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ON THE DEFINITION OF THE SOLUTION TO A SEMILINEAR ELLIPTIC PROBLEM WITH A STRONG SINGULARITY AT $u = 0$

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Dedicated to our friend Carlo Sbordone
for his seventieth birthday

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dedicated to Carlo Sbordone for his 70th birthday

Abstract. In this paper we present new results related to the ones obtained in our previous papers on the singular semilinear elliptic problem

$$\begin{cases} u \geq 0 & \text{in } \Omega, \\ -div\, A(x)Du = F(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $F(x,s)$ is a Carathéodory function which can takes the value $+\infty$ when $s = 0$. Three new topics are investigated. First, we present definitions which we prove to be equivalent to the definition given in our paper [10]. Second, we study the set $\{x \in \Omega : u(x) = 0\}$, which is the set where the right-hand side of the equation could be singular in Ω, and we prove that actually, at almost every point x of this set, the right-hand side is non singular since one has $F(x,0) = 0$. Third, we consider the case where a zeroth order term μu, with μ a nonnegative bounded Radon measure which also belongs to $H^{-1}(\Omega)$, is added to the left-hand side of the singular problem considered above. We explain how the definition of solution given in [10] has to be modified in such a case, and we explicitly give the a priori estimates that every such solution satisfies (these estimates are at the basis of our existence, stability and uniqueness results). Finally we give two counterexamples which prove that when a zeroth order term μu of the above type is added to the left-hand side of the problem, the strong maximum principle in general does not hold anymore.

CONTENTS

Key words and phrases. Semilinear elliptic problems, singularity at $u = 0$, zeroth order term with coefficient a measure, strong maximum principle.

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1. Introduction

This is a great pleasure for us to dedicate the present paper to our friend Carlo Sbordone on the occasion of his seventieth birthday. Indeed the first results on the problem we deal with here were presented (on the precise day of the birthday of some of us) at the *Seventh european conference on elliptic and parabolic problems* held in Gaeta in May 2012 whose Carlo was one of the main organizers. Then a more advanced version of our work was presented at the international conference *New trends in calculus of variations and partial differential equations* organized in Naples in November 2013 to celebrate the 65th birthday of Carlo. The present work, which in some sense completes the work that we have done up to now on this problem, inserts therefore quite naturally in the *Special Issue of Nonlinear Analysis* dedicated to Carlo for his 70th birthday. Happy birthday, Carlo!

In our papers [8], [9], [10] and [11] we indeed studied the following semilinear elliptic problem with a singularity at $u = 0$ which consists in finding a function u which satisfies

$$\begin{cases} u \geq 0 & \text{in } \Omega, \\ -div\, A(x)Du = F(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 1$, where A is a coercive matrix with coefficients in $L^\infty(\Omega)$, and where $F : (x,s) \in \Omega \times [0,+\infty[\to F(x,s) \in [0,+\infty]$ is a Carathéodory function which satisfies

$$0 \leq F(x,s) \leq \frac{h(x)}{\Gamma(s)} \text{ a.e. } x \in \Omega,\ \forall s > 0, \tag{1.2}$$

with

$$\begin{cases} h \geq 0,\ h \in L^r(\Omega),\ r = \frac{2N}{N+2} \text{ if } N \geq 3,\ r > 1 \text{ if } N = 2,\ r = 1 \text{ if } N = 1, \\ \Gamma : s \in [0, +\infty[\longrightarrow \Gamma(s) \in [0, +\infty[\text{ is a } C^1([0, +\infty[) \text{ function} \\ \text{such that } \Gamma(0) = 0 \text{ and } \Gamma'(s) > 0\ \forall s > 0. \end{cases} \tag{1.3}$$

The model for problem (1.1) is the case where the function $F(x, s)$ is given by

$$\begin{cases} F(x, s) = \dfrac{f(x)}{s^\gamma} + g(x) \text{ a.e. } x \in \Omega,\ \forall s > 0, \\ \text{with } \gamma > 0,\ f, g \geq 0,\ f, g \in L^r(\Omega) \text{ with } r \text{ as in (1.3);} \end{cases} \tag{1.4}$$

other examples of functions $F(x, s)$ are given in (2.2) and (2.3) below.

In brief, problem (1.1) is a semilinear elliptic problem, whose specificity lies in the fact that its right-hand side $F(x, u)$, which is non negative, can have a singularity at $u = 0$, or in other terms can take the value $+\infty$ when $u(x) = 0$.

We will not try here to describe the literature concerned with problem (1.1), and we will only quote the pioneering work [4] by M.G. Crandall, P.H. Rabinowitz and L. Tartar, and the works [2] by L. Boccardo and L. Orsina, and [1] by L. Boccardo and J. Casado-Díaz. More than thirty years after [4], these two works relaunched the interest on this topics, and attracted our attention on this type of singular semilinear problems. The interested reader will find more references in [1], [2] and [4], as well as in our works [8] and [10].

Let us now explain how the present paper follows along the lines of our previous works [8], [9], [10] and [11]. We begin by describing the main results of these papers.

In our paper [8], we studied the case of problem (1.1) with a *mild singularity*, namely the case where in place of (1.2) one makes the more restrictive assumption

$$0 \leq F(x, s) \leq h(x)(\frac{1}{s} + 1) \text{ a.e. } x \in \Omega,\ \forall s > 0, \tag{1.5}$$

with h as in (1.3); in the model case (1.4), this corresponds to take γ in the range $0 < \gamma \leq 1$.

In this case, using formally u as test function in (1.1), one easily obtains that a solution u to (1.1) has “naturally” to stay in the space $H_0^1(\Omega)$. We thus introduced in [8] a notion of solution to problem (1.1) with a mild singularity which is a variant of the usual notion of “weak solution” to the linear version of problem (1.1). This definition is recalled in Subsection 3.1 below as “Definition 3.1 of [8]”.

In the framework of this definition, we proved in [8] the existence of a solution to problem (1.1) and its stability with respect of variations of the function $F(x, s)$. We also proved that this solution is unique when the function $F(x, s)$ is nonincreasing with respect to s (or more exactly “almost nonincreasing” with respect to s, see [8]).

Moreover, still in the case of a mild singularity, namely in the case where (1.5) holds true, we performed in [8] the homogenization of problem (1.1) when the problem (1.1) is posed in a sequence of domains Ω^ε obtained by perforating a fixed domain Ω by an increasing number of very small holes with vanishing diameters, in such a way that a “strange term” μu appears in the left-hand side of the limit equation; this “strange term” is nothing but the memory of the fact that the solution u^ε has to take the value zero on the whole boundary of Ω^ε, and in particular on the boundary of the holes.

In our paper [10], we then studied the case of *strong singularities*, namely the case where the function $F(x,s)$ only satisfies the general assumptions (1.2) and (1.3), which of course include the much more restrictive assumption (1.5).

In this case the solution u to problem (1.1) in general does not belong anymore to $H_0^1(\Omega)$ (see [13]), because roughly speaking the solution u does not belong to $H^1(\Omega)$ "up to the boundary", even if in some sense u vanishes at the boundary. We therefore introduced in [10] a new notion of solution. This definition, which is recalled in Definition 2.9 below (see Subsection 2.4), is based on the fact that a solution u to problem (1.1), which does not in general belongs to $H_0^1(\Omega)$, nevertheless satisfies

$$G_k(u) \in H_0^1(\Omega) \text{ and } \varphi T_k(u) \in H_0^1(\Omega)\ \forall k > 0,\ \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{1.6}$$

(where as usual $T_k(s) = \inf(s,k)$ and $G_k(s) = (s-k)^+$ for $s \geq 0$); in the best of our knowledge, these properties of a solution to problem (1.1) had not been noticed before. Also, as far as the partial differential equation of (1.1) is concerned, the fact that a solution u does not in general satisfy $u \in H_0^1(\Omega)$ but satisfies (1.6) led us to introduce a new space $\mathcal{V}(\Omega)$ of test functions which allowed us to give a (weak) formulation of the equation in the spirit of the notion of "solution defined by transposition" introduced for other problems by J.-L. Lions and E. Magenes and by G. Stampacchia. For the exact formulation of this definition, see Definition 2.9 in Subsection 2.4 below.

In the framework of this definition, we were able to prove in [10] the existence of a solution to problem (1.1) and its stability with respect of variations of the function $F(x,s)$ when the function $F(x,s)$ only satisfies (1.2) and (1.3). We also proved that this solution is unique when the function $F(x,s)$ is nonincreasing with respect to s.

Moreover, in [11] we performed in this framework, under the general assumptions (1.2) and (1.3), the homogenization of problem (1.1) when this problem is posed in a sequence of domains Ω^ε which are, as in [8], obtained by perforating a fixed domain Ω by an increasing number of very small holes with vanishing diameters.

Finally, in our paper [9], we presented new results related to the ones obtained in our papers [8] and [10].

In particular we proved in Section 3 of [9] that assumption (1.2) on $F(x,s)$ can be equivalently written as

$$\begin{cases} \forall k > 0,\ \exists h_k \geq 0,\ h_k \in L^r(\Omega),\ r \text{ as in (1.3)}, \\ \text{such that } \ 0 \leq F(x,s) \leq h_k(x)\ \forall s \geq k, \end{cases} \tag{1.7}$$

even if this new formulation seems at the first glance to be much less restrictive.

We also proved in Section 5 of [9] that when in (1.2) the function h satisfies the regularity assumption $h \in L^t(\Omega)$ for $t > \frac{N}{2}$, then the solution u defined in [8] and in [10] to problem (1.1) actually belongs to $L^\infty(\Omega)$, as it is the case when (1.1) is the linear problem in which $F(x,s) = h(x)$.

In the present paper we give new results related to the definition of solution to problem (1.1) that we introduced in [10]. Let us now describe these results, which are concerned with three topics.

In order to make the present paper relatively self contained, we first recall in Definition 2.9 in Section 2 below the definition given in [10] of a solution to problem (1.1) in the case of a strong singularity, namely in the case where only the general assumptions (1.2) and (1.3) are made, and we briefly mention the results of existence, of stability, and (in the case where the function $F(x, s)$ is assumed to be nonincreasing in s) of uniqueness of such a solution that we proved in [10].

In Section 3 below we treat the first of these three topics. We indeed consider three definitions of solution to (1.1), which we prove to be equivalent to the definition introduced in [10] (and which is recalled below in Definition 2.9 of Section 2) for the case where problem (1.1) presents a strong singularity.

First we prove in Subsection 3.1 that in the case where problem (1.1) presents a mild singularity, the definitions given in [8] for the case of a mild singularity and in [10] for the case of a strong singularity are equivalent.

Second we prove in Subsection 3.2 that for a slight variant of Definition 2.9, which is actually equivalent to Definition 2.9, one gets an equivalent definition if in place of requiring this slight variant to hold true for every $k > 0$, one requires it to hold true only for a single k_0, where k_0 can be arbitrarily chosen.

Third we consider in Subsection 3.3 the case where one deals with solutions to problem (1.1) which belong to $L^\infty(\Omega)$; in this case Definition 2.9 can be replaced by an equivalent version of it which is much simpler to write.

Finally we observe in Subsection 3.4 that all the results of [10] (but apparently not the homogenization result of [11]) can be obtained by replacing the space $\mathcal{V}(\Omega)$ of test functions described in Subsection 2.3 below by its vectorial subspace $\mathcal{W}(\Omega)$ generated by the test functions w of the form $w = \varphi^2$ where $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

In Section 4 below we treat the second of these three topics. We indeed study the set $\{u = 0\} = \{x \in \Omega : u(x) = 0\}$ where the solution u to problem (1.1) in the sense of Definition 2.9 below takes the value zero, which is the set of the points of Ω where the right-hand side of the partial differential equation of problem (1.1) could be singular. We prove that, up to a set of zero measure, this set is a subset of the set $\{x \in \Omega : F(x, 0) = 0\}$, where the right-hand side is of course non singular. There is therefore almost no point of Ω where the right-hand side of problem (1.1) is singular.

We also recall in Section 4 that a stronger result, namely the fact that on every ball strictly contained in Ω, $u(x)$ is almost everywhere greater than a (strictly) positive constant, has been obtained by L. Boccardo and L. Orsina in [2] by using the strong maximum principle. We finally emphasize the fact that we never use neither this property of u nor the strong maximum principle, and this neither in the present paper nor in any of our previous papers [8], [9], [10] and [11].

We finally treat in Section 5 below the last of these three topics by considering the case where problem (1.1) is replaced by the problem

$$\begin{cases} u \geq 0 & \text{in } \Omega, \\ -div\, A(x)Du + \mu u = F(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

which involves in its left-hand side a zeroth order term μu where μ is a nonnegative finite Radon measure which also belongs to $H^{-1}(\Omega)$, namely

$$\mu \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega). \tag{1.9}$$

Such a zeroth order term naturally appears as a "strange term" when performing the homogenization of problem (1.1) with many small holes with vanishing diameters, see our papers [8] and [11].

After having recalled in the brief Subsection 5.1 the variational framework which has to be used for the linear problem obtained by taking $F(x,u) = f(x) \in L^2(\Omega)$ in (1.8), we first explain in Subsection 5.2 how Definition 2.9 presented in Subsection 2.4 below has to be adapted to the case of problem (1.8) where μ is no more equal to zero: see Definition 5.1 below. In the framework of Definition 5.1, we then state results of existence and stability with respect to variations of the function $F(x,s)$, as well as of uniqueness when the function $F(x,s)$ is nonincreasing in s. These results are the analogues of the results obtained in [10] in the case where μ was zero. We also state (and sketch the proofs of) the a priori estimates which hold true for every solution to (1.8) in the sense of Definition 5.1 below. These a priori estimates are the analogues of the a priori estimates obtained in Section 5 of [10] for solutions in the sense of Definition 2.9 below in the case where μ was zero.

We conclude Section 5 by presenting in Subsection 5.3 two counterexamples to the strong maximum principle for the problem (1.8) with μ satisfying (1.9). The first counterexample deals with the case of the linear problem obtained by taking $F(x,s) = f(x)$, and has been communicated to us by Gianni Dal Maso, to whom we express our warmest thanks. The second one, which is a variant of the first one, deals with the singular semilinear problem (1.8) itself.

2. Recalling the setting of the problem, the definition of the solution and the results obtained in [10]

2.1. Notation

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 1$.

We denote by $\mathcal{D}(\Omega)$ the space of the $C^\infty(\Omega)$ functions whose support is compact and included in Ω, and by $\mathcal{D}'(\Omega)$ the space of distributions on Ω.

We denote by $\mathcal{M}_b^+(\Omega)$ the space of nonnegative bounded Radon measures on Ω.

Since Ω is bounded, $\|Dw\|_{(L^2(\Omega))^N}$ is a norm equivalent to $\|w\|_{H^1(\Omega)}$ on $H_0^1(\Omega)$. We set

$$\|w\|_{H_0^1(\Omega)} = \|Dw\|_{(L^2(\Omega))^N} \ \ \forall w \in H_0^1(\Omega).$$

For every $s \in \mathbb{R}$ and every $k > 0$ we define as usual

$$s^+ = \max\{s, 0\}, \ \ s^- = \max\{0, -s\},$$

$$T_k(s) = \max\{-k, \min\{s, k\}\}, \quad G_k(s) = s - T_k(s).$$

For every measurable function $l : x \in \Omega \to l(x) \in [0, +\infty]$ we denote

$$\{l = 0\} = \{x \in \Omega : l(x) = 0\}, \quad \{l > 0\} = \{x \in \Omega : l(x) > 0\}.$$

Finally, in the present paper, we denote by φ and $\overline{\varphi}$ functions which belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$, while we denote by ϕ and $\overline{\phi}$ functions which belong to $\mathcal{D}(\Omega)$.

2.2. Assumptions

As said in the Introduction we study in this paper solutions to the following singular semilinear problem

$$\begin{cases} u \geq 0 & \text{in } \Omega, \\ -div\, A(x)Du = F(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where a model for the function $F(x,s)$ is given by (1.4), or more generally by

$$\begin{cases} F(x,s) = f(x)\dfrac{\big(a+\sin(\frac{1}{s})\big)}{\exp(-\frac{1}{s})} + g(x)\dfrac{\big(b+\sin(\frac{1}{s})\big)}{s^\gamma} + l(x) \\ \text{a.e. } x \in \Omega,\ \forall s > 0, \end{cases} \tag{2.2}$$

where $\gamma > 0$, $a > 1$, $b > 1$ and where the functions f, g and l are nonnegative, or even more generally by

$$\begin{cases} F(x,s) = f(x)\dfrac{(a+\sin(S(s)))}{\exp(-S(s))} + g(x)\dfrac{\big(b+\sin(\frac{1}{s})\big)}{s^\gamma} + l(x) \\ \text{a.e. } x \in \Omega,\ \forall s > 0, \end{cases} \tag{2.3}$$

where $\gamma > 0$, $a > 1$, $b > 1$, where the function S satisfies

$$S \in C^1(]0,+\infty[), \quad S'(s) < 0\ \forall s > 0, \quad S(s) \to +\infty \text{ as } s \to 0, \tag{2.4}$$

and where the functions f, g and l are nonnegative and belong to $L^r(\Omega)$ with r defined in (2.7 *i*) below (see Remark 2.1 *viii*) of [10] as far as the latest example (2.3) is concerned).

In this Section, we give the precise assumptions that we make on the data of problem (2.1).

We assume that Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 1$ (no regularity is assumed on the boundary $\partial\Omega$ of Ω), that the matrix A is bounded and coercive, i.e. satisfies

$$A(x) \in (L^\infty(\Omega))^{N\times N},\ \exists \alpha > 0,\ A(x) \geq \alpha I \quad \text{a.e. } x \in \Omega, \tag{2.5}$$

and that the function F satisfies

$$\begin{cases} F : (x,s) \in \Omega \times [0,+\infty[\to F(x,s) \in [0,+\infty] \text{ is a Carathéodory function,} \\ \text{i.e. } F \text{ satisfies} \\ i)\ \forall s \in [0,+\infty[,\ x \in \Omega \to F(x,s) \in [0,+\infty] \text{ is measurable,} \\ ii)\ \text{for a.e. } x \in \Omega,\ s \in [0,+\infty[\to F(x,s) \in [0,+\infty] \text{ is continuous,} \end{cases} \tag{2.6}$$

$$\begin{cases} i)\ \exists h,\ h(x) \geq 0 \ \text{ a.e. } x \in \Omega,\ h \in L^r(\Omega), \\ \text{with } r = \dfrac{2N}{N+2} \ \text{ if } N \geq 3,\ r > 1 \text{ if } N = 2,\ r = 1 \text{ if } N = 1, \\ ii)\ \exists \Gamma : s \in [0,+\infty[\to \Gamma(s) \in [0,+\infty[, \\ \Gamma \in C^1([0,+\infty[),\ \Gamma(0) = 0,\ \Gamma'(s) > 0\ \forall s > 0, \\ iii)\ 0 \leq F(x,s) \leq \dfrac{h(x)}{\Gamma(s)} \quad \text{a.e. } x \in \Omega, \forall s > 0. \end{cases} \tag{2.7}$$

Moreover, but only when we will be concerned with comparison and uniqueness results (Proposition 7.1 and Theorem 4.3 of [10]), we will assume that the function $F(x,s)$ is nonincreasing in s, i.e. that

$$F(x,s) \leq F(x,t) \ \text{ a.e. } x \in \Omega, \ \forall s, \forall t, \ 0 \leq t \leq s. \tag{2.8}$$

Remark 2.1. (About assumptions (2.6) **and** (2.7)**).** In the present Remark we point out some features of the previous assumptions. We refer to Remark 2.1 of [10] for further details and observations.

- *i*) If a function $\Gamma(s)$ satisfies (2.7 *ii*), then Γ is (strictly) increasing and satisfies $\Gamma(s) > 0$ for every $s > 0$; note that the function Γ can be either bounded or unbounded.
- *ii*) The function $F(x,s)$ is a nonnegative Carathéodory function with values in $[0,+\infty]$ and not only in $[0,+\infty[$. But, in view of conditions (2.7 *ii*) and (2.7 *iii*), for almost every $x \in \Omega$, the function $F(x,s)$ can take the value $+\infty$ only when $s = 0$ (or, in other terms, $F(x,s)$ is finite for almost every $x \in \Omega$ when $s > 0$).
- *iii*) Let us observe that the functions $F(x,s)$ given in examples (1.4), (2.2) and (2.3) satisfy assumption (2.7); indeed for these examples one has
$$0 \leq F(x,s) \leq \overline{h}(x)\left(\frac{1}{\overline{\Gamma}(s)} + 1\right)$$
for some $\overline{h}(x)$ and $\overline{\Gamma}(s)$ which satisfy (2.7 *i*) and (2.7 *ii*); taking $\Gamma(s) = \overline{\Gamma}(s)/(1+\overline{\Gamma}(s))$ it is clear that $\Gamma(s)$ satisfies (2.7 *ii*) and that $F(x,s)$ satisfies (2.7). □

Remark 2.2. (Sobolev's embedding). The function h which appears in hypothesis (2.7 *i*) is an element of $H^{-1}(\Omega)$. Indeed, when $N \geq 3$, the exponent $r = 2N/(N+2)$ is nothing but the Hölder's conjugate $(2^*)'$ of the Sobolev's exponent 2^*, i.e.

$$\text{when } N \geq 3, \quad \frac{1}{r} = 1 - \frac{1}{2^*}, \ \text{ where } \frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}. \tag{2.9}$$

Making an abuse of notation, we will set

$$\begin{cases} 2^* = \text{any } p \text{ with } 1 < p < +\infty \text{ when } N = 2, \\ 2^* = +\infty \text{ when } N = 1. \end{cases} \tag{2.10}$$

With this abuse of notation, assumption (2.7 *i*) is the fact that h belongs to $L^r(\Omega) = L^{(2^*)'}(\Omega) \subset H^{-1}(\Omega)$ for all $N \geq 1$ since Ω is bounded.

This result is indeed a consequence of Sobolev's, Trudinger Moser's and Morrey's inequalities, which (with this abuse of notation) assert that

$$\|v\|_{L^{2^*}(\Omega)} \leq C_S \|Dv\|_{(L^2(\Omega))^N} \quad \forall v \in H_0^1(\Omega) \text{ when } N \geq 1, \tag{2.11}$$

where C_S is a constant which depends only on N when $N \geq 3$, which depends on p and on Q when $N = 2$, and which depends on Q when $N = 1$, when Q is any bounded open set such that $\Omega \subset Q$. □

Remark 2.3. (About assumption (2.8)**).** Let us emphasize that we use assumption (2.8), namely the fact that the function $F(x,s)$ is nonincreasing in s, only when we are concerned with comparison and uniqueness results (Proposition 7.1 and Theorem 4.3 of [10]). In contrast, all the others results of [10], of the present paper and of our papers [8], [9] and [11] never use this assumption. □

2.3. The space $\mathcal{V}(\Omega)$ of test functions

In order to introduce the notion of solution to problem (2.1) that we will use in the present paper, we recall the definition introduced in [10] of the space $\mathcal{V}(\Omega)$ of test functions and a notation introduced in [10].

Definition 2.4. (Definition of $\mathcal{V}(\Omega)$) (Definition 2.1 of [10]**).** The space $\mathcal{V}(\Omega)$ is the space of the functions v which satisfy

$$v \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{2.12}$$

$$\begin{cases} \exists I \text{ finite}, \ \exists \hat{\varphi}_i, \, \exists \hat{g}_i, \, i \in I, \exists \hat{f}, \\ \text{with } \ \hat{\varphi}_i \in H_0^1(\Omega) \cap L^\infty(\Omega), \, \hat{g}_i \in (L^2(\Omega))^N, \, \hat{f} \in L^1(\Omega), \\ \text{such that } \ -div\, {}^t\!A(x)Dv = \displaystyle\sum_{i\in I} \hat{\varphi}_i(-div\, \hat{g}_i) + \hat{f} \text{ in } \mathcal{D}'(\Omega). \quad \square \end{cases} \tag{2.13}$$

In the definition of $\mathcal{V}(\Omega)$ we use the notation $\hat{\varphi}_i$, $\hat{g}_i$, and $\hat{f}$ to help the reader to identify the functions which enter in the definition of the functions of $\mathcal{V}(\Omega)$.

Note that $\mathcal{V}(\Omega)$ is a vector space.

Definition 2.5. (Notation $\langle\langle\, , \rangle\rangle_\Omega$) (Definition 3.2 of [10]**).** When $v \in \mathcal{V}(\Omega)$ with

$$-div\, {}^t\!A(x)Dv = \sum_{i\in I} \hat{\varphi}_i(-div\, \hat{g}_i) + \hat{f} \quad \text{in } \mathcal{D}'(\Omega),$$

where I, $\hat{\varphi}_i$, $\hat{g}_i$ and $\hat{f}$ are as in (2.13), and when z satisfies

$$z \in H^1_{\text{loc}}(\Omega) \cap L^\infty(\Omega) \text{ with } \varphi z \in H_0^1(\Omega) \, \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

we use the following notation:

$$\langle\langle -div\, {}^t\!A(x)Dv, z \rangle\rangle_\Omega = \sum_{i\in I} \int_\Omega \hat{g}_i D(\hat{\varphi}_i z) + \int_\Omega \hat{f} z. \quad \square \tag{2.14}$$

Remark 2.6. (On notation (2.14)**).**

In (2.13), the product $\hat{\varphi}_i(-div\, \hat{g}_i)$ with $\hat{\varphi}_i \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $\hat{g}_i \in (L^2(\Omega))^N$ is, as usual, the distribution on Ω defined by

$$\begin{cases} \forall \phi \in \mathcal{D}(\Omega), \\ \langle \hat{\varphi}_i(-div\, \hat{g}_i), \phi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \langle -div\, \hat{g}_i, \, \hat{\varphi}_i \phi \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \displaystyle\int_\Omega \hat{g}_i D(\hat{\varphi}_i \phi), \end{cases} \tag{2.15}$$

and the equality $-div\, {}^t\!A(x)Dv = \displaystyle\sum_{i\in I} \hat{\varphi}_i(-div\, \hat{g}_i) + \hat{f}$ holds in $\mathcal{D}'(\Omega)$.

In notation (2.14), the right-hand side is correctly defined since $\hat{\varphi}_i z \in H_0^1(\Omega)$ and since $z \in L^\infty(\Omega)$. In contrast the left-hand side $\langle\langle -div\, {}^t\!A\, Dv, z \rangle\rangle_\Omega$ is just a notation. $\square$

Remark 2.7. If $y \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then $\varphi y \in H_0^1(\Omega)$ for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, so that for every $v \in \mathcal{V}(\Omega)$, $\langle\langle -div\, {}^t\!A(x)Dv, y \rangle\rangle_\Omega$ is defined. Actually one has

$$\begin{cases} \forall v \in \mathcal{V}(\Omega), \ \forall y \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ \langle\langle -div\, {}^t\!A(x)Dv, y \rangle\rangle_\Omega = \langle -div\, {}^t\!A(x)Dv, y \rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \end{cases} \tag{2.16}$$

see Remark 3.4 of [10] for the proof. $\square$

Remark 2.8. (Examples of functions which belong to $\mathcal{V}(\Omega)$). Let us recall some examples of functions which belong to $\mathcal{V}(\Omega)$ (see Remark 3.5 of [10] for the details of the proofs):

- *i)* If $\varphi_1, \varphi_2 \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then $\varphi_1\varphi_2 \in \mathcal{V}(\Omega)$.
- *ii)* In particular, if $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then $\varphi^2 \in \mathcal{V}(\Omega)$.
- *iii)* If $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ has a compact support which is included in Ω, then $\varphi \in \mathcal{V}(\Omega)$.
- *iv)* In particular every $\phi \in \mathcal{D}(\Omega)$ belongs to $\mathcal{V}(\Omega)$. □

2.4. Definition of a solution to problem (2.1)

We now give the definition of a solution to problem (2.1) that we will use in the present paper.

Definition 2.9. (Definition of a solution to problem (2.1)) (Definition 3.6 of [10])). Assume that the matrix A and the function F satisfy (2.5), (2.6) and (2.7). We say that u is a solution to problem (2.1) if u satisfies

$$\begin{cases} i)\, u \in L^2(\Omega) \cap H^1_{\mathrm{loc}}(\Omega), \\ ii)\, u(x) \geq 0 \ \text{ a.e. } x \in \Omega, \\ iii)\, G_k(u) \in H_0^1(\Omega) \ \ \forall k > 0, \\ iv)\, \varphi T_k(u) \in H_0^1(\Omega) \ \ \forall k > 0, \ \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{2.17}$$

$$\begin{cases} \forall v \in \mathcal{V}(\Omega),\ v \geq 0, \\ \text{with } -div\, {}^tA(x)Dv = \displaystyle\sum_{i\in I} \hat{\varphi}_i(-div\, \hat{g}_i) + \hat{f} \text{ in } \mathcal{D}'(\Omega), \\ \text{where } \hat{\varphi}_i \in H_0^1(\Omega) \cap L^\infty(\Omega),\ \hat{g}_i \in (L^2(\Omega))^N,\ \hat{f} \in L^1(\Omega), \\ \text{one has} \\ i)\ \displaystyle\int_\Omega F(x,u)\, v < +\infty, \\ ii)\ \displaystyle\int_\Omega {}^tA(x)DvDG_k(u) + \sum_{i\in I}\int_\Omega \hat{g}_i D(\hat{\varphi}_i T_k(u)) + \int_\Omega \hat{f} T_k(u) = \\ = \langle -div\, {}^tA(x)Dv, G_k(u)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle\langle -div\, {}^tA(x)Dv, T_k(u)\rangle\rangle_\Omega = \\ = \displaystyle\int_\Omega F(x,u)\, v \ \forall k > 0. \quad \square \end{cases} \tag{2.18}$$

Remark 2.10. (About Definition 2.9).

- *i)* Note that (2.1) is only formal. In contrast, Definition 2.9 gives a precise meaning to the solution to problem (2.1) and provides a mathematically correct framework for this notion.

  In this Definition 2.9, the requirement (2.17) prescribes the "space" (which is not a vectorial space) to which the solution should belong, while the requirement (2.18), and specially (2.18 *ii*), precises the sense of the partial differential equation of (2.1). This definition of solution is close in spirit to the definition of solution defined by transposition introduced by J.-L. Lions and E. Magenes and by G. Stampacchia.

- *ii*) Note that the statement (2.17 *iii*) formally contains the boundary condition "$u = 0$ on $\partial\Omega$". Indeed $G_k(u) \in H_0^1(\Omega)$ for every $k > 0$ formally implies that "$G_k(u) = 0$ on $\partial\Omega$", i.e. "$u \le k$ on $\partial\Omega$" for every $k > 0$, which implies "$u = 0$ on $\partial\Omega$" since $u \ge 0$ in Ω.

  See also Remark 2.14 below about the boundary condition "$u = 0$ on $\partial\Omega$".

- *iii*) Note finally that (very) formally, one has

$$\begin{cases} \text{“}\langle -div\, {}^tA(x)Dv, G_k(u)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \displaystyle\int_\Omega (-div\, {}^tA(x)Dv)\, G_k(u) = \\ = \displaystyle\int_\Omega v\,(-divA(x)DG_k(u))\text{”}, \end{cases}$$

$$\begin{cases} \text{“}\langle\langle -div\, {}^tA(x)Dv, T_k(u)\rangle\rangle_\Omega = \displaystyle\int_\Omega (-div\, {}^tA(x)Dv)\, T_k(u) = \\ = \displaystyle\int_\Omega v\,(-divA(x)DT_k(u)\text{”}, \end{cases}$$

so that (2.18 *ii*) formally means that

$$\text{“}\int_\Omega v\,(-divA(x)Du) = \int_\Omega F(x,u)\, v\text{”}\ \forall v \in \mathcal{V}(\Omega),\ v \ge 0.$$

Since every v can be written as $v = v^+ - v^-$ with $v^+ \ge 0$ and $v^- \ge 0$, one has formally (this is formal since we do not know whether v^+ and v^- belong to $\mathcal{V}(\Omega)$ when v belongs to $\mathcal{V}(\Omega)$)

$$\text{“} - div\, A(x)Du = F(x,u)\text{”}.$$

Observe that the above formal computation has no meaning in general, while (2.18 *ii*) has a perfectly correct mathematical sense when $v \in \mathcal{V}(\Omega)$ and when u satisfies (2.17). □

The following Proposition 2.11 asserts that every solution to problem (2.1) in the sense of Definition 2.9 is a solution to (2.1) in the sense of distributions (see [10] for the proof). Note that Proposition 2.11 does not say anything about the boundary condition satisfied by u (on these latest topics, see Remark 2.10 *ii*)).

Proposition 2.11. (Proposition 3.8 of [10]**).** *Assume that the matrix A and the function F satisfy* (2.5), (2.6) *and* (2.7). *Then for every solution to problem* (2.1) *in the sense of Definition 2.9 one has*

$$u \ge 0 \text{ a.e. in } \Omega,\ u \in H^1_{\mathrm{loc}}(\Omega),\ F(x,u) \in L^1_{\mathrm{loc}}(\Omega), \tag{2.19}$$

$$-div\, A(x)Du = F(x,u) \text{ in } \mathcal{D}'(\Omega). \tag{2.20}$$

Remark 2.12. (φDu belongs to $(L^2(\Omega))^N$ $\forall \varphi \in H^1(\Omega) \cap L^\infty(\Omega)$). When u satisfies (2.17), then one has

$$\varphi Du \in (L^2(\Omega))^N\ \ \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{2.21}$$

More precisely, when u satisfies (2.17 *i*) and (2.17 *iii*), assertion (2.17 *iv*) is equivalent to (2.21) (see Remark 3.9 of [10] for details). □

2.5. Existence, stability and uniqueness results, and a priori estimates

With the above Definition 2.9, we proved in [10] results of existence, of stability, and (when the function $F(x,s)$ is noninscreasing in s) of uniqueness of the solution to problem (2.1) in the sense of Definition 2.9 (see Theorems 4.1, 4.2 and 4.3 of [10]). We also proved a priori estimates which hold true for every solution to problem (2.1) in the sense of Definition 2.9 (see Section 5 of [10]).

Among these a priori estimates, we now explicitely recall the result of Proposition 5.13 of [10], since we will use it in a crucial way in Section 3 below. (This property has also been used in the proofs of Comparison Principle 7.1 and of the Uniqueness Theorem 4.3 of [10].) This a priori estimate is actually in some sense a regularity result, since it asserts that for every u solution to problem (2.1) in the sense of Definition 2.9, a certain function $\beta(u)$ actually belongs to $H_0^1(\Omega)$.

Define the function $\beta : s \in [0,+\infty[\to \beta(s) \in [0,+\infty[$ by

$$\beta(s) = \int_0^s \sqrt{\Gamma'(t)}dt, \tag{2.22}$$

where the function Γ appears in assumption (2.7).

Proposition 2.13. ($\beta(u)$ belongs to $H_0^1(\Omega)$ and a priori estimate) (Proposition 5.13 of [10]**).** *Assume that the matrix A and the function F satisfy* (2.5), (2.6) *and* (2.7)*. Then for every u solution to problem* (2.1) *in the sense of Definition 2.9 one has*

$$\beta(u) \in H_0^1(\Omega), \tag{2.23}$$

with the a priori estimate

$$\alpha \|D\beta(u)\|^2_{(L^2(\Omega))^N} \leq \|h\|_{L^1(\Omega)}. \tag{2.24}$$

Remark 2.14. (On the boundary condition $u = 0$). The fact that $\beta(u) \in H_0^1(\Omega)$ (see (2.23)) formally implies that "$\beta(u) = 0$ on $\partial\Omega$". (This assertion is mathematically correct if it is understood in the sense of traces, when $\partial\Omega$ is assumed to be sufficiently smooth in order for the traces of functions of $H^1(\Omega)$ to be defined.) Since $\beta(s)$ implies that $s = 0$ (see (2.22) and (2.7 *ii*)), $\beta(u) = 0$ on $\partial\Omega$ formally implies that "$u = 0$ on $\partial\Omega$."

See also Remark 2.10 *ii*) above about the boundary condition "$u = 0$ on $\partial\Omega$". □

3. THREE DEFINITIONS EQUIVALENT TO DEFINITION 2.9 AND A VARIANT OF THE SPACE OF TEST FUNCTIONS

In this Section we consider three definitions of solutions to problem (2.1) which we prove to be equivalent to Definition 2.9 above. We also consider the case where the space $\mathcal{V}(\Omega)$ of test functions is replaced by another space $\mathcal{W}(\Omega)$.

In Subsection 3.1 we consider the case of a mild singularity, namely the case where the function $F(x,s)$ satisfies condition (3.2) below, which is much more restrictive than (2.7), and we prove that in this case, Definition 2.9 above is equivalent to the Definition 3.1 given and used in our paper [8].

In Subsection 3.2, we consider a variant of Definition 2.9 above, in which the requirement that $\beta(u) \in H_0^1(\Omega)$ (see Proposition 2.13 above) is added to requirement (2.17). We call this variant (2.17bis), and prove that Definition 2.9 above, which is made of (2.17) and (2.18), is equivalent to (2.17bis) and (2.18). We then prove that, for this (equivalent) variant of Definition 2.9, it is equivalent to require that

(2.17bis) and (2.18) hold true for every $k > 0$ or only for a single $k_0 > 0$, where k_0 can be chosen arbitrarily.

In Subsection 3.3, we consider the case where the solution u to problem (2.1) belongs to $L^\infty(\Omega)$. Then Definition 2.9 can be written in a more simpler but still equivalent way.

Finally in Subsection 3.4, we consider the case where in Definition 2.9 above, the space $\mathcal{V}(\Omega)$ of test functions used in (2.18) is replaced by its subspace $\mathcal{W}(\Omega)$ which has a structure much more simpler than $\mathcal{V}(\Omega)$, since it is generated by the functions $w = \varphi^2$ with $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

3.1. Equivalence of two definitions in the case of a mild singularity

We prove in this Subsection that the Definition 2.9 given in the present paper of a solution to problem (2.1) coincides with the Definition 3.1 of a solution given in our paper [8] when the singularity is mild, namely when F satisfies

$$\begin{cases} 0 \le F(x,s) \le h(x)\left(\dfrac{1}{s^\gamma} + 1\right) \text{ a.e. } x \in \Omega, \forall s > 0, \\ \text{with } h(x) \ge 0 \text{ a.e. } x \in \Omega, h \in L^r(\Omega),\ r \text{ as in } (2.7i),\ 0 < \gamma \le 1 \end{cases} \tag{3.1}$$

(compare with (2.7 *iii*) above). Condition (3.1) implies that

$$\begin{cases} 0 \le F(x,s) \le \overline{h}(x)\left(\dfrac{1}{s} + 1\right) \text{ a.e. } x \in \Omega, \forall s > 0, \\ \text{with } \overline{h}(x) \ge 0 \text{ a.e. } x \in \Omega, \overline{h} \in L^r(\Omega),\ r \text{ as in } (2.7i), \end{cases} \tag{3.2}$$

as it is easily seen in view of the inequality $1/s^\gamma \le \gamma/s + (1-\gamma)$ for $s > 0$ and $0 < \gamma \le 1$, which results from Young's inequality. Note that (3.2) is nothing but the case where $\Gamma(s) = s/(1+s)$ in (2.7 *iii*).

Recall that Definition 3.1 in [8] (which is concerned with functions F which satisfy (2.6), (2.7 *i*) and (3.1)) reads as follows:

Definition 3.1 of [8]. Assume that the matrix A and the function F satisfy (2.5), (2.6), (2.7 *i*) and (3.2). We say that u is a solution to problem (2.1) in the sense of Definition 3.1 of [8] if u satisfies

$$\begin{cases} u \in H_0^1(\Omega), \\ u \ge 0 \quad \text{a.e.} \quad x \in \Omega, \end{cases} \tag{3.3}$$

$$\begin{cases} \forall \varphi \in H_0^1(\Omega),\ \varphi \ge 0, \text{ one has} \\ i) \displaystyle\int_\Omega F(x,u)\,\varphi < +\infty, \\ ii) \displaystyle\int_\Omega A(x)DuD\varphi = \int_\Omega F(x,u)\,\varphi. \quad \square \end{cases} \tag{3.4}$$

Proposition 3.1. *Assume that the matrix A and the function F satisfy* (2.5), (2.6), (2.7 *i*) *and* (3.2). *Then u is a solution to problem* (2.1) *in the sense of Definition 3.1 of* [8] *if and only if u is a solution to problem* (2.1) *in the sense of Definition 2.9 of the present paper.*

Proof.

First step. When u is a solution to problem (2.1) in the sense of Definition 3.1 of [8], i.e. when u satisfies (3.3) and (3.4), it is clear that u satisfies (2.17).

On the other hand, let $v \in \mathcal{V}(\Omega)$, $v \geq 0$. Then v in particular belongs to $H_0^1(\Omega)$ and (2.18 *i*) follows from (3.4 *i*).

Finally using (2.16) with $y = T_k(u)$, which belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$, one has

$$\forall v \in \mathcal{V}(\Omega),\ \langle\langle -div\, {}^tA(x)Dv, T_k(u)\rangle\rangle_\Omega = \langle -div\, {}^tA(x)Dv, T_k(u)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)},$$

which implies that (2.18 *ii*) immediately follows from (3.4 *ii*).

Second step. Conversely, assume that u is a solution to problem (2.1) in the sense of Definition 2.9 above.

For every $k > 0$ fixed, (2.17 *iii*) implies that $G_k(u) \in H_0^1(\Omega)$. It is then sufficient to prove that $T_k(u) \in H_0^1(\Omega)$ to have (3.3). But inequality (3.2) is nothing but (2.7) with $\Gamma(s) = s/(1+s)$ so that by (2.22) $\beta'(s) = \sqrt{\Gamma'(s)} = 1/(1+s)$ and $\beta(s) = \log(1+s)$. Proposition 2.13 above then implies that $\beta(u) = \log(1+u) \in H_0^1(\Omega)$ so that $\beta'(u)Du = Du/(1+u) \in (L^2(\Omega))^N$. Therefore $DT_k(u) \in (L^2(\Omega))^N$ and $T_k(u) \in H^1(\Omega)$. Since for every $s \geq 0$ one has $0 \leq s \leq (1+s)\log(1+s)$, one has $0 \leq T_k(u) \leq (1+T_k(u))\log(1+T_k(u)) \leq (1+k)\beta(T_k(u))$ and Lemma A.1 of [10] implies that $T_k(u) \in H_0^1(\Omega)$.

We have proved that u satisfies (3.3).

Let us now prove (3.4). Let $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$, and let ϕ_n be a sequence such that

$$\phi_n \in \mathcal{D}(\Omega),\ \phi_n \to \varphi \text{ in } H_0^1(\Omega) \text{ and a.e. in } \Omega.$$

For every n, the function v_n defined by

$$v_n = \inf(\phi_n^+, \varphi)$$

belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$, has a compact support which is contained in Ω and is nonnegative. Therefore (see Remark 2.8 *iii*)), the function v_n belongs to $\mathcal{V}(\Omega)$ and can be used as test function in (2.18 *ii*), giving

$$\begin{cases} \langle -div\, {}^tA(x)Dv_n, G_k(u)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle\langle -div\, {}^tA(x)Dv_n, T_k(u)\rangle\rangle_\Omega = \\ = \displaystyle\int_\Omega F(x,u)\, v_n. \end{cases} \tag{3.5}$$

Since $T_k(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$, using (2.16) with $v = v_n$ and $y = T_k(u)$ implies that

$$\langle -div\, {}^tA(x)Dv_n, u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \int_\Omega F(x,u)\, v_n. \tag{3.6}$$

Passing to the limit in (3.6) and using Fatou's Lemma in the right-hand side gives

$$\langle -div\, {}^tA(x)D\varphi, u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \geq \int_\Omega F(x,u)\, \varphi,$$

which proves (3.4 *i*), i.e. that $F(x,u)\,\varphi \in L^1(\Omega)$; passing again to the limit in (3.6) and using now Lebesgue's dominated convergence Theorem (since $v_n \leq \varphi$) gives

$$\langle -div\, {}^tA(x)D\varphi, u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \int_\Omega F(x,u)\, \varphi,$$

i.e. (3.4 *ii*).

Proposition 3.1 is proved. □

3.2. Equivalence of two variants of Definition 2.9 using the requirement "for every k" and the requirement "for a single k_0"

Definition 2.9 consists in the two assertions (2.17) and (2.18).

Let us first observe that Definition 2.9, i.e. the fact that u satisfies (2.17) and (2.18), is equivalent to the fact that u satisfies (2.17bis) and (2.18), where (2.17bis) is

$$\tag{2.17bis} \begin{cases} i)\, u \in L^2(\Omega) \cap H^1_{\mathrm{loc}}(\Omega), \\ ii)\, u(x) \geq 0 \;\text{ a.e. } x \in \Omega, \\ iii)\, G_k(u) \in H^1_0(\Omega) \;\; \forall k > 0, \\ iv)\, \varphi T_k(u) \in H^1_0(\Omega) \;\; \forall k > 0, \;\; \forall \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega), \\ v)\, \beta(u) \in H^1_0(\Omega), \end{cases}$$

where the function β is defined from the function Γ which appears in assumption (2.7) by $\beta(s) = \displaystyle\int_0^s \sqrt{\Gamma'(t)}dt$ (see (2.22)): indeed, in view of Proposition 2.13, every u which satisfies (2.17) and (2.18) also satisfies $\beta(u) \in H^1_0(\Omega)$ and therefore satisfies (2.17bis); the converse is straightforward.

In the present Subsection we will denote (2.17bis) and (2.18) by (2.17bis$_{\forall k}$) and (2.18$_{\forall k}$) in order to emphasize that these requirements have to hold for every $k > 0$. Therefore (2.17bis$_{\forall k}$) will denote

$$\tag{2.17bis$_{\forall k}$} \begin{cases} i)\, u \in L^2(\Omega) \cap H^1_{\mathrm{loc}}(\Omega), \\ ii)\, u(x) \geq 0 \;\text{ a.e. } x \in \Omega, \\ iii)\, G_k(u) \in H^1_0(\Omega) \;\; \forall k > 0, \\ iv)\, \varphi T_k(u) \in H^1_0(\Omega) \;\; \forall k > 0, \;\; \forall \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega), \\ v)\, \beta(u) \in H^1_0(\Omega), \end{cases}$$

and (2.18$_{\forall k}$) will denote

$$\tag{2.18$_{\forall k}$} \begin{cases} \forall v \in \mathcal{V}(\Omega),\; v \geq 0, \\ \text{with } -div\, {}^tA(x)Dv = \displaystyle\sum_{i\in I} \hat{\varphi}_i(-div\, \hat{g}_i) + \hat{f} \text{ in } \mathcal{D}'(\Omega), \\ \text{where } \hat{\varphi}_i \in H^1_0(\Omega) \cap L^\infty(\Omega), \hat{g}_i \in (L^2(\Omega))^N, \hat{f} \in L^1(\Omega), \\ \text{one has} \\ i)\, \displaystyle\int_\Omega F(x,u)\, v < +\infty, \\ ii)\, \displaystyle\int_\Omega {}^tA(x)DvDG_k(u) + \sum_{i\in I}\int_\Omega \hat{g}_i D(\hat{\varphi}_i T_k(u)) + \int_\Omega \hat{f} T_k(u) = \\ = \langle -div\, {}^tA(x)Dv, G_k(u)\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} + \langle\langle -div\, {}^tA(x)Dv, T_k(u)\rangle\rangle_\Omega = \\ = \displaystyle\int_\Omega F(x,u)\, v \;\forall k > 0. \end{cases}$$

Then we have the following equivalence result, which asserts that Definition 2.9, which is equivalent to its variant given by (2.17bis$_{\forall k}$) and (2.18$_{\forall k}$), is equivalent to the same variant where the requirement "for every $k > 0$" has been replaced by the requirement "for a single $k_0 > 0$", where k_0 can be arbitrarily chosen:

Proposition 3.2. *Assume that the matrix A and the function F satisfy* (2.5), (2.6) *and* (2.7). *Then u is a solution to problem* (2.1) *in the sense of Definition 2.9 if* *and only if for a single* $k_0 > 0$ *(which can be arbitrarily chosen) one has*

$$(2.17\text{bis}_{k_0}) \qquad \begin{cases} i)\, u \in L^2(\Omega) \cap H^1_{\text{loc}}(\Omega), \\ ii)\, u(x) \geq 0 \ \ \text{a.e. } x \in \Omega, \\ iii)\, G_{k_0}(u) \in H^1_0(\Omega), \\ iv)\, \varphi T_{k_0}(u) \in H^1_0(\Omega) \ \ \forall \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega), \\ v)\, \beta(u) \in H^1_0(\Omega), \end{cases}$$

(2.18_{k_0})

$$\begin{cases} \forall v \in \mathcal{V}(\Omega),\ v \geq 0, \\ \text{with } -div\, {}^tA(x)Dv = \displaystyle\sum_{i \in I} \hat{\varphi}_i(-div\, \hat{g}_i) + \hat{f} \ \ in\ \mathcal{D}'(\Omega), \\ \text{where } \hat{\varphi}_i \in H^1_0(\Omega) \cap L^\infty(\Omega), \hat{g}_i \in (L^2(\Omega))^N, \hat{f} \in L^1(\Omega), \\ \text{one has} \\ i)\ \displaystyle\int_\Omega F(x,u)\, v < +\infty, \\ ii)\ \displaystyle\int_\Omega {}^tA(x)DvDG_{k_0}(u) + \sum_{i \in I} \int_\Omega \hat{g}_i D(\hat{\varphi}_i T_{k_0}(u)) + \int_\Omega \hat{f} T_{k_0}(u) = \\ = \langle -div\, {}^tA(x)Dv, G_{k_0}(u) \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} + \langle\langle -div\, {}^tA(x)Dv, T_{k_0}(u) \rangle\rangle_\Omega = \\ = \displaystyle\int_\Omega F(x,u)\, v. \end{cases}$$

Proof. To prove this equivalence we only have to prove that if u satisfies (2.17bis_{k_0}) and (2.18_{k_0}) for a given $k_0 > 0$, then u satisfies $(2.17\text{bis}_{\forall k})$ and $(2.18_{\forall k})$; this proves that u is a solution to problem (2.1) in the sense of Definition 2.9. The converse is straightforward using Proposition 2.13 above.

First step. In this step we will prove that (2.17bis_{k_0}) implies $(2.17\text{bis}_{\forall k})$.

Let us thus consider some u which satisfies (2.17bis_{k_0}) for a single $k_0 > 0$. Then u satisfies in particular

$$(3.7) \qquad \begin{cases} u \in L^2(\Omega) \cap H^1_{\text{loc}}(\Omega), \\ \chi_{\{u \geq k_0\}} Du \in (L^2(\Omega))^N, \\ \chi_{\{u \leq k_0\}} \varphi Du \in (L^2(\Omega))^N \ \ \forall \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega), \\ \beta'(u) Du \in (L^2(\Omega))^N. \end{cases}$$

Since $\beta'(s) = \sqrt{\Gamma'(s)}$ for every $s > 0$, the function β', like the function Γ' (see (2.7 *ii*)) is continuous and satisfies $\beta'(s) > 0$ for every $s > 0$. Therefore one has, for every a and b with $0 < a < b < +\infty$

$$0 < \min_{a \leq t \leq b} \beta'(t) \leq \max_{a \leq t \leq b} \beta'(t) < +\infty;$$

with the last assertion of (3.7), this implies that for every a and b with $0 < a < b < +\infty$, one has

$$\chi_{\{a \leq u \leq b\}} Du \in (L^2(\Omega))^N.$$

Together with (3.7), and writing when $k > k_0$

$$\chi_{\{u \leq k\}} \varphi Du = \chi_{\{u \leq k_0\}} \varphi Du + \chi_{\{k_0 < u \leq k\}} \varphi Du,$$

this implies that

$$\begin{cases} u \in L^2(\Omega) \cap H^1_{\mathrm{loc}}(\Omega), \\ \chi_{\{u \geq k\}} Du \in (L^2(\Omega))^N \ \ \forall k > 0, \\ \chi_{\{u \leq k\}} \varphi Du \in (L^2(\Omega))^N \ \ \forall k > 0, \ \ \forall \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega), \\ \beta'(u) Du \in (L^2(\Omega))^N \ \ \forall k > 0, \end{cases}$$

which implies that

$$\begin{cases} G_k(u) \in H^1(\Omega) \ \forall k > 0, \\ \varphi T_k(u) \in H^1(\Omega) \ \forall k > 0, \ \forall \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{3.8}$$

On the other hand, since β is nondecreasing, one has, when $k > k_0$,

$$\begin{cases} 0 \leq G_k(s) \leq G_{k_0}(s) \ \ \forall s > 0, \\ 0 \leq T_k(s) = T_{k_0}(s) + T_{k-k_0}(G_{k_0}(s)) \ \ \forall s > 0, \end{cases}$$

and when $k < k_0$,

$$\begin{cases} 0 \leq G_k(s) \leq \dfrac{k_0 - k}{\beta(k)} \beta(s) + G_{k_0}(s) \ \ \forall s > 0, \\ 0 \leq T_k(s) \leq T_{k_0}(s) \ \ \forall s > 0. \end{cases}$$

Using Lemma A1 of [10] and (3.8), this implies that every u which satisfies (2.17bis$_{k_0}$) also satisfies (2.17bis$_{\forall k}$).

Second step. Note first that the value of k does not appear in (2.18 *i*).

In this step we will prove that (2.17bis$_{k_0}$) and (2.18$_{k_0}$ *ii*) imply (2.18$_{\forall k}$ *ii*).

Let us thus consider some u which satisfies (2.17bis$_{k_0}$) and (2.18$_{k_0}$).

We define for every a and b with $0 < a < b < +\infty$, the function $S_{a,b}$ as

$$S_{a,b}(s) = T_b(s) - T_a(s) \ \forall s \geq 0, \tag{3.9}$$

and we observe that since $G_k(s) + T_k(s) = s$, one has

$$S_{a,b}(s) = G_a(s) - G_b(s) \ \forall s \geq 0. \tag{3.10}$$

Since we have proved in the first step that (2.17bis$_{k_0}$) implies (2.17bis$_{\forall k}$), the function $S_{a,b}(u)$, which is the difference of $G_a(u)$ and $G_b(u)$, belongs to $H^1_0(\Omega) \cap L^\infty(\Omega)$ for every $0 < a < b < +\infty$, and therefore, for every $v \in \mathcal{V}(\Omega)$ with

$$\begin{cases} -div\, {}^t\!A(x) Dv = \displaystyle\sum_{i \in I} \hat{\varphi}_i(-div\, \hat{g}_i) + \hat{f} \text{ in } \mathcal{D}(\Omega), \\ \text{where } \hat{\varphi}_i \in H^1_0(\Omega), \hat{g}_i \in (L^2(\Omega))^N, \hat{f} \in L^1(\Omega), \end{cases}$$

one has using $S_{a,b}(u)$ as test function in the latest equation

$$\int_\Omega {}^t\!A(x) Dv DS_{a,b}(u) = \sum_{i \in I} \int_\Omega \hat{g}_i D(\hat{\varphi}_i S_{a,b}(u)) + \int_\Omega \hat{f} S_{a,b}(u),$$

or in other terms, using (3.10) and (3.9),

$$\begin{cases} \displaystyle\int_\Omega {}^t\!A(x) Dv (DG_a(u) - DG_b(u)) = \\ = \displaystyle\sum_{i \in I} \int_\Omega \hat{g}_i D(\hat{\varphi}_i (T_b(u) - T_a(u))) + \int_\Omega \hat{f} (T_b(u) - T_a(u)). \end{cases}$$

Since $G_a(u)$, $G_b(u)$, $\hat{\varphi}_i T_a(u)$ and $\hat{\varphi}_i T_b(u)$ belong to $H_0^1(\Omega)$, while $T_a(u)$ and $T_b(u)$ belong to $L^\infty(\Omega)$, we can split the differences and we obtain

$$\begin{cases} \displaystyle\int_\Omega {}^tA(x)DvDG_a(u) + \sum_{i\in I}\int_\Omega \hat{g}_i D(\hat{\varphi}_i T_a(u)) + \int_\Omega \hat{f} T_a(u) = \\ \displaystyle= \int_\Omega {}^tA(x)DvDG_b(u) + \sum_{i\in I}\int_\Omega \hat{g}_i D(\hat{\varphi}_i T_b(u)) + \int_\Omega \hat{f} T_b(u), \end{cases}$$

for every a, b with $0 < a < b$, and therefore for every $a > 0$ and $b > 0$.

Taking $a = k$ and $b = k_0$ implies that $(2.18_{\forall k}\ ii)$ holds true whenever (2.17_{k_0}) and $(2.18_{k_0}\ ii)$ hold true.

The desired result is proved. ☐

3.3. The special case where the solution is bounded

In this Subsection we consider the special case where the solution u to problem (2.1) is bounded.

Note that any solution u to problem (2.1) in the sense of Definition 2.9 belongs to $L^\infty(\Omega)$ if in place of assumption (2.7 *i*), the function h is assumed to satisfy

$$h \in L^t(\Omega),\ t > \frac{N}{2} \text{ if } N \geq 2,\ t = 1 \text{ if } N = 1$$

(see [9, Section 5]).

In the case where the solution u to problem (2.1) is bounded one has the following result, which asserts that Definition 2.9 is equivalent to (3.11) and (3.12) below.

Proposition 3.3. *Assume that the matrix A and the function F satisfy* (2.5), (2.6) *and* (2.7). *Then a bounded function u is a solution to problem* (2.1) *in the sense of Definition 2.9 if and only if one has*

$$(3.11)\qquad \begin{cases} o)\ u \in L^\infty(\Omega), \\ i)\ u \in H^1_{\text{loc}}(\Omega), \\ ii)\ u(x) \geq 0 \text{ a.e. } x \in \Omega, \\ iii)\ G_k(u) \in H_0^1(\Omega)\ \ \forall k,\ 0 < k \leq \|u\|_{L^\infty(\Omega)}, \\ iv)\ \varphi u \in H_0^1(\Omega)\ \ \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases}$$

$$(3.12)\qquad \begin{cases} \forall v \in \mathcal{V}(\Omega),\ v \geq 0, \\ \text{with } -div\, {}^tA(x)Dv = \displaystyle\sum_{i\in I} \hat{\varphi}_i(-div\, \hat{g}_i) + \hat{f} \text{ in } \mathcal{D}'(\Omega), \\ \text{where } \hat{\varphi}_i \in H_0^1(\Omega) \cap L^\infty(\Omega), \hat{g}_i \in (L^2(\Omega))^N, \hat{f} \in L^1(\Omega), \\ \text{one has} \\ i)\ \displaystyle\int_\Omega F(x,u)\, v < +\infty, \\ ii)\ \displaystyle\sum_{i\in I}\int_\Omega \hat{g}_i D(\hat{\varphi}_i u) + \int_\Omega \hat{f} u = \langle\langle -div\, {}^tA(x)Dv, u\rangle\rangle_\Omega = \\ = \displaystyle\int_\Omega F(x,u)\, v. \end{cases}$$

Proof. Let us first prove that a function u which belongs to $L^\infty(\Omega)$ and which is a solution to problem (2.1) in the sense of Definition 2.9 satisfies (3.11) and (3.12).

Indeed choosing $k \geq \|u\|_{L^\infty(\Omega)}$, one has $T_k(u) = u$, so that (3.11 *iv*) follows from (2.17 *iv*). Moreover since $G_k(u) = 0$, (3.12 *ii*) follows from (2.18 *ii*).

Let us now prove the converse, namely that if a function u satisfies (3.11) and (3.12) then it satisfies (2.17) and (2.18).

This is straightforward as far as (2.17) is concerned, since for every $k > 0$ and for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, the equality $D(\varphi T_k(u)) = T_k(u) D\varphi + \varphi Du \chi_{\{u \leq k\}}$ in $(L^2_{\text{loc}}(\Omega))^N$ implies that $\varphi T_k(u) \in H^1(\Omega)$, and since the inequality $-k\varphi \leq \varphi T_k(u) \leq +k\varphi$ then implies, with the help of Lemma A.1 of [10], that $\varphi T_k(u) \in H_0^1(\Omega)$.

As far as (2.18 *ii*) is concerned, note that for every $k > 0$ and every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, one has $T_k(u) \in H^1_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ and $G_k(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$, with $\varphi T_k(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $\varphi G_k(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Writting in (3.12 *ii*)

$$\langle\langle -div\, {}^tA(x)Dv, u\rangle\rangle_\Omega = \langle\langle -div\, {}^tA(x)Dv, T_k(u)\rangle\rangle_\Omega + \langle\langle -div\, {}^tA(x)Dv, G_k(u)\rangle\rangle_\Omega,$$

which make sense in view of these properties of $T_k(u)$ and $G_k(u)$, and then using the fact that by (2.16) one has

$$\langle\langle -div\, {}^tA(x)Dv, G_k(u)\rangle\rangle_\Omega = \langle -div\, {}^tA(x)Dv, G_k(u)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)},$$

one immediately gets (2.18 *ii*). □

Remark 3.4. Still in the case where the solution u to problem (2.1) is bounded, assertion (3.11) of Proposition 3.3 can be replaced by assertion (3.11bis) given by

$$\text{(3.11bis)} \qquad \begin{cases} o)\; u \in L^\infty(\Omega), \\ i)\; u \in H^1_{\text{loc}}(\Omega), \\ ii)\; u(x) \geq 0 \text{ a.e. } x \in \Omega, \\ iv)\; \varphi u \in H_0^1(\Omega) \quad \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ v)\; \beta(u) \in H_0^1(\Omega); \end{cases}$$

note that in (3.11bis) there is no assertion *iii*), but the assertion *v*) which in some sense replaces it.

In other terms, Definition 2.9, (3.11) and (3.12), and (3.11bis) and (3.12) are all equivalent.

Indeed, if u satisfies (3.11) and (3.12), then by Proposition 3.3, u is a solution to problem (2.1) in the sense of Definition 2.9, and therefore, by Proposition 2.13, one has $\beta(u) \in H_0^1(\Omega)$, which implies that u satisfies (3.11bis).

Conversely, if u satisfies (3.11bis), it is easily seen by using the proof made in the first step of the proof of Proposition 3.2 above that u satisfies $G_k(u) \in H_0^1(\Omega)$ for every k such that $0 < k \leq \|u\|_{L^\infty(\Omega)}$, and therefore (3.11). □

3.4. A variant of Definition 2.9 using another space $\mathcal{W}(\Omega)$ of test functions

Definition 2.9 above makes use in (2.18) of test functions v which belong to the space $\mathcal{V}(\Omega)$. Actually, as far as the results of [10] are concerned, another definition of the solution could be used, where in (2.18 *i*) and (2.18 *ii*) above the space $\mathcal{V}(\Omega)$ of test functions is replaced by the space $\mathcal{W}(\Omega)$ of test functions defined by

$$\text{(3.13)} \qquad \begin{cases} \mathcal{W}(\Omega) = \{w : w = \displaystyle\sum_{i \in I} \varphi_i \psi_i \\ \qquad\qquad \text{with } I \text{ finite }, \varphi_i \in H_0^1(\Omega) \cap L^\infty(\Omega),\ \psi_i \in H_0^1(\Omega) \cap L^\infty(\Omega)\}. \end{cases}$$

This space $\mathcal{W}(\Omega)$ is a vector space, which in view of Remark 2.8 *i)* above is a subspace of $\mathcal{V}(\Omega)$. Moreover since $4\,\varphi_i\psi_i = (\varphi_i+\psi_i)^2 - (\varphi_i-\psi_i)^2$, the space $\mathcal{W}(\Omega)$ is generated by the functions $w = \varphi^2$, with $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, which are very simple to use.

It can be seen that all the results obtained in [10] (namely the existence of a solution, its stability with respect to the variation of the right-hand side, the a priori estimates obtained in Section 5 of [10], as well as the uniqueness of the solution when the function $F(x,s)$ is nonincreasing in s) can be obtained with this (new) definition using the space $\mathcal{W}(\Omega)$ in place of $\mathcal{V}(\Omega)$.

In [10], we have nevertheless chosen to present our results in the framework of the space $\mathcal{V}(\Omega)$, because it seems to us that the use of the smaller space $\mathcal{W}(\Omega)$ would not have allowed us to treat homogenization problems with many small holes, a problem that we have solved in our paper [11].

Of course a solution defined by using the space $\mathcal{V}(\Omega)$ is also a solution defined by using the smaller space $\mathcal{W}(\Omega)$. The converse is unclear to us, except in the case where one assumes that the function $F(x,s)$ is nonincreasing in s. Indeed in this case the uniqueness of both types of solutions and their approximation by problems $(2.1)_n$ (where the function $F(x,s)$ is replaced by $F_n(x,s) = T_n(F(x,s))$ easily allow one to prove that they coincide.

Let us now verify that all the results obtained in [10] can be proved using this new definition of solution based on the use in (2.18 *i*) and (2.18 *ii*) of the space $\mathcal{W}(\Omega)$ in place of $\mathcal{V}(\Omega)$. In order to do this, it is sufficient to show that the test functions used the proofs of in Propositions 5.1, 5.4, 5.9, 5.13, and 7.1 of [10] actually belong to $\mathcal{W}(\Omega)$ defined by (3.13).

- In Proposition 5.1 of [10] (a priori estimate of $G_k(u)$ in $H_0^1(\Omega)$) we have used the test function $w = S_{k,n}(u)$, where the test function $S_{n,k}$ is defined by

$$S_{k,n}(s) = \begin{cases} 0 & \text{if } 0 \le s \le k, \\ s-k & \text{if } k \le s \le n, \\ n-k & \text{if } n \le s, \end{cases} \tag{3.14}$$

  for every k and n with $0 < k < n$. This function w can be written as the product $w = \psi_k(u) S_{k,n}(u)$ where $\psi_k(s)$ is a nondecresing C^1 function with $\psi_k(s) = 0$ for $0 \le s \le \frac{k}{2}$, $\psi_k(s) = 1$ for $s \ge k$. It is then sufficient to note that both $\psi_k(u)$ and $S_{k,n}(u)$ belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$, a fact which follows from (2.17 *iii*), to prove that $w \in \mathcal{W}(\Omega)$.
- In Proposition 5.4 of [10] (a priori estimate of $\varphi D T_k(u)$ in $(L^2(\Omega))^N$) we have used the test function $w = \varphi^2 T_k(u) = \varphi\,\varphi T_k(u)$ and then the test function $w = \varphi^2$, where $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. These functions belong to $\mathcal{W}(\Omega)$, in particular since $\varphi T_k(u)$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ in view of (2.17 *iv*).
- In Proposition 5.9 of [10] (control of the integral $\displaystyle\int_{\{u \le \delta\}} F(x,u)\,v$) we have used the test function $Z_\delta(u) w$ with $w = \displaystyle\sum_{i\in I} \varphi_i \psi_i \in \mathcal{W}(\Omega)$. This function belongs to $\mathcal{W}(\Omega)$ since $Z_\delta(u) \in H^1(\Omega) \cap L^\infty(\Omega)$ (see (5.46) of [10]) while φ_i and w_i belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$.

- In Proposition 5.13 of [10] (a priori estimate of $\beta(u)$ in $H_0^1(\Omega)$) we have used the test function $w = \Gamma(S_{\delta,k}(u))$, where the function $\Gamma(s)$ appears in assumption (2.7) and where the function $S_{\delta,k}(s)$ is defined by (3.14). This function w can be written as the product $w = \psi_\delta(u)\Gamma(S_{\delta,k}(u))$, where $\psi_\delta(s)$ is a nondecreasing C^1 function with $\psi_\delta(s) = 0$ for $0 \le s \le \frac{\delta}{2}$, $\psi_\delta(s) = 1$ for $s \ge \delta$. It is then sufficient to note that both $\psi_\delta(u)$ and $\Gamma(S_{\delta,k}(u))$ belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$, a fact which follows from (2.17 *iii*), to prove that $w \in \mathcal{W}(\Omega)$.
- In Proposition 7.1 of [10] (Comparison Principle) we have used the test function $w = (B_1(T_k^+(u_1 - u_2)))^2$. In the function $\psi = B_1(T_k^+(u_1 - u_2))$, the functions u_1 and u_2 are solutions to problem (2.1) (for the functions $F_1(x,s)$ and $F_2(x,s)$) in the sense of Definition 2.9 (where in (2.18 *i*) and (2.18 *ii*) the space $\mathcal{V}(\Omega)$ of test functions is now replaced by the space $\mathcal{W}(\Omega)$), and the function B_1 is defined from the function $\Gamma_1(s)$ which appears in assumption (2.6) satisfied by the function $F_1(x,s)$. Since $\psi = B_1(T_k^+(u_1 - u_2))$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ (see (7.4) and (7.5) of [10]), the test function $w = \psi^2$ belongs to $\mathcal{W}(\Omega)$.

Remark 3.5. Another space $\mathcal{Y}(\Omega)$ of test functions, to be used in (2.18 *i*) and (2.18 *ii*) in place of the space $\mathcal{V}(\Omega)$ (or of the space $\mathcal{W}(\Omega)$), has recently been introduced in [3]. This space is defined by

$$\begin{cases} \mathcal{Y}(\Omega) = \{y \in H^1(\Omega) : \exists \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ \qquad\qquad \text{such that } |y| \le \varphi^2 \text{ a.e. in } \Omega \;\text{ and } \displaystyle\int_{\{y \neq 0\}} \frac{|Dy|^2}{\varphi^2} < +\infty\}, \end{cases} \tag{3.15}$$

where the integral $\displaystyle\int_{\{y\neq 0\}} \frac{|Dy|^2}{\varphi^2}$ is correctly defined since one has $y = 0$ on the set where $\varphi = 0$.

It can be proved (see [3] if necessary) that the space $\mathcal{Y}(\Omega)$ is a vectorial space. Since φ^2 belongs to $\mathcal{Y}(\Omega)$ when φ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$, one has $\mathcal{W}(\Omega) \subset \mathcal{Y}(\Omega)$. Therefore it is straightforward that if u satisfies (2.17), (2.18 *i*) and (2.18 *ii*) for test functions which belong to $\mathcal{Y}(\Omega)$, then u satisfies (2.17), (2.18 *i*) and (2.18 *ii*) for test functions which belong to $\mathcal{W}(\Omega)$.

It follows from [3] that the converse is true, and therefore that using $\mathcal{W}(\Omega)$ or $\mathcal{Y}(\Omega)$ as space of test functions in (2.18 *i*) and (2.18 *ii*) provides two definitions of the solution to problem (2.1) which are equivalent. □

4. ABOUT THE SET WHERE THE SOLUTION u TAKES THE VALUE ZERO

Every solution u to problem (2.1) in the sense of Definition 2.9 is a nonnegative function, which could in principle vanish on a set of (strictly) positive measure. We will prove in this Subsection that such is not the case.

A first observation in this direction is the following: the nonnegative measurable function $F(x,u(x))$ can take the value $+\infty$ when $u(x) = 0$. For every function v which is measurable and nonnegative, the integral $\displaystyle\int_\Omega F(x,u(x))\,v$ is then correctly defined as a number which belongs to $[0, +\infty]$. But assumption (2.18 *i*) on the solution u requires that this number is finite for every $v \in \mathcal{V}(\Omega)$, $v \geq 0$. This implies (see (2.19)) that $F(x, u(x)) \in L^1_{\mathrm{loc}}(\Omega)$. Therefore, when u is a solution to problem (2.1) in the sense of Definition 2.9, we have

$$F(x, u(x)) \text{ is finite a.e. } x \in \Omega, \tag{4.1}$$

which implies that

$$\operatorname{meas}\{x \in \Omega : u(x) = 0 \text{ and } F(x, 0) = +\infty\} = 0, \tag{4.2}$$

or equivalently that

$$\begin{cases} \{x \in \Omega : u(x) = 0\} \subset \{x \in \Omega : F(x, 0) < +\infty\} \\ \text{except on a set of zero measure.} \end{cases} \tag{4.3}$$

A result which is stronger than (4.2) is given in the following Proposition 4.1 (see (4.5)), and an even stronger result, due to L. Boccardo and L. Orsina [2], will be given in Proposition 4.3 below (note however that the latest result uses the strong maximum principle).

Proposition 4.1. *Assume that the matrix A and the function F satisfy* (2.5)*,* (2.6) *and* (2.7)*. Then every solution u to problem* (2.1) *in the sense of Definition 2.9 satisfies*

$$\operatorname{meas}\{x \in \Omega : u(x) = 0 \text{ and } 0 < F(x, 0) \leq +\infty\} = 0. \tag{4.4}$$

Remark 4.2. Assertion (4.4) is equivalent to

$$\begin{cases} \{x \in \Omega : u(x) = 0\} \subset \{x \in \Omega : F(x, 0) = 0\} \\ \text{except for a set of zero measure.} \end{cases} \tag{4.5}$$

This result is stronger than (4.2) and is equivalent to

$$\begin{cases} \{x \in \Omega : 0 < F(x, 0) \leq +\infty\} \subset \{x \in \Omega : u(x) > 0\} \\ \text{except for a set of zero measure.} \quad \square \end{cases} \tag{4.6}$$

Proof of Proposition 4.1.

Proposition 5.12 of [10] asserts that every u solution to problem (2.1) in the sense of Definition 2.9 satisfies

$$\int_{\{u=0\}} F(x, u)\, v = 0 \quad \forall v \in \mathcal{V}(\Omega), v \geq 0. \tag{4.7}$$

Writing $\Omega = \{u = 0\} \cup \{u > 0\}$ implies that assertion (4.7) is equivalent to

$$\int_\Omega F(x, u)\, v = \int_{\{u>0\}} F(x, u)\, v \quad \forall v \in \mathcal{V}(\Omega), v \geq 0. \tag{4.8}$$

On the other hand, writing

$$\{u = 0\} = \Big(\{u = 0\} \cap \{F(x, 0) = 0\}\Big) \cup \Big(\{u = 0\} \cap \{0 < F(x, 0) \leq +\infty\}\Big) \tag{4.9}$$

implies that (4.7) is equivalent to

$$\int_{\{u=0\} \cap \{0 < F(x,0) \leq +\infty\}} F(x, u)\, v = 0 \quad \forall v \in \mathcal{V}(\Omega), v \geq 0. \tag{4.10}$$

Since every $\phi \in \mathcal{D}(\Omega)$ belongs to $\mathcal{V}(\Omega)$ (see Remark 2.8 *iv*), assertion (4.10) is equivalent to (4.4). $\square$

The following Proposition asserts that every solution u to problem (2.1) in the sense of Definition 2.9 is actually greater than a (strictly) positive constant on every ball which is strictly included in Ω, except in the case where $u = 0$ in the whole of Ω. Together with the fact that the function $F(x, s)$ was assumed there to be increasing in s, this was a keypoint of the paper [2] by L. Boccardo and L. Orsina, which attracted our attention on this type of semilinear singular problems. This property of u is much stronger than (4.5), but its proof uses the strong maximum principle. Note however that there are situations different from (but close to) the present one where the strong maximum principle does not hold true, see the two counterexamples given in Subsection 5.3 below. This is the reason why we stated above the weaker result (4.5), whose proof does not use the strong maximum principle.

Note finally that Proposition 4.3 and Remark 4.4 below (and their analogues in our papers [8], [9], [10] and [11]) are the only points of the present paper (and of our other papers) where the strong maximum principle is used.

Proposition 4.3. ([2]). *Assume that the matrix A and the function F satisfy* (2.5)*,* (2.6) *and* (2.7)*. Then every solution u to problem* (2.1) *in the sense of Definition 2.9 satisfies*

$$\text{either} \quad \inf_B u > 0 \text{ for every ball } B \subset\subset \Omega, \quad \text{or} \quad u = 0 \text{ in } \Omega. \tag{4.11}$$

Proof. This result is due to L. Boccardo and L. Orsina [2], even if the notion of solution used by these authors is different of the notion of solution that we use. For the sake of completeness, we give here a detailed proof.

First step. In this step we recall the statement of the strong maximum principle as it can be found in the book [12] by D. Gilbarg and N. Trudinger, or more exactly the variant of Theorem 8.19 of [12] where u is replaced by $-u$. In this variant, Theorem 8.19 of [12] reads as

$$\begin{cases} \text{Let } u \in H^1(\Omega) \text{ which satisfies } Lu \le 0. \\ \text{If for some ball } B \subset\subset \Omega \text{ one has } \inf_B u = \inf_\Omega u \le 0, \\ \text{then } u \text{ is constant in } \Omega. \end{cases} \tag{4.12}$$

In the notation (8.1) and (8.2) of [12], one has $Lu = div\, A(x)Du$, and therefore $Lu \le 0$ is nothing but $-div\, A(x)Du \ge 0$. Therefore (4.12) implies that for any open bounded set $\omega \subset \mathbb{R}^N$ one has

$$\begin{cases} \text{Let } u \in H^1(\omega) \text{ with } -div\, A(x)Du \ge 0 \text{ in } \mathcal{D}'(\omega). \\ \text{If } u \ge 0 \text{ a.e. in } \omega \text{ and if } \inf_{B_0} u = 0 \text{ for some ball } B_0 \subset\subset \omega, \\ \text{then } u = 0 \text{ in } \omega, \end{cases} \tag{4.13}$$

where we have used that the fact that when u is a constant in ω with $\inf_{B_0} u = 0$, then $u = 0$ in ω.

Second step. Consider now u which is a solution to problem (2.1) in the sense of Definition 2.9. Then by Proposition 2.11 above, the function u satisfies

$$u \in H^1_{\text{loc}}(\Omega), \quad -div\, A(x)Du \ge 0 \text{ in } \mathcal{D}'(\Omega), \quad u \ge 0 \text{ a.e. in } \Omega. \tag{4.14}$$

Since $u \ge 0$ a.e. in Ω we have the alternative:

$$\begin{cases} \text{either } \inf_B u > 0 \text{ for every ball } B \subset\subset \Omega, \\ \text{or there exists a ball } B_0 \subset\subset \Omega \text{ such that } \inf_{B_0} u = 0. \end{cases}$$

In the second case, since u belongs only to $H^1_{\mathrm{loc}}(\Omega)$, we can consider any open set ω such that $B_0 \subset\subset \omega \subset\subset \Omega$. Then $u = 0$ in ω for every such ω, and therefore $u = 0$ in Ω. This proves (4.11). ☐

Remark 4.4. If $u = 0$ is a solution to problem (2.1) in the sense of Definition 2.9, then Proposition 2.11 implies that $F(x, 0) = 0$ for almost every $x \in \Omega$.

Conversely, if $F(x, 0) \not\equiv 0$, $u = 0$ is not a solution to problem (2.1) in the sense of Definition 2.9 and Proposition 4.3 then implies that $\inf_B u > 0$ for every ball $B \subset\subset \Omega$, and in particular that

$$u(x) > 0 \text{ a.e. } x \in \Omega. \quad \square \tag{4.15}$$

5. THE CASE WITH A ZEROTH ORDER TERM μu WITH $\mu \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$

In this Section we consider the case where problem (2.1) is replaced by

$$\begin{cases} u \geq 0 & \text{in } \Omega, \\ -div\, A(x)Du + \mu u = F(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

which now involves in its left-hand side the zeroth order term μu with

$$\mu \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega), \tag{5.2}$$

where, as said in Subsection 2.1 (Notation), $\mathcal{M}_b^+(\Omega)$ denotes the space of nonnegative bounded Radon measures on Ω, and we present the variations which should be made with respect to Section 2 in the context of problem (5.1).

Let us note that problem (5.1) naturally arises when performing the homogenization of problem (2.1) (where there is no zeroth order term) posed in a domain Ω^ϵ obtained from Ω by perforating Ω by many small holes with vanishing diameters, see our paper [11]; the appearance at the limit of the "strange term" μu in Ω is then the "memory" of the Dirichlet homogeneous boundary condition on $\partial\Omega^\epsilon$ which "tends to invade" the whole of Ω.

We begin the present Section by recalling in the brief Subsection 5.1 the variational framework which has to be used for the linear problem (5.3) below, namely problem (5.1) above in the case where $F(x,u) = f(x) \in L^2(\Omega)$. We then explain in Subsection 5.2 (see Definition 5.1 below) how the above Definition 2.9 of a solution to problem (2.1) has to be adapted to the case of problem (5.1) in view of the presence of the zeroth order term μu. In Subsection 5.2 we also state results of existence, stability and uniqueness of a solution to problem (5.1) in the sense of Definition 5.1 (see Theorems 5.3 to 5.5 and Remark 5.6). We then give explicitely (see Propositions and Remarks 5.7 to 5.17) a priori estimates which hold true for any solution to problem (5.1) in the sense of Definition 5.1; these a priori estimates are the analogues in this new setting of the estimates obtained in [10] for any solution to problem (2.1) in the sense of Definition 2.9. Finally, in Subsection 5.3, we first present a counterexample due to Gianni Dal Maso to the strong maximum principle for a linear problem with a zeroth order term μu involving a measure $\mu \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$, and then, by a variant of it, a counterexample to the strong maximum principle for the singular semilinear problem (5.1) with such a measure.

5.1. Recalling the variational framework for the linear problem with a zeroth order term μu with $\mu \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$

Let us recall here the weak formulation of the problem (5.1) in the case where $F(x,s) = f(x) \in L^2(\Omega)$, or in other terms the correct mathematical formulation of the problem of finding a function u which satisfies

$$\begin{cases} -div\, A(x)Du + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.3}$$

where μ satisfies (5.2) and f satisfies

$$f \in L^2(\Omega). \tag{5.4}$$

When $\nu \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$ and when $y \in H_0^1(\Omega)$, it is well known$^{(1)}$ (see e.g. [6] Section 1 and [7] Subsection 2.2 for more details) that y (or more exactly its quasi-continuous representative for the $H_0^1(\Omega)$ capacity) satisfies

$$\begin{cases} \forall \nu \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega),\ \forall y \in H_0^1(\Omega), \\ \text{one has } y \in L^1(\Omega; d\nu) \text{ with } \langle \nu, y \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \displaystyle\int_\Omega y\, d\nu\,; \end{cases} \tag{5.5}$$

moreover

$$\begin{cases} \forall \nu \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega),\ \forall y \in H^1_{\text{loc}}(\Omega) \cap L^\infty(\Omega), \\ \text{one has } y \in L^\infty(\Omega; d\nu) \text{ with } \|y\|_{L^\infty(\Omega;d\nu)} = \|y\|_{L^\infty(\Omega)}; \end{cases} \tag{5.6}$$

therefore when $y \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then y belongs to $L^1(\Omega; d\nu) \cap L^\infty(\Omega; d\nu)$ and therefore to $L^p(\Omega; d\nu)$ for every p, $1 \leq p \leq +\infty$.

Observing that $H_0^1(\Omega) \cap L^2(\Omega; d\mu)$ is an Hilbert space, the correct mathematical weak formulation of problem (5.3) is to find u such that

$$\begin{cases} u \in H_0^1(\Omega) \cap L^2(\Omega; d\mu), \\ \displaystyle\int_\Omega A(x)DuDv + \int_\Omega uv\, d\mu = \int_\Omega fv \quad \forall v \in H_0^1(\Omega) \cap L^2(\Omega; d\mu). \end{cases} \tag{5.7}$$

This problem has a unique solution by Lax-Milgram Lemma.

5.2. The adaptation of Definition 2.9 to the singular semilinear problem with a zeroth order term μu with $\mu \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$

We present here how Definition 2.9 should be adapted to the case of problem (5.1).

Definition 5.1. (Definition of a solution to (5.1)**) (Analogue of Definition 2.9 above).** Assume that the matrix A, the function F and the Radon measure μ satisfy (2.5), (2.6), (2.7) and (5.2). We say that u is a solution to problem (5.1) if u satisfies

$$\begin{cases} i)\ u \in L^2(\Omega) \cap H^1_{\text{loc}}(\Omega), \\ ii)\ u(x) \geq 0 \text{ a.e. } x \in \Omega, \\ iii)\ G_k(u) \in H_0^1(\Omega) \quad \forall k > 0, \\ iv)\ \varphi T_k(u) \in H_0^1(\Omega)\ \forall k > 0,\ \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{5.8}$$

---

(1) the reader who would not like to use these results can continue reading the present Section just assuming that μ is an $L^r(\Omega)$ function, $\mu \geq 0$, with r as in (2.7), and not only an element of $\mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$.

(5.9)

$$\begin{cases}\forall v \in \mathcal{V}(\Omega),\ v \geq 0,\\ \text{with } -div\, {}^tA(x)Dv = \displaystyle\sum_{i\in I} \hat{\varphi}_i(-div\, \hat{g}_i) + \hat{f} \text{ in } \mathcal{D}'(\Omega),\\ \text{where } \hat{\varphi}_i \in H_0^1(\Omega)\cap L^\infty(\Omega), \hat{g}_i \in (L^2(\Omega))^N, \hat{f}_i \in L^1(\Omega),\\ \text{one has}\\ i) \displaystyle\int_\Omega F(x,u)\, v < +\infty,\\ ii) \displaystyle\int_\Omega {}^tA(x)DvDG_k(u) + \sum_{i\in I}\int_\Omega \hat{g}_i D(\hat{\varphi}_i T_k(u)) + \int_\Omega \hat{f} T_k(u) + \int_\Omega uv\, d\mu =\\ = \langle -div\, {}^tA(x)Dv, G_k(u)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle\langle -div\, {}^tA(x)Dv, T_k(u)\rangle\rangle_\Omega + \displaystyle\int_\Omega uv\, d\mu =\\ = \displaystyle\int_\Omega F(x,u)\, v \quad \forall k > 0. \quad \square\end{cases}$$

Note that the only difference between Definition 5.1 and Definition 2.9 lies in the presence in Definition 5.1 of the measure μ, which appears only in the term $\int_\Omega uv\, d\mu$ in the two first lines of (5.9 *ii*). This term has a meaning, as shown by the following Remark.

Remark 5.2. (The integral $\int_\Omega uv\, d\mu$ has a meaning). Assumption (5.8) and $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$ actually imply that

$$uv \in L^1(\Omega; d\mu), \tag{5.10}$$

since one can write

$$uv = T_k(u)v + G_k(u)v,$$

where $T_k(u)v$ belongs to $H_0^1(\Omega)$ by (5.8 *iv*) and $G_k(u)$ belongs to $H_0^1(\Omega)$ by (5.8 *iii*). This implies that both $T_k(u)v$ and $G_k(u)$ belong to $L^1(\Omega; d\mu)$ in view of (5.5). Moreover, v, which belongs to $H_0^1(\Omega)\cap L^\infty(\Omega)$, belongs to $L^\infty(\Omega; d\mu)$ in view of (5.6). This proves (5.10) and gives a meaning to $\int_\Omega uv\, d\mu$.

Actually one can prove (see (5.16) in Remark 5.8 below) that every solution u to problem (5.1) in the sense of Definition 5.1 satisfies the regularity result $u \in L^2(\Omega; d\mu)$. Since $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$ also belongs to $L^2(\Omega; d\mu)$ in view of (5.5) and (5.6), this is another proof of (5.10). $\square$

With this Definition, all the results and proofs of [8], [9] and [10] continue to hold true once the necessary adaptations have been made, with the sole but notable exception of the results of Proposition 4.3 and Remark 4.4 above which are no more valid, since their proofs use the strong maximum principle, which in general does not hold true for the operator $-div\, A(x)Du + \mu u$, see the two counterexamples in Subsection 5.3 below.

Note also that Definition 5.1 is the definition of a solution to problem (5.1) that we use in our homogenization paper [11].

In this framework one has the following results of existence, stability and uniqueness for the solution to problem (5.1) in the sense of Definition 5.1; these results are extensions to problem (5.1) of the corresponding results obtained in [10] for problem (2.1).

Theorem 5.3. (Existence) (Analogue of Theorem 4.1 of [10]**).** *Assume that the matrix A, the function F and the measure μ satisfy* (2.5), (2.6), (2.7) *and* (5.2). *Then there exists at least one solution u to problem* (5.1) *in the sense of Definition 5.1.*

Theorem 5.4. (Stability) (Analogue of Theorem 4.2 of [10]**).** *Assume that the matrix A and the measure μ satisfy* (2.5) *and* (5.2). *Let F_n be a sequence of functions and F_∞ be a function which all satisfy assumptions* (2.6) *and* (2.7) *for the same h and the same Γ. Assume moreover that*

$$\text{a.e. } x \in \Omega,\ F_n(x, s_n) \to F_\infty(x, s_\infty) \ \text{ if } s_n \to s_\infty, s_n \geq 0, s_\infty \geq 0. \tag{5.11}$$

Let u_n be any solution to problem $(5.1)_n$ in the sense of Definition 5.1, where $(5.1)_n$ is the problem (5.1) *with $F(x,s)$ replaced by $F_n(x,s)$.*

Then there exists a subsequence, still labelled by n, and a function u_∞, which is a solution to problem $(5.1)_\infty$ in the sense of Definition 5.1, such that

$$\begin{cases} u_n \to u_\infty \text{in } L^2(\Omega) \text{ strongly, in } H^1_{\text{loc}}(\Omega) \text{ strongly and a.e. in } \Omega, \\ G_k(u_n) \to G_k(u_\infty) \ \text{ in } H^1_0(\Omega) \text{ strongly } \forall k > 0, \\ \varphi T_k(u_n) \to \varphi T_k(u_\infty) \text{ in } H^1_0(\Omega) \text{ strongly } \forall k > 0,\ \forall \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{5.12}$$

Theorem 5.5. (Uniqueness) (Analogue of Theorem 4.3 of [10]**).** *Assume that the matrix A, the function F and the measure μ satisfy* (2.5), (2.6), (2.7) *and* (5.2). *Assume moreover that the function $F(x,s)$ is nonincreasing with respect to s, i.e. satisfies assumption* (2.8). *Then the solution to problem* (5.1) *in the sense of Definition 5.1 is unique.*

Remark 5.6. (Well posedness of problem (5.1)**).** When assumptions (2.5), (2.6), (2.7), (5.2) as well as (2.8) hold true, Theorems 5.3, 5.4 and 5.5 together assert that problem (5.1) is well posed in the sense of Hadamard in the framework of Definition 5.1. □

Moreover, every solution to problem (5.1) in the sense of Definition 5.1 satisfies the following a priori estimates; these a priori estimates are extensions to the solutions to problem (5.1) of the corresponding results obtained in [10].

Proposition 5.7. (A priori estimate of $G_k(u)$ in $H^1_0(\Omega)$) (Analogue of Proposition 5.1 of [10]**).** *Assume that the matrix A, the function F and the measure μ satisfy* (2.5), (2.6), (2.7) *and* (5.2). *Then for every u solution to problem* (5.1) *in the sense of Definition 5.1, one has*

$$uG_k(u) \in L^1(\Omega; d\mu), \tag{5.13}$$

$$\|DG_k(u)\|^2_{(L^2(\Omega))^N} + \frac{2}{\alpha} \int_\Omega uG_k(u) d\mu \leq \frac{C_S^2}{\alpha^2} \frac{\|h\|^2_{L^r(\Omega)}}{\Gamma(k)^2} \quad \forall k > 0, \tag{5.14}$$

where C_S is the (generalized) Sobolev's constant defined in (2.11).

The proof of Proposition 5.7 is similar to the proof of Proposition 5.1 of [10]. It formally uses the test function $G_k(u)$, and correctly the test function $S_{k,n}(u)$, where the function $S_{k,n}$ is defined by (3.14) above.

Remark 5.8. (Regularity property $u \in L^2(\Omega; d\mu)$). Since

$$uG_k(u) = (T_k(u) + G_k(u))G_k(u) = kG_k(u) + |G_k(u)|^2,$$

and since $kG_k(u)$ belongs to $L^1(\Omega; d\mu)$ by (5.8 *iii*) and (5.5), assertion (5.13) implies that

$$G_k(u) \in L^2(\Omega; d\mu). \tag{5.15}$$

On the other hand, since $T_k(u)$ belongs to $L^\infty(\Omega; d\mu)$ by (5.8 *i*) and (5.6), and since μ belongs to $\mathcal{M}_b^+(\Omega)$, $T_k(u)$ also belongs to $L^2(\Omega; d\mu)$.

This implies that every solution u to problem (5.1) in the sense of Definition 5.1 satisfies the (regularity) property

$$u \in L^2(\Omega; d\mu). \tag{5.16}$$

Moreover one deduces from (5.14) the a priori estimate

$$\begin{cases} \displaystyle\int_\Omega u^2 d\mu = \int_\Omega (T_k(u) + G_k(u))^2 d\mu \leq 2\int_\Omega \left((T_k(u))^2 + (G_k(u))^2\right) d\mu \leq \\ \displaystyle\leq 2k^2\mu(\Omega) + 2\int_\Omega u\, G_k(u) d\mu \leq 2k^2\mu(\Omega) + \frac{C_S^2}{\alpha}\frac{\|h\|^2_{L^r(\Omega)}}{\Gamma(k)^2} \ \forall k > 0. \end{cases} \tag{5.17}$$

Taking in (5.17) $k = k_0$ for some $k_0 > 0$ fixed or minimizing its right-hand side in k provides an a priori estimate of $\|u\|^2_{L^2(\Omega, d\mu)}$ which does not depend on k. ☐

Remark 5.9. (A priori estimate of u in $L^2(\Omega)$) (Analogue of Remark 5.2 of [10]). Observe that by the same proof as in Remark 5.2 of [10] one deduces from (5.14) that every solution u to problem (5.1) in the sense of Definition 5.1 satisfies the following a priori estimate in $L^2(\Omega)$

$$\|u\|_{L^2(\Omega)} \leq k|\Omega|^{\frac{1}{2}} + C_P(\Omega)\frac{C_S}{\alpha}\frac{\|h\|_{L^r(\Omega)}}{\Gamma(k)} \ \forall k > 0, \tag{5.18}$$

where $C_p(\Omega)$ is the Poincaré's constant defined by

$$\|y\|_{L^2(\Omega)} \leq C_p(\Omega)\|Dy\|_{(L^2(\Omega))^N} \ \forall y \in H_0^1(\Omega). \tag{5.19}$$

Taking in (5.18) $k = k_0$ for some k_0 fixed or minimizing its right-hand side in k provides an a priori estimate of $\|u\|_{L^2(\Omega)}$ which does not depend on k. ☐

Proposition 5.10. (A priori estimate of $\varphi DT_k(u)$ in $(L^2(\Omega))^N$ for $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$) (Analogue of Propostion 5.4 of [10]). *Assume that the matrix A, the function F and the measure μ satisfy* (2.5), (2.6), (2.7) *and* (5.2). *Then for every u solution to problem* (5.1) *in the sense of Definition 5.1 one has*

(5.20)

$$\begin{cases} \displaystyle\|\varphi DT_k(u)\|^2_{(L^2(\Omega))^N} + \frac{2}{\alpha}\int_{\{u<k\}} u^2\varphi^2 d\mu \leq \\ \displaystyle\leq \frac{32k^2}{\alpha^2}\|A\|^2_{(L^\infty(\Omega))^{N\times N}}\|D\varphi\|^2_{(L^2(\Omega))^N} + \frac{C_S^2}{\alpha^2}\frac{\|h\|^2_{L^r(\Omega)}}{\Gamma(k)^2}\|\varphi\|^2_{L^\infty(\Omega)} + \frac{2k^2}{\alpha}\|\varphi\|^2_{L^2(\Omega;d\mu)} \\ \forall k > 0, \ \ \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases}$$

where C_S is the (generalized) Sobolev's constant defined in (2.11).

The proof of Proposition 5.10 is similar to the proof of Proposition 5.4 of [10]. Using the same test functions $\varphi^2 T_k(u)$ and φ^2, one indeed proves that

(5.21)

$$\begin{cases} \|\varphi D T_k(u)\|^2_{(L^2(\Omega))^N} + \dfrac{2}{\alpha}\displaystyle\int_\Omega u T_k(u)\varphi^2 d\mu \leq \\ \leq \dfrac{32k^2}{\alpha^2}\|A\|^2_{(L^\infty(\Omega))^{N\times N}}\|D\varphi\|^2_{(L^2(\Omega))^N} + \dfrac{C_S^2}{\alpha^2}\dfrac{\|h\|^2_{L^r(\Omega)}}{\Gamma(k)^2}\|\varphi\|^2_{L^\infty(\Omega)} + \dfrac{2}{\alpha}\displaystyle\int_\Omega uk\varphi^2 d\mu \\ \forall k > 0,\ \forall \varphi \in H_0^1(\Omega)\cap L^\infty(\Omega), \end{cases}$$

which immediately implies (5.20) by writing

$$\int_\Omega uk\varphi^2 d\mu = \int_{\{u<k\}} uk\varphi^2 d\mu + \int_{\{u\geq k\}} uk\varphi^2 d\mu \leq k^2\|\varphi\|^2_{L^2(\Omega;d\mu)} + \int_{\{u\geq k\}} uk\varphi^2 d\mu.$$

Remark 5.11. (A priori estimate of $\varphi T_k(u)$ in $H_0^1(\Omega)$) (Analogue of Remark 5.5 of [10]**).** From the a priori estimate (5.20) and from $D(\varphi T_k(u)) = \varphi D T_k(u) + T_k(u) D\varphi$, one deduces that every solution u to problem (5.1) in the sense of Definition 5.1 satisfies the following a priori estimate of $\varphi T_k(u)$ in $H_0^1(\Omega)$

(5.22)

$$\begin{cases} \|\varphi T_k(u)\|^2_{H_0^1(\Omega)} = \|D(\varphi T_k(u))\|^2_{(L^2(\Omega))^N} \leq \\ \leq 2\|\varphi D T_k(u)\|^2_{(L^2(\Omega))^N} + 2\|T_k(u) D\varphi\|^2_{(L^2(\Omega))^N} \leq \\ \leq \left(\dfrac{64k^2}{\alpha^2}\|A\|^2_{(L^\infty(\Omega))^{N\times N}} + 2k^2\right)\|D\varphi\|^2_{(L^2(\Omega))^N} + 2\dfrac{C_S^2}{\alpha^2}\dfrac{\|h\|^2_{L^r(\Omega)}}{\Gamma(k)^2}\|\varphi\|^2_{L^\infty(\Omega)} + \\ + \dfrac{4k^2}{\alpha}\|\varphi\|^2_{L^2(\Omega;d\mu)} \quad \forall k > 0,\ \forall \varphi \in H_0^1(\Omega)\cap L^\infty(\Omega). \quad \square \end{cases}$$

Remark 5.12. (A priori estimate of $\varphi D u$ in $(L^2(\Omega))^N$) (Analogue of Remark 5.6 of [10]**).** Adding the inequality (which immediately results from (5.14))

$$\|\varphi D G_k(u)\|^2_{(L^2(\Omega))^N} + \frac{2}{\alpha}\int_\Omega u G_k(u)\varphi^2 d\mu \leq \frac{C_S^2}{\alpha^2}\frac{\|h\|^2_{L^r(\Omega)}}{\Gamma(k)^2}\|\varphi\|^2_{L^\infty(\Omega)},$$

to (5.21) in which one writes $2uk \leq u^2 + k^2$, one obtains

(5.23)

$$\begin{cases} \|\varphi D u\|^2_{(L^2(\Omega))^N} + \dfrac{1}{\alpha}\displaystyle\int_\Omega u^2\varphi^2 d\mu \leq \\ \leq \dfrac{32k^2}{\alpha^2}\|A\|^2_{(L^\infty(\Omega))^{N\times N}}\|D\varphi\|^2_{(L^2(\Omega))^N} + 2\dfrac{C_S^2}{\alpha^2}\dfrac{\|h\|^2_{L^r(\Omega)}}{\Gamma(k)^2}\|\varphi\|^2_{L^\infty(\Omega)} + \dfrac{k^2}{\alpha}\|\varphi\|^2_{L^2(\Omega;d\mu)} \\ \forall k > 0,\ \forall \varphi \in H_0^1(\Omega)\cap L^\infty(\Omega). \end{cases}$$

Taking in (5.23) $k = k_0$ for some k_0 fixed or minimizing its right-hand side in k provides an a priori estimate of $\|\varphi D u\|_{(L^2(\Omega))^N}$ which does not depend on k. $\square$

Remark 5.13. (A priori estimate of u in $H^1_{\text{loc}}(\Omega)$) (Analogue of Remark 5.7 of [10]**).** Using the fact that for every $\phi \in \mathcal{D}(\Omega)$ one has $D(\phi u) = \phi D u + (T_k(u) + G_k(u)) D\phi$, which implies that $|D(\phi u)| \leq |\phi D u| + k|D\phi| + \|D\phi\|_{(L^\infty(\Omega))^N}|G_k(u)|$, and then using the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ and the a priori estimates (5.23) and (5.14) together with Poincaré's inequality (5.19), one deduces that every solution u to problem (5.1) in the sense of Definition 5.1 satisfies the following a priori estimate of $\|\phi u\|_{H_0^1(\Omega)}$, i.e. of $\|u\|_{H^1_{\mathrm{loc}}(\Omega)}$,

(5.24)

$$\begin{cases} \|\phi u\|^2_{H_0^1(\Omega)} + \dfrac{3}{\alpha}\displaystyle\int_\Omega u^2\phi^2 d\mu = \|D(\phi u)\|^2_{(L^2(\Omega))^N} + \dfrac{3}{\alpha}\displaystyle\int_\Omega u^2\phi^2 d\mu \leq \\ \leq 3\left(\dfrac{32k^2}{\alpha^2}\|A\|^2_{(L^\infty(\Omega))^{N\times N}}\|D\phi\|^2_{(L^2(\Omega))^N} + 2\dfrac{C_S^2}{\alpha^2}\dfrac{\|h\|^2_{L^r(\Omega)}}{\Gamma(k)^2}\|\phi\|^2_{L^\infty(\Omega)} + \right. \\ \left. + \dfrac{k^2}{\alpha}\|\phi\|^2_{L^2(\Omega;d\mu)} + k^2\|D\phi\|^2_{(L^2(\Omega))^N} + C_P^2(\Omega)\dfrac{C_S^2}{\alpha^2}\dfrac{\|h\|^2_{L^r(\Omega)}}{\Gamma(k)^2}\|D\phi\|^2_{(L^\infty(\Omega))^N}\right) \\ \forall k > 0,\ \ \forall \phi \in \mathcal{D}(\Omega). \end{cases}$$

Taking in (5.24) $k = k_0$ for some k_0 fixed or minimizing its right-hand side in k provides an a priori estimate of $\|\phi u\|^2_{H_0^1(\Omega)}$ for every fixed $\phi \in \mathcal{D}(\Omega)$, i.e. an a priori estimate of $\|u\|^2_{H^1_{\mathrm{loc}}(\Omega)}$, which does not depend on k. ☐

Proposition 5.14. (Control of the integral $\displaystyle\int_{\{u\leq\delta\}} F(x,u)v$) (Analogue of Proposition 5.9 of [10]**).** *Assume that the matrix A, the function F and the measure μ satisfy* (2.5)*,* (2.6)*,* (2.7) *and* (5.2)*. Then for every u solution to problem* (5.1) *in the sense of Definition 5.1 and for every v such that*

$$\begin{cases} v \in \mathcal{V}(\Omega),\ v \geq 0, \\ \text{with } -div\,{}^tA(x)Dv = \displaystyle\sum_{i\in I}\hat\varphi_i(-div\,\hat g_i) + \hat f \text{ in } \mathcal{D}'(\Omega), \\ \text{where } \hat\varphi_i \in H_0^1(\Omega)\cap L^\infty(\Omega), \hat g_i \in L^2(\Omega)^N, \hat f \in L^1(\Omega), \end{cases} \tag{5.25}$$

one has

$$\begin{cases} \forall\delta > 0,\ \displaystyle\int_\Omega F(x,u)Z_\delta(u)v \leq \\ \leq \dfrac{3}{2}\left(\displaystyle\int_\Omega\left|\sum_{i\in I}\hat g_i D\hat\varphi_i + \hat f\right|\right)\delta + \displaystyle\int_\Omega Z_\delta(u)\sum_{i\in I}\hat g_i Du\,\hat\varphi_i + \delta\int_\Omega v d\mu, \end{cases} \tag{5.26}$$

where for $\delta > 0$, the function $Z_\delta : s \in [0,+\infty[\to Z_\delta(s) \in [0,+\infty[$ is defined by

$$Z_\delta(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq \delta, \\ -\dfrac{s}{\delta} + 2 & \text{if } \delta \leq s \leq 2\delta, \\ 0 & \text{if } 2\delta \leq s. \end{cases} \tag{5.27}$$

The proof of Proposition 5.14 is similar to the proof of Proposition 5.9 of [10]: we use as test function $Z_\delta(u)v$ and since $0 \leq sZ_\delta(s) \leq \delta$ for every $s \geq 0$, we estimate the term $\displaystyle\int_\Omega uZ_\delta(u)\,v\,d\mu$ by $\displaystyle\int_\Omega uZ_\delta(u)\,v\,d\mu \leq \delta\int_\Omega v\,d\mu$.

Remark 5.15. Since $Z_\delta(s) \geq \chi_{\{s\leq\delta\}}(s)$ for every $s \geq 0$, estimate (5.26) provides an estimate of the integral $\displaystyle\int_{\{u\leq\delta\}} F(x,u)v$ as announced in the title of Proposition 5.14. ☐

As a consequence of Proposition 5.14 we have:

Proposition 5.16. **($F(x,0)=0$ in $\{u=0\}$) (Analogue of Proposition 5.12 of** [10]**).** *Assume that the matrix A, the function F and the measure μ satisfy* (2.5)*,* (2.6)*,* (2.7) *and* (5.2)*. Then for every u solution to problem* (5.1) *in the sense of Definition 5.1 one has*

$$\int_{\{u=0\}} F(x,u)v = 0 \quad \forall v \in \mathcal{V}(\Omega), v \geq 0, \tag{5.28}$$

and

$$F(x,0) = 0 \text{ a.e. in } \{x \in \Omega : u(x) = 0\}. \tag{5.29}$$

The following a priori estimate is actually in some sense a regularity result, since it asserts that for every u solution to problem (5.1) in the sense of Definition 5.1, a certain function $\beta(u)$ actually belongs to $H_0^1(\Omega)$.

Proposition 5.17. **(A priori estimate of $\beta(u)$ in $H_0^1(\Omega)$) (Analogue of Proposition 5.13 of** [10]**).** *Assume that the matrix A, the function F and the measure μ satisfy* (2.5)*,* (2.6)*,* (2.7) *and* (5.2)*. Define the function $\beta : s \in [0,+\infty[\to \beta(s) \in [0,+\infty[$ by*

$$\beta(s) = \int_0^s \sqrt{\Gamma'(t)}dt, \tag{5.30}$$

where Γ is the function which appears in assumption (2.7)*. Then for every u solution to problem* (5.1) *in the sense of Definition 5.1 one has*

$$\beta(u) \in H_0^1(\Omega), \tag{5.31}$$

as well as

$$u\Gamma(u) \in L^1(\Omega; d\mu), \tag{5.32}$$

with the a priori estimate

$$\alpha \|D\beta(u)\|^2_{(L^2(\Omega))^N} + \int_\Omega u\Gamma(u)d\mu \leq \|h\|_{L^1(\Omega)}. \tag{5.33}$$

The proof of Proposition 5.17 is similar to the proof of Proposition 5.13 of [10]. It formally uses the test function $\Gamma(u)$, and correctly the test function $\Gamma(S_{\delta,k}(u))$, where the function $S_{\delta,k}$ is defined by (3.14) above.

Remark 5.18. Of course, due to the zeroth order term μu, many other small adaptations have to be made here or there, in particular in the proofs. As a single example, let us just mention that in the second step of the proof of the equivalence result of Proposition 3.1 above, one has first to use an approximation of the test function $\varphi \in H_0^1(\Omega)$ by functions which belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$, e.g. by $\varphi_m = T_m(\varphi)$, and then to approximate these functions φ_m by functions $v_n = \inf(\phi_n^+, \varphi_m)$ where $\phi_n \in \mathcal{D}(\Omega)$ tends to φ_m in $H_0^1(\Omega)$ strongly. □

5.3. Two counterexamples to the strong maximum principle for problems with a zeroth order term μu with $\mu \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$

In this Subsection we first present a counterexample which shows that the strong maximum principle (namely (4.11) and (4.15) above) in general does not hold true for the solutions to the linear problem (5.3) with non homogeneous boundary condition (or, in order to be mathematically correct, to the non homogeneous problem (5.7)) when the operator involves a zeroth order term μu with $\mu \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$

when $N \geq 3$. This counterexample was communicated to us by Gianni Dal Maso, to whom we express our warmest thanks. At the end of this Subsection we then give a second counterexample (inspired by the previous one) to the strong maximum principle in the case of the semilinear problem (5.1) itself.

A counterexample to the strong maximum principle for the non homogeneous linear problem (5.3) with a zeroth order term μu with $\mu \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$

Let Ω be the unit ball

$$\Omega = \{x \in \mathbb{R}^N : |x| < 1\}, \quad N \geq 1, \tag{5.34}$$

and let μ be the (radial) function defined by

$$\mu(|x|) = \frac{2N}{|x|^2} \quad \forall x \in \Omega. \tag{5.35}$$

Consider the problem

$$\begin{cases} -\Delta u + \mu(|x|)u = 0 & \text{in } \Omega, \\ u = 1 & \text{on } \partial\Omega, \end{cases} \tag{5.36}$$

or, in a mathematically correct sense, its weak formulation (cf. (5.7) above)

$$\begin{cases} u \in H^1(\Omega) \cap L^2(\Omega; \mu(|x|)dx), \ u - 1 \in H_0^1(\Omega), \\ \displaystyle\int_\Omega DuDv + \int_\Omega uv\, \mu(|x|)dx = 0 \quad \forall v \in H_0^1(\Omega) \cap L^2(\Omega; \mu(|x|)dx). \end{cases} \tag{5.37}$$

It is easy to check that for $N \geq 1$ the (radial) function u defined by

$$u(x) = |x|^2 \quad \forall x \in \Omega \tag{5.38}$$

is the unique solution to (5.37), and that this solution satisfies $u \geq 0$ in Ω.

Since $u(0) = 0$, and since u does not coincide with 0 in Ω, the strong maximum principle does not hold true for problem (5.36) when $N \geq 1$: indeed the analogue of (4.13) above for the operator $-\Delta + \mu(|x|)$ does not hold true.

Observe moreover that when $N \geq 3$, the function μ satisfies

$$\mu \in L^{(2^*)'}(\Omega) \subset L^1(\Omega), \tag{5.39}$$

since $(2^*)' = 2N/(N+2)$ and since

$$\int_0^1 \left(\frac{2N}{\rho^2}\right)^{\frac{2N}{N+2}} \rho^{N-1} d\rho < +\infty \text{ when } N \geq 3;$$

therefore when $N \geq 3$ one has

$$\mu \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega). \tag{5.40}$$

Note that (5.39) does not hold true when $N = 1$ and $N = 2$.

Therefore the strong maximum principle (namely (4.11) and (4.15) above) in general does not hold true for a solution to the linear problem (5.7) with non homogeneous boundary condition with $\mu \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$ when $N \geq 3$.

Remark 5.19. Actually, the situation described in the explicit counterexample given by (5.34), (5.35), (5.37) and (5.38) is not an isolated case. Indeed G. Dal Maso and U. Mosco proved (see Theorem 5.1 of [5]) that when $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ is any open set with $0 \in \Omega$, and when $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is any (radial) function such that

$$\mu \in L^1_{\mathrm{loc}}([0, +\infty[), \ \mu \geq 0, \tag{5.41}$$

then any local weak solution u to problem (5.36), i.e. any u such that

$$\begin{cases} u \in H^1_{\mathrm{loc}}(\Omega) \cap L^2_{\mathrm{loc}}(\Omega; \mu(|x|)dx), \\ \displaystyle\int_\Omega DuDv + \int_\Omega uv\, \mu(|x|)dx = 0 \ \forall v \in H^1(\Omega) \cap L^2(\Omega; \mu(|x|)dx), \ \operatorname{supp} v \subset\subset \Omega, \end{cases} \tag{5.42}$$

is continuous at $x = 0$; they moreover proved that when

$$\int_0 \rho\, \mu(\rho) d\rho = +\infty, \tag{5.43}$$

then $u(0) = 0$.

Since there is a large set of functions which satisfy both (5.41) and (5.43), this provides a large set of counterexamples to the strong maximum principle for problems of the type (5.3) with a radial singular measure μ. In particular the strong maximum principle does not hold true for the radial measures having the singularity $C/|x|^\lambda$ with $C > 0$ and $\lambda \geq 2$ when $N \geq 3$ (note that $C/|x|^\lambda$ satisfies (5.41) and (5.43) if and only if $\lambda \geq 2$). (Note also that, as far as hypothesis (5.2) is concerned, when $N \geq 3$, the function $C/|x|^\lambda$ belongs to $L^{\frac{2N}{N+2}}(\Omega) \subset H^{-1}(\Omega) \cap L^1(\Omega)$ if and only if $\lambda < (N+2)/2$, and that $2 < (N+2)/2$ when $N \geq 3$.)

In contrast, it was recently proved by L. Orsina and A. Ponce in [14], where the above counterexample (5.34), (5.35), (5.37) and (5.38) is also presented, that the strong maximum principle holds true for the operator $-\Delta u + \mu u$, with μ non necessarily radial, when $\mu \in L^p(\Omega)$ with $p > N/2$ (see the comments after Theorem 1 of [14]). In particular the strong maximum principle holds true for the radial measures having the singularity $C/|x|^\lambda$, $C > 0$, $\lambda < 2$. □

A counterexample to the strong maximum principle for the singular semilinear problem (5.1) with a zeroth order term μu with $\mu \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$

Let us finish this Subsection (and the present paper) by a counterexample which proves that the strong maximum principle still fails in the case of the singular semilinear problem (5.1) itself. This counterexample is a variant of the above counterexample (5.34), (5.35), (5.37) and (5.38).

Remark 5.20. Let us explicitely note that the counterexample that we will give below continues to hold (taking $\overline{f} = 0$) in the case where $f \equiv 0$, or in other words for the linear problem (5.3) with a measure $\mu \in \mathcal{M}_b^+(\Omega) \cap H^{-1}(\Omega)$ and homogeneous Dirichlet boundary condition on $\partial\Omega$. □

Let $R > 0$ and let Ω be the ball

$$\Omega = \{x \in \mathbb{R}^N : |x| < R\}, \quad N \geq 3, \tag{5.44}$$

and let γ be any exponent with

$$\gamma > 0. \tag{5.45}$$

We will give (see (5.57) and (5.58) below) explicit radial functions μ, f, g and u which satisfy

$$\begin{cases} \mu \in L^p(\Omega) \ \forall p < \dfrac{N}{2}, \ \ \mu \geq 0 \text{ in } \Omega, \\ f \in L^\infty(\Omega), \ \ f \geq 0 \text{ in } \Omega, \\ g \in L^\infty(\Omega), \ \ g \geq 0 \text{ in } \Omega, \end{cases} \tag{5.46}$$

$$u \in C^1(\overline{\Omega}), \ \ u = 0 \text{ on } \partial\Omega, \tag{5.47}$$

$$u(|x|) > 0 \ \forall x \in \Omega \setminus \{0\}, \ \ u(0) = 0, \tag{5.48}$$

$$-\Delta u + \mu(|x|)u = \frac{f(|x|)}{u^\gamma} + g(|x|) \text{ in } \mathcal{D}'(\Omega). \tag{5.49}$$

Note that since $(2^*)' = 2N/(N+2)$ and since $N/2 > 2N/(N+2)$, the function μ satisfies

$$\mu \in L^{(2^*)'}(\Omega) \subset H^{-1}(\Omega) \cap L^1(\Omega), \tag{5.50}$$

and that the radial functions f, g and u will actually be piecewise smooth functions in Ω (this is not the case for μ in a neighborhood of the origin, since μ takes the value $+\infty$ at this point).

Note also that in view of the facts that u belongs to $H_0^1(\Omega)$ and that $f(|x|)/u^\gamma$ belongs to $L^\infty(\Omega)$ (see (5.57) below), the function u will be a solution to problem (5.1) in the sense of Definition 5.1 (and also in the classical sense), and that u will therefore be the unique solution to this problem since the function $F(x,s) = f(|x|)/s^\gamma$ is non increasing in s.

In view of (5.48) and (5.49), this will provide a counterexample to the strong maximum principle (namely to (4.11) and to (4.15) above) in the case of problem (5.1) with

$$F(x,s) = \frac{f(x)}{s^\gamma} + g(x),$$

for every $\gamma > 0$.

In order to define the functions μ, f, g and u we fix a few notation. We choose a constant θ such that

$$0 < \theta < 1, \tag{5.51}$$

and we define the constant $m = m(N,\theta)$ by

$$m = m(N,\theta) = (N-2) + 2\theta^N - N\theta^2; \tag{5.52}$$

note that for $0 \leq \theta \leq 1$ the function $m(N,\theta)$ is decreasing with respect to θ and satisfies $m(N,1) = 0$; therefore one has

$$m > 0 \ \ \forall \theta, \ \ 0 \leq \theta < 1. \tag{5.53}$$

We then define three constants a, b and c by

$$a = \frac{N-2}{m} - 1, \ \ b = \frac{2}{m}(\theta R)^N, \ \ c = \frac{N}{m}(\theta R)^2. \tag{5.54}$$

It is easy to see that a, b and c solve the following system of 3 linear equations with 3 unknowns

$$\begin{cases} (a+1) + \dfrac{b}{(\theta R)^N} - \dfrac{c}{(\theta R)^2} = 0, \\ 2(a+1) = (N-2)\dfrac{b}{(\theta R)^N}, \\ (a+1) + \theta^N \dfrac{b}{(\theta R)^N} - \theta^2 \dfrac{c}{(\theta R)^2} = 1; \end{cases} \tag{5.55}$$

indeed the two first equations are satisfied for every $m \neq 0$, while the third one is nothing but the definition (5.52) of $m = m(N,\theta)$. Moreover, since $\theta^{N-2} < N/2$ when $0 < \theta < 1$, it immediately follows from the definition (5.52) of m and from (5.53) that

$$a > 0. \tag{5.56}$$

We then define μ, f, g and u as the (radial) functions defined by

$$\begin{cases} \mu(|x|) = \dfrac{\overline{\mu}}{|x|^2}\, \chi_{\{|x|<\theta R\}}(|x|), \\ f(|x|) = \overline{f}\, |x|^{2\gamma}\, \chi_{\{|x|<\theta R\}}(|x|), \\ g(|x|) = (-2N + \overline{\mu} - \overline{f})\, \chi_{\{|x|<\theta R\}}(|x|) + 2aN\, \chi_{\{\theta R<|x|<R\}}(|x|), \\ u(|x|) = |x|^2\, \chi_{\{|x|<\theta R\}}(|x|) + (-a|x|^2 - \dfrac{b}{|x|^{N-2}} + c)\, \chi_{\{\theta R<|x|\le R\}}(|x|), \end{cases} \tag{5.57}$$

where the constants $\overline{\mu}$ and $\overline{f}$ satisfy

$$\overline{\mu} \geq 0, \;\; \overline{f} \geq 0, \;\; -2N + \overline{\mu} - \overline{f} \geq 0. \tag{5.58}$$

Using (5.58) and (5.56), it is straightforward to verify that μ, f and g satisfy (5.46) and therefore (5.50). As far as u is concerned, it is clear that u satisfies (5.47) if both $u(\rho)$ and $\dfrac{du}{d\rho}(\rho)$ are continuous at $\rho = \theta R$ and if $u(R) = 0$, which in view of

$$\frac{du}{d\rho}(\rho) = -2a\rho + (N-2)\frac{b}{\rho^{N-1}} \;\; \forall \theta, \;\; \theta R < \rho < R,$$

is nothing but

$$\begin{cases} (\theta R)^2 = -a(\theta R)^2 - \dfrac{b}{(\theta R)^{N-2}} + c, \\ 2\theta R = -2a\theta R + (N-2)\dfrac{b}{(\theta R)^{N-1}}, \\ -aR^2 - \dfrac{b}{R^{N-2}} + c = 0, \end{cases}$$

a system of 3 linear equations which is equivalent to (5.55), the solution of which, as said above, is given by the definition (5.54) of a, b and c. Since u belongs to $C^1(\overline{\Omega})$, computing $-\Delta u$ in Ω does not produce any Dirac mass at the interface $|x| = \theta R$, and a standard computation in $\{x : |x| < \theta R\}$ and in $\{x : \theta R < |x| < R\}$ proves that u satisfies (5.49). Finally, since

$$\frac{d^2u}{d\rho^2}(\rho) = -2a - \frac{(N-1)(N-2)b}{\rho^N} \;\; \forall \theta, \;\; \theta R < \rho < R,$$

the function $u(\rho)$ is a smooth concave function in $\{\rho : \theta R < \rho < R\}$ with $u(\theta R) > 0$, $\dfrac{du}{d\rho}(\theta R) > 0$ and $u(R) = 0$. Therefore one has $u(\rho) > 0$ for $\theta R < \rho < R$, and (5.48) is proved. The proof is complete.

This counterexample is of course susceptible of (and robust with respect to) many variations. In particular the measure μ given by (5.57) can be replaced by the measure $\hat{\mu}$ given by

$$\hat{\mu}(|x|) = \frac{\overline{\mu}}{|x|^2}\, \chi_\Omega(|x|) = \mu(|x|) + \frac{\overline{\mu}}{|x|^2}\, \chi_{\{\theta R < |x| < R\}}(|x|)$$

if the function g given by (5.57) is replaced by the function $\hat{g}$ given by

$$\hat{g}(|x|) = g(|x|) + \frac{\overline{\mu}}{|x|^2}(-a|x|^2 - \frac{b}{|x|^{N-2}} + c)\, \chi_{\{\theta R < |x| < R\}}(|x|).$$

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