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A new result, for the box dimension of the graph of the Weierstrass function

Claire David

April 29, 2017

1 Introduction

The determination of the box and Hausdorff dimension of the graph of the Weierstrass function has, since long been, a topic of interest. Let us recall that, given $\lambda \in ]0, 1[$, and $b$ such that $\lambda b > 1 + \frac{3\pi}{2}$, the Weierstrass function

$$x \in \mathbb{R} \mapsto \sum_{n=0}^{+\infty} \lambda^n \cos (\pi b^n x)$$

is continuous everywhere, while nowhere differentiable. The original proof, by K. Weierstrass [Wei72], can also be found in [Tit77]. It has been completed by the one, now a classical one, in the case where $b > 1$, by G. Hardy [Har11].

After the works of A. S. Besicovitch and H. D. Ursell [BU37], it is Benoît Mandelbrot [Man77] who particularly highlighted the fractal properties of the graph of the Weierstrass function. He also conjectured that the Hausdorff dimension of the graph is $D_W = 2 + \frac{\ln \lambda}{\ln b}$. Interesting discussions in relation to this question have been given in the book of K. Falconer [Fal85]. A series of results for the box dimension can be found in the works of J.-L. Kaplan et al. [KMPY84], where the authors show that it is equal to the Lyapunov dimension of the equivalent attracting torus, and in those by T-Y. Hu and K-S. Lau [HL93]. As for the Hausdorff dimension, a proof was given by B. Hunt [Hun98] in 1998 in the case where arbitrary phases are included in each cosinusoidal term of the summation. Recently, K. Barański, B. Bárány and J. Romanowska [BBR17] proved that, for any value of the real number $b$, there exists a threshold value $\lambda_b$ belonging to the interval $]0, 1[$ such that the aforementioned dimension is equal to $D_W$ for every $b$ in $]\lambda_b, 1[$. Results by W. Shen [She15] go further than the ones of [BBR17]. In [Kel17], G. Keller proposes what appears as a much simpler and very original proof.

In our work [Dav17], where we build a Laplacian on the graph of the Weierstrass function, we came across a simpler means of computing the box dimension of the graph, using a sequence a graphs that approximate the studied one. Results are exposed in the sequel.
2 Framework of the study

In this section, we recall results that are developed in [Dav17].

**Notation.** In the following, \( \lambda \) and \( N_b \) are two real numbers such that:

\[
0 < \lambda < 1 \quad , \quad N_b \in \mathbb{N} \quad \text{and} \quad \lambda N_b > 1
\]

We will consider the \((1-\text{periodic})\) Weierstrass function \( W \), defined, for any real number \( x \), by:

\[
W(x) = \sum_{n=0}^{+\infty} \lambda^n \cos(2 \pi N_b^n x)
\]

We place ourselves, in the sequel, in the Euclidean plane of dimension 2, referred to a direct orthonormal frame. The usual Cartesian coordinates are \((x, y)\).

The restriction \( \Gamma_W \) to \([0,1]\times\mathbb{R}, \) of the graph of the Weierstrass function, is approximated by means of a sequence of graphs, built through an iterative process. To this purpose, we introduce the iterated function system of the family of \( C^\infty \) contractions from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \):

\[
\{T_0, ..., T_{N_b-1}\}
\]

where, for any integer \( i \) belonging to \( \{0, ..., N_b - 1\} \), and any \((x, y)\) of \( \mathbb{R}^2 \):

\[
T_i(x, y) = \left( \frac{x + i}{N_b}, \lambda y + \cos\left(2 \pi \left( \frac{x + i}{N_b} \right)\right) \right)
\]

**Property 2.1.**

\[
\Gamma_W = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_W)
\]

**Definition 2.1.** For any integer \( i \) belonging to \( \{0, ..., N_b - 1\} \), let us denote by:

\[
P_i = (x_i, y_i) = \left( \frac{i}{N_b - 1}, \frac{1}{1 - \lambda} \cos\left( \frac{2 \pi i}{N_b - 1} \right) \right)
\]

the fixed point of the contraction \( T_i \).

We will denote by \( V_0 \) the ordered set (according to increasing abscissa), of the points:

\[
\{P_0, ..., P_{N_b-1}\}
\]

The set of points \( V_0 \), where, for any \( i \) of \( \{0, ..., N_b - 2\} \), the point \( P_i \) is linked to the point \( P_{i+1} \), constitutes an oriented graph (according to increasing abscissa), that we will denote by \( \Gamma_{W_0} \). \( V_0 \) is called the set of vertices of the graph \( \Gamma_{W_0} \).
For any natural integer \( m \), we set:

\[
V_m = \bigcup_{i=0}^{N_b-1} T_i(V_{m-1})
\]

The set of points \( V_m \), where two consecutive points are linked, is an oriented graph (according to increasing abscissa), which we will denote by \( \Gamma_{W_m} \). \( V_m \) is called the set of vertices of the graph \( \Gamma_{W_m} \).

We will denote, in the sequel, by

\[
N^S_m = 2 N'^m_b + N_b - 2
\]

the number of vertices of the graph \( \Gamma_{W_m} \), and we will write:

\[
V_m = \{ S^m_0, S^m_1, \ldots, S^m_{N_m-1} \}
\]

Figure 1: The polygons \( \mathcal{P}_{1,0}, \mathcal{P}_{1,1}, \mathcal{P}_{1,2} \), in the case where \( \lambda = \frac{1}{2} \), and \( N_b = 3 \).
Figure 2: The graphs $\Gamma_{W_0}$ (in green), $\Gamma_{W_1}$ (in red), $\Gamma_{W_2}$ (in orange), $\Gamma_{W}$ (in cyan), in the case where $\lambda = \frac{1}{2}$, and $N_b = 3$.

**Definition 2.2.** Consecutive vertices on the graph $\Gamma_W$ 

Two points $X$ et $Y$ de $\Gamma_W$ will be called **consecutive vertices** of the graph $\Gamma_W$ if there exists a natural integer $m$, and an integer $j$ of $\{0, \ldots, N_b - 2\}$, such that:

$$X = (T_{i_1} \circ \ldots \circ T_{i_m})(P_j) \quad \text{et} \quad Y = (T_{i_1} \circ \ldots \circ T_{i_m})(P_{j+1}) \quad \{i_1, \ldots, i_m\} \in \{0, \ldots, N_b - 1\}^m$$

or:

$$X = (T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_m})(P_{N_b-1}) \quad \text{et} \quad Y = (T_{i_1+1} \circ T_{i_2} \circ \ldots \circ T_{i_m})(P_0)$$

**Definition 2.3.** For any natural integer $m$, the $N_b^S$ consecutive vertices of the graph $\Gamma_{W_m}$ are, also, the vertices of $N_b^m$ simple polygons $P_{m,j}$, $0 \leq j \leq N_b^m - 1$, with $N_b$ sides. For any integer $j$ such that $0 \leq j \leq N_b^m - 1$, one obtains each polygon by linking the point number $j$ to the point number $j + 1$ if $j = i \mod N_b$, $0 \leq i \leq N_b - 2$, and the point number $j$ to the point number $j - N_b + 1$ if $j = -1 \mod N_b$. These polygons generate a Borel set of $\mathbb{R}^2$.

**Definition 2.4.** Word, on the graph $\Gamma_W$ 

Let $m$ be a strictly positive integer. We will call **number-letter** any integer $M_i$ of $\{0, \ldots, N_b - 1\}$, and **word of length** $|M| = m$, on the graph $\Gamma_W$, any set of number-letters of the form:

$$M = (M_1, \ldots, M_m)$$

We will write:

$$T_M = T_{M_1} \circ \ldots \circ T_{M_m}$$
Definition 2.5. Edge relation, on the graph $\Gamma_W$

Given a natural integer $m$, two points $X$ and $Y$ of $\Gamma_{W_m}$ will be called \textit{adjacent} if and only if $X$ and $Y$ are two consecutive vertices of $\Gamma_{W_m}$. We will write:

$$X \sim_Y$$

This edge relation ensures the existence of a word $M = (M_1, \ldots, M_m)$ of length $m$, such that $X$ and $Y$ both belong to the iterate:

$$T_M V_0 = (T_{M_1} \circ \ldots \circ T_{M_m}) V_0$$

Given two points $X$ and $Y$ of the graph $\Gamma_W$, we will say that $X$ and $Y$ are adjacent if and only if there exists a natural integer $m$ such that:

$$X \sim_Y$$

Proposition 2.2. Adresses, on the graph of the Weierstrass function

Given a strictly positive integer $m$, and a word $M = (M_1, \ldots, M_m)$ of length $m \in N^*$, on the graph $\Gamma_{W_m}$, for any integer $j$ of $\{1, \ldots, N_b - 2\}$, any $X = T_M(P_j)$ de $V_m \setminus V_0$, i.e. distinct from one of the $N_b$ fixed point $P_i$, $0 \leq i \leq N_b - 1$, has exactly two adjacent vertices, given by:

$$T_M(P_{j + 1}) \text{ et } T_M(P_{j - 1})$$

where:

$$T_M = T_{M_1} \circ \ldots \circ T_{M_m}$$

By convention, the adjacent vertices of $T_M(P_0)$ are $T_M(P_1)$ and $T_M(P_{N_b - 1})$, those of $T_M(P_{N_b - 1})$, $T_M(P_{N_b - 2})$ and $T_M(P_0)$.

Definition 2.6. $m^{th}$-order subcell, $m \in N^*$, related to a pair of points of the graph $\Gamma_W$

Given a strictly positive integer $m$, and two points $X$ and $Y$ of $V_m$ such that $X \sim_Y$, we will call $m^{th}$-order subcell, related to the pair of points $(X, Y)$, the polygon, the vertices of which are $X$, $Y$, and the intersection points of the edge between the vertices at the extremities of the polygon, i.e. the respective intersection points of polygons of the type $P_{m, j - 1}$ and $P_{m, j}$, $1 \leq j \leq N_b^m - 1$, on the one hand, and of the type $P_{m, j}$ and $P_{m, j + 1}$, $0 \leq j \leq N_b^m - 2$, on the other hand.

Notation. For any integer $j$ belonging to $\{0, \ldots, N_b - 1\}$, any natural integer $m$, and any word $M$ of length $m$, we set:

$$T_M(P_j) = (x(T_M(P_j)), y(T_M(P_j))) \quad T_M(P_{j + 1}) = (x(T_M(P_{j + 1})), y(T_M(P_{j + 1})))$$

$$L_m = x(T_M(P_{j + 1})) - x(T_M(P_j)) = \frac{1}{(N_b - 1) N_b^m} \quad h_{j, m} = y(T_M(P_{j + 1})) - y(T_M(P_j))$$
Figure 3: A $m^{th}$-order subcell, in the case where $\lambda = \frac{1}{2}$, and $N_b = 7$.

**Proposition 2.3. An upper bound and lower bound, for the box-dimension of the graph $\Gamma_{W}$**

For any integer $j$ belonging to $\{0, 1, \ldots, N_b - 2\}$, each natural integer $m$, and each word $M$ of length $m$, let us consider the rectangle, the width of which is:

$$L_m = x (T_M (P_{j+1})) - x (T_M (P_j)) = \frac{1}{(N_b - 1) N_b^m}$$

and height $|h_{j,m}|$, such that the points $T_M (P_{j+1})$ and $T_M (P_{j+1})$ are two vertices of this rectangle.

Then:

$$L_m^{2 - D_W} (N_b - 1)^{2 - D_W} \left\{ \frac{2}{1 - \lambda} \min_{0 \leq j < N_b - 1} \sin \left( \frac{\pi (2 j + 1)}{N_b - 1} \right) - \frac{\pi}{N_b (N_b - 1) (\lambda N_b - 1)} \right\} \leq |h_{j,m}|$$

and:

$$|h_{j,m}| \leq \eta_{2-D_W} L_m^{2 - D_W}$$

where the real constant $\eta_{2-D_W}$ is given by:

$$\eta_{2-D_W} = 2 \pi^2 (N_b - 1)^{2 - D_W} \left\{ \frac{(2 N_b - 1) \lambda (N_b^2 - 1)}{(N_b - 1)^2 (1 - \lambda) (\lambda N_b^2 - 1)} + \frac{2 N_b}{(\lambda N_b^2 - 1) (\lambda N_b^2 - 1)} \right\}$$
There exists thus a positive constant
\[ C = \max \left\{ (N_b - 1)^{3-D_{W}} \left| \frac{2}{1 - \lambda} \min_{0 \leq j \leq N_b - 1} \sin \left( \frac{\pi (2j + 1)}{N_b - 1} \right) - \frac{\pi}{N_b (N_b - 1) (\lambda N_b - 1)} \right|, \eta_{2-D_{W}} \right\} \]
such that the graph $\Gamma_W$ on $L_m$ can be covered by at least and at most:
\[ N_m \left\{ C \left( \frac{L_m}{N_m} \right)^{1-D_{W}} + 1 \right\} = C L_m^{1-D_{W}} N_m^{D_{W}} + N_m \]
squares, the side length of which is $\frac{L_m}{N_m}$.

**Proof.** For any pair of integers $(i_m, j)$ of $\{0, \ldots, N_b - 2\}^2$:
\[ T_{i_m} (P_j) = \left( \frac{x_j + i_m}{N_b}, \lambda y_j + \cos \left( 2 \pi \left( \frac{x_j + i_m}{N_b} \right) \right) \right) \]
For any pair of integers $(i_m, i_m-1, j)$ of $\{0, \ldots, N_b - 2\}^3$:
\[ T_{i_{m-1}} (T_{i_m} (P_j)) = \left( \frac{x_j + i_m + i_m - 1}{N_b}, \lambda^2 y_j + \lambda \cos \left( 2 \pi \left( \frac{x_j + i_m - 1}{N_b} \right) \right) + \cos \left( 2 \pi \left( \frac{x_j + i_m}{N_b} \right) \right) \right) \]
\[ = \left( \frac{x_j + i_m}{N_b^2} + \frac{i_m - 1}{N_b}, \lambda^2 y_j + \lambda \cos \left( 2 \pi \left( \frac{x_j + i_m}{N_b^2} \right) \right) + \cos \left( 2 \pi \left( \frac{x_j + i_m}{N_b^2} + \frac{i_m - 1}{N_b} \right) \right) \right) \]
For any pair of integers $(i_m, i_m-1, i_m-2, j)$ of $\{0, \ldots, N_b - 2\}^4$:
\[ T_{i_{m-2}} (T_{i_{m-1}} (T_{i_m} (P_j))) = \left( \frac{x_j + i_m + i_m - 1}{N_b^3}, \lambda^3 y_j + \lambda^2 \cos \left( 2 \pi \left( \frac{x_j + i_m}{N_b^3} \right) \right) + \cos \left( 2 \pi \left( \frac{x_j + i_m}{N_b^3} + \frac{i_m - 1}{N_b^2} + \frac{i_m - 2}{N_b} \right) \right) \right) \]
Given a strictly positive integer $m$, and two points $X$ and $Y$ of $V_m$ such that:
\[ X \sim_m Y \]
there exists a word $M$ of length $|M| = m$, on the graph $\Gamma_W$, and an integer $j$ of $\{0, \ldots, N_b - 2\}^2$, such that:
\[ X = T_M (P_j), \quad Y = T_M (P_{j+1}) \]
Let us write $T_M$ under the form:
\[ T_M = T_{i_m} \circ T_{i_{m-1}} \circ \cdots \circ T_1 \]
where $(i_1, \ldots, i_m) \in \{0, \ldots, N_b - 1\}^m$.

One has then:
\[ x (T_M (P_j)) = \frac{x_j}{N_b^m} + \sum_{k=1}^{m} \frac{i_k}{N_b^k}, \quad x (T_M (P_{j+1})) = \frac{x_{j+1}}{N_b^m} + \sum_{k=1}^{m} \frac{i_k}{N_b^k} \]
and:

\[
\begin{align*}
    y(T_M(P_j)) &= \lambda^m y_j + \sum_{k=1}^{m} \lambda^{m-k} \cos \left( 2\pi \left( \frac{x_j}{N^k_b} + \sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N^{k-\ell}_b} \right) \right) \\
    y(T_M(P_{j+1})) &= \lambda^m y_{j+1} + \sum_{k=1}^{m} \lambda^{m-k} \cos \left( 2\pi \left( \frac{x_{j+1}}{N^k_b} + \sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N^{k-\ell}_b} \right) \right)
\end{align*}
\]

This leads to:

\[
    h_{j,m} - \lambda^m (y_{j+1} - y_j) = \sum_{k=1}^{m} \lambda^{m-k} \left\{ \cos \left( 2\pi \left( \frac{x_{j+1}}{N^k_b} + \sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N^{k-\ell}_b} \right) \right) - \cos \left( 2\pi \left( \frac{x_j}{N^k_b} + \sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N^{k-\ell}_b} \right) \right) \right\}
\]

\[
= -2 \sum_{k=1}^{m} \lambda^{m-k} \sin \left( \pi \left( \frac{x_{j+1} - x_j}{N^k_b} \right) \right) \sin \left( 2\pi \left( \frac{x_{j+1} + x_j}{2N^k_b} + \sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N^{k-\ell}_b} \right) \right)
\]

Taking into account:

\[
\lambda^m (y_{j+1} - y_j) = \lambda^m \frac{1}{1-\lambda} \left( \cos \left( \frac{2\pi (j+1)}{N_b-1} \right) - \cos \left( \frac{2\pi j}{N_b-1} \right) \right) = -2 \lambda^m \frac{1}{1-\lambda} \sin \left( \frac{\pi}{N_b-1} \right) \sin \left( \frac{\pi (2j+1)}{N_b-1} \right)
\]

one has:

\[
h_{j,m} + 2 \lambda^m \frac{1}{1-\lambda} \sin \left( \frac{\pi}{N_b-1} \right) \sin \left( \frac{\pi (2j+1)}{N_b-1} \right) = -2 \sum_{k=1}^{m} \lambda^{m-k} \sin \left( \frac{\pi}{N_b^{k+1}(N_b-1)} \right) \sin \left( \frac{\pi (2j+1)}{N_b^{k+1}(N_b-1)} + 2\pi \sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N^{k-\ell}_b} \right)
\]

Thus:

\[
\left| y(T_M(P_j)) - y(T_M(P_{j+1})) - 2 \frac{\lambda^m}{1-\lambda} \sin \left( \frac{\pi}{N_b-1} \right) \sin \left( \frac{\pi (2j+1)}{N_b-1} \right) \right| \leq \sum_{k=1}^{m} \frac{2\pi \lambda^{m-k}}{\pi \lambda^m (1 - \frac{1}{\lambda N_b})} N_b^{k+1}(N_b-1)
\]

\[
= \frac{\lambda N_b N_b(N_b-1)}{\pi \lambda^m (1 - \frac{1}{\lambda N_b})}
\]

which leads to:

\[
\left| y(T_M(P_j)) - y(T_M(P_{j+1})) \right| \geq 2 \frac{\lambda^m}{1-\lambda} \sin \left( \frac{\pi}{N_b-1} \right) \sin \left( \frac{\pi (2j+1)}{N_b-1} \right) - \frac{\pi \lambda^m}{N_b(N_b-1)(\lambda N_b-1)}
\]

or:

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The predominant term is thus:

\[
y(T_M(P_{j+1})) - y(T_M(P_j)) \geq 2 \frac{\lambda^m}{1 - \lambda} \sin \left( \frac{\pi}{N_b - 1} \right) \sin \left( \frac{\pi (2j + 1)}{N_b - 1} \right) - \frac{\pi \lambda^m}{N_b(N_b - 1)(\lambda N_b - 1)}
\]

Due to the symmetric roles played by \(T_M(P_j)\) and \(T_M(P_{j+1})\), one may only consider the case when:

\[
y(T_M(P_j)) - y(T_M(P_{j+1})) \geq 2 \frac{\lambda^m}{1 - \lambda} \sin \left( \frac{\pi}{N_b - 1} \right) \sin \left( \frac{\pi (2j + 1)}{N_b - 1} \right) - \frac{\pi \lambda^m}{N_b(N_b - 1)(\lambda N_b - 1)} \geq 0
\]

The predominant term is thus:

\[
\lambda^m = e^{m(D_W - 2) \ln N_b} = N_b^{m(D_W - 2)} = L_m^{2-D_W} (N_b - 1)^{2-D_W}
\]

One also has:

\[
|h_{j,m}| \leq \frac{2 \lambda^m \pi^2 (2j + 1)}{1 - \lambda (N_b - 1)^2} + \frac{2 m \lambda^{m-k}}{N_b - 1} \left\{ \frac{2j + 1}{(N_b - 1) N_b^k} + 2 \frac{k}{\ell = 0} \sum \frac{i_{m-\ell} \lambda^{-k}}{(N_b - 1) N_b^{2k-\ell}} \right\} - \frac{\pi}{N_b(N_b - 1) N_b^k}
\]

\[
= \frac{2 \lambda^m \pi^2 (2j + 1)}{1 - \lambda (N_b - 1)^2} + \frac{2 \lambda^{m-k}}{N_b - 1} \sum_{k=1}^{m} \frac{(2j + 1) \lambda^{-k}}{(N_b - 1) N_b^{2k-1}} + \frac{2 m}{\ell = 0} \sum \frac{i_{m-\ell} \lambda^{-k}}{(N_b - 1) N_b^{2k-\ell}}
\]

\[
\leq \frac{2 \lambda^m \pi^2 (2N_b - 1)}{1 - \lambda (N_b - 1)^2} + \frac{2 \lambda^m (2N_b - 1)}{N_b - 1} \frac{1 - \lambda^{-m} N_b^{-2m}}{\lambda N_b^2 - 1}
\]

\[
+ \frac{2 \lambda^m}{N_b - 1} \frac{2 \lambda^{-1} N_b^{-2} (N_b - 1) (1 - \lambda^{-m} N_b^{-2m})}{(1 - N_b^{-1}) (1 - \lambda^{-1} N_b^{-2})}
\]

\[
+ \frac{2 \lambda^m}{N_b - 1} \frac{2 \lambda^{-1} N_b^{-3} (N_b - 1) (1 - \lambda^{-m} N_b^{-3m})}{(1 - N_b^{-1}) (1 - \lambda^{-1} N_b^{-3})}
\]

\[
\leq \frac{2 \lambda^m \pi^2 (2N_b - 1)}{1 - \lambda (N_b - 1)^2} + \frac{2 \lambda^m (2N_b - 1)}{N_b - 1} \frac{1 - \lambda^{-m} N_b^{-2m}}{\lambda N_b^2 - 1}
\]

\[
+ \frac{4 \pi^2 N_b \lambda^m}{N_b - 1} \left\{ \frac{1}{\lambda N_b^2 - 1} - \frac{1}{\lambda N_b^3 - 1} \right\}
\]

\[
= 2 \pi^2 \lambda^m \left\{ \frac{(2N_b - 1) \lambda (N_b^2 - 1)}{(N_b - 1)^2 (1 - \lambda) (\lambda N_b^2 - 1)} + \frac{2N_b}{(\lambda N_b^2 - 1) (\lambda N_b^3 - 1)} \right\}
\]
Since:

\[ x(T_M(P_{j+1})) - x(T_M(P_j)) = \frac{1}{(N_b - 1)N_b^m} \]

and:

\[ D_W = 2 + \frac{\ln \lambda}{\ln N_b}, \quad \lambda = e^{(D_W-2)\ln N_b} = N_b^{(D_W-2)} \]

one has thus:

\[ |h_{j,m}| \leq 2\pi^2 L_m^{2-D_W} (N_b - 1)^{2-D_W} \left\{ \frac{(2N_b - 1)\lambda (N_b^2 - 1)}{(N_b - 1)^2 (1 - \lambda) (\lambda N_b^2 - 1)} + \frac{2N_b}{(\lambda N_b^2 - 1)(\lambda N_b^3 - 1)} \right\} \]

References


