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1 Introduction

The determination of the box and Hausdorff dimension of the graph of the Weierstrass function has, since long been, a topic of interest. Let us recall that, given $\lambda \in]0,1[$, and b such that $\lambda\, b > 1 + \frac{3\,\pi}{2}$, the Weierstrass function

$$x \in \mathbb{R} \mapsto \sum_{n=0}^{+\infty} \lambda^n \cos\left(\pi\, b^n\, x\right)$$

is continuous everywhere, while nowhere differentiable. The original proof, by K. Weierstrass [Wei72], can also be found in [Tit77]. It has been completed by the one, now a classical one, in the case where $\lambda\, b > 1$, by G. Hardy [Har11].

After the works of A. S. Besicovitch and H. D. Ursell [BU37], it is Benoît Mandelbrot [Man77] who particularly highlighted the fractal properties of the graph of the Weierstrass function. He also conjectured that the Hausdorff dimension of the graph is $D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln b}$. Interesting discussions in relation to this question have been given in the book of K. Falconer [Fal85]. A series of results for the box dimension can be found in the works of J.-L. Kaplan et al. [KMPY84], where the authors show that it is equal to the Lyapunov dimension of the equivalent attracting torus, and in those by T-Y. Hu and K-S. Lau [HL93]. As for the Hausdorff dimension, a proof was given by B. Hunt [Hun98] in 1998 in the case where arbitrary phases are included in each cosinusoidal term of the summation. Recently, K. Barańsky, B. Bárány and J. Romanowska [BBR17] proved that, for any value of the real number b, there exists a threshold value λ_b belonging to the interval $\left]\frac{1}{b}, 1\right[$ such that the aforementioned dimension is equal to $D_{\mathcal{W}}$ for every b in $]\lambda_b, 1[$. Results by W. Shen [She15] go further than the ones of [BBR17]. In [Kel17], G. Keller proposes what appears as a much simpler and very original proof.

In our work [Dav17], where we build a Laplacian on the graph of the Weierstrass function, we came across a simpler means of computing the box dimension of the graph, using a sequence a graphs that approximate the studied one. Results are exposed in the sequel.

2 Framework of the study

In this section, we recall results that are developed in [Dav17].

Notation. In the following, λ and N_b are two real numbers such that:

$$0 < \lambda < 1 \quad , \quad N_b \in \mathbb{N} \quad \text{and} \quad \lambda\, N_b > 1$$

We will consider the (1−periodic) Weierstrass function $\mathcal{W}$, defined, for any real number x, by:

$$\mathcal{W}(x) = \sum_{n=0}^{+\infty} \lambda^n \, \cos\left(2\,\pi\, N_b^n\, x\right)$$

We place ourselves, in the sequel, in the Euclidean plane of dimension 2, referred to a direct orthonormal frame. The usual Cartesian coordinates are (x, y).

The restriction $\Gamma_{\mathcal{W}}$ to $[0,1[\times\mathbb{R}$, of the graph of the Weierstrass function, is approximated by means of a sequence of graphs, built through an iterative process. To this purpose, we introduce the iterated function system of the family of C^∞ contractions from $\mathbb{R}^2$ to $\mathbb{R}^2$:

$$\{T_0, ..., T_{N_b-1}\}$$

where, for any integer i belonging to $\{0, ..., N_b - 1\}$, and any (x, y) of $\mathbb{R}^2$:

$$T_i(x,y) = \left(\frac{x+i}{N_b}, \lambda\, y + \cos\left(2\,\pi\, \left(\frac{x+i}{N_b}\right)\right)\right)$$

Property 2.1.

$$\Gamma_{\mathcal{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}})$$

Definition 2.1. For any integer i belonging to $\{0, ..., N_b - 1\}$, let us denote by:

$$P_i = (x_i, y_i) = \left(\frac{i}{N_b - 1}, \frac{1}{1-\lambda}\, \cos\left(\frac{2\,\pi\, i}{N_b - 1}\right)\right)$$

the fixed point of the contraction T_i.

We will denote by V_0 the ordered set (according to increasing abscissa), of the points:

$$\{P_0, ..., P_{N_b-1}\}$$

The set of points V_0, where, for any i of $\{0, ..., N_b - 2\}$, the point P_i is linked to the point P_{i+1}, constitutes an oriented graph (according to increasing abscissa)), that we will denote by $\Gamma_{\mathcal{W}_0}$. V_0 is called the set of vertices of the graph $\Gamma_{\mathcal{W}_0}$.

For any natural integer m, we set:

$$V_m = \bigcup_{i=0}^{N_b-1} T_i\left(V_{m-1}\right)$$

The set of points V_m, where two consecutive points are linked, is an oriented graph (according to increasing abscissa), which we will denote by $\Gamma_{\mathcal{W}_m}$. V_m is called the set of vertices of the graph $\Gamma_{\mathcal{W}_m}$. We will denote, in the sequel, by

$$\mathcal{N}_m^{\mathcal{S}} = 2\,N_b^m + N_b - 2$$

the number of vertices of the graph $\Gamma_{\mathcal{W}_m}$, and we will write:

$$V_m = \left\{\mathcal{S}_0^m, \mathcal{S}_1^m, \ldots, \mathcal{S}_{\mathcal{N}_m-1}^m\right\}$$

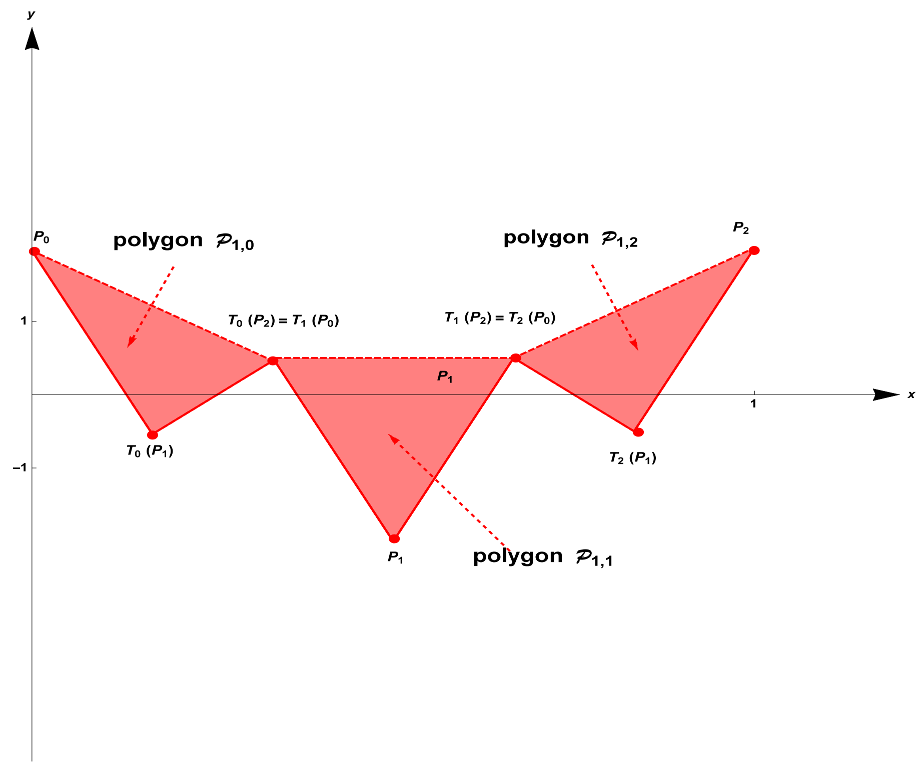

Figure 1: The polygons $\mathcal{P}_{1,0}$, $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, in the case where $\lambda = \dfrac{1}{2}$, and $N_b = 3$.

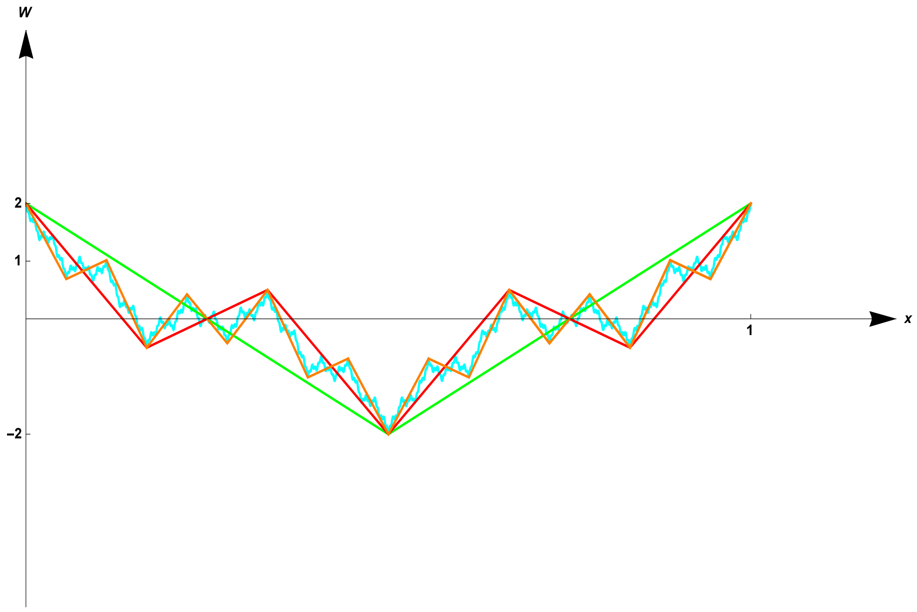

Figure 2: The graphs $\Gamma_{\mathcal{W}_0}$ (in green), $\Gamma_{\mathcal{W}_1}$ (in red), $\Gamma_{\mathcal{W}_2}$ (in orange), $\Gamma_{\mathcal{W}}$ (in cyan), in the case where $\lambda = \dfrac{1}{2}$, and $N_b = 3$.

Definition 2.2. Consecutive vertices on the graph $\Gamma_{\mathcal{W}}$

Two points X et Y de $\Gamma_{\mathcal{W}}$ will be called ***consecutive vertices*** of the graph $\Gamma_{\mathcal{W}}$ if there exists a natural integer m, and an integer j of $\{0, ..., N_b - 2\}$, such that:

$$X = (T_{i_1} \circ \ldots \circ T_{i_m})(P_j) \quad \text{et} \quad Y = (T_{i_1} \circ \ldots \circ T_{i_m})(P_{j+1}) \qquad \{i_1, \ldots, i_m\} \in \{0, ..., N_b - 1\}^m$$

or:

$$X = (T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_m})(P_{N_b - 1}) \quad \text{et} \quad Y = (T_{i_1 + 1} \circ T_{i_2} \ldots \circ T_{i_m})(P_0)$$

Definition 2.3. For any natural integer m, the $\mathcal{N}_m^{\mathcal{S}}$ consecutive vertices of the graph $\Gamma_{\mathcal{W}_m}$ are, also, the vertices of N_b^m simple polygons $\mathcal{P}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 1$, with N_b sides. For any integer j such that $0 \leqslant j \leqslant N_b^m - 1$, one obtains each polygon by linking the point number j to the point number $j + 1$ if $j = i \bmod N_b$, $0 \leqslant i \leqslant N_b - 2$, and the point number j to the point number $j - N_b + 1$ if $j = -1 \bmod N_b$. These polygons generate a Borel set of $\mathbb{R}^2$.

Definition 2.4. Word, on the graph $\Gamma_{\mathcal{W}}$

Let m be a strictly positive integer. We will call **number-letter** any integer $\mathcal{M}_i$ of $\{0, \ldots, N_b - 1\}$, and **word of length** $|\mathcal{M}| = m$, on the graph $\Gamma_{\mathcal{W}}$, any set of number-letters of the form:

$$\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_m)$$

We will write:

$$T_{\mathcal{M}} = T_{\mathcal{M}_1} \circ \ldots \circ T_{\mathcal{M}_m}$$

Definition 2.5. Edge relation, on the graph $\Gamma_{\mathcal{W}}$

Given a natural integer m, two points X and Y of $\Gamma_{\mathcal{W}_m}$ will be called ***adjacent*** if and only if X and Y are two consecutive vertices of $\Gamma_{\mathcal{W}_m}$. We will write:

$$X \underset{m}{\sim} Y$$

This edge relation ensures the existence of a word $\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_m)$ of length m, such that X and Y both belong to the iterate:

$$T_{\mathcal{M}}\, V_0 = (T_{\mathcal{M}_1} \circ \ldots \circ T_{\mathcal{M}_m})\; V_0$$

Given two points X and Y of the graph $\Gamma_{\mathcal{W}}$, we will say that X and Y are ***adjacent*** if and only if there exists a natural integer m such that:

$$X \underset{m}{\sim} Y$$

Proposition 2.2. ***Adresses, on the graph of the Weierstrass function***

Given a strictly positive integer m, and a word $\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_m)$ of length $m \in \mathbb{N}^\star$, on the graph $\Gamma_{\mathcal{W}_m}$, for any integer j of $\{1, ..., N_b - 2\}$, any $X = T_{\mathcal{M}}(P_j)$ de $V_m \setminus V_0$, i.e. distinct from one of the N_b fixed point P_i, $0 \leqslant i \leqslant N_b - 1$, has exactly two adjacent vertices, given by:

$$T_{\mathcal{M}}(P_{j+1}) \quad et \quad T_{\mathcal{M}}(P_{j-1})$$

where:

$$T_{\mathcal{M}} = T_{\mathcal{M}_1} \circ \ldots \circ T_{\mathcal{M}_m}$$

By convention, the adjacent vertices of $T_{\mathcal{M}}(P_0)$ are $T_{\mathcal{M}}(P_1)$ and $T_{\mathcal{M}}(P_{N_b-1})$, those of $T_{\mathcal{M}}(P_{N_b-1})$, $T_{\mathcal{M}}(P_{N_b-2})$ and $T_{\mathcal{M}}(P_0)$.

Definition 2.6. m^{th}–order subcell, $m \in \mathbb{N}^\star$, related to a pair of points of the graph $\Gamma_{\mathcal{W}}$

Given a strictly positive integer m, and two points X and Y of V_m such that $X \underset{m}{\sim} Y$, we will call m^{th}**–order subcell, related to the pair of points** (X, Y), the polygon, the vertices of which are X, Y, and the intersection points of the edge between the vertices at the extremities of the polygon, i.e. the respective intersection points of polygons of the type $\mathcal{P}_{m,j-1}$ and $\mathcal{P}_{m,j}$, $1 \leqslant j \leqslant N_b^m - 1$, on the one hand, and of the type $\mathcal{P}_{m,j}$ and $\mathcal{P}_{m,j+1}$, $0 \leqslant j \leqslant N_b^m - 2$, on the other hand.

Notation. For any integer j belonging to $\{0, ..., N_b - 1\}$, any natural integer m, and any word $\mathcal{M}$ of length m, we set:

$$T_{\mathcal{M}}\left(P_j\right) = \left(x\left(T_{\mathcal{M}}\left(P_j\right)\right), y\left(T_{\mathcal{M}}\left(P_j\right)\right)\right) \quad , \quad T_{\mathcal{M}}\left(P_{j+1}\right) = \left(x\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right), y\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right)\right)$$

$$L_m = x\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right) - x\left(T_{\mathcal{M}}\left(P_j\right)\right) = \frac{1}{\left(N_b - 1\right) N_b^m} \quad , \quad h_{j,m} = y\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right) - y\left(T_{\mathcal{M}}\left(P_j\right)\right)$$

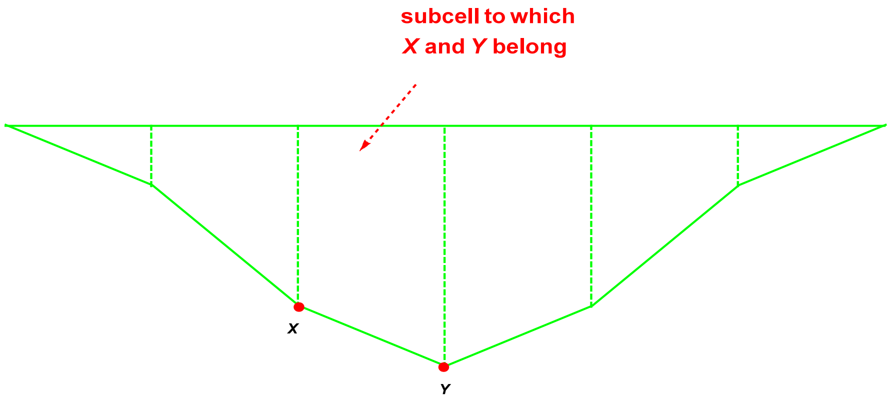

Figure 3: A m^{th}−order subcell, in the case where $\lambda = \frac{1}{2}$, and $N_b = 7$.

Proposition 2.3. ***An upper bound and lower bound, for the box-dimension of the graph*** $\Gamma_{\mathcal{W}}$

For any integer j belonging to $\{0, 1, \ldots, N_b - 2\}$, each natural integer m, and each word $\mathcal{M}$ of length m, let us consider the rectangle, the width of which is:

$$L_m = x\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right) - x\left(T_{\mathcal{M}}\left(P_j\right)\right) = \frac{1}{\left(N_b - 1\right) N_b^m}$$

and height $|h_{j,m}|$, such that the points $T_{\mathcal{M}}\left(P_{j+1}\right)$ and $T_{\mathcal{M}}\left(P_{j+1}\right)$ are two vertices of this rectangle. Then:

$$L_m^{2-D_{\mathcal{W}}} \left(N_b - 1\right)^{2-D_{\mathcal{W}}} \left|\left\{\frac{2}{1-\lambda} \min_{0 \leqslant j \leqslant N_b - 1} \sin\left(\frac{\pi\,(2\,j+1)}{N_b - 1}\right) - \frac{\pi}{N_b\,(N_b - 1)\,(\lambda\,N_b - 1)}\right\}\right| \leqslant |h_{j,m}|$$

and:

$$|h_{j,m}| \leqslant \eta_{2-D_{\mathcal{W}}}\, L_m^{2-D_{\mathcal{W}}}$$

where the real constant $\eta_{2-D_{\mathcal{W}}}$ is given by :

$$\eta_{2-D_{\mathcal{W}}} = 2\,\pi^2 \left(N_b - 1\right)^{2-D_{\mathcal{W}}} \left\{\frac{(2\,N_b - 1)\,\lambda\,(N_b^2 - 1)}{(N_b - 1)^2\,(1 - \lambda)\,(\lambda\,N_b^2 - 1)} + \frac{2\,N_b}{(\lambda\,N_b^2 - 1)\,(\lambda\,N_b^3 - 1)}\right\}$$

There exists thus a positive constant

$$C = \max\left\{ (N_b-1)^{2-D_{\mathcal{W}}} \left| \left\{ \frac{2}{1-\lambda} \min_{0\leqslant j\leqslant N_b-1} \sin\left(\frac{\pi\,(2\,j+1)}{N_b-1}\right) - \frac{\pi}{N_b\,(N_b-1)\,(\lambda\,N_b-1)} \right\} \right|, \eta_{2-D_{\mathcal{W}}} \right\}$$

such that the graph $\Gamma_{\mathcal{W}}$ on L_m can be covered by at least and at most:

$$N_m \left\{ C \left(\frac{L_m}{N_m}\right)^{1-D_{\mathcal{W}}} + 1 \right\} = C\, L_m^{1-D_{\mathcal{W}}}\, N_m^{D_{\mathcal{W}}} + N_m$$

squares, the side length of which is $\dfrac{L_m}{N_m}$.

Proof. For any pair of integers (i_m, j) of $\{0, ..., N_b-2\}^2$:

$$T_{i_m}(P_j) = \left(\frac{x_j+i_m}{N_b}, \lambda\, y_j + \cos\left(2\,\pi\left(\frac{x_j+i_m}{N_b}\right)\right)\right)$$

For any pair of integers (i_m, i_{m-1}, j) of $\{0, ..., N_b-2\}^3$:

$$\begin{aligned} T_{i_{m-1}}(T_{i_m}(P_j)) &= \left(\frac{\frac{x_j+i_m}{N_b}+i_{m-1}}{N_b}, \lambda^2\, y_j + \lambda\,\cos\left(2\,\pi\left(\frac{x_j+i_m}{N_b}\right)\right) + \cos\left(2\,\pi\left(\frac{\frac{x_j+i_m}{N_b}+i_{m-1}}{N_b}\right)\right)\right) \\ &= \left(\frac{x_j+i_m}{N_b^2} + \frac{i_{m-1}}{N_b}, \lambda^2\, y_j + \lambda\,\cos\left(2\,\pi\left(\frac{x_j+i_m}{N_b}\right)\right) + \cos\left(2\,\pi\left(\frac{x_j+i_m}{N_b^2}+\frac{i_{m-1}}{N_b}\right)\right)\right) \end{aligned}$$

For any pair of integers $(i_m, i_{m-1}, i_{m-2}, j)$ of $\{0, ..., N_b-2\}^4$:

$$\begin{aligned} T_{i_{m-2}}\left(T_{i_{m-1}}(T_{i_m}(P_j))\right) = &\left(\frac{x_j+i_m}{N_b^3} + \frac{i_{m-1}}{N_b^2} + \frac{i_{m-2}}{N_b}, \right. \\ &\lambda^3\, y_j + \lambda^2\,\cos\left(2\,\pi\left(\tfrac{x_j+i_m}{N_b}\right)\right) \\ &\left. +\lambda\,\cos\left(2\,\pi\left(\frac{x_j+i_m}{N_b^2}+\frac{i_{m-1}}{N_b}\right)\right) + \cos\left(2\,\pi\left(\frac{x_j+i_m}{N_b^3}+\frac{i_{m-1}}{N_b^2}+\frac{i_{m-2}}{N_b}\right)\right)\right) \end{aligned}$$

Given a strictly positive integer m, and two points X and Y of V_m such that:

$$X \underset{m}{\sim} Y$$

there exists a word $\mathcal{M}$ of length $|\mathcal{M}| = m$, on the graph $\Gamma_{\mathcal{W}}$, and an integer j of $\{0, ..., N_b-2\}^2$, such that:

$$X = T_{\mathcal{M}}(P_j) \quad , \quad Y = T_{\mathcal{M}}(P_{j+1})$$

Let us write $T_{\mathcal{M}}$ under the form:

$$T_{\mathcal{M}} = T_{i_m} \circ T_{i_{m-1}} \circ \ldots \circ T_{i_1}$$

where $(i_1, \ldots, i_m) \in \{0, ..., N_b-1\}^m$.

One has then:

$$x\,(T_{\mathcal{M}}(P_j)) = \frac{x_j}{N_b^m} + \sum_{k=1}^{m} \frac{i_k}{N_b^k} \quad , \quad x\,(T_{\mathcal{M}}(P_{j+1})) = \frac{x_{j+1}}{N_b^m} + \sum_{k=1}^{m} \frac{i_k}{N_b^k}$$

and:

$$\left\{ \begin{array}{rcl} y\left(T_{\mathcal{M}}\left(P_j\right)\right) & = & \lambda^m\, y_j + \displaystyle\sum_{k=1}^m \lambda^{m-k}\, \cos\left(2\,\pi\, \left(\frac{x_j}{N_b^k} + \sum_{\ell=0}^k \frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right) \\ y\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right) & = & \lambda^m\, y_{j+1} + \displaystyle\sum_{k=1}^m \lambda^{m-k}\, \cos\left(2\,\pi\, \left(\frac{x_{j+1}}{N_b^k} + \sum_{\ell=0}^k \frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right) \end{array} \right.$$

This leads to:

$$\begin{array}{rcl} h_{j,m} - \lambda^m\, \left(y_{j+1} - y_j\right) & = & \displaystyle\sum_{k=1}^m \lambda^{m-k}\, \left\{\cos\left(2\,\pi\, \left(\frac{x_{j+1}}{N_b^k} + \sum_{\ell=0}^k \frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right) - \cos\left(2\,\pi\, \left(\frac{x_j}{N_b^k} - \sum_{\ell=0}^k \frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right)\right\} \\ & = & -2 \displaystyle\sum_{k=1}^m \lambda^{m-k}\, \sin\left(\pi\, \left(\frac{x_{j+1} - x_j}{N_b^k}\right)\right)\, \sin\left(2\,\pi\, \left(\frac{x_{j+1} + x_j}{2\,N_b^k} + \sum_{\ell=0}^k \frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right) \end{array}$$

Taking into account:

$$\begin{array}{rcl} \lambda^m\, \left(y_{j+1} - y_j\right) & = & \dfrac{\lambda^m}{1-\lambda}\, \left(\cos\left(\dfrac{2\,\pi\,(j+1)}{N_b - 1}\right) - \cos\left(\dfrac{2\,\pi\,j}{N_b - 1}\right)\right) \\ & = & -2\, \dfrac{\lambda^m}{1-\lambda}\, \sin\left(\dfrac{\pi}{N_b - 1}\right)\, \sin\left(\dfrac{\pi\,(2\,j+1)}{N_b - 1}\right) \end{array}$$

one has:

$$h_{j,m} + 2\, \frac{\lambda^m}{1-\lambda}\, \sin\left(\frac{\pi}{N_b - 1}\right)\, \sin\left(\frac{\pi\,(2\,j+1)}{N_b - 1}\right) = -2 \sum_{k=1}^m \lambda^{m-k}\, \sin\left(\frac{\pi}{N_b^{k+1}\,(N_b - 1)}\right)\, \sin\left(\frac{\pi\,(2\,j+1)}{N_b^{k+1}\,(N_b - 1)} + 2\,\pi \sum_{\ell=0}^k \frac{i_{m-\ell}}{N_b^{k-\ell}}\right)$$

Thus:

$$\begin{array}{rcl} \left| y\left(T_{\mathcal{M}}\left(P_j\right)\right) - y\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right) - 2\, \dfrac{\lambda^m}{1-\lambda}\, \sin\left(\dfrac{\pi}{N_b - 1}\right)\, \sin\left(\dfrac{\pi\,(2\,j+1)}{N_b - 1}\right)\right| & \leqslant & \displaystyle\sum_{k=1}^m \frac{2\,\pi\,\lambda^{m-k}}{N_b^{k+1}\,(N_b - 1)} \\ & = & \dfrac{\pi\,\lambda^m\, \left(1 - \frac{1}{\lambda^m\, N_b^m}\right)}{\lambda\, N_b\, N_b\, (N_b - 1)\, \left(1 - \frac{1}{\lambda\, N_b}\right)} \\ & \leqslant & \dfrac{\pi\,\lambda^m}{N_b\,(N_b - 1)\, (\lambda\, N_b - 1)} \end{array}$$

which leads to:

$$y\left(T_{\mathcal{M}}\left(P_j\right)\right) - y\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right) \geqslant 2\, \frac{\lambda^m}{1-\lambda}\, \sin\left(\frac{\pi}{N_b - 1}\right)\, \sin\left(\frac{\pi\,(2\,j+1)}{N_b - 1}\right) - \frac{\pi\,\lambda^m}{N_b\,(N_b - 1)\, (\lambda\, N_b - 1)}$$

or:

$$y\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right)-y\left(T_{\mathcal{M}}\left(P_{j}\right)\right) \geqslant 2\,\frac{\lambda^m}{1-\lambda}\,\sin\left(\frac{\pi}{N_b-1}\right)\,\sin\left(\frac{\pi\,(2\,j+1)}{N_b-1}\right)-\frac{\pi\,\lambda^m}{N_b\,(N_b-1)\,(\lambda\,N_b-1)}$$

Due to the symmetric roles played by $T_{\mathcal{M}}(P_j)$ and $T_{\mathcal{M}}(P_{j+1})$, one may only consider the case when:

$$\begin{aligned}
y\left(T_{\mathcal{M}}\left(P_{j}\right)\right)-y\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right) &\geqslant 2\,\frac{\lambda^m}{1-\lambda}\,\sin\left(\frac{\pi}{N_b-1}\right)\,\sin\left(\frac{\pi\,(2\,j+1)}{N_b-1}\right)-\frac{\pi\,\lambda^m}{N_b\,(N_b-1)\,(\lambda\,N_b-1)} \geqslant 0\\
&\geqslant \lambda^m\left\{\frac{2}{1-\lambda}\,\min_{0\leqslant j\leqslant N_b-1}\,\sin\left(\frac{\pi\,(2\,j+1)}{N_b-1}\right)-\frac{\pi}{N_b\,(N_b-1)\,(\lambda\,N_b-1)}\right\}
\end{aligned}$$

The predominant term is thus:

$$\lambda^m=e^{m\,(D_{\mathcal{W}}-2)\,\ln N_b}=N_b^{m\,(D_{\mathcal{W}}-2)}=L_m^{2-D_{\mathcal{W}}}\,(N_b-1)^{2-D_{\mathcal{W}}}$$

One also has:

$$\begin{aligned}
|h_{j,m}| &\leqslant \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,j+1)}{(N_b-1)^2}+2\sum_{k=1}^{m}\lambda^{m-k}\,\pi\left\{\frac{2\,j+1}{(N_b-1)\,N_b^k}+2\sum_{\ell=0}^{k}\frac{i_{m-\ell}}{N_b^{k-\ell}}\right\}\frac{\pi}{(N_b-1)\,N_b^k}\\
&= \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,j+1)}{(N_b-1)^2}+\frac{2\,\pi^2\,\lambda^m}{N_b-1}\sum_{k=1}^{m}\left\{\frac{(2\,j+1)\,\lambda^{-k}}{(N_b-1)\,N_b^{2k}}+2\sum_{\ell=0}^{k}\frac{i_{m-\ell}\,\lambda^{-k}}{N_b^{2k-\ell}}\right\}\\
&= \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,j+1)}{(N_b-1)^2}\\
&\quad+\frac{2\,\pi^2\,\lambda^m}{N_b-1}\left\{\frac{\lambda^{-1}\,N_b^{-2}\,(2\,j+1)}{(N_b-1)}\,\frac{(1-\lambda^{-m}\,N_b^{-2m})}{1-\lambda^{-1}\,N_b^{-2}}+2\sum_{k=1}^{m}\frac{(N_b-1)\,\lambda^{-k}}{N_b^{2k}}\,\frac{1-N_b^{-k-1}}{1-N_b^{-1}}\right\}\\
&\leqslant \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,N_b-1)}{(N_b-1)^2}+\frac{2\,\pi^2\,\lambda^m}{N_b-1}\,\frac{(2\,N_b-1)}{(N_b-1)}\,\frac{(1-\lambda^{-m}\,N_b^{-2m})}{\lambda\,N_b^2-1}\\
&\quad+\frac{2\,\pi^2\,\lambda^m}{N_b-1}\,2\,\frac{\lambda^{-1}\,N_b^{-2}\,(N_b-1)\,(1-\lambda^{-m}\,N_b^{-2m})}{(1-N_b^{-1})\,(1-\lambda^{-1}\,N_b^{-2})}\\
&\quad-\frac{2\,\pi^2\,\lambda^m}{N_b-1}\,2\,\frac{\lambda^{-1}\,N_b^{-3}\,(N_b-1)\,(1-\lambda^{-m}\,N_b^{-3m})}{(1-N_b^{-1})\,(1-\lambda^{-1}\,N_b^{-3})}\\
&\leqslant \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,N_b-1)}{(N_b-1)^2}+\frac{2\,\pi^2\,\lambda^m}{N_b-1}\,\frac{(2\,N_b-1)}{(N_b-1)}\,\frac{1}{\lambda\,N_b^2-1}\\
&\quad+\frac{4\,\pi^2\,N_b\,\lambda^m}{N_b-1}\left\{\frac{1}{\lambda\,N_b^2-1}-\frac{1}{\lambda\,N_b^3-1}\right\}\\
&= 2\,\pi^2\,\lambda^m\left\{\frac{(2\,N_b-1)\,\lambda\,(N_b^2-1)}{(N_b-1)^2\,(1-\lambda)\,(\lambda\,N_b^2-1)}+\frac{2\,N_b}{(\lambda\,N_b^2-1)\,(\lambda\,N_b^3-1)}\right\}
\end{aligned}$$

Since:

$$x\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right)-x\left(T_{\mathcal{M}}\left(P_{j}\right)\right)=\frac{1}{\left(N_b-1\right)N_b^m}$$

and:

$$D_{\mathcal{W}}=2+\frac{\ln\lambda}{\ln N_b}\quad,\quad\lambda=e^{(D_{\mathcal{W}}-2)\,\ln N_b}=N_b^{(D_{\mathcal{W}}-2)}$$

one has thus:

$$|h_{j,m}|\quad\leqslant\quad 2\,\pi^2\,L_m^{2-D_{\mathcal{W}}}\,\left(N_b-1\right)^{2-D_{\mathcal{W}}}\,\left\{\frac{\left(2\,N_b-1\right)\lambda\left(N_b^2-1\right)}{\left(N_b-1\right)^2\left(1-\lambda\right)\left(\lambda\,N_b^2-1\right)}+\frac{2\,N_b}{\left(\lambda\,N_b^2-1\right)\left(\lambda\,N_b^3-1\right)}\right\}$$

□

References

[BBR17] K. Barańsky, B. Bárány, and J. Romanowska. On the dimension of the graph of the classical Weierstrass function. *Advances in Math.*, 2017.

[BU37] A. S. Besicovitch and H. D. Ursell. Sets of fractional dimensions. *Notices of the AMS*, 1937.

[Dav17] Claire David. Laplacian, on the graph of the Weierstrass function, arxiv:1703.03371, 2017.

[Fal85] K. Falconer. *The Geometry of Fractal Sets.* Cambridge University Press, 1985.

[Har11] G. Hardy. Theorems connected with Maclaurin's test for the convergence of series. *The Proceedings of the Royal Society of London*, 1911.

[HL93] T.-Y. Hu and K.-S. Lau. Fractal dimensions and singularities of the weierstrass type functions. *Transactions of the American Mathematical Society*, 1993.

[Hun98] B. Hunt. The Hausdorff dimension of graphs of Weierstrass functions. *Proc. Amer. Math. Soc.*, 1998.

[Kel17] G. Keller. A simpler proof for the dimension of the graph of the classical Weierstrass function. *Ann. Inst. Poincaré*, 2017.

[KMPY84] J. Kaplan, J. Mallet-Paret, and J. Yorke. The Lyapunov dimension of a nowhere differentiable attracting torus. *Ergodic Theory Dynam. Systems*, 1984.

[Man77] B. B. Mandelbrot. *Fractals: form, chance, and dimension.* San Francisco: Freeman, 1977.

[She15] Weixiao Shen. Hausdorff dimension of the graphs of the classical Weierstrass functions, arxiv:1505.03986, 2015.

[Tit77] E. C. Titschmarsh. *The theory of functions, Second edition.* Oxford University Press, 1977.

[Wei72] K. Weierstrass. Über continuirliche funktionen eines reellen arguments, die für keinen werth des letzteren einen bestimmten differentialquotienten besitzen, in karl weiertrass mathematische werke, abhandlungen ii. *Akademie der Wissenchaften am 18 Juli 1872*, 1872.