A relaxation result for state constrained delay differential inclusions

Hélène Frankowska, Ihab Haidar

A relaxation result for state constrained delay differential inclusions

H. FRANKOWSKA[1] AND I. HAIDAR[2]

[1] CNRS, IMJ-PRG, Sorbonne Universités, UPMC Université Paris 6, Case 247, 4 Place Jussieu, 75252 Paris, France

[2] Sorbonne Universités, IMJ-PRG, UPMC Université Paris 6, 4 Place Jussieu, 75252 Paris, France

Abstract

In this paper, we consider a delay differential inclusion $\dot{x} \in F(t, x_t)$, where x_t denotes the history function of x along an interval of time. We extend the celebrated Filippov's theorem to this case. Then, we further generalize this theorem to the case when the state variable x is constrained to the closure of an open subset $K \subset \mathbb{R}^n$. Under a new "inward pointing condition", we give a relaxation result stating that the set of trajectories lying in the interior of the state constraint is dense in the set of constrained trajectories of the convexified inclusion $\dot{x} \in \operatorname{co} F(t, x_t)$.

Index Terms

Delay differential inclusions, relaxation, state constraints, inward pointing conditions.

I. INTRODUCTION

Mathematical models arising in population dynamics or engineering sciences often involve control systems with delays (see, e.g., [5], [23]). Systems with delays, express that at each

This work was partially funded by the DeMagma project of Programme Convergence from Idex Super at Sorbonnes Universités.

instant the velocity of the state depends upon the history of its evolution up to this instant [22]. Such control systems can be described as follows:

$$\begin{cases} \dot{x}(t) &= f(t, x_t, u(t)), \qquad a.e.\ t \in [t_0, T], \\ u(t) &\in U \subset \mathbb{R}^q, \qquad a.e.\ t \in [t_0, T], \\ x_{t_0} &= \varphi, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, represents the state at time t, $x_t : [-\tau, 0] \to \mathbb{R}^n$ is the standard notation for the history function defined by $x_t(\theta) = x(t + \theta)$, for $\tau > 0$ and $-\tau \leq \theta \leq 0$, $u(\cdot)$ is a Lebesgue measurable function, f is a mapping from $[0, T] \times \mathcal{C}([-\tau, 0], \mathbb{R}^n) \times U$ into $\mathbb{R}^n$, $0 \leq t_0 \leq T$, and φ is the initial condition taken in $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$. In the above, $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ denotes the Banach space of continuous functions from $[-\tau, 0]$ into $\mathbb{R}^n$, with the usual norm.

When the trajectories of (1) are subject to the state constraint

$$x(t) \in K \qquad \forall\, t \in [t_0, T], \tag{2}$$

where K is a closed subset of $\mathbb{R}^n$, the viability theory [1] provides adequate mathematical tools to study the existence of *feasible* (or *viable*) solutions of such systems. Thanks to this theory, a necessary and sufficient condition (linking the dynamical properties of system (1) to the geometry of the constraint set K) for the existence of feasible solutions is known. Under some regularity assumptions on f, this condition was first given in [20]:

$$\begin{gathered} \forall\, t \in [0, T], \forall\, \psi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n) \text{ such that } \psi(0) \in K, \\ f(t, \psi, U) \cap T_K(\psi(0)) \neq \emptyset, \end{gathered} \tag{3}$$

where $T_K(\psi(0))$ is the contingent cone to K at $\psi(0)$. Nevertheless, in the framework of this theory, convexity conditions are imposed on the set-valued map $F(t, \psi) := f(t, \psi, U)$, i.e. for every $t \in [0, T]$ and every $\psi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, $F(t, \psi)$ is a convex subset of $\mathbb{R}^n$. This convexity hypothesis may fail in some mathematical models and may be even difficult to verify.

In the case of delay-free control systems, a vast literature [7], [8], [9], [12], [14], [18], [19] allows to relax this convexity hypothesis, by assuming, as a counterpart, stronger tangential conditions and stronger regularity of F. These conditions rely on the possibility of directing the velocity into the interior of the constraint K whenever approaching the boundary of K. Known as *inward pointing conditions*, they allow to approximate relaxed feasible trajectories by feasible trajectories [14], [16] and provide estimates on the distance of a given trajectory of unconstrained control system from the set of feasible trajectories, see for instance [8], [18], [16].

In the literature, these estimates have been referred to as neighboring feasible trajectory (NFT) estimates. In the case when F is Lebesgue measurable with respect to the time and Lipschitz with respect to the state, NFT estimates result from the following inward pointing condition [15], [16]:

$$\begin{cases} \forall\, t \in [0,T], \forall\, x \in \partial K, \forall\, v \in F(t,x) \\ \qquad \text{such that } \max_{n \in N_K^1(x)} \langle n, v \rangle \geq 0, \\ \quad \exists\, w \in \text{Liminf}_{(s,y)\to(t,x)} \text{co}\, F(s,y) \\ \quad \text{satisfying } \max_{n \in N_K^1(x)} \langle n, w - v \rangle < 0, \end{cases} \tag{4}$$

where $\text{co}\, F(s,y)$ is the convex hull of $F(s,y)$, Liminf denotes the Kuratowski lower set limit (see [2]), $N_K^1(x) := N_K(x) \cap S^{n-1}$, S^{n-1} is the unit sphere and $N_K(x)$ denotes the Clarke normal cone to K at x (see [10]). The above condition takes sometimes a simpler form depending on the regularity assumptions on F and the smoothness of the boundary ∂K (see, e.g., [7], [8], [9], [16], [19]).

When the viability condition fails to be fulfilled on the boundary of K, the largest subset of initial conditions starting from which at least one viable solution exists (called *viability kernel*) is considered. In the case of delay-free control systems, viability algorithms providing constructive methods for the computation of the viability kernel, have been developed (see, e.g., [17], [27]). Thanks to these algorithms, efficient numerical methods have been established (see, e.g., [26]) and used in order to exhibit approximating viability kernels for numerous examples coming from different fields (see, e.g., [3], [6], [28], [24], [21], [25]). Two steps are needed to extended these numerical methods to delay differential inclusions: adapt the viability algorithms to this case and obtain relaxation theorems under state constraints. This latter point is the purpose of this paper.

To our knowledge, NFT estimates for delay differential inclusions are not yet obtained in the literature. Here, we propose to extend such results to this case. Inspired by the viability condition given by (3), we propose to adapt the inward pointing condition (4) to delay differential inclusions.

Let $\lambda > 0$. Define the set

$$\mathcal{K}_\lambda := \{\psi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n) : \psi \text{ is } \lambda\text{-Lipschitz and } \psi(0) \in \partial K\}\,, \tag{5}$$

and consider the following relaxed inward pointing condition:

$$(IP_{rel}^{\lambda}) \begin{cases} \forall\, t \in [0,T], \forall\, \psi \in \mathcal{K}_\lambda, \forall\, v \in F(t,\psi) \\ \text{such that } \max\limits_{n \in N_K^1(\psi(0))} \langle n, v\rangle \geq 0, \\ \exists\, w \in \text{Liminf}_{(s,\phi)\to(t,\psi)} \text{co}\, F(s,\phi) \\ \text{satisfying } \max\limits_{n \in N_K^1(\psi(0))} \langle n, w-v\rangle < 0. \end{cases}$$

Assuming (IP_{rel}^{λ}), we give a relaxation result stating that the set of feasible trajectories is dense in the set of relaxed feasible ones. This is proved by using several preliminary results. The first one is an extension of the Filippov theorem [11] to delay differential inclusions, which is an essential step to construct feasible trajectories. Then, we provide NFT estimates on the distance of a given trajectory from the set of feasible trajectories.

The paper is organized as follows. Section II presents the list of notations, definitions and assumptions in use. In Section III we state our main results. The proofs and useful technical tools are postponed to Section IV.

II. Preliminaries

In this section we list the notations and the main assumptions in use.

A. *Notations and definitions*

Consider the Euclidean space $(\mathbb{R}^n, \|\cdot\|)$, where n is a positive integer. We denote by $\langle\cdot,\cdot\rangle$ the inner product, by $B(x,r)$ the closed ball of center $x \in \mathbb{R}^n$ and radius $r > 0$ and by B the closed unit ball in $\mathbb{R}^n$ centered at 0. Let $\text{co}\, A$ stands for the convex hull of a subset $A \subset \mathbb{R}^n$. For every pair $(a,b) \in \mathbb{R}^2$, set $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$.

Given $I \subset \mathbb{R}$, $(\mathcal{C}(I,\mathbb{R}^n), \|\cdot\|_C)$ denotes the Banach space of continuous functions from I into $\mathbb{R}^n$, where $\|\cdot\|_C$ is the norm of uniform convergence. Given $\tau > 0$, $B_C(\varphi, r)$ denotes the closed ball of center $\varphi \in \mathcal{C}([-\tau,0],\mathbb{R}^n)$ and radius $r > 0$ and B_C is the closed unit ball in $\mathcal{C}([-\tau,0],\mathbb{R}^n)$ centered at 0. Given $t \in \mathbb{R}$, we denote by $B((t,\varphi),r)$ the closed ball $B(t,r) \times B_C(\varphi,r)$.

We denote by μ the Lebesgue measure on the real line, and by $\mathcal{L}^1(I,\mathbb{R}^n)$ the space of Lebesgue integrable functions from I to $\mathbb{R}^n$.

Let K be a nonempty closed subset of $\mathbb{R}^n$, $\operatorname{Int} K$ be its interior and ∂K its boundary, $\tilde{d}_K$ is the oriented distance from $x \in \mathbb{R}^n$ to K defined by

$$\tilde{d}_K(x) = \begin{cases} d_K(x) & \text{if } x \notin K \\ -d_{\mathbb{R}^n \setminus K}(x) & \text{otherwise,} \end{cases}$$

where $d_K(x) = \inf_{y \in K} \|x - y\|$.

We will use the following notion of solution:

Definition 1: Let $0 \leq t_0 \leq T$, $\tau > 0$ and $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$. A function $x \in \mathcal{C}([t_0 - \tau, T], \mathbb{R}^n)$ is called an F-trajectory, if $x(\cdot)$ is absolutely continuous on $[t_0, T]$ and

$$\dot{x}(t) \in F(t, x_t) \qquad a.e.\ t \in [t_0, T], \tag{6}$$

$$x_{t_0} = \varphi. \tag{7}$$

An F-trajectory which verifies the state constraint (2) is called feasible F-trajectory. A trajectory associated to the relaxed differential inclusion

$$\begin{cases} \dot{x}(t) \in \operatorname{co} F(t, x_t), & a.e.\ t \in [t_0, T], \\ x_{t_0} = \varphi \end{cases} \tag{8}$$

is called relaxed F-trajectory, and relaxed feasible F-trajectory if in addition (2) holds true.

B. Assumptions

Let $0 \leq t_0 \leq T$, $\tau > 0$ and $F : [t_0, T] \times \mathcal{C}([-\tau, 0], \mathbb{R}^n) \rightsquigarrow \mathbb{R}^n$ be a set-valued map with nonempty closed images. In our main theorems, we will assume the following regularity conditions on F:

(H1) for every $\psi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$

the set-valued map $F(\cdot, \psi)$ is measurable;

(H2) the set-valued map $F(t, \cdot)$ is locally Lipschitz in the following sense: $\forall\, R > 0$, $\exists\, \zeta_R(\cdot) \in \mathcal{L}^1([t_0, T], \mathbb{R}^+)$ such that, for a.e. $t \in [t_0, T]$ and any $\varphi, \psi \in RB_C$

$$F(t, \varphi) \subset F(t, \psi) + \zeta_R(t) \|\varphi - \psi\|_C B;$$

(H3) the set-valued map F has a sublinear growth, i.e. there exists $\sigma > 0$ such that, for a.e. $t \in [t_0, T]$ and any $\psi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$

$$F(t, \psi) \subset \sigma \left(1 + \|\psi\|_C\right) B;$$

(H4) for a given $\lambda > 0$, the set-valued map F is upper semicontinuous on $[t_0, T] \times \mathcal{K}_\lambda$, i.e. for all $t \in [t_0, T]$ and all $\varphi \in \mathcal{K}_\lambda$, we have $F(t, \varphi) \neq \emptyset$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(s, \psi) \subset F(t, \varphi) + \varepsilon B \quad \forall\, (s, \psi) \in B((t, \varphi), \delta).$$

III. Main results

A. Filippov's Theorem

The following theorem extends the celebrated Filippov's theorem [11] to differential inclusions of type (6).

Theorem 1: Let $\beta > 0$ and $\delta_0 \geq 0$ and assume (H1), (H2). *Let $y \in \mathcal{C}([t_0 - \tau, T], \mathbb{R}^n)$ be such that $y(\cdot)$ is absolutely continuous on $[t_0, T]$. Set $R = \max_{t \in [t_0 - \tau, T]} \|y(t)\|$,*

$$\gamma_1(t) = d_{F(t, y_t)}(\dot{y}(t)),\ \gamma_2(t) = \exp\left\{\int_{t_0}^{t} \zeta_{R+\beta}(s) ds\right\},\ \gamma_3(t) = \gamma_2(t)\left(\delta_0 + \int_{t_0}^{t} \gamma_1(s) ds\right). \quad (9)$$

If $\gamma_3(T) < \beta$, then for all $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ with $\|\varphi - y_{t_0}\|_C \leq \delta_0$, there exists $x \in \mathcal{C}([t_0 - \tau, T], \mathbb{R}^n)$ such that $x(\cdot)$ is an F-trajectory and for all $t \in [t_0, T]$

$$\|x_t - y_t\|_C \leq \gamma_3(t)$$

and for almost every $t \in [t_0, T]$,

$$\|\dot{x}(t) - \dot{y}(t)\| \leq \zeta_{R+\beta}(t)\gamma_3(t) + \gamma_1(t).$$

The following theorem establishes the possibility of approximating any relaxed F-trajectory by an F-trajectory starting from the same initial condition.

Theorem 2: Let $y(\cdot)$ be a relaxed F-trajectory. Assume (H1), (H2) and (H3). *Then for every $\delta > 0$ there exists an F-trajectory $x(\cdot)$* satisfying $x_{t_0} = y_{t_0}$ and $\sup_{t \in [t_0, T]} \|x(t) - y(t)\| \leq \delta$.

B. Neighboring feasible trajectories theorems

Let $\lambda > 0$. Consider the following inward pointing condition:

$$(IPC^\lambda) \begin{cases} \forall\, t \in [0, T], \forall\, \psi \in \mathcal{K}_\lambda, \forall\, v \in F(t, \psi) \\ \text{such that } \max_{n \in N_K^1(\psi(0))} \langle n, v \rangle \geq 0, \\ \exists\, w \in \operatorname{Liminf}_{(s, \phi) \to (t, \psi)} F(s, \phi) \\ \text{satisfying } \max_{n \in N_K^1(\psi(0))} \langle n, w - v \rangle < 0, \end{cases}$$

where $\mathcal{K}_\lambda$ is defined by (5). Before stating our first NFT theorem, a crucial result is given by the following lemma which shows that (IPC^λ) implies an uniform inward pointing condition on a neighborhood of $\mathcal{K}_\lambda$.

Lemma 1: Let $\lambda > 0$ and assume (H1)–(H4) *and* (IPC^λ)*. Then $\forall\, R > 0$, $\exists\, \rho > 0$ and $\eta > 0$ such that for every $t \in [0, T]$, $\psi \in (\mathcal{K}_\lambda + \eta B_C) \cap RB_C$ and for every $v \in F(t, \psi)$ with*

$$\max_{n \in N_K^1(x), x \in \partial K \cap B(\psi(0), \eta)} \langle n, v\rangle \geq 0$$

we can find $w \in F(t, \psi)$ satisfying

$$\begin{cases} \langle n, w\rangle \leq -\rho \text{ and } \langle n, w - v\rangle \leq -\rho \\ \forall\, n \in N_K^1(x), \forall\, x \in \partial K \cap B(\psi(0), \eta). \end{cases} \tag{10}$$

The following theorem shows the existence of a feasible F-trajectory and provides an estimate of the distance (in the norm of uniform convergence) of this trajectory from a specified F-trajectory.

Theorem 3: Assume (H1)–(H3)*. Let $\tau > 0$, $r_0 > 0$ and $\lambda_0 > 0$ and suppose that, for*

$$\lambda = \max\{\lambda_0, (1 + (1 + \lambda_0\tau + r_0)e^{\sigma T})\sigma\}, \tag{11}$$

assumptions (H4) *and* (IPC^λ) *hold true. Then there exists a constant $C > 0$ such that for any $t_0 \in [0, T]$ and every F-trajectory $\hat{x}(\cdot)$ on $[t_0 - \tau, T]$ with λ_0-Lipschitz $\hat{x}_{t_0}$ and $\hat{x}(t_0) \in K \cap r_0 B$, and for any $\varepsilon_0 > 0$, we can find a feasible F-trajectory on $[t_0 - \tau, T]$ satisfying $x_{t_0} = \hat{x}_{t_0}, x((t_0, T]) \subset \operatorname{Int} K$ and*

$$\|x_t - \hat{x}_t\|_C \leq C\left(\max_{t \in [t_0, T]} d_K(\hat{x}(t)) + \varepsilon_0\right). \tag{12}$$

Theorem 3 together with Theorem 2 imply that under the inward pointing condition (IPC^λ), the set of F-trajectories lying in the interior of the constraint set K, for $t \in (t_0, T]$ and starting at $\hat{x}_{t_0}$, is dense in the set of feasible relaxed F-trajectories. This results from the following corollary:

Corollary 1: Under all the assumptions of Theorem 3, for any feasible relaxed F-trajectory $\bar{x}(\cdot)$ with λ_0-Lipschitz $\bar{x}_{t_0}$ and $\bar{x}(t_0) \in K \cap r_0 B$, and any $\delta > 0$, there exists a feasible F-trajectory $x(\cdot)$ such that $x_{t_0} = \bar{x}_{t_0}$, $x((t_0, T]) \in \operatorname{Int} K$ and $\|x_t - \bar{x}_t\|_C < \delta$ for all $t \in [t_0, T]$.

Now, assume the relaxed inward pointing condition given by (IP^λ_{rel}). As before, we have the following lemma which is similar to Lemma 1 but in the framework of the relaxed set-valued map.

Lemma 2: Let $\lambda > 0$ and assume (H1)–(H4) *and* (IP^{λ}_{rel})*. Then $\forall\, R > 0$, $\exists\, \rho > 0$ and $\eta > 0$ such that for every $t \in [0,T]$, $\psi \in (\mathcal{K}_\lambda + \eta B_C) \cap RB_C$ and for every $v \in co\, F(t,\psi)$ with*

$$\max_{n \in N^1_K(x), x \in \partial K \cap B(\psi(0),\eta)} \langle n, v\rangle \geq 0,$$

we can find $w \in co\, F(t,\psi)$ satisfying

$$\begin{cases} \langle n, w\rangle \leq -\rho \text{ and } \langle n, w - v\rangle \leq -\rho \\ \forall\, n \in N^1_K(x), \forall\, x \in \partial K \cap B(\psi(0),\eta). \end{cases}$$

The following theorem is related to Theorem 3, however neither one is contained in another.

Theorem 4: Assume (H1)–(H3)*. Let $\tau > 0$, $r_0 > 0$ and $\lambda_0 > 0$ and suppose that, for λ given by* (11)*, assumptions* (H4) *and* (IP^{λ}_{rel}) *hold true. Then there exists a constant $C > 0$ such that for any $t_0 \in [0,T]$ and every relaxed F-trajectory $\hat{x}(\cdot)$ on $[t_0 - \tau, T]$ with λ_0-Lipschitz $\hat{x}_{t_0}$ and $\hat{x}(t_0) \in K \cap r_0 B$, and for any $\varepsilon_0 > 0$, we can find a relaxed feasible F-trajectory on $[t_0 - \tau, T]$ satisfying $x_{t_0} = \hat{x}_{t_0}$, $x((t_0,T]) \subset \operatorname{Int} K$ and*

$$\|x_t - \hat{x}_t\|_C \leq C \left(\max_{t \in [t_0,T]} d_K(\hat{x}(t)) + \varepsilon_0 \right). \tag{13}$$

The proof of Theorem 4 is a straightforward consequence of Lemma 2, together with Theorem 3 applied with $\operatorname{co} F$ instead of F. Theorem 4 and the constructive argument of [8, Proof of Lemma 5.2] imply the following Corollary:

Corollary 2: Under all the assumptions of Theorem 4, for any relaxed feasible F-trajectory $\bar{x}(\cdot)$ with λ_0-Lipschitz $\bar{x}_{t_0}$ and $\bar{x}(t_0) \in K \cap r_0 B$, and any $\delta > 0$, there exists a feasible F-trajectory $x(\cdot)$ such that $x_{t_0} = \bar{x}_{t_0}$, $x((t_0,T]) \in \operatorname{Int} K$ and $\|x_t - \bar{x}_t\|_C < \delta$ for all $t \in [t_0,T]$.

C. Neighboring feasible trajectories theorem: constant delay case

Consider the constant-delay differential inclusion

$$\begin{cases} \dot{x}(t) & \in & \mathcal{F}(t, x(t-\tau)), \quad a.e.\ t \in [t_0,T], \\ x_{t_0} & = & \varphi, \end{cases} \tag{14}$$

where $\mathcal{F} : [0,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a set-valued map having closed nonempty images and $\varphi \in \mathcal{C}([-\tau,0],\mathbb{R}^n)$. Let $\lambda > 0$. Consider the following inward pointing condition:

$$(IP^{\lambda}_{eq}) \begin{cases} \forall\, t \in [0,T], \forall\, x \in \partial K, \forall\, y \in x + \tau\lambda B, \\ \forall\, v \in \mathcal{F}(t,y) \text{ such that } \max_{n \in N^1_K(x)} \langle n, v\rangle \geq 0, \\ \exists\, w \in \operatorname{Liminf}_{(s,z)\to(t,y)} \operatorname{co} \mathcal{F}(s,z) \\ \text{satisfying } \max_{n \in N^1_K(x)} \langle n, w - v\rangle < 0. \end{cases}$$

Assume the following regularity conditions on $\mathcal{F}$:

(A1) for every $x \in \mathbb{R}^n$ the set-valued map $\mathcal{F}(\cdot, x)$ is measurable;

(A2) the set-valued map $\mathcal{F}(t, \cdot)$ is locally Lipschitz, i.e. $\forall\, R > 0$, $\exists\, \zeta_R(\cdot) \in \mathcal{L}^1([t_0, T], \mathbb{R}^+)$ such that, for a.e. $t \in [t_0, T]$ and any $x, y \in RB$

$$\mathcal{F}(t, x) \subset \mathcal{F}(t, y) + \zeta_R(t)\|x - y\|B;$$

(A3) there exists $\sigma > 0$ such that, for a.e. $t \in [t_0, T]$ and any $x \in \mathbb{R}^n$ $\mathcal{F}(t, x) \subset \sigma(1 + |x|)B$;

(A4) for a given $\lambda > 0$, the set-valued map $\mathcal{F}$ is upper semicontinuous on $[t_0, T] \times (\partial K + \tau\lambda B)$.

Theorem 5: Assume (A1)–(A3). *Let $\tau > 0$, $r_0 > 0$ and $\lambda_0 > 0$ and suppose that, for λ given by* (11)*, assumptions* (A4) *and (IP^{λ}_{eq}) hold true. Then there exists a constant $C > 0$ such that for any $t_0 \in [0, T]$ and every $\mathcal{F}$-trajectory $\hat{x}(\cdot)$ on $[t_0 - \tau, T]$ with λ_0-Lipschitz $\hat{x}_{t_0}$ and $\hat{x}(t_0) \in K \cap r_0 B$, and for any $\varepsilon_0 > 0$, we can find a feasible $\mathcal{F}$-trajectory on $[t_0 - \tau, T]$ satisfying $x_{t_0} = \hat{x}_{t_0}$, $x((t_0, T]) \subset \operatorname{Int} K$ and*

$$\|x_t - \hat{x}_t\|_C \leq C\left(\max_{t\in[t_0,T]} d_K(\hat{x}(t)) + \varepsilon_0\right). \tag{15}$$

IV. Proofs

A. Proof of Theorem 1

We need the following lemma from [13]:

Lemma 3: Let X be a separable Banach space, $\mathcal{G}$ be a set-valued map from $[t_0, T] \times X$ into closed nonempty subsets of X and $z : [t_0, T] \to X$ be a continuous function such that

1) *$\forall\, x \in X$ the set-valued map $\mathcal{G}(\cdot, x)$ is measurable.*
2) *$\exists\, \beta > 0$, $\bar{\zeta}(\cdot) \in \mathcal{L}^1([t_0, T], \mathbb{R}^+)$ such that for almost all $t \in [t_0, T]$ the map $\mathcal{G}(t, \cdot)$ is $\bar{\zeta}(t)$-Lipschitzian on $z(t) + \beta B_X$, where B_X is the closed unit ball in X centered at 0.*

Let $x \in \mathcal{C}([t_0, T], X)$ be such that $\|x - z\|_C \leq \beta$. Then the set-valued map $t \rightsquigarrow \mathcal{G}(t, x(t))$ is measurable.

In addition to Lemma 3, the proof of Theorem 1 requires the following two lemmas. The first one states that, for every $x \in \mathcal{C}([t_0 - \tau, T], \mathbb{R}^n)$ taken in a neighborhood of the reference trajectory y, the map $t \rightsquigarrow F(t, x_t)$ is measurable.

Lemma 4: Let $\beta > 0$. Assume (H1), (H2) *and let y be as in Theorem 1. Let $x \in \mathcal{C}([t_0 - \tau, T], \mathbb{R}^n)$ be such that $\|x(t) - y(t)\| \leq \beta$, for every $t \in [t_0 - \tau, T]$. Then the set-valued map $t \rightsquigarrow F(t, x_t)$ is measurable.*

Proof. Since x and y are continuous on $[t_0 - \tau, T]$, we can easily prove (see [22, Lemma 2.1] for more details) that x_t and y_t are also continuous functions of t on $[t_0, T]$. By (H2), $F(t, \cdot)$ is $\bar{\zeta}(t)$-Lipschitzian on $y_t + \beta B_C$ with the Lipschitz constant $\bar{\zeta}(\cdot) = \zeta_{R+\beta}(\cdot)$. Then, by Lemma 3 (with $X = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, $\mathcal{G} = F$ and $z(\cdot) = y(\cdot + \theta)$, $\theta \in [-\tau, 0]$), we obtain that the set-valued function $[t_0, T] \ni t \rightsquigarrow F(t, x_t)$ is measurable, which concludes the proof. □

The following lemma proves that, starting from a reference trajectory y, we can construct a sequence $(x_n)_{n\geq 0}$ in $\mathcal{C}([t_0 - \tau, T], \mathbb{R}^n)$ approximating a solution of (6)–(7).

Lemma 5: Let $\beta > 0$ *and* $\delta_0 \geq 0$. *Assume* (H1), (H2) *and let* y, γ_1, γ_2, γ_3 *be as in Theorem 1. If* $\gamma_3(T) < \beta$, *then for any* $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ *with* $\|\varphi - y_{t_0}\|_C \leq \delta_0$ *there exist sequences* $x_n \in \mathcal{C}([t_0 - \tau, T], \mathbb{R}^n)$ *and* $f_n \in \mathcal{L}^1([t_0, T], \mathbb{R}^n), for\ n \geq 0$, *such that*

$$x_{0,t} = y_t, \qquad f_0 = \dot{y}, \quad t \in [t_0, T], \tag{16}$$

$$\|f_1(t) - f_0(t)\| = \gamma_1(t), \quad a.e.\ t \in [t_0, T]; \tag{17}$$

and for $n \geq 1$

$$\begin{cases} x_n(t) = \varphi(0) + \displaystyle\int_{t_0}^{t} f_n(s)ds, & t \in [t_0, T], \\ x_{n,t_0} = \varphi, \end{cases} \tag{18}$$

$$f_n(t) \in F(t, x_{n-1,t}), \quad t \in [t_0, T], \tag{19}$$

with

$$\|f_{n+1}(t) - f_n(t)\| \leq \zeta_{R+\beta}(t)\|x_{n,t} - x_{n-1,t}\|_C, \tag{20}$$

for almost every $t \in [t_0, T]$.

Proof. By Lemma 4, the set-valued map $t \rightsquigarrow F(t, y_t)$ is measurable. Since the function $t \to \gamma_1(t)$ is measurable (see, [13, Lemma 1.5]), the set-valued map U_1 defined by

$$U_1(t) := \{v \in F(t, y_t) : \|v - f_0(t)\| = \gamma_1(t)\}$$

is measurable (see, e.g., [2]). Hence, by the measurable selection theorem the set-valued map U_1 admits a measurable selection $f_1 : [t_0, T] \mapsto \mathbb{R}^n$. From the definition of U_1, we have $f_1(t) \in F(t, y_t)$ for $t \in [t_0, T]$ and

$$\|f_1(t) - f_0(t)\| = \gamma_1(t) \qquad a.e.\ t \in [t_0, T]. \tag{21}$$

Let $\varphi \in \mathcal{C}([-\tau,0],\mathbb{R}^n)$ be such that $\|\varphi - y_{t_0}\|_C \le \delta_0$ and define $x_1 \in \mathcal{C}([t_0-\tau,T],\mathbb{R}^n)$ by

$$\begin{cases} x_1(t) &= \varphi(0) + \int_{t_0}^{t} f_1(s)ds, \quad t \in [t_0,T], \\ x_1(t_0+\theta) &= \varphi(\theta), \qquad\qquad\qquad \theta \in [-\tau,0]. \end{cases}$$

Observe that

$$\|x_{1,t} - y_t\| \le \delta_0 + \int_{t_0}^{t} \gamma_1(s)ds, \quad t \in [t_0,T]. \tag{22}$$

Indeed, for $\theta \in [-\tau,0]$ and $t \in [t_0,T]$ such that $t+\theta \ge t_0$, we have

$$\begin{aligned} \|x_1(t+\theta) - y(t+\theta)\| &\le \|\varphi(0) - y(t_0)\| + \|\int_{t_0}^{t+\theta} f_1(s) - \dot{y}(s)ds\| \\ &\le \|\varphi - y_{t_0}\|_C + \int_{t_0}^{t+\theta} \gamma_1(s)ds \le \delta_0 + \int_{t_0}^{t} \gamma_1(s)ds. \end{aligned}$$

In the case when $t+\theta < t_0$, we have $\|x_1(t+\theta) - y(t+\theta)\| = \|\varphi - y_{t_0}\|_C \le \delta_0$.

With x_1, we associate the set-valued map $[t_0,T] \ni t \rightsquigarrow U_2(t)$ defined by $U_2(t) := \{v \in F(t,x_{1,t}) : \|v - \dot{x}_1(t)\| = d_{F(t,x_{1,t})}(\dot{x}_1(t))\}$. By (22), $\|x_{1,t} - y_t\|_C \le \beta$ and, since $F(t,\cdot)$ is $\zeta_{R+\beta}$-Lipschitz on $y_t + \beta B_C$, we deduce (using the same arguments as before) the existence of a measurable selection $f_2 : [t_0,T] \mapsto \mathbb{R}^n$ of U_2 such that $\|f_2(t) - f_1(t)\| \le \zeta_{R+\beta}(t)\|x_{1,t} - x_{0,t}\|_C$, for almost every $t \in [t_0,T]$. Then, we conclude that (18)–(20) hold true for $n=1$.

Assume that we already have constructed $x_n \in \mathcal{C}([t_0-\tau,T],\mathbb{R}^n)$ and $f_n \in \mathcal{L}^1([t_0,T],\mathbb{R}^n)$, for $n = 1,\cdots,N$, verifying (18), (19) and (20). Before extending to $n = N+1$, we prove that the constructed sequence x_n verifies the following:

Claim 1: $\|x_{n,t} - y_t\|_C \le \beta,\ \forall\, t \in [t_0,T], \forall\, n = 1,\cdots,N.$

In fact, for $n=1$, the claim follows directly from (22). For $n \ge 2$, (20) implies the following inequalities:

$$\|x_{n,t} - x_{n-1,t}\|_C \le \int_{t_0}^{t} \|f_n(s_1) - f_{n-1}(s_1)\|ds_1 \le \int_{t_0}^{t} \zeta_{R+\beta}(s_1)\|x_{n-1,s_1} - x_{n-2,s_1}\|_C ds_1,$$

which can be repeated recursively (see the proof of [13, Theorem 1.2] for more details) to obtain the following property:

$$\|x_{n,t} - x_{n-1,t}\|_C \le (\delta_0 + \int_{t_0}^{t} \gamma_1(s)ds)\frac{[\ln(\gamma_2(t))]^n}{n!}, \tag{23}$$

for every $n \in \{2,\cdots,N\}$. From (22) and the last inequality, we get (see the proof of [13, Theorem 1.2] for more details)

$$\|x_{N,t} - y_t\|_C \le \sum_{i=1}^{N} \|x_{i,t} - x_{i-1,t}\|_C \le \gamma_3(t) \le \beta. \tag{24}$$

Again, define the set-valued map $[t_0, T] \ni t \rightsquigarrow U_{N+1}(t)$ by $U_{N+1}(t) := \{v \in F(t, x_{N,t}) : \|v - \dot{x}_N(t)\| = d_{F(t,x_{N,t})}(\dot{x}_N(t))\}$. Knowing that $\|x_{N,t} - y_t\|_C \leq \beta$ and using the same reasoning as before, we deduce the existence of $f_{N+1} : [t_0, T] \mapsto \mathbb{R}^n$, a measurable selection of U_{N+1}, such that $\|f_{N+1}(t) - f_N(t)\| \leq \zeta_{R+\beta}(t)\|x_{N,t} - x_{N-1,t}\|_C$, for almost every $t \in [t_0, T]$. The function x_{N+1}, associated to f_{N+1}, is defined by (18), for $n = N + 1$. □

Proof of Theorem 1. By (23), for each $t \in [t_0, T]$, the sequence $\{x_{n,t}\}$ is Cauchy in the Banach space $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$. Thus, for each $t \in [t_0, T]$ we may define $x_t \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ as the limit of $x_{n,t}$. In addition, by (20), for almost every $t \in [t_0, T]$ the sequence $\{f_n(t)\}$ is Cauchy in $\mathbb{R}^n$. Furthermore, from (20) and (21), it follows that for $n \geq 1$

$$\begin{aligned} \|f_n(t) - \dot{y}(t)\| &\leq \sum_{i=0}^{n-1} \|f_{i+1}(t) - f_i(t)\| \leq \gamma_1(t) + \zeta_{R+\beta}(t) \sum_{i=1}^{n-1} \|x_{i,t} - x_{i-1,t}\|_C \\ &\leq \gamma_1(t) + \zeta_{R+\beta}(t)\gamma_3(t) \qquad a.e.\ t \in [t_0, T], \end{aligned} \tag{25}$$

from which we conclude that the sequence $\{f_n\}$ is integrably bounded. Thus we may define $f \in \mathcal{L}^1([t_0, T], \mathbb{R}^n)$ by $f(t) = \lim_{n\to+\infty} f_n(t)$. By arguments similar to [13, Theorem 1.2], we obtain that $x(\cdot)$ is an F-Trajectory satisfying $\dot{x}(t) = f(t)$ a.e. in $[t_0, T]$. Passing to the limits in (24) and (25) yields the desired estimations on x and $\dot{x}$. □

B. Proof of Theorem 2

Fix $\delta > 0$ and let $R := \max_{t\in[t_0-\tau,T]} \|y(t)\|$. It is not restrictive to assume that $t_0 = 0$ and $\int_0^T \zeta_{R+\delta}(t)dt > 0$. Choose any positive α such that

$$\alpha < \min\left\{\frac{\delta}{2}, \frac{\delta}{2\gamma_2(T)\int_0^T \zeta_{R+\delta}(t)dt}\right\}, \tag{26}$$

where $\gamma_2(\cdot)$ is as in Theorem 1, with $\beta = \delta$. Let $n \geq 1$ be so large such that $\sigma(1 + R)/n < \alpha/2$, where σ is given by (H3). Let I_j be the interval $[(j-1)\frac{T}{n}, j\frac{T}{n}]$, for $j = 1, \cdots, n$. By Lemma 4, the set-valued map $t \rightsquigarrow F(t, y_t)$ is measurable. In addition, by (H3), $t \rightsquigarrow F(t, y_t)$ is integrably bounded because for almost every $t \in [0, T], F(t, y_t) \subset \sigma(1 + R)B$. Then, by Aumann's Theorem [4], there exists a measurable selection $f_j(t) \in F(t, y_t)$ such that

$$\int_{I_j} f_j(t)dt = \int_{I_j} \dot{y}(t)dt, \qquad j = 1, \cdots, n.$$

Let f be the function which is equal to f_j on I_j, and define the continuous function $z:[-\tau,T]\to\mathbb{R}^n$ by

$$\begin{cases} z(t) &= y(0)+\int_0^t f(s)ds, \qquad t\in[0,T],\\ z_0 &= y_0. \end{cases}$$

Observe that

$$\|z_t-y_t\|_C<\alpha,\quad \forall\, t\in[0,T].$$

Indeed, for every $t\in[0,T]$ and every $\theta\in[-\tau,0]$ such that $t+\theta\geq 0$, there exists $j\in\{1,\cdots,n\}$ for which $t+\theta\in I_j$ and

$$\|z(t+\theta)-y(t+\theta)\|=\|\int_0^{t+\theta}(f(s)-\dot{y}(s))\,ds\|\leq\int_{I_j}\|f(s)-\dot{y}(s)\|ds<\alpha.$$

If $t+\theta<0$, then $\|z(t+\theta)-y(t+\theta)\|=0$.
Since for almost every $t\in[0,T]$, $F(t,\cdot)$ is $\zeta_{R+\delta}(t)$-Lipschitz on $y_t+\delta B_C$, we obtain

$$d_{F(t,z_t)}(\dot{z}(t))\leq\sup\{d_{F(t,z_t)}(\xi):\xi\in F(t,y_t)\}+d_{F(t,y_t)}(\dot{z}(t))\leq\zeta_{R+\delta}(t)\|z_t-y_t\|_C\leq\alpha\zeta_{R+\delta}(t).$$

Inequality (27) together with (26) imply that $\gamma_2(T)\int_0^T\alpha\zeta_{R+\delta}(t)dt<\delta/2$. Then, by Theorem 1 applied with $\beta=\delta$ and $\delta_0=0$, there exists a trajectory x of (6) satisfying $x_0=z_0=y_0$ and

$$\|x_t-z_t\|_C\leq\gamma_2(T)\int_0^T\alpha\zeta_{R+\delta}(s)ds<\frac{\delta}{2}.$$

Finally, we obtain

$$\|x_t-y_t\|_C\leq\|x_t-z_t\|_C+\|z_t-y_t\|_C<\frac{\delta}{2}+\frac{\delta}{2}=\delta,$$

which concludes the proof. □

C. *Proof of Lemma 1.*

We proceed in three steps.

Step 1. Let $R>0$, $\bar{t}\in[0,T]$ and $\bar{\psi}\in\mathcal{K}_\lambda\cap RB_C$ be fixed. Knowing that F is locally bounded and using exactly the same argument as [16, Lemma 3.5], we prove the existence of $\rho_{\bar{t},\bar{\psi}}>0$ such that for all $v\in F(\bar{t},\bar{\psi})$ with $\max\limits_{n\in N_K^1(\bar{\psi}(0))}\langle n,v\rangle\geq 0$, there exists $w\in\text{Liminf}_{(s,\phi)\to(\bar{t},\bar{\psi})}F(s,\phi)$ satisfying

$$\max\left\{\langle n,w\rangle,\langle n,w-v\rangle \mid n\in N_K^1(\bar{\psi}(0))\right\}\leq-2\rho_{\bar{t},\bar{\psi}}. \tag{27}$$

Step 2. We show the existence of $\eta_{\bar{t},\bar{\psi}} > 0$ such that for every $t \in B(\bar{t}, \eta_{\bar{t},\bar{\psi}})$, for every $\psi \in \mathcal{K}_\lambda \cap B_C(\bar{\psi}, \eta_{\bar{t},\bar{\psi}}) + \eta_{\bar{t},\bar{\psi}} B_C$ and for every $v \in F(t,\psi)$ with

$$\max_{n \in N_K^1(x), x \in \partial K \cap B(\psi(0), \eta_{\bar{t},\bar{\psi}})} \langle n, v \rangle \geq 0,$$

there exists $w \in F(t,\psi)$ satisfying

$$\begin{cases} \langle n, w \rangle \leq -\rho_{\bar{t},\bar{\psi}} \text{ and } \langle n, w - v \rangle \leq -\rho_{\bar{t},\bar{\psi}} \\ \forall\, n \in N_K^1(x), \forall\, x \in \partial K \cap B(\psi(0), \eta_{\bar{t},\bar{\psi}}). \end{cases} \tag{28}$$

Suppose by contradiction that there exist $t_i \to \bar{t}$, $\psi_i \to \bar{\psi}$ in $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$, $v_i \in F(t_i, \psi_i)$, $x_i \to_{\partial K} \bar{\psi}(0)$ and $n_i \in N_K^1(x_i)$ such that $\langle n_i, v_i \rangle \geq 0$ and for every $w_i \in F(t, \psi_i)$ we can find $x_i' \to_{\partial K} \bar{\psi}(0)$, $n_i' \in N_K^1(x_i')$ satisfying

$$\langle n_i', w_i \rangle \vee \langle n_i', w_i - v_i \rangle > -\rho_{\bar{t},\bar{\psi}}. \tag{29}$$

Since F is upper semicontinuous at every point of $[0,T] \times \mathcal{K}_\lambda$, taking subsequences and keeping the same notations we may assume that v_i converge to some $v \in F(\bar{t}, \bar{\psi})$, $n_i \to n$ and $n_i' \to n'$. Since the map $x \rightsquigarrow N_K^1(x)$ is upper semicontinuous, we have $n, n' \in N_K^1(\bar{\psi}(0))$ and $\langle n, v \rangle \geq 0$. Then $\max_{n \in N_K^1(\bar{\psi}(0))} \langle n, v \rangle \geq 0$. Consider w as in (27) corresponding to this v and let $w_i \in F(t, \psi_i)$ be such that $w_i \to w$. From (29) we deduce that $\langle n', w \rangle \vee \langle n', w - v \rangle \geq -\rho_{\bar{t},\bar{\psi}}$, contradicting the choice of w.

Step 3. Consider a covering of $[0,T] \times (\mathcal{K}_\lambda \cap 2RB_C)$ by the open balls $\mathring{B}((\bar{t}, \bar{\psi}), \eta_{\bar{t},\bar{\psi}})$ satisfying the following requirement:

$$[0,T] \times (\mathcal{K}_\lambda \cap 2RB_C) \subset \bigcup_{(\bar{t},\bar{\psi}) \in [0,T] \times \mathcal{K}_\lambda \cap 2RB_C} \mathring{B}((\bar{t}, \bar{\psi}), \eta_{\bar{t},\bar{\psi}})$$

such that for every $t \in B(\bar{t}, \eta_{\bar{t},\bar{\psi}})$, for every $\psi \in \mathcal{K}_\lambda \cap B_C(\bar{\psi}, \eta_{\bar{t},\bar{\psi}}) + \eta_{\bar{t},\bar{\psi}} B_C$ and for every $v \in F(t, \psi)$ with

$$\max_{n \in N_K^1(x), x \in \partial K \cap B(\psi(0), \eta_{\bar{t},\bar{\psi}})} \langle n, v \rangle \geq 0,$$

there exists $w \in F(t, \psi)$ satisfying (28).

We claim that the set $\mathcal{K}_\lambda \cap 2RB_C$ is compact. Indeed, Thanks to Ascoli's Theorem, we know that a subset of $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ is compact if and only if it is closed, bounded, and equicontinuous. The set $\mathcal{K}_\lambda$ is closed (the uniform limit of λ-Lipschitz functions is λ-Lipschitz) and equicontinuous (by assumption). The boundedness follows from the fact that for all $\theta \in [-\tau, 0]$

$$\|\psi(\theta)\| \leq \|\psi(\theta) - \psi(0)\| + \|\psi(0)\| \leq \lambda\tau + R.$$

Now, consider a finite subcovering

$$[0,T] \times (\mathcal{K}_\lambda \cap 2RB_C) \subset \bigcup_{i=1\cdots,N} \mathring{B}((t_i,\psi_i),\eta_{t_i,\psi_i}).$$

Then, for $\rho = \min\{\rho_{t_i,\psi_i}, i = 1\cdots,N\}$, for some $0 < \eta < \min\{R, \eta_{t_i,\psi_i}, i = 1\cdots,N\}$ and for all $(t,\psi) \in [0,T] \times (\mathcal{K}_\lambda + \eta B_C) \cap RB_C$, there exists $1 \le i \le N$ such that $(t,\psi) \in B(t_i,\eta_{t_i,\psi_i}) \times (\mathcal{K}_\lambda \cap B_C(\psi_i,\eta_{t_i,\psi_i}) + \eta_{t_i,\psi_i} B_C)$. This complete the proof. □

D. Proof of Theorem 3.

The proof is inspired by the construction proposed in [15].

Lemma 6: Assume (H1)–(H3)*. Let* $\tau > 0$*,* $r > 0$ *and* $\lambda_0 > 0$ *and suppose that, for* λ *given by* (11)*, assumptions* (H4) *and* (IPC^λ) *hold true. Then there exist positive constants* δ *and* c *such that for every* $\bar{t} \in [0,T]$ *and every* F*-trajectory* $\hat{x}(\cdot)$ *on* $[\bar{t}-\tau,T]$ *with* λ_0*-Lipschitz* $\hat{x}_{\bar{t}}$ *and* $\hat{x}(\bar{t}) \in K \cap rB$*, and for any* $\varepsilon > 0$*, we can find an* F*-trajectory on* $[\bar{t}-\tau,T]$ *satisfying*

$$\begin{cases} x_{\bar{t}} = \hat{x}_{\bar{t}}, \\ x(t) \in \operatorname{Int} K, \qquad \forall\, t \in (\bar{t}, (\bar{t}+\delta)\wedge T] \\ \|x_t - \hat{x}_t\|_C \le c \max\limits_{t\in[\bar{t},T]} d_K(\hat{x}(t)) + \varepsilon. \end{cases}$$

Proof. Let $R := (1+\lambda_0\tau + r)e^{\sigma T}$, $\tilde{R} := (1+R)\sigma$ and $\bar{R} := 2\tilde{R}T + R$, where σ is as in (H3). Fix $\bar{t} \in [0,T]$ and an F-trajectory $\hat{x}(\cdot)$ on $[\bar{t}-\tau,T]$ such that $\hat{x}_{\bar{t}}$ is λ_0-Lipschitz and $\hat{x}(\bar{t}) \in K \cap rB$. Let $\delta > 0$ and $0 < \beta < \rho$ be such that $\delta < \frac{\eta}{4\sigma(1+R)}$ (where ρ and η are given by Lemma 1) and

$$\nu(\delta) := 2\tilde{R}\exp\left(\int_{\bar{t}}^{T} \zeta_{\bar{R}+\beta}(s)ds\right)\omega_{\zeta_{\bar{R}}}(\delta) < \beta,$$

where $\zeta_{\bar{R}}(\cdot)$ and $\zeta_{\bar{R}+\beta}(\cdot)$ are as in (H2), and $\omega_{\zeta_{\bar{R}}}(\cdot)$ is the modulus of continuity of the map $t \mapsto \int_{\bar{t}}^{t} \zeta_{\bar{R}}(s)ds$. Consider $\Gamma > 0$ such that $\Gamma(\rho - \nu(\delta)) > 1$ and call $\bar{\delta} := (\bar{t}+\delta)\wedge T$.

We proceed in four steps.

Step 1. We have $\|\hat{x}_t\|_C \le R$ for every $t \in [\bar{t},T]$. Indeed, for $t \in [\bar{t},T]$ and $\theta \in [-\tau,0]$, we have

$$\hat{x}(t+\theta) = \begin{cases} \hat{x}(\bar{t}) + \displaystyle\int_{\bar{t}}^{t+\theta} \dot{\hat{x}}(s)ds, & \text{if } t+\theta \ge \bar{t} \\ \hat{x}_{\bar{t}}(t+\theta-\bar{t}), & \text{if } t+\theta < \bar{t}. \end{cases}$$

Then,

$$\|\hat{x}(t+\theta)\| \le \|\hat{x}_{\bar{t}}\|_C + \int_{\bar{t}}^{\max\{\bar{t},t+\theta\}} \sigma(1+\|\hat{x}_s\|_C)ds.$$

Hence

$$\|\hat{x}_t\|_C \leq \|\hat{x}_{\bar{t}}\|_C + \int_{\bar{t}}^{\max\{\bar{t},t+\theta\}} \sigma(1+\|\hat{x}_s\|_C)ds.$$

Thanks to Gronwall's Lemma, we can easily verify that for any $t \in [\bar{t}, T]$

$$\|\hat{x}_t\|_C \leq (1+\|\hat{x}_{\bar{t}}\|_C)e^{\sigma(t-\bar{t})}, \tag{30}$$

from which conclude that $\|\hat{x}_t\|_C \leq R$ for every $t \in [\bar{t}, T]$.

Step 2. If $\tilde{d}_K(\hat{x}(\bar{t})) < -\frac{\eta}{4}$, then $x(\cdot) = \hat{x}(\cdot)$ satisfies our lemma. Indeed, if $\tilde{d}_K(\hat{x}(\bar{t})) < -\frac{\eta}{4}$, then for all $t \in [\bar{t}, \bar{\delta}]$, we have

$$\begin{aligned} \tilde{d}_K(\hat{x}(t)) &\leq \tilde{d}_K(\hat{x}(\bar{t})) + |\tilde{d}_K(\hat{x}(t)) - \tilde{d}_K(\hat{x}(\bar{t}))| < -\frac{\eta}{4} + \|\hat{x}(t) - \hat{x}(\bar{t})\| \\ &\leq -\frac{\eta}{4} + \int_{\bar{t}}^{t} \|\dot{\hat{x}}(s)\|ds < -\frac{\eta}{4} + (1+R)\sigma\delta < 0, \end{aligned}$$

and $x(\cdot) = \hat{x}(\cdot)$ is as required.

Step 3. If $-\frac{\eta}{4} \leq \tilde{d}_K(\hat{x}(\bar{t})) \leq 0$, we define the measurable set

$$\mathcal{S} := \{s \in [\bar{t}, \bar{\delta}] : \exists\, x \in \partial K \cap B(\hat{x}(s), \eta), n \in N_K^1(x), \langle n, \dot{\hat{x}}(s)\rangle \geq 0\}.$$

Fix any $\varepsilon > 0$ and $\varepsilon' > 0$ such that

$$0 < \varepsilon' < \frac{\varepsilon}{2\tilde{R}\left[1 + \exp\left(\int_0^T \zeta_{\bar{R}+\beta}(s)ds\right)\int_0^T \zeta_{\bar{R}}(s)ds\right]},$$

and let $\kappa \in [\bar{t}, \bar{\delta}]$ be defined as follows:

- If $\mu(\mathcal{S}) \leq \Gamma \max_{s\in[\bar{t},T]} d_K(\hat{x}(s)) + \varepsilon'$ then set $\kappa = \bar{\delta}$.
- If $\mu(\mathcal{S}) > \Gamma \max_{s\in[\bar{t},T]} d_K(\hat{x}(s)) + \varepsilon'$ then take κ be the smallest number in $[\bar{t}, \bar{\delta}]$ such that

$$\mu(\mathcal{S} \cap [\bar{t}, \kappa]) = \Gamma \max_{s\in[\bar{t},T]} d_K(\hat{x}(s)) + \varepsilon'.$$

For each $s \in \mathcal{S}$, we have $\hat{x}_s \in (\mathcal{K}_\lambda + \eta B_C) \cap RB_C$. Indeed, let $z \in \partial K$ be such that $\|z - \hat{x}(s)\| = d_K(\hat{x}(s))$. Let us define the function $\psi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ by

$$\psi(\theta) = \hat{x}(s+\theta) - \hat{x}(s) + z, \quad \theta \in [-\tau, 0].$$

The function ψ belongs to $\mathcal{K}_\lambda$. In addition, we have

$$\|\hat{x}_s(\theta) - \psi(\theta)\| = \|\hat{x}(s) - z\| = d_K(\hat{x}(s)) \leq d_K(\hat{x}(\bar{t})) + |d_K(\hat{x}(s)) - d_K(\hat{x}(\bar{t}))| < \eta.$$

Then $\hat{x}_s \in (\mathcal{K}_\lambda + \eta B_C) \cap RB_C$. Thanks, to Lemma 1, for each $s \in \mathcal{S}$, we can find $w \in F(s, \hat{x}_s)$ satisfying (10). By the measurable selection theorem [2], there exists a measurable function $w : \mathcal{S} \to \mathbb{R}^n$ such that $w(s) \in F(s, \hat{x}_s)$, and for a.e. $s \in \mathcal{S}$

$$\begin{cases} \langle n, w(s) \rangle \le -\rho, \text{ and } \langle n, w(s) - \dot{\hat{x}}(s) \rangle \le -\rho \\ \forall\, x \in \partial K \cap B(\hat{x}(s), \eta), \forall\, n \in N_K^1(x). \end{cases} \tag{31}$$

Define the absolutely continuous function $y : [\bar{t} - \tau, T] \mapsto \mathbb{R}^n$ by $y_{\bar{t}} = \hat{x}_{\bar{t}}$ and

$$\dot{y}(t) := \begin{cases} w(t) & \text{if} \quad t \in \mathcal{S} \cap [\bar{t}, \kappa] \\ \dot{\hat{x}}(t) & \text{if} \quad t \in ([\bar{t}, T] \setminus \mathcal{S}) \cap [\bar{t}, \kappa]. \end{cases} \tag{32}$$

For $t \in [\bar{t}, T]$ and $\theta \in [-\tau, 0]$, we have

$$\begin{aligned} \|y_t(\theta) - \hat{x}_t(\theta)\| &\le \| \int_{\bar{t}}^{\max\{\bar{t}, t+\theta\}} \Big(\dot{y}(s) - \dot{\hat{x}}(s) \Big) ds \| \le \int_{\bar{t}}^{t} \|\dot{y}(s) - \dot{\hat{x}}(s)\| ds \\ &= \int_{\mathcal{S} \cap [\bar{t}, t \wedge \kappa]} \|w(s) - \dot{\hat{x}}(s)\| ds \le 2\tilde{R}\mu(\mathcal{S} \cap [\bar{t}, t \wedge \kappa]) \end{aligned}$$

implying that,

$$\|y_t - \hat{x}_t\|_C \le 2\tilde{R}\mu(\mathcal{S} \cap [\bar{t}, t \wedge \kappa]). \tag{33}$$

Moreover, for a.e. $t \in [\bar{t}, T]$,

$$\begin{aligned} d_{F(t, y_t)}(\dot{y}(t)) &\le \sup\{d_{F(t, \hat{x}_t)}(\xi) : \xi \in F(t, y_t)\} + d_{F(t, \hat{x}_t)}(\dot{y}(t)) \\ &\le \zeta_{\bar{R}}(t)\|y_t - \hat{x}_t\| \le 2\tilde{R}\mu(\mathcal{S} \cap [\bar{t}, t \wedge \kappa])\zeta_{\bar{R}}(t). \end{aligned}$$

Hence, thanks to Theorem 1 applied with $\delta_0 = 0$, there exists an F-trajectory $x(\cdot)$ on $[\bar{t} - \tau, T]$ such that $x_{\bar{t}} = y_{\bar{t}}$ and for every $t \in [\bar{t}, T]$

$$\|x_t - y_t\|_C \le 2\tilde{R} \exp\left(\int_0^T \zeta_{\bar{R}+\beta}(s) ds \right) \omega_{\zeta_{\bar{R}}}(|t - \bar{t}|)\mu(\mathcal{S} \cap [\bar{t}, t \wedge \kappa]).$$

By the definition of κ, the inequality (33) and the triangle inequality, we have

$$\|x_t - \hat{x}_t\|_C \le \frac{\varepsilon}{\varepsilon'}\mu(\mathcal{S} \cap [\bar{t}, \kappa]) \le c \max_{t \in [\bar{t}, T]} d_K(\hat{x}(t)) + \varepsilon,$$

for a constant c independent from $\bar{t}$ and $\hat{x}(\cdot)$.

Step 4. We show next that $\{x(t) : t \in (\bar{t}, \bar{\delta}]\} \subset \operatorname{Int} K$. We distinguish two different cases:

Case 1. $t \in (\bar{t}, \kappa]$: The mean-value theorem (see, e.g., [10, Theorem 2.3.7]), affirms the existence of some $z(t) \in [y(\bar{t}), y(t)]$ and $\xi(t) \in \partial \tilde{d}_K(z(t))$, such that

$$\tilde{d}_K(y(t)) = \tilde{d}_K(y(\bar{t})) + \langle \xi(t), y(t) - y(\bar{t}) \rangle.$$

Then,

$$\begin{aligned}
\tilde{d}_K(x(t)) &\le \tilde{d}_K(y(t)) + \|x(t)-y(t)\| \le \tilde{d}_K(y(t)) + \nu(\delta)\mu(\mathcal{S}\cap[\bar{t},t\wedge\kappa]) \\
&\le \tilde{d}_K(y(\bar{t})) + \langle \xi(t), y(t)-y(\bar{t})\rangle + \nu(\delta)\mu(\mathcal{S}\cap[\bar{t},t\wedge\kappa]) \\
&\le \int_{\mathcal{S}\cap[\bar{t},t]} \langle \xi(t), w(s)\rangle ds + \int_{[\bar{t},t]\setminus\mathcal{S}} \langle \xi(t), \dot{\hat{x}}(s)\rangle ds + \nu(\delta)\mu(\mathcal{S}\cap[\bar{t},t\wedge\kappa]).
\end{aligned} \tag{34}$$

As in the proof of [15, Theorem 5], it follows that $\xi(t)$ is a convex combination of m vectors $\xi_\alpha \in N_K^1(y^\alpha)$, with $y^\alpha \in \pi_{\partial K}(z(t))$, such that for all $s \in [\bar{t},\kappa]$ and $\alpha \in \{1,\cdots,m\}$ we have $\|y^\alpha - \hat{x}(s)\| \le \eta$, where $1 \le m \le n+1$. Then, from (34) together with (31), we obtain that

$$\begin{aligned}
\tilde{d}_K(x(t)) &\le \sum_{\alpha=1}^m \lambda_\alpha \int_{\mathcal{S}\cap[\bar{t},t]} \langle \xi_\alpha, w(s)\rangle ds + \sum_{\alpha=1}^m \lambda_\alpha \int_{[\bar{t},t]\setminus\mathcal{S}} \langle \xi_\alpha, \dot{\hat{x}}(s)\rangle ds + \nu(\delta)\mu(\mathcal{S}\cap[\bar{t},t\wedge\kappa]) \\
&\le (\nu(\delta)-\rho)\,\mu(\mathcal{S}\cap[\bar{t},t]) < 0.
\end{aligned}$$

Case 2. $t \in (\kappa,\bar{\delta}]$: By the mean-value theorem, for some $z(t) \in [\hat{x}(t), y(t)]$ and $\xi(t) \in \partial\tilde{d}_K(z(t))$,

$$\tilde{d}_K(y(t)) = \tilde{d}_K(\hat{x}(t)) + \langle \xi(t), y(t)-\hat{x}(t)\rangle.$$

Then,

$$\begin{aligned}
\tilde{d}_K(x(t)) &\le \tilde{d}_K(y(t)) + \nu(\delta)\mu(\mathcal{S}\cap[\bar{t},t\wedge\kappa]) \\
&= \tilde{d}_K(\hat{x}(t)) + \langle \xi(t), y(t)-\hat{x}(t)\rangle + \nu(\delta)\mu(\mathcal{S}\cap[\bar{t},t\wedge\kappa]) \\
&= \tilde{d}_K(\hat{x}(t)) + \int_{\bar{t}}^t \langle \xi(t), \dot{y}(s)-\dot{\hat{x}}(s)\rangle ds + \nu(\delta)\mu(\mathcal{S}\cap[\bar{t},t\wedge\kappa]) \\
&= \tilde{d}_K(\hat{x}(t)) + \int_{\mathcal{S}\cap[\bar{t},t]} \langle \xi(t), w(s)-\dot{\hat{x}}(s)\rangle ds + \nu(\delta)\mu(\mathcal{S}\cap[\bar{t},t\wedge\kappa])
\end{aligned} \tag{35}$$

As in the first case, $\xi(t)$ is a convex combination of m vectors $\xi_\alpha \in N_K^1(y^\alpha)$, with $y^\alpha \in \pi_{\partial K}(z(t))$, such that for all $s \in [\bar{t},\kappa]$ and $\alpha \in \{1,\cdots,m\}$ we have $\|y^\alpha - \hat{x}(s)\| \le \eta$, where $1 \le m \le n+1$. Then, from (35), we obtain that

$$\begin{aligned}
\tilde{d}_K(x(t)) &\le \tilde{d}_K(\hat{x}(t)) + \nu(\delta)\mu(\mathcal{S}\cap[\bar{t},t\wedge\kappa]) + \sum_{\alpha=1}^m \lambda_\alpha \int_{\mathcal{S}\cap[\bar{t},t]} \langle \xi_\alpha, w(s)-\dot{\hat{x}}(s)\rangle ds \\
&\le \tilde{d}_K(\hat{x}(t)) - \rho\mu(\mathcal{S}\cap[\bar{t},\kappa]) + \nu(\delta)\mu(\mathcal{S}\cap[\bar{t},t\wedge\kappa]) \\
&\le \tilde{d}_K(\hat{x}(t)) - (\rho-\nu(\delta))\left[\Gamma \max_{s\in[\bar{t},T]} d_K(\hat{x}(s)) + \varepsilon'\right] < 0.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 3. Let $r := (1+\lambda_0\tau+r_0)e^{\sigma T}$ and $\delta = \delta(r), c = c(r) > 0$ be as in Lemma 6. Fix $t_0 \in [0,T]$ and an $F-$trajectory $\hat{x}(\cdot)$ such that $\hat{x}_{t_0}$ is λ_0-Lipschitz and $\hat{x}(t_0) \in K \cap r_0 B$. Let N be the smallest integer satisfying $(t_0 + N\delta) \wedge T = T$. Set $t_i = (t_0 + i\delta) \wedge T$ for all $i = 1, \cdots N$. Fix any $\varepsilon_0 > 0$. Lemma 6 assures that, for any sequence of positive numbers $\varepsilon_1, \cdots, \varepsilon_{N-1}$, there exists a sequence of $F-$trajectories $\{x_0(\cdot) = \hat{x}(\cdot), x_i(\cdot) : i = 1, \cdots, N\}$ such that for all $i = 1, \cdots, N$

$$\begin{cases} x_i(t) = x_{i-1}(t), & \forall\, t \in [t_0 - \tau, t_{i-1}] \\ x(t) \in \operatorname{Int} K, & \forall\, t \in (t_0, t_i] \\ \|x_{i,t} - x_{i-1,t}\|_C \leq c \max\limits_{t\in[t_0,T]} d_K(x_{i-1}(t)) + \varepsilon_{i-1}. \end{cases}$$

Lemma 6 is applied recursively on the interval $[(t_{i-1} - \tau) \wedge T, T]$ with reference trajectory $x_{i-1}(\cdot)$ restricted to this interval, for $i = 1, \cdots, N$. Note that, at each stage of this recursive construction, the same constant δ and c are used; this is justified by the fact that $x_{i-1,t_{i-1}} \in rB_C$, for all $i = 1, \cdots, N$. Call $x(\cdot) = x_N(\cdot)$, then $x_{t_0} = \hat{x}_{t_0}$ and $x(t) \in \operatorname{Int} K$ for every $t \in (t_0, T]$. Using the same arguments as in [15, Theorem 5], we prove the existence of some $C > 0$, independent from $t_0, \hat{x}(\cdot)$ and ε_0 such that

$$\|x_t - \hat{x}_t\|_C \leq C \left(\max_{t\in[t_0,T]} d_K(\hat{x}(t)) + \varepsilon_0 \right).$$

This completes the proof. $\square$

E. Proof of Corollary 1.

Fix a relaxed feasible F-trajectory $\bar{x}(\cdot)$ such that $\bar{x}_{t_0}$ is λ_0-Lipschitz and $\bar{x}(t_0) \in K \cap r_0 B$, and $\delta > 0$. Let C be as in Theorem 3. By Theorem 2, there exists an F-trajectory $\hat{x}(\cdot)$ on $[t_0 - \tau, T]$ satisfying $\hat{x}_{t_0} = \bar{x}_{t_0}$ and $\|\hat{x}_t - \bar{x}_t\|_C \leq \delta/3C$ for all $t \in [t_0, T]$. By Theorem 3, for every $\varepsilon_0 > 0$, there exists a feasible F-trajectory $x(\cdot)$ such that $x_{t_0} = \hat{x}_{t_0}$, $x((t_0, T]) \in \operatorname{Int} K$ and $\|x_t - \hat{x}_t\|_C < C \left(\max\limits_{t\in[t_0,T]} d_K(\hat{x}(t)) + \varepsilon_0 \right)$. Remark that $d_K(\hat{x}(t)) \leq d_K(\bar{x}(t)) + \|\hat{x}_t - \bar{x}_t\|_C < \delta/3C$. Set $\varepsilon_0 = \delta/3C$. Then, $\|x_t - \bar{x}_t\|_C \leq \delta$ for all $t \in [t_0, T]$. $\square$

F. Proof of Lemma 2.

Lemma 7: Let $\lambda > 0$ and assume (H1)–(H4) *and (IP^{λ}_{rel}). Then for every $R > 0$, $\bar{\psi} \in \mathcal{K}_\lambda \cap RB_C$ and every $\bar{t} \in [0,T]$ there exists $\rho_{\bar{t},\bar{\psi}} > 0$ such that $\forall\, v \in co\, F(\bar{t}, \bar{\psi})$ with* $\max\limits_{n\in N^1_K(\bar{\psi}(0))} \langle n, v \rangle \geq 0$

we can find $w \in \mathrm{Liminf}_{(s,\phi)\to(\bar{t},\bar{\psi})} co\, F(s,\phi)$ *satisfying*

$$\max\left\{\langle n, w\rangle, \langle n, w-v\rangle \mid n \in N_K^1(\bar{\psi}(0))\right\} \leq -2\rho_{\bar{t},\bar{\psi}}. \tag{36}$$

Proof. The proof follows the same lines as [16, Proof of Lemma 3.7]. □

Proof of Lemma 2. Let $R > 0$, $\bar{t} \in [0,T]$ and $\bar{\psi} \in \mathcal{K}_\lambda \cap RB_C$ be fixed and let $\rho_{\bar{t},\bar{\psi}}$ be as in Lemma 7. We claim the existence of $\eta_{\bar{t},\bar{\psi}} > 0$ such that for every $t \in B(\bar{t}, \eta_{\bar{t},\bar{\psi}})$, every $\psi \in \mathcal{K}_\lambda \cap B_C(\bar{\psi}, \eta_{\bar{t},\bar{\psi}}) + \eta_{\bar{t},\bar{\psi}} B_C$ and every $v \in \mathrm{co}\, F(t,\psi)$ with

$$\max_{n\in N_K^1(x), x\in\partial K\cap B(\psi(0),\eta_{\bar{t},\bar{\psi}})} \langle n, v\rangle \geq 0,$$

there exists $w \in \mathrm{co}\, F(t,\psi)$ satisfying

$$\begin{cases} \langle n, w\rangle \leq -\rho_{\bar{t},\bar{\psi}} \text{ and } \langle n, w-v\rangle \leq -\rho_{\bar{t},\bar{\psi}} \\ \forall\, n \in N_K^1(x), \forall\, x \in \partial K \cap B(\psi(0), \eta_{\bar{t},\bar{\psi}}). \end{cases}$$

Suppose by contradiction that there exist $t_i \to \bar{t}, \psi_i \to \bar{\psi}$, $v_i \in \mathrm{co}\, F(t_i,\psi_i), x_i \to_{\partial K} \bar{\psi}(0)$ and $n_i \in N_K^1(x_i)$ such that $\langle n_i, v_i\rangle \geq 0$ and for every $w_i \in \mathrm{co}\, F(t,\psi_i)$ we can find $x_i' \to_{\partial K} \bar{\psi}(0), n_i' \in N_K^1(x_i')$ satisfying

$$\langle n_i', w_i\rangle \vee \langle n_i', w_i - v_i\rangle > -\rho_{\bar{t},\bar{\psi}}. \tag{37}$$

Since F is upper semicontinuous at every point of $[0,T]\times\mathcal{K}_\lambda$, so is $\mathrm{co}\, F$. taking subsequences and keeping the same notations we may assume that v_i converge to some $v \in \mathrm{co}\, F(\bar{t},\bar{\psi})$, $n_i \to n$ and $n_i' \to n'$. Since the map $x \rightsquigarrow N_K^1(x)$ is upper semicontinuous, we have $n, n' \in N_K^1(\bar{\psi}(0))$ and $\langle n, v\rangle \geq 0$. Then $\max_{n\in N_K^1(\bar{\psi}(0))} \langle n, v\rangle \geq 0$. Consider w as in (36) corresponding to this v and let $w_i \in \mathrm{co}\, F(t,\psi_i)$ be such that $w_i \to w$. From (37) we deduce that $\langle n', w\rangle \vee \langle n', w-v\rangle \geq -\rho_{\bar{t},\bar{\psi}}$, contradicting the choice of w. The rest of the proof is similar to *Step 3* of Lemma 1. □

G. Proof of Theorem 5.

The proof of Theorem 5 is a straightforward consequence of Theorem 4. Indeed, fix $t_0 \in [0,T]$ and let us introduce for every $(t,\psi) \in [t_0,T]\times\mathcal{C}([-\tau,0],\mathbb{R}^n)$ the set-valued map F defined by $F(t,\psi) := \mathcal{F}(t,\psi(-\tau))$. It is easy to see that under the assumptions (A1)–(A4), the set-valued map F verifies (H1)–(H4). In addition, we can show that, under (IP_{eq}^λ), condition (IP_{rel}^λ) holds true. In fact, let $\psi \in \mathcal{K}_\lambda$. By definition of $\mathcal{K}_\lambda$, we have $\psi(0) \in \partial K$ and $\psi(-\tau) \in \psi(0) + \lambda\tau B$.

Then, from (IP_{eq}^{λ}), for every $v \in F(t,\psi) \equiv \mathcal{F}(t,\psi(-\tau))$ such that $\max_{n\in N_K^1(\psi(0))} \langle n, v\rangle \geq 0$ there exists $w \in \text{Liminf}_{(s,z)\to(t,\psi(-\tau))}\text{co}\,\mathcal{F}(s,z)$ satisfying

$$\max_{n\in N_K^1(\psi(0))} \langle n, w-v\rangle < 0.$$

From the inclusion

$$\text{Liminf}_{(s,z)\to(t,\psi(-\tau))}\text{co}\,\mathcal{F}(s,z) \subset \text{Liminf}_{(s,\phi)\to(t,\psi)}\text{co}\,F(s,\phi),$$

we deduce that (IP_{rel}^{λ}) holds true. Hence, Theorem 4 concludes the proof. $\square$

REFERENCES

[1] J. P. Aubin. *Viability theory*. Birkhäuser, Basel, 1991.

[2] J. P. Aubin and H. Frankowska. *Set-Valued Analysis*. Birkhaüser, Boston, 1990.

[3] J.P. Aubin and P. Saint-Pierre. An introduction to viability theory and management of renewable resources. In J. Kropp and J. Scheffran, editors, *Advanced Methods for Decision Making and Risk Management*, pages 43–80. Nova Science Publishers, 2007.

[4] R. J. Aumann. Integrals of set-valued functions. *J. Mathematical Analysis and Applications*, 12:1–12, 1965.

[5] N. Bekiaris-Liberis and M. Krstic. *Nonlinear Control Under Nonconstant Delays*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.

[6] C. Bernard and S. Martin. Building strategies to ensure language coexistence in presence of bilingualism. *Applied Mathematics and Computation*, 218(17):8825–8841, 2012.

[7] P. Bettiol, A. Bressan, and R. B. Vinter. On trajectories satisfying a state constraint: $W^{1,1}$ estimates and counter-examples. *SIAM J. Control Optimal*, 48:4664–4679, 2010.

[8] P. Bettiol, H. Frankowska, and R. B. Vinter. L^{∞} estimates on trajectories confined to a closed subset. *J. Differential Equations*, 252:1912–1933, 2012.

[9] P. Bettiol, H. Frankowska, and R.-B. Vinter. Improved sensitivity relations in state constrained optimal control. *Applied Mathematics & Optimization*, 71(2):353–377, 2015.

[10] F. H. Clarke. *Optimization and nonsmooth analysis*. SIAM, Philadelphia, 1990.

[11] A. F. Filippov. Classical solutions of differential equations with multivalued right hand side. *SIAM J. Control Optimal*, 5:609–621, 1967.

[12] F. Forcellini and F. Rampazzo. On nonconvex differential inclusions whose state is constrained in the closure of an open set. applications to dynamic programming. *J. Differential and Integral Equations*, 12(4):471–497, 1999.

[13] H. Frankowska. A priori estimates for operational differential inclusions. *J. Differential Equations*, 84:100–128, 1990.

[14] H. Frankowska, E. M. Marchini, and M. Mazzola. A relaxation result for state constrained inclusions in infinite dimension. *Mathematical Control and Related Fields*, 6(1):113–141, 2016.

[15] H. Frankowska and M. Mazzola. Discontinuous solutions of Hamilton-Jacobi-Bellman equation under state constraints. *Calculus of Variations and Partial Differential Equations*, 46:725–747, 2013.

[16] H. Frankowska and M. Mazzola. On relations of the adjoint state to the value function for optimal control problems with state constraints. *Nonlinear Differential Equations and Applications*, 20:361–383, 2013.

[17] H. Frankowska and M. Quincampoix. Viability kernels of differential inclusions with constraints: algorithm and applications. *J. Math. Systems, Estimation and Control*, 1:371–388, 1991.

[18] H. Frankowska and F. Rampazzo. Filippov's and Filippov–Wazewski's theorems on closed domains. *J. Differential Equations*, 161:449–478, 2000.

[19] H. Frankowska and R. B. Vinter. Existence of neighboring feasible trajectories: Applications to dynamic programming for state-constrained optimal control problems. *J. Optimization Theory and Applications*, 104(1):21–40, 2000.

[20] G. Haddad. Monotone viable trajectories for functional differential inclusions. *J. Differential Equations*, 42:1–24, 1981.

[21] I. Haidar, I. Alvarez, and A.C. Prévot. Mathematical modeling of an urban pigeon population subject to local management strategies. *Mathematical Biosciences*, 288:71 – 83, 2017.

[22] J.K. Hale and S.M. Verduyn Lunel. *Introduction to functional differential equations*, volume 99. Springer-Verlag, 1993.

[23] Y. Kuang. *Delay differential equations with application in population dynamics*. Academic Press, Boston, 1993.

[24] M. De Lara and V. Martinet. Multi-criteria dynamic decision under uncertainty: A stochastic viability analysis and an application to sustainable fishery management. *Mathematical Biosciences*, 217(2):118 – 124, 2009.

[25] H.M. Regan, Y. Ben-Haim, B. Langford, W. Wilson, and P. Lundgerg. Robust decision-making under severe uncertainty for conservation management. *Ecological Applications*, 15(4):1471–1477, 2005.

[26] J.-B. Rouquier, I. Alvarez, R. Reuillon, and P.-H. Wuillemin. A kd-tree algorithm to discover the boundary of a black box hypervolume. *Annals of Mathematics and Artificial Intelligence*, 75(3):335–350, 2015.

[27] P. Saint-Pierre. Approximation of the viability kernel. *Applied Mathematics & Optimisation*, 29(2):187–209, 1994.

[28] M. Sicard, N. Perrot, R. Reuillon, S. Mesmoudi, I. Alvarez, and S. Martin. A viability approach to control food processes: Application to a camembert cheese ripening process. *Food Control*, 23(2):312 – 319, 2012.