Liftings for Differential Privacy
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Abstract

Recent developments in formal verification have identified approximate liftings (also known as approximate couplings) as a clean, compositional abstraction for proving differential privacy. There are two styles of definitions for this construction. Earlier definitions require the existence of one or more witness distributions, while a recent definition by Sato uses universal quantification over all sets of samples. These notions have different strengths and weaknesses: the universal version is more general than the existential ones, but the existential versions enjoy more precise composition principles.

We propose a novel, existential version of approximate lifting, called $\star$-lifting, and show that it is equivalent to Sato’s construction for discrete probability measures. Our work unifies all known notions of approximate lifting, giving cleaner properties, more general constructions, and more precise composition theorems for both styles of lifting, enabling richer proofs of differential privacy. We also clarify the relation between existing definitions of approximate lifting, and generalize our constructions to approximate liftings based on $f$-divergences.

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1 Introduction

Differential privacy [7] is a rigorous notion of statistical privacy that delivers strong individual guarantees for privacy-preserving computations. Informally, differential privacy guarantees to every individual that their (non)-participation in a database will have a small (in a rigorous, quantitative sense) effect on the results obtained by third parties when querying the database. The formal definition of differential privacy is parametrized by two non-negative real numbers, $(\epsilon, \delta)$. These parameters quantify the effect of individuals on the output of the private query; smaller values give stronger privacy guarantees. The main strengths of differential privacy lie in its theoretical elegance, minimal assumptions, and flexibility for many applications.

Motivated by the importance of differential privacy, programming language researchers have developed approaches based on dynamic analysis, type systems, and program logics for formally proving differential privacy for programs. (We refer the interested reader to a recent
survey [6] for an overview of this growing field.) In this paper, we consider approaches based on relational program logics [2,5,10,11]. To capture the quantitative nature of differential privacy, these systems rely on a quantitative generalization of probabilistic couplings (see, e.g., [9,13,14]), called approximate liftings or $(\epsilon, \delta)$-liftings. Existing works have considered several potential definitions. While all definitions support compositional reasoning and enable program logics that can verify complex examples from the privacy literature, the various notions of approximate liftings have different strengths and weaknesses.

Broadly speaking, one class of definitions require the existence of one or two witness distributions that ‘couple’ the two executions of programs. The earliest definition [3] supports accuracy-based reasoning for the Laplace mechanism, while subsequent definitions [2,10] support more precise composition principles from differential privacy and can be generalized to other notions of distance on distributions. These definitions, and their associated program logics, were designed for discrete distributions.

In the course of extending these ideas to continuous distributions, Sato [11] proposes a radically different notion of approximate lifting, which does not rely on witness distributions. Instead, it uses a universal quantification over all sets of samples. Sato shows that this definition is strictly more general than the existential versions, but it is unclear (a) whether the gap can be closed and (b) whether his construction satisfies the same composition principles enjoyed by some existential definitions.

As a consequence, there is currently no single approximate lifting with the properties needed to support all existing formalized proofs of differential privacy. Furthermore, some of the most involved privacy proofs cannot be formalized at all, as their proofs require a combination of tools from several kinds of approximate liftings.

Outline of the paper

After reviewing the necessary mathematical preliminaries in Section 2, we introduce our main technical contribution: a new, existential definition of approximate lifting. This construction, which we call $\star$-lifting, is a generalization of an existing definition by Barthe and Olmedo [2], Olmedo [10]. The key idea is to allow the witness distributions to have a larger domain, broadening the class of approximate liftings. By a maximum flow/minimum cut argument, we show that $\star$-liftings are equivalent to Sato’s lifting over discrete distributions. This equivalence can be viewed as an approximate version of Strassen’s theorem [12], a classical result in probability theory describing the existence of probabilistic couplings. We present the definition of $\star$-lifting and the proof of equivalence in Section 3.

Then, we show that $\star$-liftings satisfy desirable theoretical properties. We are able to leverage the equivalence of liftings in two ways. In one direction, Sato’s definition gives simpler proofs of more general properties of $\star$-liftings. In the other direction, $\star$-liftings—like other existential definitions—can smoothly incorporate composition principles from the theory of differential privacy. Our connection shows that Sato’s definition can use these principles in the discrete case. We describe the key theoretical properties of $\star$-liftings in Section 4.

Finally, we provide a thorough comparison of $\star$-lifting with existing definitions of approximate lifting in Section 5 and describe how to construct $\star$-liftings for more general version of approximate liftings based on $f$-divergences in Section 6.

Overall, the equivalence of $\star$-liftings and Sato’s lifting, along with the natural theoretical properties satisfied by the common notion, suggest that these definitions are two views on the same concept: an approximate version of probabilistic coupling.
2 Background

To model probabilistic behavior, we work with discrete sub-distributions.

Definition 1. A sub-distribution over a set $A$ is defined by its mass function $\mu : A \rightarrow \mathbb{R}^+$, which gives the probability of the singleton events $a \in A$. This mass function must be s.t. $|\mu| \triangleq \sum_{a \in A} \mu(a)$ is well-defined and at most 1. In particular, the support $\text{supp}(\mu) \triangleq \{a \in A | \mu(a) \neq 0\}$ must be discrete (i.e. finite or countably infinite). When the weight $|\mu|$ is equal to 1, we call $\mu$ a (proper) distribution. We let $D(A)$ denote the set of sub-distributions over $A$. The probability of an event $E(x)$ w.r.t. $\mu$, written $\mathbb{P}_{x \sim \mu}[E(x)]$ or $\mathbb{P}_\mu[E]$, is defined as $\sum_{x \in A \mid E(x)} \mu(x)$.

Simple examples of sub-distributions include the null sub-distribution $\Theta^A \in D(A)$, which maps each element of $A$ to 0, and the Dirac distribution centered on $x$, written $\mathbb{1}_x$, which maps $x$ to 1 and all other elements to 0. One can equip distributions with a monadic structure using the Dirac distributions $\mathbb{1}_x$ for the unit and distribution expectation $\mathbb{E}_{x \sim \mu}[f(x)]$ for the bind; if $\mu$ is a distribution over $A$ and $f$ has type $A \rightarrow \mathbb{D}(B)$, then the bind defines a sub-distribution over $B$: $\mathbb{E}_{a \sim \mu}[f(a)] : b \mapsto \sum_a \mu(a) \cdot f(a)(b)$.

If $f : A \rightarrow B$, we can lift $f$ to a function $f^2 : D(A) \rightarrow D(B)$ as follows: $f^2(\mu) \triangleq \mathbb{E}_{a \sim \mu}[\mathbb{1}_f(a)]$ — or, equivalently, $f^2(\mu) : b \mapsto \mathbb{P}_{a \sim \mu}[a \in f^{-1}(b)]$. For instance, when working with sub-distributions over pairs, this allows to obtain the probabilistic versions $\pi_1^*$ and $\pi_2^*$ (called marginals) of the usual projections $\pi_1$ and $\pi_2$. One can check that the first and second marginals $\pi_1^*(\mu)$ and $\pi_2^*(\mu)$ of a distribution $\mu$ over $A \times B$ are also given by the following equations: $\pi_1^*(\mu)(a) = \sum_{b \in B} \mu(a, b)$ and $\pi_2^*(\mu)(b) = \sum_{a \in A} \mu(a, b)$. When $f : A \rightarrow \mathbb{D}(B)$, we will abuse notation and write the lifting $f^2 : D(A) \rightarrow D(B)$ to mean $f^2(\mu) \triangleq \mathbb{E}_{x \sim \mu}[f(x)]$.

Finally, if $\alpha : A \rightarrow \mathbb{R}^+$, we write $\alpha[X] \in \mathbb{R}^+ \cup \{\infty\}$ for $\sum_{x \in A} \alpha(x)$. Moreover, if $\alpha : A \times B \rightarrow \mathbb{R}^+$, we write $\alpha[X,Y]$ (resp. $\alpha[x,Y]$, $\alpha[X,y]$) for $\alpha[X \times Y]$ (resp. $\alpha[x \times Y$, $\alpha[X \times \{y\}]$). Note that for a sub-distribution $\mu \in D(A)$ and an event $E \subseteq A$, $\mathbb{P}_\mu[E] = \mu(E)$.

We now review the definition of differential privacy.

Definition 2 (Dwork et al. [7]). A probabilistic computation $M : A \rightarrow \mathbb{D}(B)$ satisfies $(\epsilon, \delta)$-differential privacy w.r.t. an adjacency relation $\phi \subseteq A \times A$ iff for every pair of inputs $a, a' \in A$ such that $a \phi a'$ and every subset of outputs $E \subseteq B$,

$$\mathbb{P}_{M(a)}[E] \leq e^\epsilon \cdot \mathbb{P}_{M(a')}[E] + \delta.$$

It is useful to define a notion of distance on distributions, reflecting differential privacy.

Definition 3 (Barthe and Olmedo [2], Barthe et al. [3], Olmedo [10]). Let $\epsilon \geq 0$. The $\epsilon$-DP divergence $\Delta_\epsilon(\mu_1, \mu_2)$ between two sub-distributions $\mu_1, \mu_2 \in D(B)$ is defined as

$$\sup_{E \subseteq B} (\mathbb{P}_{\mu_1}[E] - e^\epsilon \cdot \mathbb{P}_{\mu_2}[E]).$$

Then, differential privacy admits an alternative characterization based on DP divergence.

Lemma 4. A probabilistic computation $M : A \rightarrow \mathbb{D}(B)$ satisfies $(\epsilon, \delta)$-differential privacy w.r.t. an adjacency relation $\phi \subseteq A \times A$ iff $\Delta_\epsilon(M(a), M(a')) \leq \delta$ for every pair of inputs $a, a' \in A$ such that $a \phi a'$.

Our new definition of approximate lifting is inspired by a version of approximate liftings involving two witness distributions, proposed by Barthe and Olmedo [2], Olmedo [10].
Liftings for Differential Privacy

Definition 5 (Barthe and Olmedo [2, Olmedo [10]). Let \( \mu_1 \in \mathcal{D}(A) \) and \( \mu_2 \in \mathcal{D}(B) \) be sub-distributions, \( \epsilon, \delta \in \mathbb{R}^+ \) and \( \mathcal{R} \) be a binary relation over \( A \) & \( B \). An \((\epsilon, \delta)\)-approximate 2-lifting of \( \mu_1 \) & \( \mu_2 \) for \( \mathcal{R} \) is a pair \((\mu_{\circ, \circ}, \mu_{\circ, \circ})\) of sub-distributions over \( A \times B \) s.t.

1. \( \pi_1^1(\mu_{\circ, \circ}) = \mu_1 \) and \( \pi_2^2(\mu_{\circ, \circ}) = \mu_2 \);
2. \( \Delta_{\epsilon}(\mu_{\circ, \circ}, \mu_{\circ, \circ}) \leq \delta \); and
3. \( \text{supp}(\mu) \subseteq \mathcal{R} \).

We write \( \mu_1 \mathcal{R}_{\epsilon, \delta}^{(2)} \mu_2 \) if there exists an \((\epsilon, \delta)\)-approximate (2-)lifting of \( \mu_1 \) & \( \mu_2 \) for \( \mathcal{R} \); the \((2)\) indicates that there are two witnesses in this definition of lifting.

Combined with Lemma 3, a probabilistic computation \( M : A \rightarrow \mathcal{D}(B) \) is \((\epsilon, \delta)\)-differentially private if and only if for every two adjacent inputs \( a \neq a' \), there is an approximate lifting of the equality relation: \( M(a) = \mathcal{R}_{\epsilon, \delta}^{(2)} M(a') \).

2-liftings can be generalized by varying the notion of distance given by \( \Delta_{\epsilon} \); we will return to this point in Section 3. These liftings also satisfy useful theoretical properties, but some of the properties are not as general as we would like. For example, it is known that 2-liftings satisfy the following mapping property.

Theorem 6 (Barthe et al. [4]). Let \( \mu_1 \in \mathcal{D}(A_1) \), \( \mu_2 \in \mathcal{D}(A_2) \), \( f_1 : A_1 \rightarrow B_1 \), \( f_2 : A_2 \rightarrow B_2 \) surjective maps and \( \mathcal{R} \) a binary relation on \( B_1 \) & \( B_2 \). Then

\[
\pi_1^1(\mu_1) \mathcal{R}_{\epsilon, \delta}^{(2)} \pi_2^1(\mu_2) \iff \mu_1 \mathcal{S}_{\epsilon, \delta}^{(2)} \mu_2
\]

where \( a_{1} \mathcal{S} a_{2} \iff f_1(a_{1}) \mathcal{R} f_2(a_{2}) \).

This property can be used to pull back an approximate lifting on two distributions over \( B_1, B_2 \) to an approximate lifting on two distributions over \( A_1, A_2 \). For applications in program logics, \( B_1, B_2 \) could be the domain of a program variable, \( A_1, A_2 \) could be the set of memories, and \( f_1, f_2 \) could project a memory to a program variable. While the mapping theorem is quite useful, it is puzzling why it only applies to surjective maps. For instance, this theorem cannot be used when the maps \( f_1, f_2 \) embed a smaller space into a larger space.

For another example, there exist 2-liftings of the following form, sometimes called the optimal subset coupling.

Theorem 7 (Barthe et al. [4]). Let \( \mu \in \mathcal{D}(A) \) and consider two subsets \( P_1 \subseteq P_2 \subseteq A \). Suppose that \( P_2 \) is a strict subset of \( A \). Then, we have the following equivalence:

\[
P_{\mu}[P_2] \leq e' \cdot P_{\mu}[P_1] \iff \mu \mathcal{R}_{\epsilon, \delta}^{(2)} \mu,
\]

where \( a_1 \mathcal{R} a_2 \iff a_1 \in P_1 \iff a_2 \in P_2 \).

In this construction, it is puzzling why the larger subset \( P_2 \) must be a strict subset of the domain \( A \). For example, this theorem does not apply for \( P_2 = A \), but we may be able to construct the approximate lifting if we simply embed \( A \) into a larger space \( B \)—even though \( \mu \) has support over \( A! \) Furthermore, it is not clear why the subsets must be nested, nor is it clear why we can only relate \( \mu \) to itself.

These shortcomings suggest that the definition of 2-liftings may be problematic. While the distance condition appears to be the most constraining requirement, the marginal and support conditions are responsible for the main issues.
Witnesses can only use pairs in the relation.

For some relations $\mathcal{R}$, there may be elements $a$ such that $a \mathcal{R} b$ does not hold for any $b$, or vice versa. It can be impossible find witnesses with the correct marginals on these elements, even if the distance condition can be easily satisfied. For instance, we can sometimes construct a pair $\mu_a$ and $\mu_b$ satisfying the distance requirement, but where $\mu_a$ needs additional mass to achieve the marginal requirement for an element $b$. Adding this mass anywhere preserves the distance bound, but there may not be an element $a$ such that $a \mathcal{R} b$.

No canonical choice of witnesses.

A related problem is that the marginal requirement only constrains one marginal of each witness distribution. Along the other component, the witnesses may place the mass anywhere on any pair in the relation. As a result, witnesses to an approximate lifting $\mu_1 \mathcal{R}_{\epsilon,\delta}^{(2)} \mu_2$ may have mass outside of $\text{supp}(\mu_1) \times \text{supp}(\mu_2)$, even though it seems that only elements in the support should be relevant to the lifting.

### 3 $\star$-Liftings and Strassen’s Theorem

To improve the theoretical properties of 2-liftings, we propose a simple extension: allow witnesses to be distributions over a larger set.

- **Notation 8.** Let $A$ be a set. We write $A^*$ for $A \cup \{\star\}$.

- **Definition 9 ($\star$-lifting).** Let $\mu_1 \in \mathcal{D}(A)$ and $\mu_2 \in \mathcal{D}(B)$ be sub-distributions, $\epsilon, \delta \in \mathbb{R}^+$ and $\mathcal{R}$ be a binary relation over $A \times B$. An $(\epsilon, \delta)$-approximate $\star$-lifting of $\mu_1$ and $\mu_2$ for $\mathcal{R}$ is a pair of sub-distributions $\eta_1 \in \mathcal{D}(A \times B^*)$ and $\eta_2 \in \mathcal{D}(A^* \times B)$ s.t.

  1. $\pi_1^\star(\eta_1) = \mu_1$ and $\pi_2^\star(\eta_2) = \mu_2$;
  2. $\text{supp}(\eta_1|_{A \times B}), \text{supp}(\eta_2|_{A^* \times B}) \subseteq \mathcal{R}$; and
  3. $\Delta_\epsilon(\eta_1, \eta_2) \leq \delta$, where $\eta_1^\star$ is the canonical lifting of $\eta_1$ to $A^* \times B^*$.

We write $\mu_1 \mathcal{R}_{\epsilon,\delta}^{(\star)} \mu_2$ if there exists an $(\epsilon, \delta)$-approximate lifting of $\mu_1$ and $\mu_2$ for $\mathcal{R}$.

By adding an element $\star$, we address both problems discussed at the end of the previous section. First, for every $a \in A$, witnesses may place mass at $(a, \star)$; for every $b \in B$, witnesses may place mass at $(\star, b)$. Second, $\star$ can serve as a generic element where all mass that lies outside the supports $\text{supp}(\mu_1) \times \text{supp}(\mu_2)$ may be placed, while preserving the marginal and distance requirements, giving more control over the form of the witnesses.

- **Lemma 10.** Let $\mu_1 \in \mathcal{D}(A)$ and $\mu_2 \in \mathcal{D}(B)$ be distributions such that $\mu_1 \mathcal{R}_{\epsilon,\delta}^{(\star)} \mu_2$. Then, there are witnesses with support contained in $\text{supp}(\mu_1)^* \times \text{supp}(\mu_2)^*$.

**Proof.** See Appendix, p. 14

### 3.1 Basic Properties

$\star$-liftings satisfy all basic properties satisfied by other notions of lifting. We start by proving that this new definition of lifting still characterizes differential privacy.

- **Lemma 11.** A randomized algorithm $P : A \rightarrow \mathcal{D}(B)$ is $(\epsilon, \delta)$-differentially private for $\phi$ iff for all $a_1, a_2 \in A$, $a_1 \phi a_2$ implies $P(a_1) =_{\epsilon,\delta} P(a_2)$.

**Proof.** See Appendix, p. 14
The next lemma establishes several other basic properties of $\star$-liftings: monotonicity, and closure under relational and sequential composition.

**Lemma 12.** Let $\mu_1 \in \mathcal{D}(A)$, $\mu_2 \in \mathcal{D}(B)$, and $\mathcal{R}$ be a binary relation over $A \& B$. If $\mu_1 \mathcal{R}^{(\star)}_{\epsilon,\delta} \mu_2$, then for any $\epsilon' \geq \epsilon, \delta' \geq \delta$ and $\mathcal{S} \supseteq \mathcal{R}$, we have $\mu_1 \mathcal{S}^{(\star)}_{\epsilon',\delta'} \mu_2$.

Let $\mu_1 \in \mathcal{D}(A)$, $\mu_2 \in \mathcal{D}(B)$, $\mu_3 \in \mathcal{D}(C)$ and $\mathcal{R}$ (resp. $\mathcal{S}$) be a binary relation over $A \& B$ (resp. over $B \& C$). If $\mu_1 \mathcal{R}^{(\star)}_{\epsilon,\delta} \mu_2$ and $\mu_2 \mathcal{S}^{(\star)}_{\epsilon',\delta'} \mu_3$, then $\mu_1 \mathcal{S} \circ \mathcal{R}^{(\star)}_{\epsilon+\epsilon', \delta+\delta'} \mu_3$.

For $i \in \{1, 2\}$, let $\mu_i \in \mathcal{D}(A_i)$ and $\eta_i : A_i \rightarrow \mathcal{D}(B_i)$. Let $\mathcal{R}$ (resp. $\mathcal{S}$) be a binary relation over $A_1 \& A_2$ (resp. over $B_1 \& B_2$). If $\mu_1 \mathcal{R}^{(\star)}_{\epsilon,\delta} \mu_2$ for some $\epsilon, \delta \geq 0$ and for any $(a_1,a_2) \in \mathcal{R}$, $\eta_1(a_1) \mathcal{S}^{(\star)}_{\epsilon',\delta'} \eta_2(a_2)$ for some $\epsilon', \delta' \geq 0$, then

$$\mathbb{E}_{\mu_1}[\eta_1] \mathcal{S}^{(\star)}_{\epsilon+\epsilon', \delta+\delta'} \mathbb{E}_{\mu_2}[\eta_2].$$

**Proof.** See Appendix, p. [14]

### 3.2 Equivalence with Sato’s Definition

In recent work on verifying differential privacy over general, continuous distributions, Sato [11] proposes an alternative definition of approximate lifting. In the special case of discrete distributions, where measurability of events can be forgotten, his definition can be stated as follows.

**Definition 13 (Sato [11]).** Let $\mu_1 \in \mathcal{D}(A)$ and $\mu_2 \in \mathcal{D}(B)$, $\mathcal{R}$ be a binary relation over $A \& B$ and $\epsilon, \delta \geq 0$. Then, there is an $(\epsilon, \delta)$-approximate lifting of $\mu_1 \& \mu_2$ for $\mathcal{R}$ if

$$\forall X \subseteq A. \mu_1[X] \leq e^{\epsilon} \cdot \mu_2[\mathcal{R}(X)] + \delta.$$

Notice that this definition has no witness distributions at all; instead, it uses a universal quantifier over all subsets. We can show that $\star$-liftings are equivalent to Sato’s definition in the case of discrete distributions. This equivalence is reminiscent of Strassen’s theorem from probability theory, which characterizes the existence of probabilistic couplings.

**Theorem 14 (Strassen [12]).** Let $\mu_1 \in \mathcal{D}(A)$, $\mu_2 \in \mathcal{D}(B)$ be two proper distributions, and $\mathcal{R}$ let be a binary relation over $A \& B$. Then there exists a joint distribution $\mu \in \mathcal{D}(A \times B)$ with support in $\mathcal{R}$ such that $\pi_1^\star(\mu) = \mu_1$ and $\pi_2^\star(\mu) = \mu_2$ if and only if

$$\forall X \subseteq A. \mu_1[X] \leq \mu_2[\mathcal{R}(X)].$$

Our result (Theorem [19]) can be viewed as a generalization of Strassen’s theorem to approximate couplings. The key ingredient in our proof is the max-flow min-cut theorem for countable networks; we begin by reviewing the basic setting.

**Definition 15 (Flow network).** A flow network is a structure $((V,E), \top, \bot, c)$ s.t. $\mathcal{N} = (V,E)$ is a loop-free directed graph without infinite simple path (or rays), $\top$ and $\bot$ are two distinct distinguished vertices of $\mathcal{N}$ s.t. no edge starts from $\bot$ and ends at $\top$, and $c : E \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ is a function assigning to each edge of $\mathcal{N}$ a capacity. The capacity $c$ is extended to $V^2$ by assigning capacity 0 to any pair $(u,v)$ s.t. $(u,v) \notin E$.

**Definition 16 (Flow).** Given a flow network $\mathcal{N} \overset{\Delta}{=} ((V,E), \top, \bot, c)$, a function $f : V^2 \rightarrow \mathbb{R}$ is a flow for $\mathcal{N}$ iff

1. $\forall u,v \in V. f(u,v) \leq c(u,v)$,
2. $\forall u,v \in V. f(u,v) = -f(v,u)$, and
and both the supremum and infimum are reached.

Definition 17 (Cut). Given a flow network \( N \doteq ((V, E), \top, \bot, c) \), a cut for \( N \) is any set \( C \subseteq V \) that partition \( V \) s.t. \( \top \in V \) but \( \bot \notin V \). The cut-set \( \mathcal{E}(C) \) of a cut \( C \) is defined as: \( \{(u, v) \in E \mid u \in S, v \notin S\} \). The capacity \( |C| \in \mathbb{R}_+ \cup \{\infty\} \) of a cut is defined as \( |C| \doteq \sum_{(u, v) \in \mathcal{E}(C)} c(u, v) \).

For flow networks with finitely many vertices and edges, the maximum flow is equal to the minimum cut. Aharoni et al. [1] consider when this is the case for a countable network. For the flow networks that we consider in this paper—where there are no infinite directed paths—equality holds.

Theorem 18 (Weak Countable Max-Flow Min-Cut). Let \( N \) be a network flow. Then,

\[
\sup \{|f| \mid f \text{ is a flow for } N\} = \inf \{|C| \mid C \text{ is a cut for } N\}
\]

and both the supremum and infimum are reached.

We are now ready to prove an approximate version of Strassen’s theorem, thereby showing equivalence between \( \ast \)-liftings and Sato’s liftings.

Theorem 19. Let \( \mu_1 \in \mathcal{D}(A) \) and \( \mu_2 \in \mathcal{D}(B) \), \( R \) be a binary relation over \( A \& B \) and \( \epsilon, \delta \in \mathbb{R}_+ \). Then, \( \mu_1 \mathcal{F}_{\epsilon, \delta}^{( \ast )} \mu_2 \) iff \( \forall X \subseteq A. \mu_1(X) \leq \epsilon^\ast \cdot \mu_2(\mathcal{R}(X)) + \delta \).

Proof. We only detail the reverse direction. We can assume that \( A \) and \( B \) are countable; in the case where \( A \) and \( B \) are not both countable, we first consider the restriction of \( \mu_1 \) and \( \mu_2 \) to their respective supports—which are countable sets—and construct witnesses to the \( \ast \)-lifting. The witnesses can then be extended to a coupling of \( \mu_1 \) and \( \mu_2 \) by adding a null mass to the extra points.

Let \( \omega \doteq |\mu_2| + e^{-\epsilon} \cdot \delta \) and let \( \top \) and \( \bot \) be fresh symbols. For any set \( X \), define \( X^\top \) and \( X^\bot \) resp. as \( \{x^\top \mid x \in X\} \) and \( \{x^\bot \mid x \in X\} \). Let \( N \) be the flow network of Figure 1 whose resp. source and sink are \( \top \) and \( \bot \), whose set of vertices \( V \) is \( \{\top, \bot\} \cup (A^\top \cup (B^\top)^\perp \cup (A^\bot \cup (B^\bot)^\top) \) and whose set of edges \( E \) is \( E_{\top} \cup E_{\bot} \cup E_{\bot} \cup E_{\ast} \) with

\[
E_{\top} \doteq \{\top \mapsto \mu_1(a) \ast^\top \mid a \in A\} \quad E_{\bot} \doteq \{b^\top \mapsto e^{-\epsilon} \mu_2(b) \bot \mid b \in B\} \quad E_{\bot} \doteq \{a^\top \mapsto b^\bot \mid a \mathcal{R} b \vee a = \ast \vee b = \ast\} \quad E_{\ast} \doteq \{\top \mapsto \omega - e^{-\epsilon} |\mu_1| \ast^\top, \bot^\top \mapsto e^{-\epsilon} \delta \bot^\bot\}.
\]

Let \( C \) be a cut of \( N \) — in the following, we use \( C \) independently for the cut \( C \) and its cut-set \( \mathcal{E}(C) \). We check \( |C| \geq \omega \). If \( C \cap E_{\ast} \neq \emptyset \) then \( |C| = \infty \). Note that \( C \cap E_{\ast} = \emptyset \) implies
We clearly have \( \text{Theorem 13} \), which restrict the post-condition to be equality. There are two versions: the

\[
\text{Lemma 21.}
\]

\[\mu \text{ maps must be surjective.} \]

Our main theorem can be used to show a variety of natural properties of networks:

\[\E \in \text{maximum flow} \]

\[
\text{Hence, } E_\T \cup \{(\perp, \perp)\} \text{ is a minimum cut with capacity } \omega. \text{ By Theorem } 18 \text{, we obtain a maximum flow } f \text{ with mass } \omega. \text{ Note that the flow } f \text{ saturates the capacity of all edges in } E_\T, E_\perp, \text{ and } E_* \text{. Let } \hat{f} : (a, b) \in A^* \times B^* \mapsto f(a^\top, b^\perp). \text{ We now define the following distributions:
}

\[\eta_a : A \times B^* \mapsto \mathbb{R}^+ \quad \eta_b : A^* \times B \mapsto \mathbb{R}^+
\]

\[\text{(a, b) } \mapsto e^\epsilon \cdot \hat{f}(a, b) \quad (a, b) \mapsto \hat{f}(a, b)
\]

We clearly have \( \pi_{\mathbb{A}}^1(\eta_a) = \mu_1 \) and \( \pi_{\mathbb{B}}^2(\eta_b) = \mu_2 \). Moreover, by construction of the flow network \( \mathcal{N} \), \( \text{supp}(\hat{f}_{A \times B}) \subseteq \mathcal{R} \). Hence, \( \text{supp}(\eta_a|_{A \times B}), \text{supp}(\eta_b|_{A \times B}) \subseteq \mathcal{R} \). It remains to show that \( \Delta_\epsilon(\eta_a, \eta_b) \leq \delta \). Let \( X \) be a subset of \( A^* \times B^* \). Let \( \mathcal{X}_a = \{(a, \ast), X \}\}, \mathcal{X}_b = \{(b, \ast), X \}\}\}, \text{ and } \mathcal{X} = X \cap (A \times B) \text{. Then,
}

\[
\pi_{\mathbb{A}}[X] - e^\epsilon \cdot \pi_{\mathbb{B}}[X] = \epsilon^{\epsilon} \left( \hat{f}[\mathcal{X}] + \hat{f}[\mathcal{X}_a \times \{\ast}\}]\right) - \epsilon^{\epsilon} \left( \hat{f}[\mathcal{X}] + \hat{f}[\{\ast\} \times \mathcal{X}_b]\right)
\]

\[
\leq \epsilon^{\epsilon} \cdot \hat{f}[\mathcal{X}_a \times \{\ast}\}] \leq \epsilon^{\epsilon} \cdot \hat{f}[A \times \{\ast}\}] = \delta.
\]

The last equality holds by Kirchhoff’s law: \( \hat{f}[A \times \{\ast}\}] = \sum_{a \in A} f(a^\top, \ast^\perp) = f(\ast^\perp, \perp) = e^{\epsilon - \epsilon} \cdot \delta. \]

\[\]

### 4 Properties of \( \ast \)-Liftings

Our main theorem can be used to show a variety of natural properties of \( \ast \)-liftings. To begin, we can generalize the mapping property from Theorem \[6\] lifting the requirement that the maps must be surjective.

\[\text{Lemma 20.} \text{ Let } \mu_1 \in \text{D}(A_1), \mu_2 \in \text{D}(A_2), f_1 : A_1 \mapsto B_1, f_2 : A_2 \mapsto B_2 \text{ and } \mathcal{R} \text{ a binary relation on } B_1 \text{ and } B_2. \text{ Let } S \text{ such that } a_1, a_2 \iff f_1(a_1) \mathcal{R} f_2(a_2). \text{ Then}
\]

\[f_1^\epsilon(\mu_1) \mathcal{R}_\epsilon(\mu_2) \iff \mu_1, S_{\epsilon, \delta} \mu_2.
\]

\[\text{Proof.} \text{ See Appendix, p. } 15
\]

Similarly, we can generalize the existing rules for up-to-bad reasoning (cf. Barthe et al. \[4\] Theorem 13), which restrict the post-condition to be equality. There are two versions: the conditional event is either on the left side, or the right side. Note that the resulting index \( \delta \) are different in the two cases.

\[\text{Lemma 21.} \text{ Let } \mu_1 \in \text{D}(A), \mu_2 \in \text{D}(B), \theta \subseteq A \text{ and } \mathcal{R} \subseteq A \times B. \text{ Assume that } \mu_1(\theta_\epsilon \iff \mathcal{R})_{\epsilon, \delta} \mu_2 \text{ for some parameters } \epsilon, \delta \geq 0. \text{ Then, } \mu_1, \mathcal{R}_{\epsilon, \delta} \mu_2, \text{ where } \delta \triangleq \delta + \mu_1(\theta).
\]

\[\text{Proof.} \text{ See Appendix, p. } 15
\]
Then, we have the following equivalence:

\[ a \iff \mu_1(\theta_a) \Rightarrow \mathcal{R}^{(s)}_{\epsilon,\delta} \mu_2 \text{ for some parameters } \epsilon, \delta \geq 0. \]

Then, \( \mu_1 \mathcal{R}^{(s)}_{\epsilon,\delta} \mu_2 \), where \( \delta \triangleq \delta + \epsilon' \cdot \mu_2[\theta_a] \).

**Proof.**  See Appendix, p. 16  

As a consequence, an approximately lifted relation can be conjuncted with a one-sided predicate if the \( \delta \) parameter is increased. This principle is useful for constructing approximate liftings that express accuracy bounds: when \( \theta_{a,<} \) is an event that happens with high probability, we can assume that \( \theta_{a,<} \) holds if we increase the \( \delta \) parameter of the approximate lifting.

**Lemma 23.** Let \( \mu_1 \in \mathbb{D}(A) \), \( \mu_2 \in \mathbb{D}(B) \), \( \theta_a \subseteq A \), \( \theta_b \subseteq B \) and \( \mathcal{R} \subseteq A \times B \). Assume that \( \mu_1(\theta_a) \Rightarrow \mathcal{R}^{(s)}_{\epsilon,\delta} \mu_2 \). Then, \( \mu_1(\theta_{a,b} \wedge \mathcal{R})^{(s)}_{\epsilon,\delta} \mu_2 \) and \( \mu_1(\theta_{b,a} \wedge \mathcal{R})^{(s)}_{\epsilon,\delta} \mu_2 \) where \( \delta_a \triangleq \delta + \mu_1[\theta_a] \) and \( \delta_b \triangleq \delta + \epsilon' \cdot \mu_2[\theta_b] \).

**Proof.**  See Appendix, p. 16  

\( \ast \)-liftings also support a significant generalization of optimal subset coupling. Unlike the known construction for 2-liftings (Theorem 7), the two subsets need not be nested, and either subset may be the entire domain. Furthermore, the distributions \( \mu_1, \mu_2 \) need not be the same, or even have the same domain. Finally, the equivalence is valid for any parameters \((\epsilon, \delta)\), not just \( \delta = 0 \).

**Theorem 24.** Let \( \mu_1 \in \mathbb{D}(A_1) \), \( \mu_2 \in \mathbb{D}(A_2) \) and consider two subsets \( P_1 \subseteq A_1, P_2 \subseteq A_2 \). Then, we have the following equivalence:

\[ \mathbb{P}_{\mu_1}[P_1] \leq \epsilon' \cdot \mathbb{P}_{\mu_2}[P_2] + \delta \wedge \mathbb{P}_{\mu_1}[A_1 - P_1] \leq \epsilon' \cdot \mathbb{P}_{\mu_2}[A_2 - P_2] + \delta \iff \mu_1 \mathcal{R}^{(s)}_{\epsilon,\delta} \mu_2, \]

where \( a_1 \mathcal{R} a_2 \iff a_1 \in P_1 \iff a_2 \in P_2 \).

**Proof.**  Immediate by Theorem 19  

We can then recover the existing notion of optimal subset coupling [1] for \( \ast \)-liftings, as a special case of the previous theorem.

**Corollary 25 (Barthe et al. [4]).** Let \( \mu \in \mathbb{D}(A) \) and consider two nested subsets \( P_2 \subseteq P_1 \subseteq A \). Then, we have the following equivalence:

\[ \mathbb{P}_\mu[P_1] \leq \epsilon' \cdot \mathbb{P}_\mu[P_2] \iff \mu_1 \mathcal{R}^{(s)}_{\epsilon,\delta} \mu_2, \]

where \( a_1 \mathcal{R} a_2 \iff a_1 \in P_1 \iff a_2 \in P_2 \).

**Proof.**  Immediate by Theorem 24 noting that

\[ \mathbb{P}_\mu[A - P_1] \leq \epsilon' \cdot \mathbb{P}_\mu[A - P_2] \]

is automatic since \( P_2 \subseteq P_1 \) implies \( \mathbb{P}_\mu[A - P_1] \leq \mathbb{P}_\mu[A - P_2] \).

Finally, we can directly extend known composition theorems from differential privacy to \( \ast \)-liftings. This connection is quite useful for lifting existing results from the privacy literature—which can be quite sophisticated—to approximate liftings.

**Lemma 26.** Pose \( \mathbb{R}^+_2 \ast \mathbb{R}^+ \ast \mathbb{R}^+ \) and let \( (\mathbb{R}^+_2)^* \) be the set of finite sequences over \( \mathbb{R}^+_2 \). Let \( r : (\mathbb{R}^+_2)^* \rightarrow \mathbb{R}^+_2 \) be a DP-composition operator, i.e. \( r \) is an operator such that for any sets \( A, D \) and family \( \{ f_i : D \times A \rightarrow \mathbb{D}(A) \}_{i<n} \) of functions, if for every \( a \in A \) and \( i<n \),
\( f_i(-, a) : D \rightarrow D(A) \) is \((\epsilon_i, \delta_i)\)-differentially private for some parameters \( \epsilon_i, \delta_i \geq 0 \) and fixed adjacency relation \( \phi \), then, for any \( a \in A \), \( F(-, a) \) is \((\epsilon^*, \delta^*)\)-differentially private for \( \phi \), where \( F : (d, a) \rightarrow (\bigcup_{i<n} (f_i(d, -))^\dagger(1_a)) \) is the the \( n \)-fold composition of the \([f_i]_{i<n}\) and \((\epsilon^*, \delta^*) \hat{=} r([([\epsilon_i, \delta_i])_{i<n}]). \)

Let \( n \in \mathbb{N} \) and assume given two families of sets \( \{A_i\}_{i \leq n} \) and \( \{B_i\}_{i \leq n} \), together with a family of binary relations \( \{\mathcal{R}(i) \subseteq A_i \times B_i\}_{i \leq n} \). Fix two families of functions \( \{g_i : A_i \rightarrow D(A_{i+1})\}_{i<n} \) and \( \{h_i : B_i \rightarrow D(B_{i+1})\}_{i<n} \) s.t. for any \( i < n \) and \((a, b) \in \mathcal{R}(i)\) we have:

1. \( g_i(a) (\mathcal{R}(i+1))^{(\ast)}(a_i) \), \( h_i(b) \) for some parameters \( \epsilon_i, \delta_i \geq 0 \), and
2. \( g_i(a) \) and \( h_i(b) \) are proper distributions.

Then, for \((a_0, b_0) \in \mathcal{R}_0\), there exists a \( \ast \)-lifting

\[
G(a_0) \mathcal{R}(n)^{(\ast)} \overset{\ast}{\rightarrow} H(b_0)
\]

where \((\epsilon^*, \delta^*) \hat{=} r([([\epsilon_i, \delta_i])_{i<n}])\), and \( G : A_0 \rightarrow D(A_n) \) and \( H : B_0 \rightarrow D(B_n) \) are the \( n \)-fold compositions of \([g_i]_{i \leq n}\) and \([h_i]_{i \leq n}\) respectively — i.e. \( G(a) \hat{=} (\bigcup_{i<n} g_i^\dagger)(1_a) \) and \( H(b) \hat{=} (\bigcup_{i<n} h_i^\dagger)(1_b) \).

For some of the more sophisticated composition results (notably, the advanced composition theorem by Dwork et al. [8]), Lemma 20 is not quite strong enough and requires a slight adaptation of the notion of \( \ast \)-lifting. We refer to the full version of the paper for more details.

## 5 Comparison with Existing Approximate Liftings

Now that we have seen \( \ast \)-liftings, we briefly consider other definitions of approximate liftings. We have already seen 2-liftings, which involve two witnesses (Definition 3). Evidently, \( \ast \)-liftings strictly generalize 2-liftings.

**Theorem 27.** For all binary relations \( \mathcal{R} \) over \( A \& B \) and parameters \( \epsilon, \delta \geq 0 \), we have

\[
\mathcal{R}^{(\ast)}_{\epsilon, \delta} \subseteq \mathcal{R}^{(\ast)}_{\epsilon, \delta}.
\]

**Proof.** The inclusion \( \mathcal{R}^{(2)}_{\epsilon, \delta} \subseteq \mathcal{R}^{(\ast)}_{\epsilon, \delta} \) is immediate. We have a strict inclusion \( \mathcal{R}^{(2)}_{\epsilon, \delta} \not\subseteq \mathcal{R}^{(\ast)}_{\epsilon, \delta} \) even for \( \delta = 0 \) by considering the optimal subset coupling from Theorem 7. Consider a distribution \( \mu \) over set \( A \), and let \( P_1 \subseteq P_2 = A \). There is an \((\epsilon, 0)\)-approximate \( \ast \)-lifting (by Theorem 24), but a \((\epsilon, 0)\)-approximate 2-lifting does not exist if \( \mu \) has non-zero mass outside of \( P_1 \); the first witness \( \mu_{\ast_1} \) must place non-zero mass at \( (a_1, a_2) \) with \( a_1 \notin P_1 \) in order to have \( \pi_1^\dagger(\mu_{\ast_1}) = \mu \), but we must have \( a_2 \notin P_2 \) for the support requirement, and there is no such \( a_2 \).

It is more interesting to compare \( \ast \)-liftings with the original definitions of \((\epsilon, \delta)\)-approximate lifting, by Barthe et al. [3]. They introduce two notions, a symmetric lifting and an asymmetric lifting, each using a single witness distribution. We will focus on the asymmetric version.

**Definition 28 (Barthe et al. [3]).** Let \( \mu_1 \in D(A) \) and \( \mu_2 \in D(B) \) be sub-distributions, \( \epsilon, \delta \in \mathbb{R}^+ \) and \( \mathcal{R} \) be a binary relation over \( A \& B \). An \((\epsilon, \delta)\)-approximate 1-lifting of \( \mu_1 \& \mu_2 \) for \( \mathcal{R} \) is a sub-distribution \( \mu \in D(A \times B) \) s.t.

1. \( \pi_1^\dagger(\mu) \leq \mu_1 \) and \( \pi_2^\dagger(\mu) \leq \mu_2 \);
2. \( \Delta_\epsilon(\mu_1, \pi_1^\dagger(\mu)) \leq \delta \); and
3. \( \text{supp}(\mu) \subseteq \mathcal{R} \).
In the first point we take the point-wise order on sub-distributions: if $\mu$ and $\mu'$ are sub-distributions over $X$, then $\mu \leq \mu'$ when $\mu(x) \leq \mu'(x)$ for all $x \in X$. We will write $\mu_1 R_{\epsilon,\delta}^{(1)} \mu_2$ if there exists an $(\epsilon, \delta)$-approximate 1-lifting of $\mu_1$ and $\mu_2$ for $R$; the (1) indicates that there is one witness for this lifting.

1-liftings bear a close resemblance to probabilistic couplings from probability theory, which also have a single witness. However, 1-liftings are less well-understood theoretically than 2-liftings—basic properties such as mapping (Theorem 20) are not known to hold; the subset coupling (Theorem 7) is not known to exist. Somewhat surprisingly, 1-liftings are equivalent to $\ast$-liftings (and hence by Theorem 19, also to Sato’s approximate lifting).

Theorem 29. For all binary relations $R$ over $A \& B$ and parameters $\epsilon, \delta \geq 0$, we have $R_{\epsilon,\delta}^{(1)} = R_{\epsilon,\delta}^{(\ast)}$.

Proof. See Appendix, p. 16

$\Box$

6 $\ast$-Lifting for $f$-Divergences

The definition of $\ast$-lifting can be extended to lifting constructions based on general $f$-divergences, as previously proposed by Barthe and Olmedo [2], Olmedo [10]. Roughly, a $f$-divergence a function $\Delta_f(\mu_1, \mu_2)$ that measures the difference between two probability distributions $\mu_1$ and $\mu_2$. Much like we generalized their definition for $(\epsilon, \delta)$-liftings, we can define $\ast$-lifting with $f$-divergences. Before going any further, let us first define formally $f$-divergences. We denote by $F$ the set of non-negative convex functions vanishing at 1: $F = \{ f: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \mid f(1) = 0 \}$. We also adopt the following notational conventions: $0 \cdot f(0/0) \equiv 0$, and $0 \cdot f(x/0) \equiv x \cdot \lim_{t \to 0^+} t \cdot f(1/t)$; we write $L_f$ for the limit.

Definition 30. Given $f \in F$, the $f$-divergence $\Delta_f(\mu_1, \mu_2)$ between two distributions $\mu_1$ and $\mu_2$ in $\mathbb{D}(A)$ is defined as:

$$\Delta_f(\mu_1, \mu_2) = \sum_{a \in A} \mu(a) f\left( \frac{\mu_1(a)}{\mu_2(a)} \right).$$

Examples of $f$-divergences include statistical distance ($f(t) = \frac{1}{2} |t - 1|)$, Kullback-Leibler divergence ($f(t) = \ln(t) - t + 1$), and Hellinger distance ($f(t) = \frac{1}{2} (\sqrt{t} - 1)^2$).

Definition 31 ($\ast$-lifting for $f$-divergences). Let $\mu_1 \in \mathbb{D}(A)$ and $\mu_2 \in \mathbb{D}(B)$ be distributions, $R$ a binary relation over $A \& B$, and $f \in F$. An $(f; \delta)$-approximate lifting of $\mu_1$ and $\mu_2$ for $R$ is a pair of distributions $\eta_1 \in \mathbb{D}(A \times B)$ and $\eta_2 \in \mathbb{D}(A \times B)$ s.t.

- $\pi_1(\eta_1) = \mu_1$ and $\pi_2(\eta_2) = \mu_2$;
- $\text{supp}(\eta_1|_{A \times B}), \text{supp}(\eta_2|_{A \times B}) \subseteq R$; and
- $\Delta_f(\eta_1, \eta_2) \leq \delta$,

where $\eta_1$ is the canonical lifting of $\eta_*$ to $A^* \times B^*$. We will write: $\mu_1 R_{f;\delta}^{(\ast)} \mu_2$ if there exists an $(f; \delta)$-approximate lifting of $\mu_1$ and $\mu_2$ for $R$.

$\ast$-liftings for $f$-divergences compose sequentially.

Lemma 32. Suppose $f$ has divergence statistical distance, Kullback-Leibler, or Hellinger distance. For $i \in \{1, 2\}$, let $\mu_i \in \mathbb{D}(A_i)$ and $\eta_i: A_i \rightarrow \mathbb{D}(B_i)$. Let $R$ (resp. $S$) be a binary relation over $A_1 \& A_2$ (resp. over $B_1 \& B_2$). If $\mu_1 R_{f;\delta}^{(\ast)} \mu_2$ for some $\delta \geq 0$ and for any $(a_1, a_2) \in R$ we have $\eta_1(a_1) S_{f;\delta'}^{(\ast)} \eta_2(a_2)$ for some $\delta' \geq 0$, then

$$\mathbb{E}_{\mu_1}[\eta_1] S_{f;\delta+\delta'}^{(\ast)} \mathbb{E}_{\mu_2}[\eta_2].$$
Much like the $\star$-liftings we saw before, $\star$-liftings for $f$-divergences have witness distributions with support determined by the support of $\mu_1$ and $\mu_2$ (cf. Lemma 10).

**Lemma 33.** Let $\mu_1 \in D(A)$ and $\mu_2 \in D(B)$ be distributions such that $\mu_1 R_{f, \delta}^{(\star)} \mu_2$. Then, there are witnesses with support contained in $\text{supp}(\mu_1)^* \times \text{supp}(\mu_2)^*$.

**Proof.** See Appendix, p. 17

Finally, the mapping property from Lemma 20 holds also for these $\star$-liftings. While the proof of Lemma 20 relies on the equivalence for Sato’s definition, there is no such equivalence (or definition) for general $f$-divergences. Therefore, we must work directly with the witnesses of the approximate lifting.

**Lemma 34.** Let $\mu_1 \in D(A_1)$, $\mu_2 \in D(A_2)$, $g_1 : A_1 \to B_1$, $g_2 : A_2 \to B_2$ and $R$ a binary relation on $B_1 \& B_2$. Let $S$ such that $a_1 S a_2 \iff g_1(a_1) R g_2(a_2)$. Then

$$g_1^\sharp(\mu_1) R_{f, \delta}^{(\star)} g_2^\sharp(\mu_2) \iff \mu_1 S_{f, \delta}^{(\star)} \mu_2.$$  

**Proof.** See Appendix, p. 18

## 7 Conclusion

We have proposed a new definition of approximate lifting that unifies existing constructions and satisfies an approximate variant of Strassen’s theorem. Our notion is useful both to simplify the soundness proof of existing program logics and to strengthen some of their proof rules. We see at least two important directions for future work. First, adapting existing program logics (for instance, apRHL [3]) to use $\star$-liftings, and formalizing examples that were out of reach of previous systems. Second, our notion of $\star$-liftings only applies when distributions have discrete support. It would be interesting to see if $\star$-liftings—and the approximate Strassen’s theorem—can be generalized to the continuous setting.

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## References


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We will construct witnesses $\eta$.

This is the definition of $\eta$.

Proof of Lemma 10. Let $\mu_a$ and $\mu_b$ be any pair of witnesses to the approximate lifting. We will construct witnesses $\eta_a, \eta_b$ with the desired support. For ease of notation, let $S_i = \text{supp}(\mu_i)$ for $i \in \{1, 2\}$. Define:

$$
\eta_a(a, b) = \begin{cases} 
\mu_a(a, b) & : (a, b) \in S_1 \times S_2 \\
\mu_a[a, B^* - S_2] & 
\end{cases} \\
\eta_b(a, b) = \begin{cases} 
\mu_b(a, b) & : (a, b) \in S_1 \times S_2 \\
\mu_b[a^* - S_1, b] & : a = * 
\end{cases}
$$

Evidently, $\eta_a$ and $\eta_b$ have support in $S_1^* \times S_2^*$. Additionally, it is straightforward to check that $\pi_1(\eta_a) = \pi_1(\mu_a) = \mu_1$ and $\pi_2(\eta_b) = \pi_2(\mu_b) = \mu_2$ so $\eta_a$ and $\eta_b$ have the desired marginals.

It only remains to check the distance condition. By the definition of the distance $\Delta_\epsilon$, we know that there are non-negative values $\delta(a, b)$ such that (i) $\overline{\mu_a}(a, b) \leq e^\epsilon \overline{\mu_a}(a, b) + \delta(a, b)$ and (ii) $\sum_{a,b} \delta(a, b) \leq \delta$. We can define new constants:

$$
\zeta(a, b) = \begin{cases} 
\delta(a, b) & : (a, b) \in S_1 \times S_2 \cup \{*\} \times B \\
\delta[a, B^* - S_2] & : b = * 
\end{cases}
$$

Since $\overline{\mu_a}(*, b) = \overline{\mu_a}(*, b) = 0$ for all $b \in B^*$, and $\overline{\mu_a}(a, b) = \overline{\mu_a}(a, b) = 0$ for all $b \notin S_2$, point (i) holds for the witnesses $\eta_a, \eta_b$ and constants $\zeta(a, b)$. Since $\sum_{a,b} \zeta(a, b) = \sum_{a,b} \delta(a, b) \leq \delta$, point (ii) holds as well. Hence, $\Delta_\epsilon(\overline{\mu_a}, \overline{\mu_b}) \leq \delta$ and we have witnesses for the desired approximate lifting.

Proof of Lemma 11. ($\implies$) Let $P$ be $(\epsilon, \delta)$-differentially private for $\phi$ and $a_1, a_2 \in A$ s.t. $a_1 \phi a_2$. Let $X$ be a subset of $B$. Then, by definition of differential privacy, we have $P(x|X) \leq e^\epsilon \cdot P(a)|X| + \delta = e^\epsilon \cdot P(a)(\{\epsilon\}(X)) + \delta$. Hence, by application of Theorem 19 we have $P(a_1) = e^\epsilon \cdot P(a_2)$.

($\impliedby$) By application of Theorem 19 we have that

$$
\forall a_1, a_2 \in A, \forall X \subseteq B, (a_1, a_2) \in \phi \implies P(a)|X| \leq e^\epsilon \cdot P(a)|X| + \delta.
$$

This is the definition of $P$ being $(\epsilon, \delta)$-differentially private for $\phi$.

Proof of Lemma 12. Immediate.

Let $\tau \triangleq \epsilon + e^\epsilon$ and $\delta \triangleq \delta + e^\epsilon \cdot \delta'$. By Theorem 19 it is sufficient to show that $\mu_1(X) \leq e^\epsilon \cdot \mu_1(S(R(X))) + \delta$ for any set $X$. We have:

$$
\mu_1[X] \leq e^\epsilon \cdot \mu_2[R(X)] + \delta \leq e^\epsilon \cdot e^\epsilon' \cdot \mu_3[S(R(X))] + \delta' + \delta = e^{\epsilon+\epsilon'} \cdot \mu_3[S(R(X))] + e^\epsilon \cdot \delta' + \delta.
$$

(Theorem 19)
We know that $\exists \langle \mu_a, \mu_b \rangle \in \mathcal{R}_C^{\circ} \langle \mu_1 \& \mu_2 \rangle$. Likewise, for $a \triangleq (a_1, a_2) \in \mathcal{R}$, $\exists \langle \eta_{a_1,a}, \eta_{b,a} \rangle \in \mathcal{S}_{C,\delta}^{\circ} \langle \eta_1(a_1) \& \eta_2(a_2) \rangle$. Let $\eta_a$ and $\eta_b$ be the following distribution constructors:

\[
\eta_a : a \mapsto \begin{cases} 
\eta_{a,a} & \text{if } a \in \mathcal{R} \\
0 & \text{otherwise}
\end{cases} \quad \eta_b : a \mapsto \begin{cases} 
\eta_{b,a} & \text{if } a \in \mathcal{R} \\
0 & \text{otherwise}
\end{cases}
\]

and let $\xi_a \triangleq \mathbb{E}_{\mu_a}[\eta_a]$ (resp. $\xi_b \triangleq \mathbb{E}_{\mu_b}[\eta_b]$). We now prove that:

\[
\langle \xi_a, \xi_b \rangle \in \mathcal{S}_{C+e^c+\delta+\delta'}^\circ (\mathbb{E}_{\mu_1}[\eta_1] \& \mathbb{E}_{\mu_2}[\eta_2]).
\]

The marginal and support conditions are immediate. The distance condition is obtained by an immediate application of the previous point.

**Proof of Theorem 19.** To show the forward direction of Theorem 19 let $X \subseteq A$ and $\mathcal{R}(X)^C = B - \mathcal{R}(X)$. Then, we have

\[
\mu_1[X] = \pi_X^\circ(\eta_a)[X] = \eta_a[X, B^*] = \pi_{\mathcal{R}_C}[X, B^*] = \pi_X[\mathcal{R}, \mathcal{R}(X)^C] \leq e^c \cdot \pi_X[\mathcal{R}, \mathcal{R}(X)^C] + \delta
\]

\[
\leq e^c \cdot \pi_X[A^*, \mathcal{R}(X)] + e^c \cdot \pi_X[A^*, \{\ast\}] + \delta
\]

\[
= e^c \cdot \pi_X^\circ(\eta_a)[X] + \delta = e^c \cdot \mu_2[\mathcal{R}(X)] + \delta,
\]

as desired.

**Proof of Lemma 20.** (\(\implies\)) Assume that $f_1^\circ(\mu_1) \in \mathcal{R}_{c,\delta}^{\circ} f_2^\circ(\mu_2)$ and let $X \subseteq A_1$. Then,

\[
\mu_1[X] \leq \mu_1[f_1^{-1}(f_1(X))] \leq f_1^\circ(\mu_1)[f_1(X)]
\]

\[
\leq e^c \cdot f_2^\circ(\mu_2)[\mathcal{R}(f_1(X))] + \delta \quad \text{(Theorem 19)}
\]

\[
= e^c \cdot \mu_2[f_2^{-1}(\mathcal{R}(f_1(X)))] + \delta \leq e^c \cdot \mu_2[\mathcal{S}(X)] + \delta.
\]

Hence, by Theorem 19 $\mu_1 S_{c,\delta}^{\circ} \mu_2$.

(\(\impliedby\)) Assume that $\mu_1 S_{c,\delta}^{\circ} \mu_2$ and let $X \subseteq A_2$. Then,

\[
f_1^\circ(\mu_1)[X] = \mu_1[f_1^{-1}(X)]
\]

\[
\leq f_2^\circ(\mu_2)[\mathcal{F}(f_1^{-1}(X))] + \delta \leq f_2^\circ(\mu_2)[\mathcal{R}(X)] + \delta.
\]

Hence, by Theorem 19 $f_1^\circ(\mu_1) \mathcal{R}_{c,\delta}^{\circ} f_2^\circ(\mu_2)$.

**Proof of Lemma 21.** By Theorem 19 it is sufficient to prove that

\[
\mu_1[X] \leq e^c \cdot \mu_2[\mathcal{R}(X)] + \mu_1[\theta^\circ] + \delta
\]

for any $X \subseteq A$. By direct computation:

\[
\mu_1[X] = \mu_1[X \cap \theta] + \mu_1[X \cap \theta^\circ] \leq \mu_1[X \cap \theta] + \mu_1[\theta^\circ]
\]

\[
\leq e^c \cdot \mu_2[\theta_0 \implies \mathcal{R}(X \cap \theta)] + \delta + \mu_1[\theta^\circ]
\]

\[
= \mathcal{R}(X \cap \theta) \subseteq \mathcal{R}(X)
\]

\[
\leq e^c \cdot \mu_2[\mathcal{R}(X)] + \mu_1[\theta^\circ] + \delta.
\]
Proof of Lemma 22. By Theorem 19 it is sufficient to prove that

\[ \mu_1[X] \leq e^\epsilon \cdot \mu_2[R(X)] + e^\epsilon \cdot \mu_2[\theta^\delta] + \delta \]

for any \( X \subseteq A \). Let \( X \) be such a set, then:

\[ \mu_1[X] \leq e^\epsilon \cdot \mu_2[(\theta_{\delta}) \implies R(X)] + \delta \]

\[ \leq e^\epsilon \cdot (\mu_2[\theta_{\delta}) \implies R(X) \cap \theta] + \mu_2[\theta^\delta] + \delta \]

\[ \leq \mu_2[(\theta_{\delta}) \implies R(X) \cap \theta] + e^\epsilon \cdot \mu_2[\theta^\delta] + \delta \]

\[ \leq \mu_2[R(X)] + e^\epsilon \cdot \mu_2[\theta^\delta] + \delta. \]

Proof of Lemma 23. From \( \mu_1 \mathcal{R}_{e,\delta}^{(s)} \mu_2 \) and Lemma 12 we have \( \mu_1 S^{(s)} \mu_2 \) where \( S \triangleq \theta_{a,\epsilon} \implies \theta_{a,\epsilon} \land R \). Hence, by Lemma 21 we obtain \( \mu_1 (\theta_{a,\epsilon} \land R)^{(s)}_{e,\delta} \mu_2 \). Using similar reasoning with \( \theta_{b,\epsilon} \implies \theta_{b,\epsilon} \land R \) and Lemma 22 we have \( \mu_1 (\theta_{b,\epsilon} \land R)^{(s)}_{e,\delta} \mu_2 \).

Proof of Theorem 29. Suppose that \((\mu_L, \mu_R)\) are witnesses to \( \mu_1 \mathcal{R}_{e,\delta}^{(s)} \mu_2 \). Define the witness \( \eta \in \mathcal{D}(A \times B) \) as the point-wise minimum: \( \eta(a, b) \triangleq \min(\mu_L(a, b), \mu_R(a, b)) \). We will show that \( \eta \) is a witness to \( \mu_1 \mathcal{R}_{e,\delta}^{(1)} \mu_2 \).

The support condition follows from the support condition for \((\mu_L, \mu_R)\). The marginal conditions \( \pi^1_\eta(\eta) \leq \mu_1 \) and \( \pi^2_\eta(\eta) \leq \mu_2 \) also follow by the marginal conditions for \((\mu_L, \mu_R)\). The only thing to check is the distance condition. By the distance condition on \((\mu_L, \mu_R)\), there exist non-negative values \( \delta(a, b) \) such that

\[ \mu_L(a, b) \leq \exp(\epsilon)\mu_R(a, b) + \delta(a, b) \]

and \( \sum_{a,b} \delta(a, b) \leq \delta \). So, \( \mu_R(a, b) \geq \exp(-\epsilon)(\mu_L(a, b) - \delta(a, b)) \). Now let \( S \subseteq A \) be any subset. Then:

\[ \mu_1(S) - \exp(\epsilon)\pi^1_\eta(S) = \sum_{a \in S} \mu_1(a) - \exp(\epsilon) \sum_{b \in B} \min(\mu_L(a, b), \mu_R(a, b)) \]

\[ \leq \sum_{a \in S} \mu_1(a) - \exp(\epsilon) \sum_{b \in B} \exp(-\epsilon)(\mu_L(a, b) - \delta(a, b)) \]

\[ = \sum_{a \in S, b \in B} \delta(a, b) \leq \delta. \]

Thus, \( \eta \) witnesses \( \mu_1 \mathcal{R}_{e,\delta}^{(1)} \mu_2 \), so \( \mathcal{R}_{e,\delta}^{(s)} \leq \mathcal{R}_{e,\delta}^{(1)} \).

The other direction is more interesting. Let \( \eta \in \mathcal{D}(A \times B) \) be the witness for \( \mathcal{R}_{e,\delta}^{(1)} \). By the distance condition \( \Delta_e(\mu_1, \pi^1_\eta) \leq \delta \), there exist non-negative values \( \delta(a) \) such that

\[ \mu_1(a) \leq \exp(\epsilon)\pi^1_\eta(a) + \delta(a) \]

with equality when \( \delta(a) \) is strictly positive, and \( \sum_{a \in A} \delta(a) \leq \delta \). Define two witnesses \( \mu_L \in \mathcal{D}(A \times B^*), \mu_R \in \mathcal{D}(A^* \times B) \) as follows:

\[ \mu_L(a, b) \triangleq \begin{cases} \eta(a, b) \cdot \frac{\mu_1(a) - \delta(a)}{\pi^1_\eta(a)} : b \neq * \\ \mu_1(a) - \sum_{b \in B} \mu_L(a, b) : b = * \end{cases} \]

\[ \mu_R(a, b) \triangleq \begin{cases} \eta(a, b) : a \neq * \\ \mu_2(b) - \sum_{a \in A} \mu_R(a, b) : a = * \end{cases} \]
(As usual, if any denominator is zero, we take the probability to be zero as well.)

The support condition follows from the support condition of \( \eta \). The marginal conditions hold by definition. Note that all probabilities are non-negative. For \( \mu_L \), note that if \( \delta(a) > 0 \) then \( \mu_L(a) = \mu_1(a) - \delta(a) = \exp(\epsilon)\pi^1_1\eta(a) \geq 0 \) and hence

\[
\mu_L(a, \star) = \mu_1(a) - \delta(a) \geq 0.
\]

assuming \( \pi^1_1\eta(a) > 0 \); if \( \pi^1_1\eta(a) = 0 \) then \( \mu_L(a, \star) = 0 \). For \( \mu_R \), non-negativity holds because \( \pi^1_1\eta \leq \mu_2 \).

We show the distance bound. Note that when \( a, b \neq \star \), by definition \( \mu_L(a, b) \) and \( \mu_R(a, b) \) are both strictly positive or both equal to zero, and \( \eta(a, b) \) is strictly positive or equal to zero accordingly. If \( \mu_L(a, b), \mu_R(a, b), \eta(a, b) \) are all strictly positive, then we know

\[
\frac{\mu_L(a, b)}{\eta(a, b)} = \frac{\mu_1(a) - \delta(a)}{\pi^1_1\eta(a)} \leq \exp(\epsilon).
\]

Thus we always have

\[
\mu_L(a, b) \leq \exp(\epsilon)\eta(a, b) = \exp(\epsilon)\mu_R(a, b).
\]

We can also bound the mass on points \((a, \star)\). Let \( S \subseteq A \) be any subset. Then:

\[
\overline{\mu_L}(S \times \{\star\}) = \sum\limits_{a \in S} \mu_1(a) - \mu_1(a) \sum\limits_{b \in B} \frac{\eta(a, b)}{\pi_1^1\eta(a)} + \delta(a) \sum\limits_{b \in B} \frac{\eta(a, b)}{\pi_1^1\eta(a)} = \mu_1(S) - \mu_1(S) + \delta(S) \leq \exp(\epsilon)\overline{\mu_R}(S \times \{\star\}) + \delta.
\]

So \( \Delta_\epsilon(\overline{\mu_L}, \overline{\mu_R}) \leq \delta \) as desired, and we have witnesses to \( \mu_1 \mathcal{R}_{\epsilon, \delta} \mu_2 \). Hence, \( \mathcal{R}^{(1)}_{\epsilon, \delta} \subseteq \mathcal{R}^{(\star)}_{\epsilon, \delta} \). ♦

Proof of Lemma 33. Let \( \mu_a \) and \( \mu_b \) be any pair of witnesses to the approximate lifting. We will construct witnesses \( \eta_a, \eta_b \) with the desired support. For ease of notation, let \( S_i = \text{supp}(\mu_i) \) for \( i \in \{1, 2\} \). Define:

\[
\eta_a(a, b) = \begin{cases} 
\mu_a(a, b) & : (a, b) \in S_1 \times S_2 \\
\mu_a[a, B^* - S_2] & : b = \star
\end{cases}
\]

\[
\eta_b(a, b) = \begin{cases} 
\mu_b(a, b) & : (a, b) \in S_1 \times S_2 \\
\mu_b[A^* - S_1, b] & : a = \star
\end{cases}
\]

Evidently, \( \eta_a \) and \( \eta_b \) have support in \( S_1^* \times S_2^* \). Additionally, it is straightforward to check that \( \pi^1_1(\eta_a) = \pi^1_1(\mu_a) = \mu_1 \) and \( \pi^2_2(\eta_b) = \pi^2_2(\mu_b) = \mu_2 \) so \( \eta_a \) and \( \eta_b \) have the desired marginals.

It only remains to check the distance condition. We can compute:

\[
\Delta_f(\overline{\eta_a}, \overline{\eta_b}) = \sum\limits_{(a, b) \in S_1 \times S_2} \eta_a(a, b) \cdot f\left(\frac{\eta_a(a, b)}{\eta_b(a, b)}\right) + \sum\limits_{a \in S_1} \eta_a(a, \star) \cdot f\left(\frac{\eta_a(a, \star)}{\eta_b(a, \star)}\right) + \sum\limits_{b \in S_2} \eta_b(\star, b) \cdot f\left(\frac{\eta_b(\star, b)}{\eta_b(a, b)}\right)
\]

\[
= \sum\limits_{(a, b) \in S_1 \times S_2} \mu_a(a, b) \cdot f\left(\frac{\mu_a(a, b)}{\mu_b(a, b)}\right) + \sum\limits_{a \in S_1} \eta_a(a, \star) \cdot L_f + \sum\limits_{b \in S_2} \eta_b(\star, b) \cdot f(0)
\]

\[
= \sum\limits_{(a, b) \in S_1 \times S_2} \mu_a(a, b) \cdot f\left(\frac{\mu_a(a, b)}{\mu_b(a, b)}\right) + \sum\limits_{a \in S_1} \mu_a(a, \star) \cdot L_f + \sum\limits_{b \in S_2} \mu_b(\star, b) \cdot f(0)
\]

\[
+ \sum\limits_{a \in S_1} \sum\limits_{b' \in B^* - S_2} \mu_a(a, b') \cdot L_f + \sum\limits_{b \in S_2} \sum\limits_{a' \in A^* - S_1} \mu_b(\star, a') \cdot f(0)
\]

\[
\leq \sum\limits_{a \in S_1} \mu_a(a, \star) \cdot L_f + \sum\limits_{b \in S_2} \mu_b(\star, b) \cdot f(0)
\]
Now, note that for all $b' \in B^* - S_2$, we know $\mu_b(a, b') = 0$. Similarly, for all $a' \in A^* - S_1$, we know $\mu_a(a', b) = 0$. Hence, the last line is equal to

$$
\Delta f(\overline{\pi}, \overline{\nu}) = \sum_{(a, b) \in S_1 \times S_2} \mu_b(a, b) \cdot f\left( \frac{\mu_a(a, b)}{\mu_b(a, b)} \right) + \sum_{a \in S_1} \sum_{b' \in B^* - S_2} \mu_b(a, b') \cdot f\left( \frac{\mu_a(a, b')}{\mu_b(a, b')} \right) + \sum_{b \in S_2} \sum_{a' \in A^* - S_1} \mu_a(a', b) \cdot f\left( \frac{\mu_a(a', b)}{\mu_b(a', b)} \right)
$$

$$
= \Delta f(\overline{\pi}, \overline{\nu}) \leq \delta.
$$

Thus, $\eta_a$ and $\eta_b$ witness the desired $*$-lifting.

\textbf{Proof of Lemma 34.} For the reverse direction, take the witnesses $\mu_{a, \mu} \in \mathcal{D}(A^* \times A^*)$ and define witnesses $\nu_{a, \nu} \triangleq (g_1^* \times g_2^*)(\mu_{a, \mu})$ and $\nu_{b, \nu} \triangleq (g_1^* \times g_2^*)(\mu_{b, \mu})$, where $g_1^* \times g_2^*$ takes a pair $(a_1, a_2)$ to the pair $(g_1(a_1), g_2(a_2))$ and maps $*$ to $\ast$. The support and marginal requirements are clear. The only thing to check is the distance condition, but this follows from monotonicity of $f$-divergences—under the mapping $g_1^* \times g_2^* : A^* \times A^* \to B^* \times B^*$, the $f$-divergence can only decrease (see, e.g., [2]).

For the forward direction, let $\nu_{a, \nu}, \nu_{b, \nu} \in \mathcal{D}(B^* \times B^*)$ be the witnesses to the second lifting. By Lemma 33, we may assume without loss of generality that $\text{supp}(\nu_{a, \nu})$ and $\text{supp}(\nu_{b, \nu})$ are contained in

$$
\text{supp}(g_1^*(\mu_1)) \times \text{supp}(g_2^*(\mu_2)) \subseteq g_1(A)^* \times g_2(A)^*.
$$

We aim to construct a pair of witnesses $\mu_{a, \mu}, \mu_{b, \mu} \in \mathcal{D}(A^* \times A^*)$ to the first lifting. The basic idea is to define $\mu_{a, \mu}$ and $\mu_{b, \mu}$ based on equivalence classes of elements in $A$ mapping to a particular $b \in B$, and then smooth out the probabilities within each equivalence class. To begin, for $a \in A$, define $[a]_g \triangleq g^{-1}(g(a))$ and $\alpha_i(a) \triangleq \text{Pr}_{\mu_{a, \mu}}([a]_g)$. We take $\alpha_i(a) = 0$ when $\mu_{a, \mu}([a]_g) = 0$, and we let $\alpha_i(\ast) = 0$. We define $\mu_{a, \mu}$ and $\mu_{b, \mu}$ as

$$
\begin{align*}
\mu_{a, \mu} : (a_1, a_2) &\mapsto \alpha_{c}(a_1, a_2) \cdot \nu_{c}(g_1(a_1), g_2(a_2)) \\
\mu_{b, \mu} : (a_1, a_2) &\mapsto \alpha_{b}(a_1, a_2) \cdot \nu_{b}(g_1(a_1), g_2(a_2))
\end{align*}
$$

where

$$
\begin{align*}
\alpha_c(a_1, a_2) &= \begin{cases} 
\alpha_1(a_1) \cdot \alpha_2(a_2) : a_2 \neq \ast \\
\alpha_1(a_1) : a_2 = \ast
\end{cases}, &
\alpha_b(a_1, a_2) &= \begin{cases} 
\alpha_1(a_1) \cdot \alpha_2(a_2) : a_1 \neq \ast \\
\alpha_2(a_2) : a_1 = \ast
\end{cases}.
\end{align*}
$$

The support and marginal conditions follow from the support and marginal conditions of $\nu_{a, \nu}$, $\nu_{b, \nu}$. For instance:

$$
\begin{align*}
\sum_{a_2 \in A^*} \mu_{a, \mu}(a_1, a_2) &= \sum_{a_2 \in A^*} \alpha_{c}(a_1, a_2) \cdot \nu_{c}(g_1(a_1), g_2(a_2)) \\
&= \alpha_1(a_1) \nu_{c}(g_1(a_1), \ast) + \sum_{a_2 \in A} \alpha_1(a_1) \alpha_2(a_2) \nu_{c}(g_1(a_1), g_2(a_2)) \\
&= \alpha_1(a_1) \left( \nu_{c}(g_1(a_1), \ast) + \sum_{b_2 \in g_2(A)} \nu_{c}(g_1(a_1), b_2) \sum_{a_2 \in g_1^{-1}(b_2)} \alpha_2(a_2) \right) \\
&= \alpha_1(a_1) \sum_{b_2 \in B^*} \nu_{c}(g_1(a_1), b_2) = \alpha_1(a_1) \mu_1([a_1]_{g_1}) = \mu_1(a_1).
\end{align*}
$$
We will handle each term separately. Evidently by further rearranging:

$$P_0 \triangleq \mu_p(\ast, \ast) \cdot f \left( \frac{\mu_0(\ast, \ast)}{\mu_p(\ast, \ast)} \right)$$

$$P_1 \triangleq \sum_{(a_1, a_2) \in A \times A} \mu_p(a_1, a_2) \cdot f \left( \frac{\mu_0(a_1, a_2)}{\mu_p(a_1, a_2)} \right)$$

$$P_2 \triangleq \sum_{a_1 \in A} \mu_p(a_1, \ast) \cdot f \left( \frac{\mu_0(a_1, \ast)}{\mu_p(a_1, \ast)} \right)$$

$$P_3 \triangleq \sum_{a_2 \in A} \mu_p(\ast, a_2) \cdot f \left( \frac{\mu_0(\ast, a_2)}{\mu_p(\ast, a_2)} \right)$$

We will handle each term separately. Evidently $P_0 = 0$. For $P_1$, we have:

$$P_1 = \sum_{(a_1, a_2) \in A \times A} \alpha_0(a_1, a_2) \nu_0(g_1(a_1), g_2(a_2)) \cdot f \left( \frac{\alpha_0(a_1, a_2) \nu_0(g_1(a_1), g_2(a_2))}{\alpha_0(a_1, a_2) \nu_0(g_1(a_1), g_2(a_2))} \right)$$

$$= \sum_{(a_1, a_2) \in S = \emptyset} \alpha_0(a_1, a_2) \nu_0(g_1(a_1), g_2(a_2)) \cdot L_f$$

$$+ \sum_{(a_1, a_2) \in S \neq \emptyset} \alpha_0(a_1, a_2) \nu_0(g_1(a_1), g_2(a_2)) \cdot f \left( \frac{\nu_0(g_1(a_1), g_2(a_2))}{\nu_0(g_1(a_1), g_2(a_2))} \right)$$

where the sets $S = \emptyset$ and $S \neq \emptyset$ are defined as:

$$S = \emptyset \triangleq \{(a_1, a_2) \mid \nu_0(g_1(a_1), g_2(a_2)) = 0\}$$

$$S \neq \emptyset \triangleq \{(a_1, a_2) \mid \nu_0(g_1(a_1), g_2(a_2)) \neq 0\}$$

By further rearranging:

$$P_1 = \sum_{(b_1, b_2) \in (g_1 \times g_2)(S = \emptyset)} \nu_0(b_1, b_2) \cdot L_f \left( \sum_{a_1 \in g_1^{-1}(b_1)} \alpha_0(a_1) \right) \left( \sum_{a_2 \in g_2^{-1}(b_2)} \alpha_0(a_2) \right)$$

$$+ \sum_{(b_1, b_2) \in (g_1 \times g_2)(S \neq \emptyset)} \nu_0(b_1, b_2) \cdot f \left( \frac{\nu_0(b_1, b_2)}{\nu_0(b_1, b_2)} \right) \left( \sum_{a_1 \in g_1^{-1}(b_1)} \alpha_0(a_1) \right) \left( \sum_{a_2 \in g_2^{-1}(b_2)} \alpha_0(a_2) \right)$$

$$= \sum_{(b_1, b_2) \in (g_1 \times g_2)(S = \emptyset)} \nu_0(b_1, b_2) \cdot L_f + \sum_{(b_1, b_2) \in (g_1 \times g_2)(S \neq \emptyset)} \nu_0(b_1, b_2) \cdot f \left( \frac{\nu_0(b_1, b_2)}{\nu_0(b_1, b_2)} \right)$$

$$= \sum_{(b_1, b_2) \in (g_1 \times g_2)(A \times A)} \nu_0(b_1, b_2) \cdot f \left( \frac{\nu_0(b_1, b_2)}{\nu_0(b_1, b_2)} \right)$$

The final equality is because without loss of generality, we can assume (by Lemma 33) that $\nu_0, \nu_0$ are zero outside of the support of $g_1^0(\mu_1)$ and $g_2^0(\mu_2)$, which have support contained in $(g_1 \times g_2)(A \times A)$.

The remaining two terms $P_2$ and $P_3$ are simpler to bound. For $P_2$, note that $\mu_p(a, \ast) = 0$
for all \( a \in A \). Thus:
\[
P_2 = \sum_{a_1 \in A} \alpha_c(a_1, \star) \nu_c(g_1(a_1), \star) \cdot L_f = \sum_{b_1 \in g_1(A)} \sum_{a_1 \in g_1^{-1}(b_1)} \alpha_1(a_1) \nu_c(b_1, \star) \cdot L_f
\]
\[
= \sum_{b_1 \in B(A)} \nu_c(b_1, \star) \cdot L_f = \sum_{b_1 \in B} \nu_c(b_1, \star) \cdot L_f = \sum_{b_1 \in B} \nu_c(b_1, \star) \cdot f(\frac{\nu_c(b_1, \star)}{\nu_c(b_1, 1) + \nu_c(b_1, 2)}).
\]
where the last equality is because \( \nu_c(b, \star) = 0 \) for all \( b \in B \).

Similarly for \( P_3 \), using \( \nu_c(\star, a) = \nu_c(\star, b) = 0 \) for all \( a \in A \) and \( b \in B \), we have:
\[
P_3 = \sum_{a_2 \in A} \alpha_c(\star, a_2) \nu_c(\star, g_2(a_2)) \cdot f(0) = \sum_{b_2 \in g_2(A)} \sum_{a_2 \in g_2^{-1}(b_2)} \alpha_2(a_2) \nu_\nu(\star, b_2) \cdot f(0)
\]
\[
= \sum_{b_2 \in B(A)} \nu_c(\star, b_2) \cdot f(0) = \sum_{b_2 \in B} \nu_c(\star, b_2) \cdot f(0) = \sum_{b_2 \in B} \nu_c(\star, b_2) \cdot f(\frac{\nu_c(\star, b_2)}{\nu_c(\star, 1) + \nu_c(\star, 2)}).
\]
Putting everything together, we conclude
\[
\Delta_f(\nu_c, \nu_c) = \Delta_\nu(\nu_c, \nu_c) \leq \delta
\]
by assumption, so \( \mu_\nu, \mu_\nu \) witness the desired approximate lifting.

\section{Symmetric \( \star \)-lifting}

The approximate liftings we have considered so far are all asymmetric. For instance, the approximate lifting \( \mu_1 \mathcal{R}_{\epsilon, \delta} \mu_2 \) may not imply the lifting \( \mu_2 \mathcal{R}^{-1}(\mathcal{R}(\mathcal{R}^{-1}))^{(\epsilon, \delta)} \mu_1 \). Given witnesses \((\mu_L, \mu_R)\) to the first lifting, we may consider the witnesses \((\nu_L, \nu_R) \equiv (\mu_L^T, \mu_R^T)\) where the transpose map \((\cdot)^T : \mathcal{D}(A \times B) \to \mathcal{D}(B \times A)\) is defined in the obvious way. Then \((\nu_L, \nu_R)\) almost witness the second lifting, except that the distance bound is in the opposite direction:
\[
\Delta_\nu(\nu_R, \nu_L) = \Delta_\nu(\mu_L^T, \mu_R^T) = \Delta_\nu(\mu_L, \mu_R) \leq \delta.
\]

In general, we cannot bound \( \Delta_\nu(\nu_L, \nu_R) \) and the symmetric lifting \( \mu_2 \mathcal{R}^{-1}(\mathcal{R}(\mathcal{R}^{-1}))^{(\epsilon, \delta)} \mu_1 \) may not hold. To recover symmetry, we can define a symmetric version of \( \star \)-lifting.

\begin{definition}[Symmetric \( \star \)-lifting] Let \( \mu_1 \in \mathcal{D}(A) \) and \( \mu_2 \in \mathcal{D}(B) \) be sub-distributions, \( \epsilon, \delta \in \mathbb{R}^+ \) and \( \mathcal{R} \) be a binary relation over \( A \times B \). An \((\epsilon, \delta)\)-approximate symmetric \( \star \)-lifting of \( \mu_1 \) & \( \mu_2 \) for \( \mathcal{R} \) is a pair of sub-distributions \( \eta_c \in \mathcal{D}(A \times B^*) \) and \( \eta_c \in \mathcal{D}(A^* \times B) \) s.t.
\begin{enumerate}
\item \( \pi_1^T(\eta_c) = \mu_1 \) and \( \pi_2^T(\eta_c) = \mu_2 \);
\item \( \text{supp}(\eta_c|_{A \times B}), \text{supp}(\eta_c|_{A \times B}) \subseteq \mathcal{R} \); and
\item \( \Delta_\nu(\mathcal{R}, \eta_c) \leq \delta, \Delta_\nu(\mathcal{R}, \eta_c) \leq \delta, \) where \( \mathcal{R}^c \) is the canonical lifting of \( \eta_c \) to \( A^* \times B^* \).
\end{enumerate}

We write \( \mu_1 \mathcal{R}^{(*)}_{\epsilon, \delta} \mu_2 \) if there exists an \((\epsilon, \delta)\)-approximate symmetric lifting of \( \mu_1 \) & \( \mu_2 \) for \( \mathcal{R} \).

Symmetric \( \star \)-lifting is a special case of \( \star \)-lifting that can capture differential privacy under when the adjacency relation \( \phi \) is symmetric: a probabilistic computation \( M : A \to \mathcal{D}(B) \) is \((\epsilon, \delta)\)-differentially private if and only if for every two adjacent inputs \( a, a' \), there is an approximate lifting of the equality relation: \( M(a) \approx_{\epsilon, \delta} M(a') \). Unfortunately, the more advanced properties in Section 4 do not all hold when moving to symmetric liftings. However, we can show that symmetric \( \star \)-liftings are equivalent to the symmetric version of 1-witness lifting proposed by Barthe et al. 3.
Definition 37 (Barthe et al. [3]). Let \( \mu_1 \in \mathcal{D}(A) \) and \( \mu_2 \in \mathcal{D}(B) \) be sub-distributions, \( \epsilon, \delta \in \mathbb{R}^+ \) and \( \mathcal{R} \) be a binary relation over \( A \& B \). An \((\epsilon, \delta)\)-approximate symmetric 1-lifting of \( \mu_1 \& \mu_2 \) for \( \mathcal{R} \) is a sub-distribution \( \mu \in \mathcal{D}(A \times B) \) s.t.

1. \( \pi_1^1(\mu) \leq \mu_1 \) and \( \pi_2^1(\mu) \leq \mu_2 \);
2. \( \Delta_\epsilon(\mu_1, \pi_1(\mu)) \leq \delta \) and \( \Delta_\epsilon(\mu_2, \pi_2(\mu)) \leq \delta \); and
3. \( \text{supp}(\mu) \subseteq \mathcal{R} \).

We will write \( \mu_1 \bar{\mathcal{R}}_{\epsilon, \delta}^{(1)} \mu_2 \) if there exists an \((\epsilon, \delta)\)-approximate symmetric 1-lifting of \( \mu_1 \& \mu_2 \) for \( \mathcal{R} \); the (1) indicates that there is one witness for this lifting.

Theorem 38 (cf. the asymmetric result Theorem 29). For all binary relations \( \mathcal{R} \) over \( A \& B \) and parameters \( \epsilon, \delta \geq 0 \), we have \( \mathcal{R}_{\epsilon, \delta}^{(1)} = \mathcal{R}_{\epsilon, \delta}^{(\alpha)} \).

Proof. Suppose that \((\mu_L, \mu_R)\) are witnesses to \( \mu_1 \bar{\mathcal{R}}_{\epsilon, \delta}^{(1)} \mu_2 \). Define the witness \( \eta \in \mathcal{D}(A \times B) \) as the point-wise minimum: \( \eta(a, b) \triangleq \min(\mu_L(a, b), \mu_R(a, b)) \). We will show that \( \eta \) is a witness to \( \mu_1 \bar{\mathcal{R}}_{\epsilon, \delta}^{(1)} \mu_2 \).

The support condition follows from the support condition for \((\mu_L, \mu_R)\). The marginal conditions \( \pi_1^1(\eta) \leq \mu_1 \) and \( \pi_2^1(\eta) \leq \mu_2 \) also follow by the marginal conditions for \((\mu_L, \mu_R)\). The only thing to check is the distance condition. By the distance condition on \((\mu_L, \mu_R)\), there exist non-negative values \( \delta(a, b) \) such that

\[
\mu_L(a, b) \leq \exp(\epsilon)\mu_R(a, b) + \delta(a, b)
\]

and \( \sum_{a, b} \delta(a, b) \leq \delta \). So, \( \mu_R(a, b) \geq \exp(-\epsilon)(\mu_L(a, b) - \delta(a, b)) \). Similarly, there are non-negative values \( \delta'(a, b) \) such that

\[
\mu_R(a, b) \leq \exp(\epsilon)\mu_L(a, b) + \delta'(a, b)
\]

and \( \sum_{a, b} \delta'(a, b) \leq \delta \). So, \( \mu_L(a, b) \geq \exp(-\epsilon)(\mu_R(a, b) - \delta'(a, b)) \).

Now let \( S \subseteq A \) be any subset. Then:

\[
\mu_1(S) - \exp(\epsilon)\pi_1^1(\eta)(S) = \sum_{a \in S} \mu_1(a) - \exp(\epsilon) \sum_{b \in B} \min(\mu_L(a, b), \mu_R(a, b))
\leq \sum_{a \in S} \mu_1(a) - \exp(\epsilon) \sum_{b \in B} \exp(-\epsilon)(\mu_L(a, b) - \delta(a, b))
= \sum_{a \in S, b \in B} \delta(a, b) \leq \delta.
\]

The other marginal is similar. For any subset \( T \subseteq B \) we have

\[
\mu_2(T) - \exp(\epsilon)\pi_2^1(\eta)(T) = \sum_{b \in T} \mu_2(b) - \exp(\epsilon) \sum_{a \in A} \min(\mu_L(a, b), \mu_R(a, b))
\leq \sum_{b \in T} \mu_2(b) - \exp(\epsilon) \sum_{a \in A} \exp(-\epsilon)(\mu_R(a, b) - \delta'(a, b))
= \sum_{b \in T, a \in A} \delta'(a, b) \leq \delta.
\]

Thus, \( \eta \) witnesses \( \mu_1 \bar{\mathcal{R}}_{\epsilon, \delta}^{(1)} \mu_2 \).
The other direction is more interesting. Let \( \eta \in \mathcal{D}(A \times B) \) be the single witness to \( \overline{R}_{\epsilon,\delta}^{(1)} \).

By the distance conditions \( \Delta_o(\mu_1, \pi_1^1 \eta) \leq \delta \) and \( \Delta_o(\mu_2, \pi_2^1 \eta) \leq \delta \), there exist non-negative values \( \delta(a) \) and \( \delta'(b) \) such that

\[
\begin{align*}
\mu_1(a) &\leq \exp(\epsilon)\pi_1^1 \eta(a) + \delta(a) \\
\mu_2(b) &\leq \exp(\epsilon)\pi_2^1 \eta(b) + \delta'(b),
\end{align*}
\]

there is equality when \( \delta(a) \) or \( \delta'(b) \) are strictly positive, and both \( \sum_{a \in A} \delta(a) \) and \( \sum_{b \in B} \delta'(b) \) are at most \( \delta \). Define two witnesses \( \mu_L \in \mathcal{D}(A \times B^*) \), \( \mu_R \in \mathcal{D}(A^* \times B) \) as follows:

\[
\mu_L(a, b) \triangleq \begin{cases} 
\eta(a, b) \cdot \frac{\mu_1(a) - \delta(a)}{\pi_1^1 \eta(a)} & : b \neq * \\
\mu_1(a) - \sum_{b \in B} \mu_L(a, b) & : b = *
\end{cases}
\]

\[
\mu_R(a, b) \triangleq \begin{cases} 
\eta(a, b) \cdot \frac{\mu_2(b) - \delta'(b)}{\pi_2^1 \eta(b)} & : a \neq * \\
\mu_2(b) - \sum_{a \in A} \mu_R(a, b) & : a = *
\end{cases}
\]

(As usual, if any denominator is zero, we take the probability to be zero as well.)

The support condition follows from the support condition of \( \eta \). The marginal conditions hold by definition. Note that all probabilities are non-negative. For instance in \( \mu_L \), note that if \( \delta(a) > 0 \) then \( \mu_L(a, \ast) = \delta(a) \geq 0 \) and hence

\[
\mu_L(a, \ast) = \mu_1(a) - \delta(a) \geq 0.
\]

assuming \( \pi_1^1 \eta(a) > 0 \); if \( \pi_1^1 \eta(a) = 0 \) then \( \mu_L(a, \ast) = 0 \). A similar argument shows that \( \mu_R \) is non-negative.

So, it remains to check the distance bounds. Note that when \( a, b \neq \ast \), by definition \( \mu_L(a, b) \) and \( \mu_R(a, b) \) are both strictly positive or both equal to zero, and \( \eta(a, b) \) is strictly positive or equal to zero accordingly. If \( \mu_L(a, b), \mu_R(a, b), \eta(a, b) \) are all strictly positive, then we know

\[
\begin{align*}
\frac{\mu_L(a, b)}{\eta(a, b)} &= \frac{\mu_1(a) - \delta(a)}{\pi_1^1 \eta(a)} \leq \exp(\epsilon) \\
\frac{\mu_R(a, b)}{\eta(a, b)} &= \frac{\mu_2(b) - \delta'(b)}{\pi_2^1 \eta(b)} \leq \exp(\epsilon).
\end{align*}
\]

We can also lower bound the ratios:

\[
\begin{align*}
\frac{\mu_L(a, b)}{\eta(a, b)} &= \frac{\mu_1(a) - \delta(a)}{\pi_1^1 \eta(a)} \geq 1 \\
\frac{\mu_R(a, b)}{\eta(a, b)} &= \frac{\mu_2(b) - \delta'(b)}{\pi_2^1 \eta(b)} \geq 1;
\end{align*}
\]

for instance when \( \delta(a) > 0 \) then the ratio is exactly equal to \( \exp(\epsilon) \geq 1 \), and when \( \delta(a) = 0 \) then the ratio is at least 1 by the marginal property \( \pi_1^1 \eta \leq \mu_1 \). So we have \( \mu_L(a, b)/\eta(a, b) \) and \( \mu_R(a, b)/\eta(a, b) \) in \([1, \exp(\epsilon)]\) when all distributions are strictly positive. Thus we always have

\[
\begin{align*}
\mu_L(a, b) &\leq \exp(\epsilon)\mu_R(a, b) \\
\mu_R(a, b) &\leq \exp(\epsilon)\mu_L(a, b).
\end{align*}
\]
We can also bound the mass on points \((a, \ast)\). Let \(S \subseteq A\) be any subset. \(\overline{\mu}(S \times \{\ast\}) \leq \exp(\epsilon)\overline{\mu}(S \times \{\ast\}) + \delta\) is clear. For the other direction:

\[
\overline{\mu}(S \times \{\ast\}) = \sum_{a \in S} \mu_1(a) - \mu_2(a) \sum_{b \in B} \frac{\eta(a,b)}{\pi_1^2(a)} + \delta(a) \sum_{b \in B} \frac{\eta(a,b)}{\pi_2^2(a)} \\
= \mu_1(S) - \mu_2(S) + \delta(S) \leq \exp(\epsilon)\overline{\mu}(S \times \{\ast\}) + \delta.
\]

The mass at points \((\ast, b)\) can be bounded in a similar way. Let \(T \subseteq B\) be any subset. Then, \(\overline{\mu}(\{\ast\} \times T) \leq \exp(\epsilon)\overline{\mu}(\{\ast\} \times T) + \delta\) is clear. For the other direction:

\[
\overline{\mu}(\{\ast\} \times T) = \sum_{a \in A} \mu_2(b) - \mu_2(b) \sum_{a \in A} \frac{\eta(a,b)}{\pi_2^2(b)} + \delta'(b) \sum_{a \in A} \frac{\eta(a,b)}{\pi_1^2(b)} \\
= \mu_2(T) - \mu_2(T) + \delta'(T) \leq \exp(\epsilon)\overline{\mu}(\{\ast\} \times T) + \delta.
\]

So \(\Delta_\epsilon(\overline{\mu}, \overline{\mu}) \leq \delta\) and \(\Delta_\epsilon(\overline{\mu}, \overline{\mu}) \leq \delta\) so we have witnesses to \(\mu_1 \overline{\mu}_{\epsilon, \delta} \mu_2\). Hence, \(\overline{\mu}_{\epsilon, \delta} = \overline{\mu}_{\epsilon, \delta}\).

The main use of symmetric approximate liftings is to support richer composition results that only apply to symmetric adjacency relations. We have the following reduction, a symmetric version of Lemma 39.

**Lemma 39.** Let \(n\) be a natural number. Suppose that there exists a function \(r : (\mathbb{R}^+ \times \mathbb{R}^+) \rightarrow (\mathbb{R}^+ \times \mathbb{R}^+)\) such that for any sets \(D, A\), symmetric adjacency relation \(\phi \subseteq D \times D\), \(n\) pairs \(\epsilon_i, \delta_i \geq 0\), and \(n\) functions \(f_i : D \times A \rightarrow \mathcal{D}(A)\) such that for every \(a, f_i(-, a) : D \rightarrow \mathcal{D}(A)\) is \((\epsilon_i, \delta_i)\)-differentially private with respect to \(\phi\), the \(n\)-fold composition \(F : D \rightarrow \mathcal{D}(A)\) is \(r((\epsilon_i, \delta_i)))\)-differentially private with respect to \(\phi\). Then for:

1. any relations \(\{\mathcal{R}(i)\}_i\) on \(A_i\) & \(B_i\) with \(i\) ranging from \(0, \ldots, n\); and
2. any functions \(\{g_i : A_i \rightarrow \mathcal{D}(A_{i+1})\}_i\), \(\{h_i : B_i \rightarrow \mathcal{D}(B_{i+1})\}_i\) with \(i\) ranging from \(0, \ldots, n-1\) and for all \((a, b) \in \mathcal{R}(i)\), we have

\[
g_i(a) \overline{\mathcal{R}}(i + 1)_{\epsilon_{i+1}, \delta_{i+1}} h_i(b)
\]

and \(g_i(a), h_i(b)\) proper distributions,

there is a symmetric \((\ast)\)-lifting

\[
G(a_0) \overline{\mathcal{R}}(n)_{\epsilon, \delta} H(b_0)
\]

for every \((a_0, b_0) \in \mathcal{R}_0\), where \(G : A_0 \rightarrow \mathcal{D}(A_n)\) and \(H : B_0 \rightarrow \mathcal{D}(B_n)\) are the \(n\)-fold (Kleisli) compositions of \(\{g_i\}_\phi\) and \(\{h_i\}_\phi\) respectively.

With this reduction we hand, we can generalize the advanced composition theorem from differential privacy to \((\ast)\)-liftings.

**Theorem 40** (Advanced composition, [3]). Consider a symmetric adjacency relation \(\phi\) on databases \(D\). Let \(f_i : D \times A \rightarrow \mathcal{D}(A)\) be a sequence of functions, such that for every \(a \in A\) the functions \(f_i(\ast, a) : D \rightarrow \mathcal{D}(A)\) are \((\epsilon, \delta)\)-differentially private with respect to \(\phi\). Then, for every \(a \in A\) and \(\omega \in (0, 1)\), running \(f_1, \ldots, f_n\) in sequence is \((\epsilon^*, \delta^*)\)-differentially private for

\[
\epsilon^* = \left(\sqrt{2n \ln(1/\omega)}\right) \epsilon + n \epsilon (\epsilon^* - 1) \quad \text{and} \quad \delta^* = n \delta + \omega.
\]
Theorem 41. Let $n$ be a natural number, $\epsilon, \delta \geq 0$, and $\omega \in (0,1)$ be real parameters. Suppose we have:

1. sets $\{A_i\}$, $\{B_i\}$, with $i$ ranging from 0, \ldots, $n$;
2. relations $\{R(i)\}$ on $A_i$ & $B_i$, with $i$ ranging from 0, \ldots, $n$; and
3. functions $\{f_i : A_i \to \Delta(A_{i+1})\}$, $\{g_i : B_i \to \Delta(B_{i+1})\}$, with $i$ ranging from 0, \ldots, $n - 1$

such that for all $(a,b) \in R(i)$, we have

$$f_i(a) \overline{R(i+1)}^\epsilon g_i(b)$$

and $f_i(a), g_i(b)$ proper distributions. Then, there is an approximate lifting of the compositions:

$$\overline{R(n)}^\epsilon \delta G(b_0)$$

for every $(a_0, b_0) \in R_0$, where $F : A_0 \to \Delta(A_n)$ and $G : B_0 \to \Delta(B_n)$ are the n-fold (Kleisli) compositions of $\{f_i\}$ and $\{g_i\}$ respectively, and the lifting parameters are:

$$\epsilon' \triangleq \epsilon \sqrt{2n \ln(1/\omega)} + n\epsilon(\epsilon' - 1) \quad \delta' \triangleq \eta \delta + \omega.$$

Proof. By Lemma 39 and the advanced composition for differential privacy (Theorem 40).

C An Elementary Proof of Weak Max-Flow Min-Cut Theorem

We give here an elementary proof of Max-Flow Min-Cut theorem for countable networks. We call it weak because it only covers graphs without rays and with a capacity function that is summable at each node. Our proof of approximate Strassen’s theorem lies in this setting. Indeed, using the notations of Theorem 19 although an infinite capacity can occur on an edge of the form $a^+ \to b^-$, we know that the flow between these two nodes cannot be above $\min\{a, b\}$ where $a$ and $b$ are the respective capacities of the edges $T \to a^+$ and $b^- \to T$. Hence, we can build a network flow whose set of flows is identical to the former one and with only finite capacities. It is immediate to check that the capacity of the network flow is summable at each node, the families $\{\alpha_a\}_{a \in (A^*)^+}$ and $\{\beta_b\}_{b \in (B^*)^-}$ being summable.

From now on, let $\mathcal{N} \triangleq ((V,E), T, \perp, c)$ be a flow network that contains only finite capacities. Moreover, assume that for any $u \in V$, the families $\{c(u,v)\}_{v \in V}$ and $\{c(v,u)\}_{v \in V}$ are summable. In the proof, we will use the edge definition of cuts (as defined in the paper above) or the equivalent following one:

Definition 42. Given a flow network $\mathcal{N} \triangleq ((V,E), T, \perp, c)$, a cut for $\mathcal{N}$ is any subset $C$ of $V$ s.t. $T \in C$ but $\perp \notin C$. The capacity $|C|$ of a cut $C$ is defined as

$$|C| \triangleq \sum_{u \in C, v \notin C} c(u,v) \in \mathbb{R}^+ \cup \{+\infty\}.$$  

We can now start our proof of Weak Max-Flow Min-Cut Theorem.

Lemma 43. There exists a flow for $\mathcal{N}$ with maximal mass.

Proof. We assume that $V$, the set of vertices, is countably infinite — the finite case is already covered by the usual max-flow min-cut theorem. Let $\{[n]\}_{n \in \mathbb{N}}$ be an enumeration of $V^2$.

Let $\mathcal{F} = \{f \mid f \text{ is a flow for } \mathcal{N}\}$. We order $\mathcal{F}$ by $f \succeq g$ iff $|f| \leq |g|$. We first show that $\mathcal{F}$ is an inductive set. Let $X = \{f_i\}_{i \in I}$ be a $\succeq$-chain for $\mathcal{F}$ for some countable set $I$. We
prove that \( X \) admits an upper bound in \( \mathcal{F} \). If \( I = \emptyset \), we take the null flow as the upper bound of \( X \). If \( I \) is finite but non empty, we take any \( f \) of \( X \), that maximizes \( |f| \), as the upper bound of \( X \). Otherwise, wlog, we can assume that \( X = \{f_i\}_{i \in \mathbb{N}} \) with \( \forall i, f_i \leq f_{i+1} \). We inductively construct a sequence \( \{t_n\}_{n \in \mathbb{N}} \) of strictly increasing functions from \( \mathbb{N} \) to itself. Let \( n \in \mathbb{N} \) and assume that \( t_k \) is constructed for any \( k < n \). Let \( t_n = t_0 \circ \cdots \circ t_{n-1} \). The sequence \( t_n \) is chosen among all the strictly monotone sequences \( \{i_n\}_n \)'s s.t. \( \{f_{i_n}(\{n\})\}_n \) converges. Such a sequence exists by the Bolzano-Weierstrass theorem, \( \{f_{i_n}(\{n\})\}_n \) being absolutely bounded by \( c(\{n\}) \). Let \( \{\omega_n\}_n \) be defined as \( \omega_n \equiv t_n(n) \). By construction, it is immediate that for any \( k \in \mathbb{N} \), \( \{\omega_n\}_n \) is a sub-sequence, at infinity, of \( \{f_{i_k(n)}\}_n \). As such, for any \( e \in \mathcal{E} \), \( \{f_{\omega_n}(e)\}_n \) admits a limit which is equal to \( \lim_{n \to \infty}[f_{t_{k+1}(n)}(e)] \).

We denote by \( f \) the point-wise limit of \( \{f_{\omega_n}\}_n \) and prove that \( f \) is an upper bound of \( E \). Let \( u, v \in V \). Then,

\[
\begin{align*}
    f(u, v) &= \lim_{n \to \infty}\left[ f_{\omega_n}(u, v) \right] \leq c(u, v) \\
    f(u, v) &= \lim_{n \to \infty}\left[ f_{\omega_n}(u, v) \right] = -\lim_{n \to \infty}\left[ f_{\omega_n}(v, u) \right] = -f(v, u).
\end{align*}
\]

Last, let \( u \in V \) s.t. \( u \notin \{\top, \bot\} \). First, note that for any \( v \in V \) and \( k \in \mathbb{N} \), \( f_{\omega_k}(u, v) \leq c(u, v) \) and \( f_{\omega_k}(u, v) \leq c(v, u) \), where \( f_{\omega_k}^+ \) and \( f_{\omega_k}^- \) are the respective positive and negative parts of \( f_{\omega_k} \). Hence, by taking \( k \to \infty \), we obtain \( f^+(u, v) \leq c(u, v) \) and \( f^-(u, v) \leq c(v, u) \). By summability of the in- and out-going capacities of \( u \), we obtain that \( \{f(u, v)\}_{v \in V} \) is summable. We now prove that \( \sum_{v \in V} f(u, v) = 0 \):

\[
\sum_{v \in V} f(u, v) = \sum_{v \in V} \lim_{n \to \infty}[f_{\omega_n}(u, v)] = \lim_{n \to \infty} \left[ \sum_{v \in V} f_{\omega_n}(u, v) \right] = 0,
\]

where the swapping lim and \( \sum \) is obtained by the dominated convergence theorem. (As noted above, for any \( v \in V \) and \( k \in \mathbb{N} \), \( f_{\omega_k}^+(u, v) \leq c(u, v) \) and \( f_{\omega_k}^-(u, v) \leq c(v, u) \) with \( \{c(u, v)\}_{v \in V} \) and \( \{c(v, u)\}_{v \in V} \) summable families.) So, \( f \) is a flow for \( N \).

Finally, we show that the flow is maximal: \( f \geq g \) for any \( g \in X \). Calculating,

\[
|f| = \sum_{v \in V} f(\top, v) = \sum_{v \in V} \lim_{n \to \infty} f_{\omega_n}(\top, v) = \lim_{n \to \infty} \sum_{v \in V} f_{\omega_n}(\top, v) = \lim_{n \to \infty} |f_{\omega_n}| = \sup_{g \in X} |g|,
\]

where swapping lim and \( \sum \) is by the dominated convergence theorem. Hence, \( \mathcal{F} \) is an inductive set, and by Zorn’s lemma, it admits a maximal element.  

**Lemma 44.** For any cut \( X \) of \( \mathcal{F} \), the mass of any flow \( f \) for \( \mathcal{F} \) is equal to the flow of \( f \) going through \( C \), i.e. to \( \sum\{f(u, v) \mid (u, v) \in X^\uparrow\} \) where \( X^\uparrow \equiv X \times X^\infty \).

**Proof.** The proof identical to the finite case.

**Theorem 45 (Weak Countable Max-Flow Min-Cut).** We have

\[
\sup\{|f| \mid f \text{ is a flow for } N\} = \inf\{|C| \mid C \text{ is a cut for } N\}
\]

and both the supremum and infimum are reached.
The two first properties of a flow are immediate. For the first one, consider where \( \forall u \in V \). For the second flow property, consider any pair \((u, v)\) in \(X\) and that the infimum is a minimum.

By Lemma 43, there exists a maximal flow for \(N\) that we denote by \(f\). Let \(R\) be s.t. \(u \not\in R \iff f(u, v) < c(u, v)\) and consider the set \(C = \{x \in N \mid f(x, x) = 0\}\), where \(R^*\) is the reflexive-transitive closure of \(R\). Assume that \(\perp \in C\). Then, there exists a path from \(\top\) to \(\perp\) following \(R\) that we can assume, wlog, simple. Let \(u_0 = \perp, u_{n+1} = \top\) be such a path — i.e. \(u_0 = \top, u_{n+1} = \perp\) and the \(u_i\)'s are pairwise disjoint. Let \(\delta \geq \min_{i \leq n} \{c(u_i, u_{i+1}) - f(u_i, u_{i+1})\}\) (that is strictly positive by assumption of \(R\)), and let \(\Omega : V^2 \to \mathbb{R}\) be defined as \(\Omega(u, v) = \sigma_{u,v} \cdot \delta\), where \(\forall i, \sigma_{u_i, u_{i+1}} = \pm 1\) and is equal to 0 otherwise — all these cases being mutual exclusive by simplicity of the path. We consider \(g = f + \Omega\) and prove that \(g\) is a flow for \(N\) s.t. \(|g| > |f|\).

The two first properties of a flow are immediate. For the first one, consider \(u, v \in V\) such that \(g(u, v) > f(u, v)\). This can only happen if \(\sigma_{u,v} > 0\). In that case, \(u = u_k\) and \(v = u_{k+1}\) for some \(k\), and we have

\[
g(u, v) = g(u_k, u_{k+1}) = f(u_k, u_{k+1}) + \delta \leq f(u_k, u_{k+1}) + c(u_k, u_{k+1}) - f(u_k, u_{k+1}) = c(u_k, u_{k+1}) = c(u, v).
\]

For the second flow property, consider any pair \((u, v)\) of vertices. Then,

\[
g(u, v) = f(u, v) + \sigma_{u,v} \cdot \delta = f(u, v) - \sigma_{v,u} \cdot \delta = -f(v, u).
\]

For the Kirchhoff law, let \(u \in V\) s.t. \(u \not\in \{\top, \perp\}\). If \(u\) is not part of the considered path, then \(\forall v \in V, g(u, v) = f(u, v)\) and the flow at node \(u\) is identical for \(f\) and \(g\). Otherwise, there exists an index \(k \in [1, n]\) s.t. \(u = u_k\) and \(f(u, v)\) and \(g(u, v)\) only differ for \(v = u_{k-1}\) and \(v = u_{k+1}\). Hence, \(\sum_{v \in V} g(u, v)\) converges and:

\[
\sum_{v \in V} g(u, v) = \sum_{v \in V} f(u, v) + \sum_{i=1}^{k-1} \sigma_{u_i, u_{i+1}} \cdot \delta + \sum_{i=1}^{k} \sigma_{u_i, u_{i-1}} \cdot \delta = \sum_{v \in V} f(u, v) = 0.
\]

We now prove that the flow \(g\) has a mass strictly greater than the one of \(f\):

\[
|g| = \sum_{v \in V} g(\top, v) = \sum_{v \in V} f(\top, v) + \Omega(\top, u_1) = |f| + \delta > |f|.
\]

By maximality of \(f\), such a flow \(g\) cannot exist. Hence, \(\perp \not\in C\) and \(C\) is a cut.

We can now conclude the proof. By Lemma 44 for any cut \(X\), \(|f| = \sum_{e \in X} f(e) \leq \sum_{e \in X^1} c(e) = |C|\). Hence, \(f \preceq \inf \{|C| \mid C\ is a cut for \(N\}\} \). Now, by definition of \(C\), any edge in \(X^1\) is saturated by \(f\). Hence, for \(e \in C^1\), \(f(e) = c(e)\) and \(|f| = |C|\). This proves that \(C\) is a minimal cut and that the infimum is a minimum.