The firefighter problem: Further steps in understanding its complexity

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Abstract

We consider the complexity of the firefighter problem where a budget of $b \geq 1$ firefighters are available at each time step. This problem is known to be NP-complete even on trees of degree at most three and $b = 1$ [14] and on trees of bounded degree $(b+3)$ for any fixed $b \geq 2$ [4].

In this paper we provide further insight into the complexity landscape of the problem by showing a complexity dichotomy result with respect to the parameters pathwidth and maximum degree of the input graph. More precisely, first, we prove that the problem is NP-complete even on trees of pathwidth at most three for any $b \geq 1$. Then we show that the problem turns out to be fixed parameter-tractable with respect to the combined parameter "pathwidth" and "maximum degree" of the input graph. Finally, we show that the problem remains NP-complete on very dense graphs, namely co-bipartite graphs, but is fixed-parameter tractable with respect to the parameter "cluster vertex deletion".

1 Introduction

The firefighter problem was introduced by Hartnell [17] and received considerable attention in a series of papers [2, 7, 11, 12, 14, 18, 20, 21, 23, 24]. In its original version, a fire breaks out at some vertex of a given graph. At each time step, one vertex can be protected by a firefighter and then the fire spreads to all unprotected neighbours of the vertices on fire. The process ends when the fire can no longer spread. At the end all vertices that are not on fire are considered as saved. The objective is at each time step to choose a vertex which is protected by a firefighter such that a maximum number of vertices in the graph is saved at the end of the process. In this paper we consider a more general version which allows us to protect $b \geq 1$ vertices at each step (the value b is called *budget*).

The original firefighter problem was proved to be NP-hard for bipartite graphs [23], cubic graphs [21] and unit disk graphs [15]. Finbow et al. [14] showed that the problem is NP-hard even on trees. More precisely, they proved the following dichotomy theorem: the problem is NP-hard even for trees of maximum degree three and it is solvable in polynomial-time for

*The preliminary results of this publication were presented at IPEC 2014 [9].

†A major part of this work was done during a three-month visit of the University of Portsmouth supported by the ERASMUS program.

graphs with maximum degree three, provided that the fire breaks out at a vertex of degree at most two. Furthermore, the problem is polynomial-time solvable for caterpillars and so-called P-trees [23]. Later, Bazgan et al. [4] extended the previous results by showing that the general firefighter problem is NP-hard even for trees of maximum degree $(b+3)$ for any fixed budget $b \geq 2$ and polynomial-time solvable on k-caterpillars. From the approximation point of view, the problem is $\frac{e}{e-1}$-approximable on trees ($\frac{e}{e-1} \approx 1.5819$) [7] and it is not $n^{1-\varepsilon}$-approximable on general graphs for any $\varepsilon > 0$ unless $P = NP$ [2]. Moreover for trees in which each non-leaf vertex has at most four neighbours, the firefighter problem is 1.3997-approximable [20]. Very recently, a significant progress has been achieved on the approximability status of the problem for trees. Chalermsook et al. claimed in [8] that the integrability gap of the standard LP relaxation can be arbitrarily close to $\frac{e}{e-1}$ and finally Adjiashvili et al. claimed to prove a PTAS for the firefighter problem on trees [1]. Costa et al. [11] extended the $\frac{e}{e-1}$-approximation algorithm on trees to the case where the fire breaks out at $f > 1$ vertices and $b > 1$ firefighters are available at each step. From a parameterized perspective, the problem is W[1]-hard with respect to the natural parameters "number of saved vertices" and "number of burned vertices" [3]. Furthermore, it admits an $O(2^{\tau}k\tau)$-size kernel where τ is the minimum vertex cover of the input graph and k the number of burned vertices [3]. Cai et al. [7] presented first fixed-parameter tractable algorithms and polynomial-size kernels for trees for each of the following parameters: "number of saved vertices", "number of saved leaves", "number of burned vertices", and "number of protected vertices".

In this paper we provide a complexity dichotomy result of the problem with respect to the parameters maximum degree and pathwidth of the input graph. In Section 2 we first provide the formal definition of the problem as well as some preliminaries. In Section 3 we extend the hardness results on trees by proving that the problem is also NP-complete on trees of pathwidth three. The presented proof is also a simpler proof of the NP-completeness of the problem on trees. In Section 4 we devise a parameterized algorithm with respect to the combined parameter "pathwidth" and "maximum degree" of the input graph. In Section 5 we show that the problem is also NP-hard on co-bipartite graphs which are very dense graphs, but fixed-parameter tractable with respect parameter "cluster vertex deletion" (cvd). This last result strengthens the previous $O(2^{\tau}k\tau)$-size kernel as it suppresses the dependence with k and the cvd number is smaller than the vertex cover number τ. The conclusion is given in Section 6.

2 Preliminaries

Graph terminology. Let $G = (V, E)$ be an *undirected graph* of order n. For a subset $S \subseteq V$, $G[S]$ is the induced subgraph of G. The *neighborhood* of a vertex $v \in V$, denoted by $N(v)$, is the set of all neighbors of v. For a vertex set $V' \subseteq V$ we define $N_{V'}(v) = N(v) \cap V'$. We denote by $N^k(v)$ the set of vertices that are at distance at most k from v. The *degree* of a vertex v is denoted by $\deg_G(v)$ and the *maximum degree* of the graph G is denoted by $\Delta(G)$.

A *linear layout* of G is a bijection $\pi : V \to \{1, \ldots, n\}$. For convenience, we express π by the list $L = (v_1, \ldots, v_n)$ where $\pi(v_i) = i$. Given a linear layout L, we denote the distance between two vertices in L by $d_L(v_i, v_j) = j - i$.

The *cutwidth* $\mathrm{cw}(G)$ of G is the minimum $k \in \mathbb{N}$ such that the vertices of G can be arranged in a linear layout $L = (v_1, \ldots, v_n)$ in such a way that, for every $i \in \{1, \ldots, n-1\}$, there are at most k edges between $\{v_1, \ldots, v_i\}$ and $\{v_{i+1}, \ldots, v_n\}$.

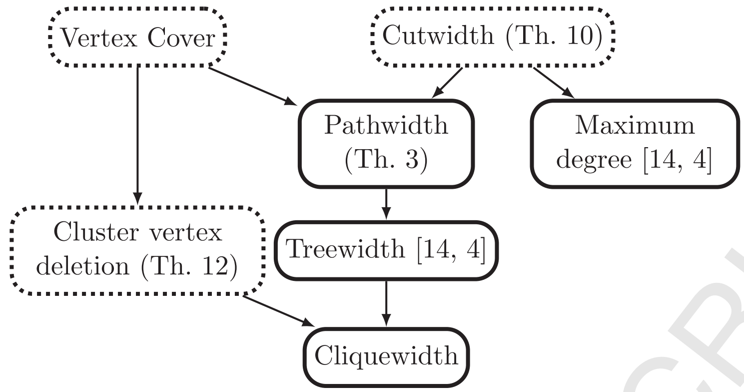

Figure 1: The parameterized complexity of the Firefighter problem with respect to some structural graph parameters. An arc from a parameter k_2 to a parameter k_1 means that there exists some function h such that $k_1 \leq h(k_2)$. For any fixed budget, a dotted rectangle means fixed-parameter tractability for this parameter and a thick rectangle means NP-hardness even for constant values of this parameter.

The *bandwidth* $\mathrm{bw}(G)$ of G is the minimum $k \in \mathbb{N}$ such that the vertices of G can be arranged in a linear layout $L = (v_1, \ldots, v_n)$ so that $|d_L(v_i, v_j)| \leq k$ for every edge $v_i v_j$ of G.

A path decomposition $\mathcal{P}$ of G is a pair $(P, \mathcal{H})$ where P is a path with node set X and $\mathcal{H} = \{H_x : x \in X\}$ is a family of subsets of V such that the following conditions are met

1. $\bigcup_{x \in X} H_x = V$.
2. For each $uv \in E$ there is an $x \in X$ with $u, v \in H_x$.
3. For each $v \in V$, the set of nodes $\{x : x \in X \text{ and } v \in H_x\}$ induces a subpath of P.

The width of a path decomposition $\mathcal{P}$ is $\max_{x \in X} |H_x| - 1$. The pathwidth $\mathrm{pw}(G)$ of a graph G is the minimum width over all possible path decompositions of G.

We may skip the argument of $\mathrm{pw}(G)$, $\mathrm{cw}(G)$, $\mathrm{bw}(G)$ and $\Delta(G)$ if the graph G is clear from the context.

A *star* is a tree consisting of one vertex, called the *center* of the star, adjacent to all the other vertices.

Problem definition. We start with an informal explanation of the propagation process for the firefighter problem. Let $G = (V, E)$ be a graph of order n with a vertex $s \in V$, let $b \in \mathbb{N}$ be a *budget*. At step $t = 0$, a fire breaks out at vertex s and s starts burning. At any subsequent step $t > 0$ the following two phases are performed in sequence:

1. *Protection phase* : The firefighter protects at most b vertices not yet on fire.
2. *Spreading phase* : Every unprotected vertex which is adjacent to a burned vertex starts burning.

Burned and protected vertices remain burned and protected until the propagation process stops, respectively. The propagation process stops when in a next step no new vertex can be burned. We call a vertex saved if it is either protected or if all paths from any burned vertex to it contain at least one protected vertex. Notice that, until the propagation process stops, there is at least one new burned vertex at each step. This leads to the following obvious lemma.

Lemma 1 *The number of steps before the propagation process stops is less or equal to the total number of burned vertices.*

A *protection strategy* (or simply *strategy*) Φ indicates which vertices to protect at each step until the propagation process stops. Since there can be at most n burned vertices, it follows from Lemma 1 that the propagation unfolds in at most n steps. We are now in position to give the formal definition of the investigated problem.

The FIREFIGHTER problem:
Input: A graph $G = (V, E)$, a vertex $s \in V$, and positive integers b and k.
Question: Is there a strategy for an instance (G, s, b, k) with respect to budget b such that at most k vertices are burned if a fire breaks out at s?

When dealing with trees, we use the following observation which is a straightforward adaptation of the one by MacGillivray and Wang for the case $b > 1$ [23, Section 4.1].

Lemma 2 *Among the strategies that maximize the number of saved vertices (or equivalently minimize the number of burned vertices) for a tree, there exists one that protects vertices adjacent to a burned vertex at each time step.*

Throughout the paper, we assume all graphs to be connected since otherwise we can simply consider the component where the initial burned vertex s belongs to.

3 Firefighting on path-like graphs

Finbow et al. [14] showed that the problem is NP-complete even on trees of degree at most three. However, the tree constructed in the proof has an unbounded pathwidth. In this section we show that the FIREFIGHTER problem is NP-complete even on trees of pathwidth three. For that purpose we use the following problem.

The CUBIC MONOTONE 1-IN-3-SAT problem:
Input: A CNF formula with no negative literals in which every clause contains exactly three variables and every variable appears in exactly three clauses.
Question: Is there a 1-perfect satisfying assigment (a truth assignment such that each clause has exactly one true literal) for the formula?

The NP-completeness of the above problem is due to its equivalence with the NP-complete EXACT COVER BY 3-SETS problem [16].

Theorem 3 *For any budget $b \geq 1$, the* FIREFIGHTER *problem is NP-complete even on trees of pathwidth three.*

Proof: Clearly, FIREFIGHTER belongs to NP. Now we provide a polynomial-time reduction from CUBIC MONOTONE 1-IN-3-SAT. We start with the case where $b = 1$ and later explain how to extend the proof for larger values of b.

In the proof, a *guard-vertex* is a star with k leaves where the center is adjacent to a vertex of a graph. It is clear that if at most k vertices can be burned, then the guard-vertex has to be saved.

Let ϕ be a formula of Cubic Monotone 1-In-3-Sat with n variables $\{x_1, \ldots, x_n\}$ and m initial clauses $\{c_1, \ldots, c_m\}$. Notice that we have $n = m$ since there is a total of $3n = 3m$ literals in ϕ. First, we extend ϕ into a new formula ϕ' by adding m new clauses as follows. For each clause c_j we add the clause $\bar{c}_j$ by taking negation of each variable of c_j. A perfect satisfying assignment for ϕ' is then a truth assignment such that each clause c_j has exactly one true literal (1-perfect) and each clause $\bar{c}_j$ has exactly two true literals (2-perfect). Clearly, we have that ϕ has a 1-perfect satisfying assignment if and only if ϕ' has a perfect one. To see this, observe that a clause c_j has exactly one true literal if and only if $\bar{c}_j$ has two true literals.

Now we construct an instance $I' = (T, s, 1, k)$ of Firefighter from ϕ' as follows (see Figure 2). We start with the construction of the tree T, the value of k will be specified later.

- Start with a vertex set $\{s = u_1, u_2, \ldots, u_p\}$ and edges of $\{su_2, u_2u_3, \ldots, u_{p-1}u_p\}$ where $p = 2n - 1$ and add two degree-one vertices v_{x_i} and $v_{\bar{x}_i}$ adjacent to u_{2i-1} for every $i \in \{1, \ldots, n\}$.

Then for each $i \in \{1, \ldots, n\}$ in two steps:

- Add a guard-vertex g_i (resp. $\bar{g}_i$) adjacent to v_{x_i} (resp. $v_{\bar{x}_i}$).
- At each vertex v_{x_i} (resp. $v_{\bar{x}_i}$) root a path of length $2 \cdot (n - i)$ at v_{x_i} (resp. $v_{\bar{x}_i}$) in which the endpoint is adjacent to three degree-one vertices (called literal-vertices) denoted by $\ell_1^{x_i}$, $\ell_2^{x_i}$, and $\ell_3^{x_i}$ (resp. $\ell_1^{\bar{x}_i}$, $\ell_2^{\bar{x}_i}$, and $\ell_3^{\bar{x}_i}$). Each literal-vertex corresponds to an occurence of the variable x_i in an initial clause of ϕ. Analogously, the literal-vertices $\ell_1^{\bar{x}_i}$, $\ell_2^{\bar{x}_i}$, and $\ell_3^{\bar{x}_i}$ represent the negative literal $\bar{x}_i$ that appears in the new clauses of ϕ'.

Notice that each literal-vertex is at distance exactly $p + 1$ from s.

- For each variable x_i (resp. $\bar{x}_i$), $i \in \{1, \ldots, n\}$, there are exactly three clauses containing x_i (resp. $\bar{x}_i$). Let c_j (resp. $\bar{c}_j$), $j \in \{1, \ldots, m\}$, be the first one of them. Then root a path $Q_j^{x_i}$ (resp. $Q_j^{\bar{x}_i}$) of length $3 \cdot (j - 1)$ at $\ell_1^{x_i}$ (resp. $\ell_1^{\bar{x}_i}$), and add a guard-vertex $g_j^{x_i}$ adjacent to the endpoint of $Q_j^{x_i}$. To the endpoint of $Q_j^{\bar{x}_i}$ (i) add a degree-one vertex $d^{\bar{x}_i}$ (a dummy-vertex) and (ii) root a path $D_j^{\bar{x}_i}$ of length 3 where the last vertex of the path is a guard vertex $g_j^{\bar{x}_i}$. Repeat the same for the two other clauses with x_i (resp. $\bar{x}_i$) and $\ell_2^{x_i}$, $\ell_3^{x_i}$ (resp. $\ell_2^{\bar{x}_i}$, $\ell_3^{\bar{x}_i}$).

To finish the construction, set $k = p + \frac{n}{2}(11n + 7)$.

In what follows, we use Lemma 2 and thus we only consider strategies that protect a vertex adjacent to a burned vertex at each time step. Recall that the budget is set to one in the instance I'. Now we show that there is a perfect satisfying assignment for ϕ' if and only if there exists a strategy for I' such that at most k vertices in T are burned.

"$\Rightarrow$" : Suppose that there is a perfect satisfying assignment τ for ϕ'. We define the following strategy Φ_τ from τ. At each step t from 1 to $p + 1$, if t is odd then protect $v_{\bar{x}_{\lceil t/2 \rceil}}$ if $x_{\lceil t/2 \rceil}$ is true, otherwise protect $v_{x_{\lceil t/2 \rceil}}$. If t is even, then protect the guard-vertex $g_{\lceil t/2 \rceil}$

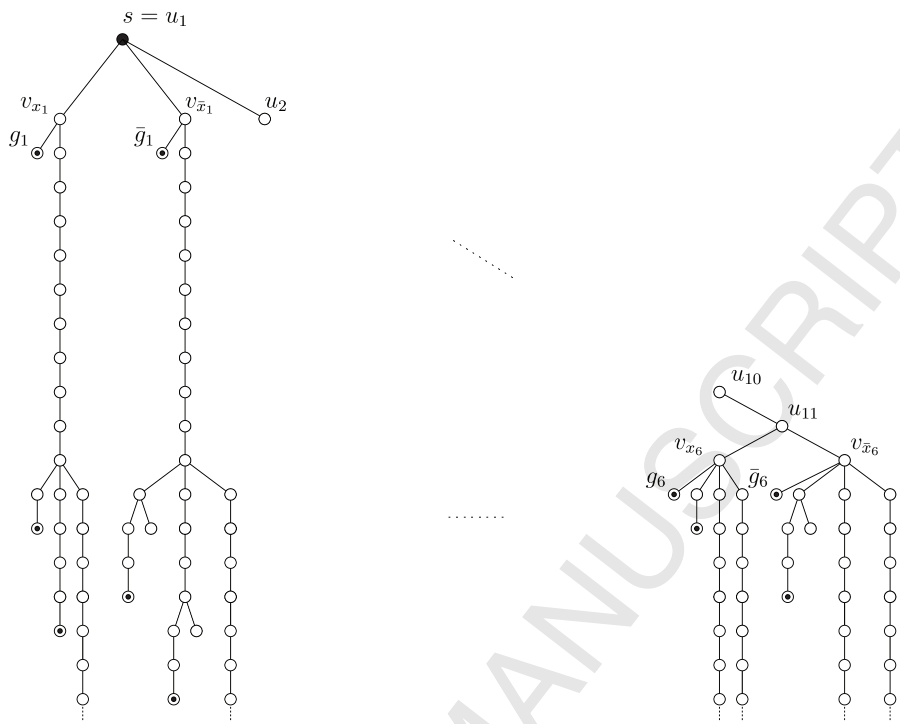

Figure 2: An example of part of a tree constructed from the formula $\phi = (x_1 \vee x_3 \vee x_6) \wedge (x_1 \vee x_2 \vee x_3) \wedge (x_3 \vee x_4 \vee x_5) \wedge (x_2 \vee x_4 \vee x_5) \wedge (x_1 \vee x_4 \vee x_6) \wedge (x_2 \vee x_6 \vee x_5)$. Guard vertices are represented by a dot within a circle.

if $v_{\bar{x}_{\lceil t/2 \rceil}}$ has been protected, otherwise protect $\bar{g}_{\lceil t/2 \rceil}$. At the end of time step $p+1$, the number of burned vertices is exactly $p + \sum_{i=1}^{n}(3 + 2(n-i) + 1) = p + 3n + n^2$. Moreover, the literal-vertices that are burned in T correspond to the true literals in ϕ'. Thus, by construction and since τ satisfies ϕ', the vertices adjacent to a burning vertex are exactly one guard-vertex $g_1^{x_a}$, two dummy vertices $d^{\bar{x}_b}, d^{\bar{x}_c}$ and $3n-1$ other vertices where $x_a \vee x_b \vee x_c$ is the first clause, $a, b, c \in \{1, \ldots, n\}$. At step $p+2$, we must protect the guard vertex $g_1^{x_a}$. During the steps $p+3$ and $p+4$, the strategy must protect one vertex lying on the path $D_1^{\bar{x}_b}$ and $D_1^{\bar{x}_c}$, respectively. Thus $3(3n-3) + 5 = 9n - 4$ more vertices are burned at the end of step $p+4$. More generally, from time step $p + 3(j-1) + 2$ to $p + 3(j-1) + 4$, for some $j \in \{1, \ldots, m\}$, the strategy Φ_τ must protect a guard-vertex $g_j^{x_a}$ and one vertex of each path $D_j^{\bar{x}_b}$ and $D_j^{\bar{x}_c}$, where x_a, x_b, x_c appear in the clause c_j, $a, b, c \in \{1, \ldots, n\}$. Thus $9(n - (j-1)) - 4$ vertices get burned. It follows that the number of burned vertices from step $p+2$ to $p + 3m + 1$ is $\sum_{j=1}^{m}[9(n - (j-1)) - 4] = \frac{9}{2}m(m+1) - 4m$. Putting all together, we arrive at a total of $p + 3n + n^2 + \frac{9}{2}m(m+1) - 4m = p + \frac{n}{2}(11n + 7) = k$ burned vertices.

"$\Leftarrow$": Conversely, assume that there is no perfect satisfying assignment for ϕ'. Observe first that any strategy Φ for I' protects either the pair v_{x_i} and $\bar{g}_i$ or the pair $v_{\bar{x}_i}$ and g_i for each $i \in \{1, \ldots, n\}$. As a contradiction, suppose that there exists $i \in \{1, \ldots, n\}$ such that Φ does not protect v_{x_i} and $v_{\bar{x}_i}$. Then in some time step both v_{x_i} and $v_{\bar{x}_i}$ get burned. Hence, it is not possible to protect both g_i and $\bar{g}_i$, and at least one will burn implying that more than k vertices would burn, a contradiction. Furthermore, v_{x_i} and $v_{\bar{x}_i}$ cannot be both protected, otherwise we would have protected a vertex not adjacent to a burned vertex at some step. Now consider the situation at the end of step $p+1$. By the previous observation, the literal-vertices that are burned in T can be interpreted as being the literals in ϕ' set to true. As previously, the number of burned vertices so far is exactly $p + \sum_{i=1}^{n}(3 + 2(n-i) + 1) = p + 3n + n^2$. Let n_g and n_d be the number of guard-vertices and dummy-vertices adjacent to a burned vertex, respectively. As it follows from the previous construction, we know that $n_g = 3 - n_d$ with $0 \leq n_g \leq 3$ and $0 \leq n_d \leq 3$. We have the following possible cases:

(1) $n_g > 1$. In this case, a guard-vertex gets burned, and hence more than k vertices would burn.

(2) $n_g = 1$. Let $g_1^{x_a}$ be that guard-vertex and let $d^{\bar{x}_b}, d^{\bar{x}_c}$ be the $n_d = 3 - n_g = 2$ dummy-vertices where x_a, x_b, x_c are variables of the first clause. At time step $p+2$, we must protect $g_1^{x_a}$. Furthermore, during the step $p+3$ (resp. $p+4$), any strategy must protect a vertex lying on the path $D_1^{\bar{x}_b}$ (resp. $D_1^{\bar{x}_c}$). Indeed, if a strategy does otherwise, then at least one guard-vertex $g_1^{\bar{x}_b}$ or $g_1^{\bar{x}_c}$ gets burned. Thus 2 dummy-vertices are burned.

(3) $n_g = 0$. Hence we have exactly $n_d = 3 - n_g = 3$ dummy-vertices $d^{\bar{x}_a}, d^{\bar{x}_b}, d^{\bar{x}_c}$ adjacent to burned vertices. Using a similar argument as before, we know that during the step $p+2$ (resp. $p+3$, $p+4$), a strategy must protect a vertex lying on the path $D_1^{\bar{x}_a}$ (resp. $D_1^{\bar{x}_b}$, $D_1^{\bar{x}_c}$). Thus 3 dummy-vertices are burned.

Notice that at step $p+5$, we end up with a similar situation as in step $p+2$. Now consider an assignment for ϕ'. Since ϕ' is not perfect satisfiable, ϕ is not 1-perfect satisfiable as well. There are two possibilities:

- There exists a clause c_j in ϕ with more than one true literal. Thus, we end up with case (1), and there is no strategy for I' such that at most k vertices are burned.
- There is a clause c_j in ϕ with only false literals. This corresponds to case (3), and the number of burned vertices would be at least $1 + p + \frac{n}{2}(11n + 7)$ (at least one extra dummy-vertex gets burned) giving us a total of at least $k+1$ burned vertices. Hence there is no strategy for I' where at most k vertices are burned.

It remains to prove that the pathwidth of T is at most three. To see this, observe that any subtree rooted at v_{x_i} or $v_{\bar{x}_i}$ has pathwidth two. Let P_{x_i} and $P_{\bar{x}_i}$ be the paths of the path-decompositions of these subtrees, respectively. We construct the path-decomposition for T as follows. For every $i \in \{1, \ldots, n-1\}$, define the node $B_i = \{u_{2i-1}, u_{2i}, u_{2i+1}\}$. Extend all nodes of the paths P_{x_i} and $P_{\bar{x}_i}$ to P'_{x_i} and $P'_{\bar{x}_i}$ by adding the vertex u_{2i-1} inside it. Finally, connect the paths P'_{x_1}, $P'_{\bar{x}_1}$ and the node B_1 to form a path and continue in this way with P'_{x_2}, $P'_{\bar{x}_2}$, B_2, P'_{x_3}, $P'_{\bar{x}_3}$, B_3, $\ldots$, B_{n-1}, P'_{x_n}, $P'_{\bar{x}_n}$.

Finally, we consider the case where $b > 1$. We start from the above reduction and alter the tree T as follows. Let w_1 be the vertex s (corresponding also to u_1). Add a path $\{w_1w_2, w_2w_3, \ldots, w_{5n}w_{5n+1}\}$ to T together with $b-1$ guard-vertices added to each w_i. First, one can easily check that the pathwidth remains unchanged, since the added component has pathwidth two and is only connected to the root s. Second, it can be seen that at each time step, only one firefighter can be placed "freely", as the other $b-1$ firefighters must protect $b-1$ guard-vertices. It follows that we end up with a similar proof as above. This completes the proof. □

As a side result, we also obtain the following.

Proposition 4 *For any budget $b \geq 1$, the* Firefighter *problem is NP-complete even on line graphs.*

Proof: We use a polynomial-time reduction from the Firefighter problem on trees. Consider an instance $I = (T, s, b, k)$ of Firefighter that consists of a tree $T = (V, E)$ and the root s. Let $v_{i,j}$ be the j^{th} vertex at level i of the tree T. We construct an instance $I' = (G, s', b, k)$ of Firefighter in which a line graph G is defined as follows: add a vertex v and adjacent it to the root s, let G be the line graph of such tree. Denote by $e_{i,j}$ the unique edge that connects $v_{i,j}$ to its parent and by s' the edge vs, the edges $e_{i,j}$ and s' correspond to the vertices in G.

As follows from the construction, there is a one-to-one correspondence between the vertices '$v_{i,j}$' in T and '$e_{i,j}$' in G. Therefore, given a strategy Φ for I we can define a corresponding strategy Φ' for I', and vice versa. Clearly, a vertex $e_{i,j}$ in G is protected (resp. saved, resp. burned) by Φ if and only if the corresponding vertex $v_{i,j}$ in T is protected (resp. saved, resp. burned) by Φ'. Hence, there exists a strategy in I' such that at most k vertices are burned if and only if there exists a strategy in I such that at most k vertices are burned. The rest follows from Theorem 3. □

4 Path-like graphs of bounded degree

As previously shown, for any fixed budget $b \geq 1$, the Firefighter problem is NP-complete on trees of bounded degree $b+3$ [14, 4] and on trees of bounded pathwidth three (Theorem 3). It is thus natural to explore the complexity of the problem when both the maximum degree and the pathwidth of the input graph are bounded.

First we introduce some definitions. Let $L = (v_1, \ldots, v_n)$ be a linear layout of a graph G such that for every $i \in \{1, \ldots, n-1\}$, there are at most $k+1$ edges between $\{v_1, \ldots, v_i\}$ and $\{v_{i+1}, \ldots, v_n\}$. A *vertex-interval* I of L is any ordered vertex subset $(v_i, \ldots, v_{i+k})$ of L for some $i, k > 0$. Denote by $\ell(I)$ (resp. $r(I)$) the leftmost (resp. rightmost) vertex in I.

Lemma 5 *Let $G = (V, E)$ be a graph of cutwidth cw and $F \subseteq V$ be a set of initially burned vertices. Then there exists a protection strategy such that the statements (1) and (2) hold after $2 \cdot \mathrm{cw} \cdot |F|$ steps of propagation:*

(1) The number of burned vertices is at most $(2 \cdot \mathrm{cw} \cdot |F|)^{2 \cdot \mathrm{cw} \cdot |F|}$.

(2) There is a set of disjoint vertex-intervals $\mathcal{I}$ such that for every $I \in \mathcal{I}$, the vertices $\ell(I)$, $r(I)$ are burned and every vertex v outside $\mathcal{I}$ which is adjacent to a vertex in I is protected.

Proof: (1) Notice that after $2 \cdot \mathrm{cw} \cdot |F|$ steps of propagation there are at most $(|F| \cdot \Delta(G))^{2 \cdot \mathrm{cw} \cdot |F|}$ burned vertices regardless of any strategy. Since we have $\Delta(G) \leq 2 \cdot \mathrm{cw}(G)$ [22], the result follows.

(2) Initially suppose that F contains only one vertex $s \in F$. Denote by F' the set of burned vertices after $2 \cdot \mathrm{cw}$ steps of propagation without protecting any vertices. Define the vertex-interval I_s such that $l(I_s)$ and $r(I_s)$ are the leftmost and the rightmost burned vertices from F', respectively. By the definition the number of edges with one vertex in I_s and the other one in $V \setminus I_s$ is less or equal to $2 \cdot \mathrm{cw}$. Thus, we can define the strategy that consists

of protecting one vertex $v \in V \setminus I_s$ adjacent to a vertex in I_s in each step $t = 1, \ldots, 2 \cdot \mathrm{cw}$. Observe that those vertices are not burned before they are protected, otherwise they would have been included in I_s.

If $|F| > 1$, we repeat the previous ideas in the sequence of the steps from $t = 1, \ldots, 2 \cdot \mathrm{cw} \cdot |F|$. In each step t we consider one by one all vertices which started to burn in step $t-1$ of the propagation and if it is necessary to make a modification of the vertex-intervals to ensure that all of them are disjoint: if any two vertex-intervals intersect each other, we replace them by a vertex-interval corresponding to the union of these two vertex-internals which obviously has the required properties. After $2 \cdot \mathrm{cw} \cdot |F|$ steps of propagation we can define the set $\mathcal{I}$ of at most $|F|$ disjoint vertex-intervals such that the left and rightmost vertices of each interval are burned and due to the properties of the graph, there are at most $2 \cdot \mathrm{cw} \cdot |F|$ vertices in $V \setminus \mathcal{I}$ which are adjacent to a vertex in an interval $I \in \mathcal{I}$ which can be protected in $2 \cdot \mathrm{cw} \cdot |F|$ steps. □

Now we show the key combinatorial characterization of the number of burned vertices in a graph of bounded cutwidth. To make the formulas easier to read, we use the Knuth's tetration operator $\uparrow\uparrow$ defined as $x \uparrow\uparrow n = \underbrace{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}_{n \text{ times}}$.

Theorem 6 *Let $G = (V, E)$ be a graph of cutwidth* cw *and $F \subseteq V$ be a set of initially burned vertices. Then there exists a protection strategy such that the total number of burned vertices is bounded by $q(\mathrm{cw}, F)$, where*

$$q(\mathrm{cw}, F) = \begin{cases} ((2\,\mathrm{cw})^{\mathrm{cw}} \cdot |F|) \uparrow\uparrow 2^{\mathrm{cw}} & \text{if } \mathrm{cw} > 0, \\ |F| & \text{otherwise.} \end{cases}$$

Proof: We will prove the statement by induction on the cutwidth. The theorem is obviously true when the cutwidth is 0, since the graph does not contain any edge. Suppose now that the statement is true for any graph of cutwidth at most c, $c \geq 0$. We show that it also holds for graphs of cutwidth $(c+1)$.

Let $H = (V, E)$ be a graph of cutwidth $(c+1)$ and $F \subseteq V$ be a set of burned vertices. We apply the strategy defined in Lemma 5 for the initial $2 \cdot (c+1) \cdot |F|$ steps of propagation. After the application of this strategy, the number of burned vertices so far is at most $|F'| = (2 \cdot (c+1) \cdot |F|)^{2 \cdot (c+1) \cdot |F|}$. Furthermore, every vertex $v \notin \bigcup_{I \in \mathcal{I}} I$ (defined in Lemma 5) is either protected or saved. Hence we only need to protect the vertices in the graph H' induced by the vertices $\bigcup_{I \in \mathcal{I}} I$. Observe that we can safely remove every edge uv from H' for which u and v are burned. Indeed, such a *burned* edge cannot have any influence during the subsequent steps of propagation. As it follows from the definition of a vertex-interval, either there is a path from $\ell(I)$ to $r(I)$ or the vertex-interval is the union of several overlapping vertex-intervals. In both cases after removing all the *burned* edges from H', the cutwidth of H' decreases by 1. Hence we can apply the induction step on H', which has cutwidth c, together with the set F' of the burned vertices. To summarise, we can define the valid strategy for H as follows: First apply the strategy defined in Lemma 5 for the initial $2 \cdot (c+1) \cdot |F|$ steps on H and then apply the strategy given by induction from this theorem for the subgraph H' of H. The total number of burned vertices in H using this new strategy is at most

$$
\begin{aligned}
q(c, F') &= \big((2c)^c \cdot |F'|\big) \uparrow\uparrow 2^c \\
&= \Big(2^c \cdot c^c \cdot (2 \cdot (c+1) \cdot |F|)^{2\cdot(c+1)\cdot|F|}\Big) \uparrow\uparrow 2^c \\
&\leq \Big(2^c \cdot (c+1)^c \cdot (2 \cdot (c+1) \cdot |F|)^{2^{c+1}\cdot(c+1)^{c+1}\cdot|F|}\Big) \uparrow\uparrow 2^c \\
&\leq \Big((2^{c+1} \cdot (c+1)^{c+1} \cdot |F|)^{2^{c+1}\cdot(c+1)^{c+1}\cdot|F|}\Big) \uparrow\uparrow 2^c \\
&\leq \big(2^{c+1} \cdot (c+1)^{c+1} \cdot |F|\big) \uparrow\uparrow 2^{c+1} \\
&= \big((2 \cdot (c+1))^{c+1} \cdot |F|\big) \uparrow\uparrow 2^{c+1} \\
&= q(c+1, F)
\end{aligned}
$$

This concludes the proof. □

Remark 7 *Notice that Theorem 6 is still valid even if the number of firefighters available at each step is not the same (for example if there are b_1 firefighters at time step one, b_2 firefighters during the second time step, etc.).*

Using the fact that $\mathrm{cw}(G) \leq \mathrm{pw}(G) \cdot \Delta(G)$ [10] for every graph G, we can easily deduce the following corollary.

Corollary 8 *Let $G = (V, E)$ be a graph of pathwidth pw and maximum degree Δ. Let $F \subseteq V$ be a set of initially burned vertices. There exists a protection strategy such that the total number of burned vertices is bounded by $q(\mathrm{pw}, \Delta, F)$, where*

$$
q(\mathrm{pw}, \Delta, F) = \begin{cases} \big((2\,\mathrm{pw}\cdot\Delta)^{\mathrm{pw}\cdot\Delta} \cdot |F|\big) \uparrow\uparrow 2^{\mathrm{pw}\cdot\Delta} & \text{if } \mathrm{pw}\cdot\Delta > 0 \\ |F| & \text{otherwise} \end{cases}
$$

We are now in position to give the main result of this section.

Theorem 9 *The* Firefighter *problem is fixed-parameter tractable with respect to the combined parameter "pathwidth" and "maximum degree" of the input graph.*

Proof: Let $I = (G, s, b, k)$ be an instance of Firefighter where G has maximum degree Δ and pathwidth pw. First we claim that the problem can be solved by a $f(k, \Delta) \cdot n^{O(1)}$-time algorithm denoted by $\mathcal{B}$ further. To see this observe that whenever $b \geq \Delta$ then one can protect all the vertices in $N(s)$ at step one. If $b < \Delta$ we simply apply the $f(k, b) \cdot n^{O(1)}$-time procedure from [3], so the claim follows.

Now consider the following algorithm denoted by $\mathcal{A}$. For each value $k' = 1, \ldots, k$, execute $\mathcal{B}$ on $I' = (G, s, b, k')$: If $\mathcal{B}$ returns "yes" then $\mathcal{A}$ halts and returns "yes". If $\mathcal{B}$ has returned the answer "no" for all $k' = 1, \ldots, k$ then $\mathcal{A}$ returns "no".

Using Corollary 8 we know that whenever $k' \geq q(\mathrm{pw}, \Delta, \{s\}) = f'(\mathrm{pw}, \Delta)$ the algorithm $\mathcal{B}$ necessarily returns "yes" and $\mathcal{A}$ stops. It follows that $\mathcal{B}$ is called at most $f'(\mathrm{pw}, \Delta)$ times. The overall running time is then bounded by

$$
\begin{aligned}
O(f'(\mathrm{pw}, \Delta) \cdot f(k, \Delta) \cdot n^{O(1)}) &= O(f'(\mathrm{pw}, \Delta) \cdot f(f'(\mathrm{pw}, \Delta), \Delta) \cdot n^{O(1)}) \\
&= f''(\mathrm{pw}, \Delta) \cdot n^{O(1)}
\end{aligned}
$$

for some function f''. This completes the proof. $\square$

Since for any graph G, we have that $\mathrm{pw}(G) \leq \mathrm{cw}(G)$ and $\Delta(G) \leq 2 \cdot \mathrm{cw}(G)$ [22] and $\mathrm{cw}(G) \leq \frac{\mathrm{bw}(G)(\mathrm{bw}(G)+1)}{2}$ [5], we easily deduce the following theorem.

Theorem 10 *The* FIREFIGHTER *problem is fixed-parameter tractable with respect to the parameter "cutwidth" and to the parameter "bandwidth".*

5 Firefighting on dense graphs

As trees are rather sparse graphs, it seems natural to ask for the tractability of the problem when the graphs are essentially made up of cliques. In the following we show that even if a graph can be partitioned into two cliques (also known as a co-bipartite graph), the problem turns out to be NP-complete. Notice that the problem is trivial for cliques.

Theorem 11 *The* FIREFIGHTER *problem is NP-complete and W[1]-hard for the parameter k even on co-bipartite graphs.*

Proof: We construct a polynomial-time reduction from the CLIQUE problem on regular graphs which has been proved to be W[1]-hard [6, Th. 2.1]. This proof is based on the one presented in [3]. We use a different construction, but the proof of correctness remains essentially the same. Let $(G = (V, E), k)$ be an instance of CLIQUE where G is a Δ-regular graph of size n. We construct the instance (G', s, b, k') of FIREFIGHTER from (G, k) as follows. We start with the construction of G'.

- Add a new vertex s adjacent to all vertices of G.
- Set $b = \max\{n - k,\ k\Delta - \binom{k}{2}\}$ and $k' = k + 1$.
- Create two cliques C_1 and C_2 of $b - (n - k) + k'$ and $b - (k\Delta - \binom{k}{2}) + k'$ vertices, respectively. We later refer to these cliques as guard-cliques.
- Make s adjacent to any $b - (n - k)$ vertices of C_1.
- Choose a subset S of $b - (k\Delta - \binom{k}{2})$ vertices of C_2 and make every vertex of V adjacent to every vertex in S.
- Remove every edge $uv \in E$ and add an edge-vertex e_{uv} adjacent to u and v. We denote by F the set of all edge-vertices.
- Finally, add edges to make V and $F \cup C_1 \cup C_2$ cliques.

Observe that the graph G' can be partitioned into two cliques $\{s\} \cup V$ and $F \cup C_1 \cup C_2$.

It is clear that if at most k' vertices can be burned in G' then some vertices of both guard-cliques must be saved. As a consequence, at step one, there are only $n - k$ firefighters that can be placed freely because of the guard-clique C_1.

We claim that (G, k) is a yes-instance of CLIQUE if and only if (G', s, k', b) is a yes-instance of FIREFIGHTER.

"$\Rightarrow$": Suppose that we have a clique $C \subseteq V$ of size k and consider the following strategy. At step one, the strategy uses the $n-k$ remaining firefighters to protect all the original vertices

V in G' except those in the clique C. At step two, all k vertices of C are burned. Since those vertices are adjacent to $b - (k\Delta - \binom{k}{2})$ vertices of the guard-clique C_2, it remains exactly $k\Delta - \binom{k}{2}$ possible vertices to protect. Observe that there are exactly $k\Delta - \binom{k}{2}$ edge-vertices adjacent to the vertices in the clique C. Hence, there is enough firefighters to protect all of them and then no more than $k + 1 = k'$ vertices are burned at the end of the process.

"$\Leftarrow$": Conversely, suppose that there is no clique of size k in G. At step one, any valid strategy has to place the $n - k$ remaining firefighters on vertices that are not edge-vertices; otherwise at least $k + 2 > k'$ vertices will burn. At step two, since there is no clique of size k, there will be at least $k\Delta - \binom{k}{2} + 1$ edge-vertices adjacent to the k burned vertices. For the same reason as before, there remains $k\Delta - \binom{k}{2}$ firefighters which is not enough to protect these edge-vertices. Therefore, given any valid strategy there will be at least $k + 2 > k'$ burned vertices. □

We note that if the budget b is fixed, then one can solve the problem in polynomial time on co-bipartite graphs. To see this, observe that there are at most 3 propagation steps in such a graph. Hence the total number of protected vertices is bounded by a constant, which implies that the problem is polynomial-time solvable [3].

As a final result we show that the FIREFIGHTER problem is fixed-parameter tractable with respect to the parameter cluster vertex deletion number, that is the minimum number of vertices that have to be deleted to obtain a disjoint union of complete graphs.

Recall that in Theorem 3 we prove NP-completeness of the FIREFIGHTER problem on trees on pathwidth three. Whenever a problem is NP-hard on graphs of bounded treewidth or pathwidth, it is a common approach to ask for the parameterized complexity of the problem with respect to the more general parameter as vertex cover. However, the class of graphs of a small vertex cover is rather limited and therefore it is natural to look for the parameters that generalise it. Among them the cluster vertex deletion number appears to be a suitable intermediate parameterisation between vertex cover and cliquewidth [13].

Theorem 12 *The* FIREFIGHTER *problem is fixed-parameter tractable with respect to the parameter "cluster vertex deletion".*

Proof: Let $(G = (V, E), s, b, k)$ be an instance of the FIREFIGHTER problem with $|V| = n$ and let ℓ be the cluster vertex deletion number of G. Let L be a minimum cluster vertex deletion set in G that can be found in time $O(2^\ell \cdot \ell^3 \log \ell + n^3)$ as it follows from [19].

It is easy to observe that in every three steps at least one new vertex from L gets burned. Hence all non-protected vertices in G are necessarily burned after 3ℓ steps. Since the number of propagation steps can be at most 3ℓ, at most $3\ell b$ vertices can be protected in G.

Let $\mathcal{C}$ be a set of disjoint cliques (of various sizes) in the graph $G[V \setminus L]$. Define an equivalence relation on $\mathcal{C}$: $C_1, C_2 \in \mathcal{C}$ are equivalent if and only if C_1, C_2 have the same set L' of neighbours in L. The cliques C_1, C_2 are called L'-dependent. For any $L' \subseteq L$ let $[L']$ be a set of all L'-dependent cliques. Obviously, there are at most 2^ℓ equivalence classes that define a partition of $\mathcal{C}$.

The non-protected vertices in a clique $C \in [L']$, for $L' \subseteq L$, can be saved: (a) either by protecting some vertices outside the clique C or (b) by protecting some vertices in C. Obviously, case (a) is independent from the strategy used for the vertices in the clique C. In case (b), the non-protected vertices in C can only be saved if all neighbours of the burned vertices from L' are protected in C. This follows from an easy observation: if a vertex $v \in L'$

with a non-protected neighbour in C starts burning, then no strategy can save non-protected vertices in C from burning, all such vertices get burned in the next step. Therefore the only way to save non-protected vertices in C is to protect all neighbours of v in C. Due to the fact that at most $3\ell b$ vertices can be protected, if v has more than $3\ell b$ neighbours in C, only protected vertices in C can be saved. Therefore if v has more than $3\ell b$ neighbours in C, the exact number of neighbours in C doesn't make any difference for any strategy, because all non-protected vertices in C will get burned. Based on this observation we define an equivalence relation on the set $[L']$.

Fix a set $L' = \{v_1, \ldots, v_q\} \subseteq L$, $q \geq 1$ and define an equivalence relation ν on $[L']$: C_1, $C_2 \in [L']$ are equivalent if and only if for each vertex $v_i \in L'$, $1 \leq i \leq q$ one of the options (i)–(ii) holds:

(i) each of the neighbour sets $N_{C_1}(v_i)$, $N_{C_2}(v_i)$ has more than $3\ell b$ vertices or,

(ii) $|N_{C_1}(v_i)| = |N_{C_2}(v_i)| \leq 3\ell b$ and for each $1 \leq j \leq q$ $(i \neq j)$ with $|N_{C_1}(v_j)| \leq 3\ell b$ (hence also $|N_{C_2}(v_j)| \leq 3\ell b$) there is the same amount of vertices in the intersection of the neighbourhoods $|N_{C_1}(v_i) \cap N_{C_1}(v_j)| = |N_{C_2}(v_i) \cap N_{C_2}(v_j)|$.

The equivalence relation ν defines a partition on the set $[L']$, let $[L'_\nu]$ be an equivalence class of the set $[L']$. Each equivalence class $[L'_\nu]$ can contain any number of ν-equivalent cliques of various sizes. Due to the restriction $3\ell b$ on size of the neighbour sets, the number of equivalence classes $[L'_\nu]$ of the set $[L']$ exponentially depends on ℓ and b only. Any protection strategy can only protect $3\ell b$ vertices, therefore any strategy can protect vertices in at most $3\ell b$ cliques in each $[L'_\nu]$ class. In the following we show that in each class $[L'_\nu]$ it is enough to focus only on the clique from $seq([L'_\nu])$, where $seq([L'_\nu])$ are at most $3\ell b$ largest cliques in $[L'_\nu]$ ordered in size (starting from the largest).

Fix an equivalence class $[L'_\nu]$ of $[L']$. In the following we discuss a strategy for $[L'_\nu]$ that saves the maximum number of vertices in $[L'_\nu]$ supposing a given amount of vertices in the set $[L'_\nu]$ can be protected. If we fix a vertex $v \in L'$ and a clique $C \in [L'_\nu]$ such that $|N_C(v)| \leq 3lb$, then each clique from $[L'_\nu]$ contains exactly $|N_C(v)|$ neighbours of v due to the definition of the ν-equivalence. This enables us to define the set $L^s \subseteq L'$:

$$L^s = \{v \mid v \in L' \text{ and } |N_C(v)| \leq 3lb \text{ for a clique } C \in [L'_\nu]\}.$$

If a vertex $u \in L' \setminus L^s$ starts burning then only the protected vertices in $[L'_\nu]$ can be saved from burning. Therefore if we have to select at most $3\ell b$ cliques from $[L'_\nu]$ without loss of generality we can restrict to the set $seq([L'_\nu])$ of cliques. If no vertex in $L' \setminus L^s$ gets burned, then protecting suitable neighbour sets of $v \in L^s$ in a clique $C \in [L'_\nu]$ can save some vertices in C. As $|\cup_{u \in L^s} N_C(u)|$ has the same value for all cliques in $[L'_\nu]$, if we want to save the most vertices in the class $[L'_\nu]$, the strategy must focus to protect the neighbour sets of L^s in the largest cliques from $[L'_\nu]$. Due to the assumption on the number of selected cliques we can again only focus on the cliques from $seq([L'_\nu])$.

To summarise, the graph G has at most 2^ℓ equivalence classes $[L']$ (one for each subset L' of L), and each set $[L']$ can be further partitioned to equivalence classes $[L'_\nu]$ depending on the number of neighbours of the vertices L' in the cliques. Hence, the graph G consists of a minimum cluster deletion set L of size ℓ and an exponential number (exponential on ℓ and b only) of equivalence classes $[L'_\nu]$ for any subset L' of L. Therefore all cliques from $G[V \setminus L]$ can be partitioned into $F(\ell, b)$ equivalence classes, where the value of $F(\ell, b)$ depends exponentially only on ℓ and b.

In the following we show that if there is a strategy with at most k burned vertices then we can find it in time which exponentially depends only on ℓ and b. We discuss different options which $3\ell b$ vertices should be protected in a way that covers all possible strategies that could potentially lead to burning at most k vertices. The rest follows from a simple linear check whether there exists a valid protection strategy which protect these vertices, see e.g. [3] and whether at most k vertices get burned.

Suppose that a set $L_1 \subseteq L$ of vertices of size ℓ_1, $1 \le \ell_1 \le \ell$, is protected. Then $\ell_2 = 3\ell b - \ell_1$ vertices can be protected in the graph $G[V \setminus L]$. Due to the previous discussion, those ℓ_2 protected vertices can be split among the $F(\ell, b)$ equivalence classes in $\binom{F(l,b)+\ell_2-1}{\ell_2}$ different way. Hence the number of different cases exponentially depends on ℓ and b only.

Now suppose that we can protect $\ell'_2 \le \ell_2$ vertices inside the equivalence class $[L'_\nu]$ for a fixed $L' \subseteq L$ and a fixed neighbour set of L' in the cliques. As it was already discussed above, for all strategies maximising the number of saved vertices in the equivalence class $[L'_\nu]$ it is enough to focus on (at most) $3\ell b$ chosen cliques from $seq([L'_\nu])$, hence the number of cliques we need to consider depends only on ℓ and b. Now a brute-force algorithm can generate all options how these ℓ'_2 vertices can protect the whole neighbour sets of the vertices from L^s in at most $3\ell b$ cliques from $seq([L'_\nu])$, taking into account also their order. If there is not enough vertices to protect a whole neighbour set, then it makes no difference which vertices are protected. Such an algorithm runs in exponential time in b, ℓ, but not n. Hence the problem is fixed-parameter tractable with respect to the parameter "cluster vertex deletion". □

6 Conclusion

The main result of this paper is that the Firefighter problem is NP-complete even on trees of pathwidth three but fixed-parameter tractable with respect to the combined parameter "pathwidth" and "maximum degree" of the input graph. The combination of these two results with the NP-completeness of the problem on trees of bounded degree [14] indicates that the complexity of the problem depends heavily on the degree and the pathwidth of the graph. We left as an open question whether the problem is polynomial-time solvable on graphs of pathwidth two.

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