0, ..., N_b-2\}^2$, such that:

$$X=T_{\mathcal{M}}\left(P_j\right) \quad , \quad Y=T_{\mathcal{M}}\left(P_{j+1}\right)$$

Let us write $T_{\mathcal{M}}$ under the form:

$$T_{\mathcal{M}}=T_{i_m}\circ T_{i_{m-1}}\circ\ldots\circ T_{i_1}$$

where $(i_1,\ldots,i_m)\in\{0, ..., N_b-1\}^m$.

One has then:

$$\left\{\begin{array}{rcl} x\left(T_{\mathcal{M}}\left(P_j\right)\right) & = & \dfrac{x_j}{N_b^m}+\displaystyle\sum_{k=1}^{m}\dfrac{i_k}{N_b^k} \\ x\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right) & = & \dfrac{x_{j+1}}{N_b^m}+\displaystyle\sum_{k=1}^{m}\dfrac{i_k}{N_b^k} \end{array}\right.$$

and:

$$\left\{\begin{array}{rcl} y\left(T_{\mathcal{M}}\left(P_j\right)\right) & = & \lambda^m\,y_j+\displaystyle\sum_{k=1}^{m-1}\lambda^{m-k}\,\cos\left(2\,\pi\,\left(\frac{x_j}{N_b^k}+\sum_{\ell=0}^{k}\frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right)+\cos\left(2\,\pi\,\left(\frac{x_j}{N_b^m}+\sum_{k=1}^{m}\frac{i_k}{N_b^k}\right)\right) \\ y\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right) & = & \lambda^m\,y_{j+1}+\displaystyle\sum_{k=1}^{m-1}\lambda^{m-k}\,\cos\left(2\,\pi\,\left(\frac{x_{j+1}}{N_b^k}+\sum_{\ell=0}^{k}\frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right)+\cos\left(2\,\pi\,\left(\frac{x_{j+1}}{N_b^m}+\sum_{k=1}^{m}\frac{i_k}{N_b^k}\right)\right) \end{array}\right.$$

This leads to:

$$
\begin{aligned}
|y\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right)-y\left(T_{\mathcal{M}}\left(P_j\right)\right)| \leqslant\; & \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,j+1)}{(N_b-1)^2}+2\,\sum_{k=1}^{m-1}\lambda^{m-k}\,\pi\left\{\frac{2\,j+1}{(N_b-1)\,N_b^k}+2\,\sum_{\ell=0}^{k}\frac{i_{m-\ell}}{N_b^{k-\ell}}\right\}\frac{\pi}{(N_b-1)\,N_b^k}\\
& +\frac{2\,\pi^2}{(N_b-1)\,N_b^m}\left(\frac{2\,j+1}{(N_b-1)\,N_b^m}+2\,\sum_{k=1}^{m}\frac{i_k}{N_b^k}\right)\\
=\; & \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,j+1)}{(N_b-1)^2}+\frac{2\,\pi^2\,\lambda^m\,\pi}{N_b-1}\sum_{k=1}^{m-1}\left\{\frac{(2\,j+1)\,\lambda^{-k}}{(N_b-1)\,N_b^{2k}}+2\,\sum_{\ell=0}^{k}\frac{i_{m-\ell}\,\lambda^{-k}}{N_b^{2k-\ell}}\right\}\\
& +\frac{2\,\pi^2}{(N_b-1)\,N_b^m}\left(\frac{2\,j+1}{(N_b-1)\,N_b^m}+2\,\sum_{k=1}^{m}\frac{i_k}{N_b^k}\right)+\frac{4\,\pi^2}{(N_b-1)\,N_b^m}\sum_{k=1}^{m}\frac{(N_b-1)}{N_b^k}\\
=\; & \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,j+1)}{(N_b-1)^2}\\
& +\frac{2\,\pi^2\,\lambda^m\,\pi}{N_b-1}\left\{\frac{(2\,j+1)}{(N_b-1)}\,\frac{\lambda^{-1}\,N_b^{-2}\,(1-\lambda^{-m+1}\,N_b^{-2m+2})}{1-\lambda^{-1}\,N_b^{-2}}+2\,\sum_{k=1}^{m}\frac{(N_b-1)\,\lambda^{-k}}{N_b^{2k}}\,\frac{1-N_b^{-k-1}}{1-N_b^{-1}}\right\}\\
& +\frac{2\,\pi^2}{(N_b-1)\,N_b^m}\left(\frac{2\,j+1}{(N_b-1)\,N_b^m}+2\,\sum_{k=1}^{m}\frac{(N_b-1)}{N_b^k}\right)+\frac{4\,\pi^2}{(N_b-1)\,N_b^m}\sum_{k=1}^{m}\frac{(N_b-1)}{N_b^k}\\
=\; & \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,j+1)}{(N_b-1)^2}+\frac{2\,\pi^2\,\lambda^m\,\pi}{N_b-1}\left\{\frac{(2\,j+1)}{(N_b-1)}\,\frac{\lambda^{-1}\,N_b^{-2}\,(1-\lambda^{-m+1}\,N_b^{-2m+2})}{1-\lambda^{-1}\,N_b^{-2}}\right.\\
& \left.+2\sum_{k=1}^{m}\frac{(N_b-1)\,\lambda^{-k}}{N_b^{2k}}\,\frac{1-N_b^{-k-1}}{1-N_b^{-1}}\right\}\\
& +\frac{2\,\pi^2}{(N_b-1)\,N_b^m}\left(\frac{2\,j+1}{(N_b-1)\,N_b^m}+2\,\frac{N_b^{-1}\,(N_b-1)\,(1-N_b^{-m+1})}{1-N_b^{-1}}\right)\\
& +\frac{4\,\pi^2}{N_b^m}\,\frac{N_b^{-1}\,(1-N_b^{-m+1})}{1-N_b^{-1}}\\
=\; & \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,j+1)}{(N_b-1)^2}\\
& +\frac{2\,\pi^2\,\lambda^m\,\pi}{N_b-1}\left\{\frac{(2\,j+1)}{(N_b-1)}\,\frac{\lambda^{-1}\,N_b^{-2}\,(1-\lambda^{-m+1}\,N_b^{-2m+2})}{1-\lambda^{-1}\,N_b^{-2}}\right.\\
& \left.+2\sum_{k=1}^{m}\frac{(N_b-1)\,\lambda^{-k}}{N_b^{2k}}\,\frac{1}{1-N_b^{-1}}-2\,\sum_{k=1}^{m}\frac{(N_b-1)\,\lambda^{-k}}{N_b^{2k}}\,\frac{N_b^{-k-1}}{1-N_b^{-1}}\right\}\\
& +\frac{2\,\pi^2}{(N_b-1)\,N_b^m}\left(\frac{2\,j+1}{(N_b-1)\,N_b^m}+\frac{2\,(N_b-1)\,(1-N_b^{-m+1})}{N_b-1}\right)+\frac{4\,\pi^2\,(1-N_b^{-m+1})}{(N_b-1)\,N_b^m}
\end{aligned}
$$

$$
\begin{aligned}
|y\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right)-y\left(T_{\mathcal{M}}\left(P_j\right)\right)| \leqslant\; & \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,j+1)}{(N_b-1)^2}\\
& +\frac{2\,\pi^2\,\lambda^m}{N_b-1}\left\{\frac{(2\,j+1)}{(N_b-1)}\,\frac{\lambda^{-1}\,N_b^{-2}\,(1-\lambda^{-m+1}\,N_b^{-2m+2})}{1-\lambda^{-1}\,N_b^{-2}}+2\,\sum_{k=1}^{m}\frac{(N_b-1)\,\lambda^{-k}}{N_b^{2k}}\,\frac{1}{1-N_b^{-1}}\right.\\
& \left.-2\sum_{k=1}^{m}(N_b-1)\,\lambda^{-k}\,\frac{N_b^{k-1}}{1-N_b^{-1}}\right\}\\
& +\frac{2\,\pi^2}{(N_b-1)\,N_b^m}\left(\frac{2\,j+1}{(N_b-1)\,N_b^m}+2\,(1-N_b^{-m+1})+2\,(1-N_b^{-m+1})\right)\\
\leqslant\; & \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,j+1)}{(N_b-1)^2}+\frac{2\,\pi^2\,\lambda^m\,(2\,j+1)}{(N_b-1)^2}\,\frac{(1-\lambda^{-m+1}\,N_b^{-2m+2})}{\lambda\,N_b^2-1}\\
& +4\,\pi^2\,\lambda^m\,\frac{1-\lambda^{-m+1}\,N_b^{-2m+2}}{\lambda\,N_b^2-1}\,\frac{N_b}{N_b-1}-4\,\pi^2\,\lambda^m\,\frac{1-\lambda^{-m+1}\,N_b^{-2m+2}}{\lambda\,N_b^2-1}\,\frac{1}{N_b-1}\\
& +\frac{2\,\pi^2}{(N_b-1)\,N_b^m}\left(\frac{2\,j+1}{(N_b-1)\,N_b^m}+4\,(1-N_b^{-m+1})\right)
\end{aligned}
$$

Since:

$$
x\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right)-x\left(T_{\mathcal{M}}\left(P_j\right)\right)=\frac{1}{(N_b-1)\,N_b^m}
$$

and:

$$D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln N_b} \quad , \quad \lambda = e^{(D_{\mathcal{W}}-2)\,\ln N_b} = N_b^{(D_{\mathcal{W}}-2)}$$

one has thus:

$$\begin{aligned}
&\left| y\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right) - y\left(T_{\mathcal{M}}\left(P_j\right)\right)\right| \leqslant \\
&\leqslant \lambda^m \left\{ \frac{2}{1-\lambda}\, \frac{\pi^2\,(2\,N_b-1)}{(N_b-1)^2} + \frac{2\,\pi^2\,(2\,N_b-1)}{(N_b-1)^2}\, \frac{1}{\lambda\,N_b^2-1} + \frac{4\,\pi^2}{(\lambda\,N_b^2-1)} \right\} \\
&\quad + \frac{2\,\pi^2}{(N_b-1)\,N_b^m} \left\{ 4 + \frac{2\,N_b-1}{(N_b-1)\,N_b^m} \right\} \\
&= e^{m\,(D_{\mathcal{W}}-2)\,\ln N_b} \left\{ \frac{2}{1-e^{(D_{\mathcal{W}}-2)\,\ln N_b}}\, \frac{\pi^2\,(2\,N_b-1)}{(N_b-1)^2} + \frac{2\,\pi^2\,(2\,N_b-1)}{(N_b-1)^2}\, \frac{1}{N_b^{(D_{\mathcal{W}}-2)}\,N_b^2-1} + \frac{4\,\pi^2}{N_b^{(D_{\mathcal{W}}-2)}\,N_b^2-1)} \right\} \\
&\quad + \frac{2\,\pi^2}{(N_b-1)\,N_b^m} \left\{ 4 + \frac{2\,N_b-1}{(N_b-1)\,N_b^m} \right\} \\
&= N_b^{m\,(D_{\mathcal{W}}-2)} \left\{ \frac{2}{1-N_b^{(D_{\mathcal{W}}-2)}}\, \frac{\pi^2\,(2\,N_b-1)}{(N_b-1)^2} + \frac{2\,\pi^2\,(2\,N_b-1)}{(N_b-1)^2}\, \frac{1}{N_b^{(D_{\mathcal{W}}-2)}\,N_b^2-1} + \frac{4\,\pi^2}{(N_b^{(D_{\mathcal{W}}-2)}\,N_b^2-1)} \right\} \\
&\quad + \frac{2\,\pi^2}{(N_b-1)\,N_b^m} \left\{ 4 + \frac{2\,N_b-1}{(N_b-1)\,N_b^m} \right\}
\end{aligned}$$

$$\begin{aligned}
&= N_b^{m\,(D_{\mathcal{W}}-2)} \left\{ \frac{2}{1-N_b^{(D_{\mathcal{W}}-2)}}\, \frac{\pi^2\,(2\,N_b-1)}{(N_b-1)^2} + \frac{2\,\pi^2\,(2\,N_b-1)}{(N_b-1)^2}\, \frac{1}{N_b^{D_{\mathcal{W}}}-1} + \frac{4\,\pi^2}{(N_b^{D_{\mathcal{W}}}-1)} \right\} \\
&\quad + \frac{2\,\pi^2}{(N_b-1)\,N_b^m} \left\{ 4 + \frac{2\,N_b-1}{(N_b-1)\,N_b^m} \right\}
\end{aligned}$$

$$\begin{aligned}
&= L_{j,m}^{2-D_{\mathcal{W}}}\,(N_b-1)^{2-D_{\mathcal{W}}} \left\{ \frac{2}{1-N_b^{(D_{\mathcal{W}}-2)}}\, \frac{\pi^2\,(2\,N_b-1)}{(N_b-1)^2} + \frac{2\,\pi^2\,(2\,N_b-1)}{(N_b-1)^2}\, \frac{1}{N_b^{D_{\mathcal{W}}}-1} + \frac{4\,\pi^2}{(N_b^{D_{\mathcal{W}}}-1)} \right\} \\
&\quad + 2\,\pi^2\,L_{j,m}\,\{4 + (2\,N_b-1)\,L_{j,m}\}
\end{aligned}$$

This way:

$$\eta^m = \left(\frac{1}{N_b^m}\right)^{2-D_{\mathcal{W}}} \left\{ \frac{2}{1-N_b^{(D_{\mathcal{W}}-2)}} \frac{\pi^2\,(2\,N_b-1)}{(N_b-1)^2} + \frac{2\,\pi^2\,(2\,N_b-1)}{(N_b-1)^2} \frac{1}{N_b^{D_{\mathcal{W}}}-1} + \frac{4\,\pi^2}{(N_b^{D_{\mathcal{W}}}-1)} \right.$$
$$\left. + \frac{2\,\pi^2}{(N_b-1)\,N_b^m} \left\{ 4 + (2\,N_b-1)\left(\frac{1}{(N_b-1)\,N_b^m}\right)\right\} \right\}$$

□

## 3.2 Effective resistance metric

**Property 3.3.** *The space dom $\Delta$, modulo constant functions, is a Hilbert space, included in the space of continuous functions on the graph $\Gamma_{\mathcal{W}}$, modulo constant functions.*

**Definition 3.2. Effective resistance metric, on the graph $\Gamma_{\mathcal{W}}$**
Given a pair of points $(X,Y)$ of the graph $\Gamma_{\mathcal{W}}$, we define, as in [Str03], the **effective resistance metric between the points $X$ and $Y$**, by:

$$R_{\Gamma_{\mathcal{W}}}(X,Y) = \left\{ \min_{\{u \,|\, u(X)=0, u(Y)=1\}} \mathcal{E}(u) \right\}^{-1}$$

In an equivalent way, $R_{\Gamma_{\mathcal{W}}}(X,Y)$ may be defined as the minimum value of the real numbers $R$ such that, for any function $u$ of dom $\Delta$:

$$|u(X)-u(Y)|^2 \leqslant R\,\mathcal{E}(u)$$

**Definition 3.3. Metric, on the graph $\Gamma_{\mathcal{W}}$**
Let us define, on the graph $\Gamma_{\mathcal{W}}$, the distance $d_{\Gamma_{\mathcal{W}}}$ defined, for any pair of points $(X,Y)$ of $\Gamma_{\mathcal{W}}$, by:

$$d_{\Gamma_{\mathcal{W}}}(X,Y) = \left\{ \min_{\{u \,|\, u(X)=0, u(Y)=1\}} \mathcal{E}(u,u) \right\}^{-1}$$

*Remark* 3.1. As it is explained in [Str06], one may note that the minimum

$$\min_{\{u \,|\, u(X)=0, u(Y)=1\}} \mathcal{E}(u)$$

is reached when the function $u$ is harmonic on the complement set, in $\Gamma_{\mathcal{W}}$, of the set $\{X\} \cup \{Y\}$ (we recall that, by definition, a harmonic function $u$ on $\Gamma_{\mathcal{W}}$ minimizes the sequence of energies $\left(\mathcal{E}_{\Gamma_{\mathcal{W}_m}}(u,u)\right)_{m\in\mathbb{N}}$.

In order to fully apprehend and understand the intrinsic meaning of these functions, one might reason by analogy with the unit interval $[0,1]$. In this case, one will note that, given two points $X$ and $Y$ of $[0,1]$ such that $X<Y$, the function $u$ is affine by pieces, taking the value zero on $[0,X]$, and the value 1 on $[Y,1]$ (see the illustration on the following figure):

$$\forall\, t \in [0,1] : \quad u(t) = \frac{t-X}{Y-X}$$

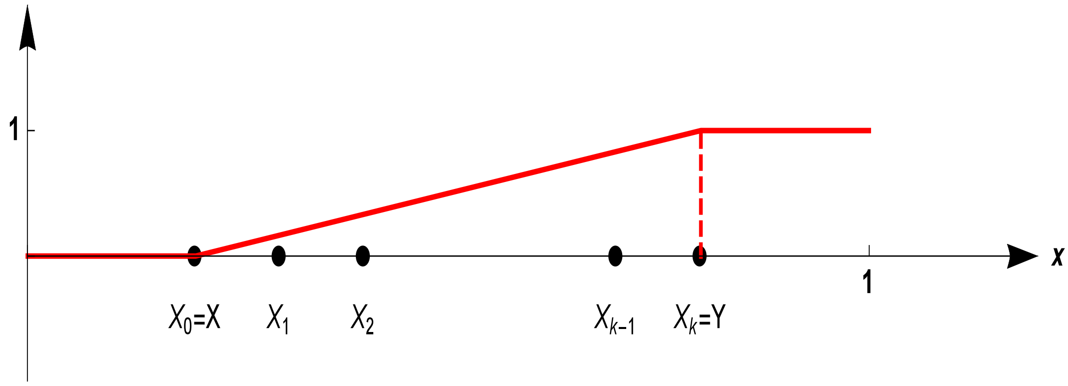


Figure 4: The graph of the function $u$ where the value $\min_{\{u\,|\,u(X)=0,u(Y)=1\}} \mathcal{E}(u)$ is reached.

Let us denote by $m$ the natural integer such that:

$$X \underset{m}{\sim} Y$$

One may introduce, the, for any integer $p$, the sequence of points $(X_j)_{0\leqslant j\leqslant 2^p}$ such that:

$$X_0 = X \quad , \quad X_{2^p} = Y$$

and, for any integer $j$ such that $0 < j < 2^p - 1$:

$$X_j \in V_{p+1} \quad , \quad X_j \underset{p+1}{\sim} X_{j+1}$$

In the case of the unit interval, the normalization constant is:

$$r^{-1} = \frac{1}{2}$$

One has then:

$$\begin{array}{rcl}\mathcal{E}(u,u) & = & \lim\limits_{p\to+\infty} \mathcal{E}_p(u,u)\\ & = & \lim\limits_{m\to+\infty} r^{-p}\,\mathcal{E}_{\Gamma_{\mathcal{W}_p}}(u)\\ & = & \lim\limits_{p\to+\infty} \sum\limits_{(X,Y)\,\in V_p^2,\,X\underset{p}{\sim}Y} r^{-p}\,\left(u_{|V_p}(X) - u_{|V_p}(Y)\right)^2\\ & = & \lim\limits_{p\to+\infty} \sum\limits_{(X,Y)\,\in V_p^2,\,X\underset{p}{\sim}Y} \frac{1}{2^p}\,\left(u_{|V_p}(X) - u_{|V_p}(Y)\right)^2\\ & = & \lim\limits_{p\to+\infty} \sum\limits_{j=0}^{k-1} \frac{1}{2^p}\left(u\left(X+\frac{j}{2^p}\right) - u\left(X+\frac{j+1}{2^p}\right)\right)^2\\ & = & \displaystyle\int_X^Y \frac{dt}{(Y-X)^2}\\ & = & \dfrac{1}{Y-X}\end{array}$$

If $d_{\mathbb{R}}$ denotes the usual Euclidean distance on $\mathbb{R}$:

$$\forall\,(X,Y)\,\in\mathbb{R}^2\,:\quad d_{\mathbb{R}}(X,Y) = |Y-X|$$

one has then:

$$\min_{\{u\,|\,u(X)=0,u(Y)=1\}} \mathcal{E}(u) \quad = \quad \frac{1}{d_{\mathbb{R}}(X,Y)}$$

Let us now consider, more generally, a fractal domain $\mathcal{F}$, in an Euclidean space of dimension $d \in \mathbb{N}^\star$, equipped with the distance $d_{\mathbb{R}^d}$. If, one has, in advance, defined an energy on $\mathcal{F}$, it is worth searching wether there exists a real number $\beta$ such that:

$$\forall (X,Y) \in \mathcal{F}^2 \, : \quad \left( \min_{\{u \,|\, u(X)=0, u(Y)=1\}} \mathcal{E}(u) \right)^{-1} \sim \left( d_{\mathbb{R}^d}(X,Y) \right)^\beta$$

In the case of the Sierpiǹski gasket $\mathcal{SG}$ (we refer to [?]), Robert S. Strichartz lays the emphasis upon the fact that, given $X \underset{m}{\sim} Y$, one has:

$$\min_{\{u \,|\, u(X)=0, u(Y)=1\}} \mathcal{E}(u) \lesssim r_{\mathcal{SG}}^m = \left(\frac{3}{5}\right)^m$$

This also corresponds thus to the order of the diameter of the $m^{th}-$order cells.

Since the Sierpiǹski gasket $\mathcal{SG}$ is obtained from the initial triangle of diameter 1 by means of three contractions, the respective ratios of which are equal to $\frac{1}{2}$, one has simply to look the real number $\beta_{\mathcal{SG}}$ such that:

$$\left(\frac{1}{2}\right)^{m\,\beta_{\mathcal{SG}}} = \left(\frac{3}{5}\right)^m$$

This leads to:

$$\beta_{\mathcal{SG}} = \frac{\ln \frac{5}{3}}{\ln 2}$$

**Definition 3.4. Dimension of the graph $\Gamma_{\mathcal{W}}$, in the effective resistance metric**

The **dimension of the graph** $\Gamma_{\mathcal{W}}$, in the effective resistance metric, is the strictly positive number $d_{Gamma_{\mathcal{W}}}$ such that, given a strictly positive real number $r$, and a point $X \in \Gamma_{\mathcal{W}}$, for the $X-$centered ball of radius $r$, denoted by $\mathcal{B}_r(X)$:

$$\mu\left(\mathcal{B}_r(X)\right) = r^{d_{\Gamma_{\mathcal{W}}}}$$

**Proposition 3.4.** *The dimension of the graph $\Gamma_{\mathcal{W}}$, in the effective resistance metric, is given by:*

i. ***First case:*** $\lambda > \dfrac{1}{N_b}$.

$$d_{\Gamma_{\mathcal{W}}} = \frac{\ln \frac{N_b}{\lambda}}{\ln N_b}$$

ii. ***Second case:*** $\lambda < \dfrac{1}{N_b}$.

$$d_{\Gamma_{\mathcal{W}}} = 2$$

*Proof.*

*Remark* 3.2. Once again, it is worth having a look at the case of the Sierpiński gasket. Robert S. Strichartz stars from the fact that the measure of $m^{th}-$order cells is $\frac{1}{3^m}$. Two consecutive points $x$ and $y$ are such that, for the effective resistance metric

$$d(x,y) \sim \left(\frac{3}{5}\right)^m$$

For the self-similar measure $\mu_{\mathcal{SG}}$, which affects the value $\frac{1}{3^m}$ to each $m^{th}-$order cell, one has simply to look for the real number $d_{\mathcal{SG}}$ such that:

$$\left(\frac{3}{5}\right)^{m\,d_{\mathcal{SG}}} = \frac{1}{3^m}$$

which leads to:

$$d_{\mathcal{SG}} = \frac{\ln 3}{\ln \frac{5}{3}}$$

One may then deduce from the above an estimate, for the effective resistance metric, of the measure of a $X-$centered ball of radius $r$, denoted by $\mathcal{B}_r(X)$:

$$\mu_{\mathcal{SG}}\left(\mathcal{B}_r(x)\right) = r^{d_{\mathcal{SG}}}$$

Let us now go back to the graph $\Gamma_{\mathcal{W}}$.

Given a natural integer $m$, and two points $X$ and $Y$ such that $X \underset{m}{\sim} Y$:

$$\min_{\{u \,|\, u(X)=0, u(Y)=1\}} \mathcal{E}(u) \lesssim r^{-m} = N_b^m$$

For the detailed calculations which enable one to obtain the normalization constants, we refer to [**?**]. For the self-similar measure $\tilde{\mu}$ introduced in the above, each $m^{th}-$order cell, i.e. each simple polygon $\mathcal{P}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 1$, with $N_b$ sides and $N_b$ vertices, has a measure of the order of:

$$(N_b - 1)\,\frac{\eta^m}{N_b^m}$$

The points $X$ and $Y$ such that $X \underset{m}{\sim} Y$ belong to a $m^{th}-$order subcell, which is the intersection of a simple polygon $\mathcal{P}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 1$, with the rectangle of which $X$ and $X$ are two vertices, of width $\frac{\eta^m}{N_b^m}$, and heigth $\eta^m$. This subcell a has a measure, the order of which is thus:

$$\frac{\eta^m}{N_b^m}$$

*i.* **First case:** $\lambda > \frac{1}{N_b}$.

One has simply to look for the real number $d_{\Gamma_{\mathcal{W}}}$ such that:

$$N_b^{m\,d_{\Gamma_{\mathcal{W}}}} = \frac{\lambda^m}{N_b^m}$$

which yields:

$$d_{\Gamma_{\mathcal{W}}} = \frac{\ln \frac{N_b}{\lambda}}{\ln N_b}$$

*ii.* **Second case:** $\lambda < \dfrac{1}{N_b}$.

One has simply to look for the real number $\beta_{\Gamma_{\mathcal{W}}}$ such that:

$$N_b^{m\, d_{\Gamma_{\mathcal{W}}}} = \frac{1}{N_b^{2m}}$$

which yields:

$$d_{\Gamma_{\mathcal{W}}} = 2$$

$\square$

# 4 Detailed study of the spectrum of the Laplacian

As exposed by R. S. Strichartz in [Str06], one may bear in mind that the eigenvalues can be grouped into two categories:

*i.* initial eigenvalues, which a priori belong to the set of forbidden values (as for instance $\Lambda = 2$) ;

*ii.* continued eigenvalues, obtained by means of spectral decimation.

We present, in the sequel, a detailed study of the spectrum of $\Delta$, in the case where $N_b = 3$, which can be easily extend to higher values of the integer $N_b$.

## 4.1 Eigenvalues and eigenvectors of $\Delta_1$

Let us recall that the vertices of the graph $\Gamma_{\mathcal{W}_1}$ are:

$$P_0 \quad , \quad T_0\,(P_1) \quad , \quad T_0\,(P_2) \quad , \quad T_1\,(P_0)$$

$$P_1 \quad , \quad T_1\,(P_2) \quad , \quad T_2\,(P_0) \quad , \quad T_2\,(P_1) \quad , \quad P_2$$

One may note that:

$$\text{Card } (V_1 \setminus V_0) = 4$$

Let us denote by $u$ an eigenfunction, for the eigenvalue $-\Lambda$. For the sake of simplicity, we set:

$$u\,(T_0\,(P_1)) = a \in \mathbb{R} \quad , \quad u\,(T_0\,(P_2)) = b \in \mathbb{R} \quad , \quad u\,(T_2\,(P_0)) = c \in \mathbb{R} \quad , \quad u\,(T_2\,(P_1)) = d \in \mathbb{R}$$

One has then:

$$\left\{ \begin{array}{rcl} u(P_0) - a + b - a &=& -\Lambda\, a \\ a - b + u(P_1) - b &=& -\Lambda\, b \\ u(P_1) - c + d - c &=& -\Lambda\, c \\ c - d + u(P_2) - d &=& -\Lambda\, d \end{array} \right.$$

One may note that the only "Dirichlet eigenvalues", i.e. the ones related to the Dirichlet problem:

$$u_{|V_0} = 0 \quad \text{i.e.} \quad u(P_0) = u(P_1) = u(P_2) = 0$$

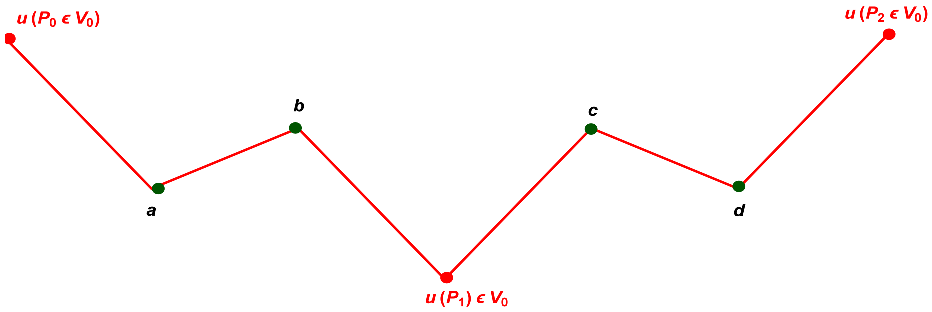


Figure 5: Successive values of an eigenfunction on $V_1$, in the case where $N_b = 3$.

are obtained for:

$$\left\{ \begin{array}{lcl} b & = & -(\Lambda-2)\,a \\ a & = & -(\Lambda-2)\,b \\ d & = & -(\Lambda-2)\,c \\ c & = & -(\Lambda-2)\,d \end{array} \right.$$

i.e.:

$$\left\{ \begin{array}{lcl} b & = & (\Lambda-2)^2\,b \\ a & = & (\Lambda-2)^2\,a \\ d & = & (\Lambda-2)^2\,d \\ c & = & (\Lambda-2)^2\,c \end{array} \right.$$

The forbidden eigenvalue $\Lambda = 2$ cannot thus be a Dirichlet one.

Let us consider the case where:

$$(\Lambda-2)^2 = 1$$

i.e.

$$\Lambda = 1 \quad \text{or} \quad \Lambda = 3$$

The value $\Lambda = 1$ leads to:

$$a = b \quad , \quad c = d$$

which yields a two-dimensional eigenspace. The multiplicity of the eigenvalue $\Lambda = 3$ is 2.

For the eigenvalue $\Lambda = 3$:

$$a = -b \quad , \quad c = -d$$

The eigenspace, for the eigenvalue 3, has dimension 2. The multiplicty of the eigenvalue $\Lambda = 3$ is 2.

Since the cardinal of $V_1 \setminus V_0$ is:

$$\mathcal{N}_1^{\mathcal{S}} - N_b = 2\,N_b - 2 = 4$$

one may note that we have the complete spectrum.

### 4.2 Eigenvalues of $\Delta_2$

Let us now look at the spectrum of $\Delta_2$. For the sake of simplicity, we will denote by $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$, the successive values of an eigenfunction at the $N_b^2 - 1$ points between $P_0$ and $P_1$, and by $a'$, $b'$, $c'$, $d'$, $e'$, $f'$, $g'$, $h'$, the successive values of an eigenfunction at the $N_b^2 - 1$ points between $P_1$ and $P_2$, as it appears on the following figure.

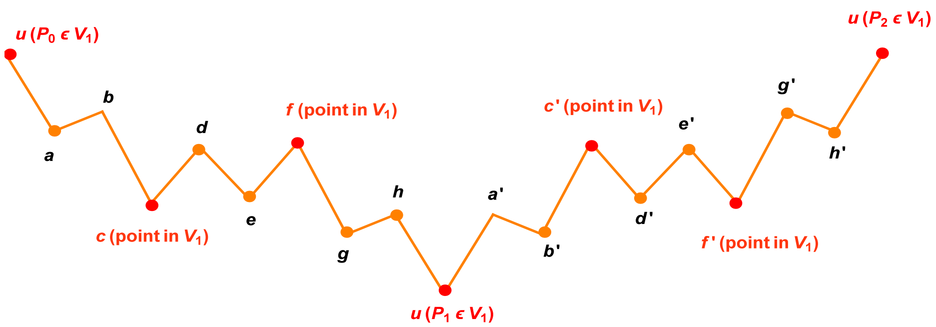


Figure 6: Successive values of an eigenfunction on $V_2$, in the case where $N_b = 3$.

One has then:

$$\left\{ \begin{array}{rcl} (2-\Lambda)\,a & = & -u(P_0) - b \\ (2-\Lambda)\,b & = & -c - a \\ (2-\Lambda)\,c & = & -b - d \\ (2-\Lambda)\,d & = & -c - e \\ (2-\Lambda)\,e & = & -d - f \\ (2-\Lambda)\,f & = & -e - g \\ (2-\Lambda)\,g & = & -f - h \\ (2-\Lambda)\,h & = & -g - u(P_1) \end{array} \right.$$

and:

$$\left\{\begin{array}{lcc}(2-\Lambda)\,a' & = & -u(P_1)-b' \\ (2-\Lambda)\,b' & = & -c'-a' \\ (2-\Lambda)\,c' & = & -b'-d' \\ (2-\Lambda)\,d' & = & -c'-e' \\ (2-\Lambda)\,e' & = & -d'-f' \\ (2-\Lambda)\,f' & = & -e'-g' \\ (2-\Lambda)\,g' & = & -f'-h' \\ (2-\Lambda)\,h' & = & -g'-u(P_2)\end{array}\right.$$

One may note that the only Dirichlet eigenvalues, in the case where:

$$u_{|V_1}=0 \quad \text{i.e.} \quad u(P_0)=u(P_1)=u(P_2)=c=f=c'=f'=0$$

are obtained for:

$$\left\{\begin{array}{ccc}(2-\Lambda)\,a & = & -b \\ (2-\Lambda)\,b & = & -c-a \\ 0 & = & -b-d \\ (2-\Lambda)\,d & = & -e \\ (2-\Lambda)\,e & = & -d \\ 0 & = & -e-g \\ (2-\Lambda)\,g & = & -h \\ (2-\Lambda)\,h & = & -g\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ccc}(2-\Lambda)\,a' & = & -b' \\ (2-\Lambda)\,b' & = & -c'-a' \\ 0 & = & -b'-d' \\ (2-\Lambda)\,d' & = & -e' \\ (2-\Lambda)\,e' & = & -d' \\ 0 & = & -e'-g' \\ (2-\Lambda)\,g' & = & -h' \\ (2-\Lambda)\,h' & = & -g'\end{array}\right.$$

i.e.:

$$\left\{\begin{array}{ccc}(2-\Lambda)\,a & = & -b \\ \left\{1-(2-\Lambda)^2\right\}\,a & = & -c \\ 0 & = & -b-d \\ (2-\Lambda)\,d & = & -e \\ (2-\Lambda)^2\,e & = & e \\ 0 & = & -e-g \\ (2-\Lambda)^2\,h & = & h \\ (2-\Lambda)\,h & = & -g\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ccc}(2-\Lambda)\,a' & = & -b' \\ \left\{1-(2-\Lambda)^2\right\}\,a' & = & -c' \\ 0 & = & -b'-d' \\ (2-\Lambda)\,d' & = & -e' \\ (2-\Lambda)^2\,e' & = & e' \\ 0 & = & -e'-g' \\ (2-\Lambda)^2\,h' & = & h' \\ (2-\Lambda)\,h' & = & -g'\end{array}\right.$$

The forbidden eigenvalue $\Lambda=2$ is not therefore a Dirichlet one.

Let us consider the case where:

$$(\Lambda-2)^2=1$$

i.e.

$$\Lambda=3 \quad \text{or} \quad \Lambda=1$$

For $\Lambda=1$, one has:

$$\left\{\begin{array}{ccccccc}a & = & -b & & & & \\ c & = & 0 & & & & \\ d & = & -b & = & a & & \\ e & = & -d & = & a & & \\ d & = & -e & = & -a & & \\ g & = & -e & = & -a & & \\ h & = & -g & = & e & = & a\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ccccccc}a' & = & -b' & & & & \\ c' & = & 0 & & & & \\ d' & = & -b' & = & a' & & \\ e' & = & -d' & = & a' & & \\ d' & = & -e' & = & -a' & & \\ g' & = & -e' & = & -a' & & \\ h' & = & -g' & = & e' & = & a'\end{array}\right.$$

The eigenspace, for $\Lambda=1$, has thus dimension 2. The multiplicity of the eigenvalue $\Lambda=1$ is 2.

For $\Lambda=3$:

$$\left\{\begin{array}{lllll} a & = & b & & \\ c & = & 0 & & \\ d & = & -b & = & -a \\ e & = & d & = & -a \\ g & = & -e & = & a \\ h & = & g & = & a \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lllll} a' & = & b' & & \\ c' & = & 0 & & \\ d' & = & -b' & = & -a' \\ e' & = & d' & = & -a' \\ g' & = & -e' & = & a' \\ h' & = & g' & = & a' \end{array}\right.$$

The eigenspace, for $\Lambda=3$, has thus dimension 2. The multiplicity of the eigenvalue $\Lambda=3$ is 2.

Let us now look at the continued eigenvalues, i.e. the ones obtained from the eigenvalues $\Lambda_1=1$ and $\Lambda_1=3$ by means of spectral decimation:

$$\Lambda_2=\phi^{-1}\left((\phi\,(\Lambda_1))^{\frac{1}{N_b}}\right)=\frac{\left\{(\phi\,(\Lambda_1))^{\frac{1}{N_b}}+1\right\}^2}{(\phi\,(\Lambda_1))^{\frac{1}{N_b}}}=\frac{\left\{\left(\frac{-2+\Lambda_1-\varepsilon\sqrt{\{\Lambda_1-2\}^2-4}}{2}\right)^{\frac{1}{N_b}}+1\right\}^2}{\left(\frac{-2+\Lambda_1-\varepsilon\sqrt{\{\Lambda_1-2\}^2-4}}{2}\right)^{\frac{1}{N_b}}}$$

where $\varepsilon\in\{-1,1\}$, for the values:

$$\Lambda_1\in\{1,3\}$$

As in [Str06], let us get rid, temporarily, of the Dirichlet conditions. We have thus:

$$\left\{\begin{array}{rcc} u(P_0)+b & = & -(\Lambda-2)\,a \\ a+c & = & -(\Lambda-2)\,b \\ b+d & = & -(\Lambda-2)\,c \\ e+f & = & -\Lambda\,d \\ e+g & = & -(\Lambda-2)\,f \\ f+h & = & -(\Lambda-2)\,g \\ g+u(P_1) & = & -(\Lambda-2)\,h \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcc} u(P_1)+b' & = & -(\Lambda-2)\,a' \\ a'+c' & = & -(\Lambda-2)\,b' \\ b'+d' & = & -(\Lambda-2)\,c' \\ e'+f' & = & -\Lambda\,d' \\ e'+g' & = & -(\Lambda-2)\,f' \\ f'+h' & = & -(\Lambda-2)\,g' \\ g'+u(P_2) & = & -(\Lambda-2)\,h' \end{array}\right.$$

For the initial eigenvalue $\Lambda_1=1$, it is worth noticing that its restriction to $V_1\setminus V_0$ must satisfy the eigensystem associated to the eigenvalue $\Lambda_1=1$, i.e.:

$$\left\{\begin{array}{lcl} u(P_0)+f & = & -(\Lambda_1-2)\,c \\ u(P_1)+c & = & -(\Lambda_1-2)\,f \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} u(P_1)+f' & = & -(\Lambda_1-2)\,c' \\ u(P_2)+c' & = & -(\Lambda_1-2)\,f' \end{array}\right.$$

or:

$$\left\{\begin{array}{lcl} u(P_0)+f & = & c \\ u(P_1)+c & = & f \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} u(P_1)+f' & = & c' \\ u(P_2)+c' & = & f' \end{array}\right.$$

i.e.:

$$u(P_0)+u(P_1)=0 \quad \text{and} \quad u(P_1)+u(P_2)=0$$

For $u(P_0) = u(P_1) = u(P_2) = 0$, it works, and the Dirichlet conditions appear to be satisfied. One has then:

$$\left\{ \begin{array}{ccc} b & = & -(\Lambda-2)\,a \\ c & = & \left\{-1+(\Lambda-2)^2\right\}\,a \\ d & = & (\Lambda-2)\,\left\{1-\left\{1-(\Lambda-2)^2\right\}\right\}\,a \\ e+f & = & -\Lambda\,(\Lambda-2)\,\left\{1-\left\{1-(\Lambda-2)^2\right\}\right\}\,a \\ e & = & (\Lambda-2)\,\left\{1-\left\{-1+(\Lambda-2)^2\right\}\right\}\,h \\ f & = & \left\{-1+(\Lambda-2)^2\right\}\,h \\ g & = & -(\Lambda-2)\,h \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ccc} b' & = & -(\Lambda-2)\,a' \\ c' & = & \left\{-1+(\Lambda-2)^2\right\}\,a' \\ d' & = & (\Lambda-2)\,\left\{1-\left\{1-(\Lambda-2)^2\right\}\right\}\,a' \\ e'+f' & = & -\Lambda\,(\Lambda-2)\,\left\{1-\left\{1-(\Lambda-2)^2\right\}\right\}\,a' \\ e' & = & (\Lambda-2)\,\left\{1-\left\{-1+(\Lambda-2)^2\right\}\right\}\,h' \\ f' & = & \left\{-1+(\Lambda-2)^2\right\}\,h' \\ g' & = & -(\Lambda-2)\,h' \end{array} \right.$$

We obtain thus an eigenspace, the dimension of which is 4.

For the eigenvalue $\Lambda_1 = 1$, the spectral decimation spectral leads to:

$$\left(\frac{2-\Lambda_2+\varepsilon_2\,\rho\,(\omega_2)^2\;e^{i\,\theta\omega_2}}{2}\right)^{N_b} = \frac{1+\varepsilon_1\,\sqrt{3}\,e^{\frac{i\,\pi}{2}}}{2}$$

which leads to the double eigenvalue:

$$\Lambda_2 = 2+\cos\frac{\pi}{9}+\sqrt{3}\,\sin\frac{\pi}{9}$$

For the eigenvalue $\Lambda_1 = 3$, the spectral decimation leads to the double eigenvalue:

$$\Lambda_2 = 2\,\left\{1+\cos\frac{\pi}{9}\right\}$$

Since the cardinal of $V_2 \setminus V_1 = 12$ is:

$$\mathcal{N}_2^{\mathcal{S}} - \mathcal{N}_1^{\mathcal{S}} = 2\,N_b^2 + N_b - 2 - (3\,N_b - 2) = 12$$

one may note that we have the complete spectrum.

### 4.3 Eigenvalues of $\Delta_3$

As previously, one can easily check that the forbidden eigenvalue $\Lambda = 2$ is not therefore a Dirichlet one.

One can also check that $\Lambda_3 = 1$ and $\Lambda_3 = 3$ are eigenvalues of $\Delta_3$, both with multiplicity 2.

From:

$$\Lambda_2 = 2\,\left\{1+\cos\frac{\pi}{9}\right\}$$

the spectral decimation leads then to the quadruple eigenvalue:

$$\Lambda_3 = 4\,\cos^2\frac{\pi}{27}$$

From:

$$\Lambda_2 = 2+\cos\frac{\pi}{9}+\sqrt{3}\,\sin\frac{\pi}{9}$$

the spectral decimation leads then to the quadruple eigenvalue:

$$\Lambda_3 = 4\,\cos^2\frac{\pi}{54}$$

### 4.4 Eigenvalues of $\Delta_m$, $m \in \mathbb{N}$, $m \geqslant 4$

As previously, one can easily check that the forbidden eigenvalue $\Lambda = 2$ is not therefore a Dirichlet one.

One can also check that $\Lambda_m = 1$ and $\Lambda_m = 3$ are eigenvalues of $\Delta_m$, both with multiplicity 2.

By induction, one may note that, due to the spectral decimation, the initial eigenvalue $\Lambda_1 = 1$ gives birth, at this $m^{th}$ step, to an eigenvalue $\Lambda_m$, of multiplicity $2^{m-1}$. In the same way, the initial eigenvalue $\Lambda_1 = 3$ gives birth, at this $m^{th}$ step, to an eigenvalue $\Lambda_m$, of multiplicity $2^{m-1}$.

Results are summarized in the following array:

| Initial eigenvalue $\Lambda_1$ | continued eigenvalue $\Lambda_2$ | continued eigenvalue $\Lambda_3$ | continued eigenvalue $\Lambda_4$ |
|---|---|---|---|
| 1 | $2+\cos\frac{\pi}{9}+\sqrt{3}\,\sin\frac{\pi}{9}$ | $4\,\cos^2\frac{\pi}{27}$ | $4\,\cos^2\frac{\pi}{81}$ |
| 3 | $2\left\{1+\cos\frac{\pi}{9}\right\}$ | $4\,\cos^2\frac{\pi}{54}$ | $2\left\{1+\cos\frac{\pi}{81}\right\}$ |

**Property 4.1.** *Let us introduce:*

$$\Lambda = \lim_{m\to+\infty} N_b^m\,\Lambda_m$$

*One may note that, due to the definition of the Laplacian $\Delta$, the limit exists.*

### 4.5 Eigenvalue counting function

**Definition 4.1. Eigenvalue counting function**

Let us introduce the eigenvalue counting function, related to $\Gamma_{\mathcal{W}} \setminus V_0$, such that, for any positive number $x$:

$$\mathcal{N}^{\Gamma_{\mathcal{W}}\setminus V_0}(x) = \text{Card}\,\{\Lambda\,\text{Dirichlet eigenvalue of } -\Delta\ :\quad \Lambda \leqslant x\}$$

**Property 4.2.** *Given a strictly positive integer, the cardinal of $V_m \setminus V_{m-1}$ is:*

$$\mathcal{N}_m^{\mathcal{S}} - \mathcal{N}_{m-1}^{\mathcal{S}} = 2\left(N_b^{m-1} - N_b^{m-2}\right)$$

*The highest eigenvalue is:*

$$N_b^m \times 3$$

*This leads to:*

$$\mathcal{N}^{\Gamma_{\mathcal{W}}}(N_b^m \times 3) = \left(N_b^{m-1} - N_b^{m-2}\right)$$

*If one looks for an asymptotic growth rate of the form*

$$\mathcal{N}^{\Gamma_{\mathcal{W}}}(x) \sim x^{\alpha}$$

*one obtains:*

$$\alpha = 1$$

*By following [Str06], one may note that the ratio*

$$\frac{\mathcal{N}^{\Gamma_{\mathcal{W}}}(x)}{x}$$

*is bounded above and away from zero, and admits a limit along any sequence of the form* $C\,N_b^m$*,* $C > 0$*,* $m \in \mathbb{N}^\star$*. This enables one to deduce the existence of a periodic function* $g$*, the period of which is equal to* $\ln N_b$*, discontinuous at the value* $3$*, such that:*

$$\lim_{x \to +\infty} \left\{ \frac{\mathcal{N}^{\Gamma_{\mathcal{W}}}(x)}{x} - g(\ln x) \right\} = 0$$

*Remark* 4.1. Existing results of J. Kigami and M. Lapidus [KL03], and also of R. S. Strichartz [Str06], yield:

$$\mathcal{N}^{\Gamma_{\mathcal{W}}}(x) = G(x)\, x^{\alpha_{\Gamma_{\mathcal{W}}}} + \mathcal{O}(1)$$

with:

$$\alpha_{\Gamma_{\mathcal{W}}} = \frac{d_{\Gamma_{\mathcal{W}}}}{d_{\Gamma_{\mathcal{W}}} + 1} = \frac{\frac{\ln \frac{N_b}{\lambda}}{\ln N_b}}{\frac{\ln \frac{N_b}{\lambda}}{\ln N_b} + 1}$$

where:

$$d_{\Gamma_{\mathcal{W}}} = \frac{\ln \frac{N_b}{\lambda}}{\ln N_b}$$

is the dimension of the graph $\Gamma_{\mathcal{W}}$ for the resistance metric.

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