An intrinsic Proper Generalized Decomposition for parametric symmetric elliptic problems

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June 28, 2017

Abstract

We introduce in this paper a technique for the reduced order approximation of parametric symmetric elliptic partial differential equations. For any given dimension, we prove the existence of an optimal subspace of at most that dimension which realizes the best approximation in mean of the error with respect to the parameter in the quadratic norm associated to the elliptic operator between the exact solution and the Galerkin solution calculated on the subspace. This is analogous to the best approximation property of the Proper Orthogonal Decomposition (POD) subspaces, excepting that in our case the norm is parameter-depending, and then the POD optimal sub-spaces cannot be characterized by means of a spectral problem. We apply a deflation technique to build a series of approximating solutions on finite-dimensional optimal subspaces, directly in the on-line step. We prove that the partial sums converge to the continuous solutions in mean quadratic elliptic norm.

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1 Introduction

The Karhunen-Loève's expansion (KLE) is a widely used tool, that provides a reliable procedure for a low dimensional representation of spatiotemporal signals (see [8, 18]). It is referred to as the principal components analysis (PCA) in statistics (see [10, 12, 23]), or called singular value decomposition (SVD) in linear algebra (see [9]). It is named the proper orthogonal decomposition (POD) in mechanical computation, where it is also widely used (see [3]). Its use allows large savings of computational costs, and make affordable the solution of problems that need a large amount of solutions of parameter-depending Partial Differential Equations (see [2, 6, 11, 16, 23, 24, 25, 26]).

However the computation of the POD expansion requires to know the function to be expanded, or at least its values at the nodes of a fine enough net. This makes it rather expensive to solve parametric elliptic Partial Differential Equations (PDEs), as it requires the previous solution of the PDE for a large enough number of values of the parameter ("snapshots") (see [13]), even if these can be located at optimal positions (see [15]). Galerkin-POD strategies are well suited to solve parabolic problems, where the POD basis is obtained from the previous solution of the underlying elliptic operator (see [14, 20]).

An alternative approach is the Proper Generalized Decomposition that iteratively computes a tensorized representation of the parameterized PDE, that separates the parameter and the independent variables, introduced in [1]. It has been interpreted as a Power type Generalized Spectral Decomposition (see [21, 22]). It has experienced a fast development, being applied to the low-dimensional tensorized solution of many applied problems. The mathematical analysis of the PGD has experienced a relevant development in the last years. The convergence of a version of the PGD for symmetric elliptic PDEs via minimization of the associated energy has been proved in [17]. Also, in [7] the convergence of a recursive approximation of the solution of a linear elliptic PDE is proved, based on the existence of optimal subspaces of rank 1 that minimize the elliptic norm of the current residual.

The present paper is aimed at the direct determination of a variety of reduced dimension for the solution of parameterized symmetric elliptic PDEs. We intend to on-line determine an optimal subspace of given dimension that yields the best approximation in mean (with respect to the parameter) of the error (in the quadratic norm associated to the elliptic operator) between the exact solution and the Galerkin solution calculated on the subspace. The optimal POD sub-spaced can no longer be characterized by means of a spectral problem for a compact self-adjoint operator (the standard POD operator) and thus the spectral theory for compact self-adjoint operators does no apply. We build recursive approximations on finite-dimensional optimal subspaces by minimizing the mean quadratic error of the current residual, similar to the one introduced in [7], that we prove to be strongly convergent in the "natural"mean quadratic elliptic norm. The method shares some properties of PGD and POD expansions: It builds a tensorized representation of the parameterized solutions, by means of optimal subspaces that minimize the residual in mean quadratic norm.

The paper is structured as follows: In Section 2 we state the general problem of finding optimal subspaces of a given dimension. We prove in Section 3 that there exists a solution for 1D optimal subspaces, characterized as a maximization problem with a non-linear normalization restriction. We extend this existence result in Section 4 to general dimensions. Finally, in Section 5 we use the results in Sections 3 and 4 to build a deflation algorithm to approximate the solution of a parametric family of elliptic problems and we show the convergence.

2 Statement of the problem

Let H be a separable Hilbert space endowed with the scalar product $(\cdot,\cdot)$. The related norm is denoted by $\|\cdot\|$.

We denote by $B_s(H)$ the space of bilinear, symmetric and continuous forms in H.

Assume given a measure space $(\Gamma,\mathcal{B},\mu)$, with standard notation, so that μ is σ-finite.

Let $a \in L^\infty(\Gamma, B_s(H); d\mu)$ be such that there exists $\alpha > 0$ satisfying

$$\alpha\,\|u\|^2 \le a(u,u;\gamma), \quad \forall u \in H,\ d\mu\text{-a.e. } \gamma \in \Gamma. \tag{1}$$

For $\mu-$a.e $\gamma \in \Gamma$, the bilinear form $a(\cdot,\cdot;\gamma)$ determines a norm uniformly equivalent to the norm $\|\cdot\|$. Moreover, $\overline{a} \in B_s(H)$ defined by

$$\overline{a}(v,w) = \int_\Gamma a(v(\gamma),w(\gamma);\gamma)\,d\mu(\gamma), \quad \forall v,\, w \in L^2(\Gamma,H;d\mu) \tag{2}$$

defines an inner product in H which generates a norm equivalent to the standard one in $L^2(\Gamma,H;d\mu)$.

Let be given a data function $f \in L^2(\Gamma,H';d\mu)$. We are interested in the variational problem:

$$\text{Find} \quad u(\gamma) \in H \quad \text{such that} \quad a(u(\gamma),v;\gamma) = \langle f(\gamma), v\rangle, \quad \forall v \in H,\ d\mu\text{-a.e. } \gamma \in \Gamma, \tag{3}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between H' and H.

By Riesz representation theorem, problem (3) admits a unique solution for $d\mu$-a.e. $\gamma \in \Gamma$. On the other hand, we claim that $\tilde{u}$ solution of

$$\tilde{u} \in L^2(\Gamma,H;d\mu), \qquad \bar{a}(\tilde{u},\bar{v}) = \int_\Gamma \langle f(\gamma),\bar{v}(\gamma)\rangle\,d\mu(\gamma), \quad \forall\,\bar{v} \in L^2(\Gamma,H;d\mu), \tag{4}$$

also satisfies (3): Indeed taking $\bar{v} = v\chi_B$, with $v \in H$ fixed and $B \in \mathcal{B}$ arbitrary, implies that there exists a subset $N_v \in \mathcal{B}$ with $\mu(N_v) = 0$ such that

$$a(\tilde{u}(\gamma),v;\gamma) = \langle f(\gamma),v\rangle, \quad \forall\,\gamma \in \Gamma \setminus N_v.$$

The separability of H implies that N_v can be chosen independent of v, which proves the claim. By the uniqueness of the solution of (3) this shows that

$$\tilde{u} = u \quad d\mu\text{-a.e. } \gamma \in \Gamma. \tag{5}$$

This proves that u belongs to $L^2(\Gamma,H;d\mu)$ and provides an equivalent definition of u. Namely, that u is the solution of (4).

Given a closed subspace Z of H, let us denote by $u_Z(\gamma)$ the solution of the Galerkin approximation of problem (3) on Z, which reads as

$$u_Z(\gamma) \in Z, \quad a(u_Z(\gamma),z;\gamma) = \langle f(\gamma),z\rangle, \quad \forall z \in Z,\ d\mu\text{-a.e. } \gamma \in \Gamma, \tag{6}$$

or equivalently as

$$u_Z \in L^2(\Gamma,Z;d\mu), \qquad \bar{a}(u_Z,z) = \int_\Gamma \langle f(\gamma),z(\gamma)\rangle\,d\mu(\gamma), \quad \forall\,z \in L^2(\Gamma,Z;d\mu). \tag{7}$$

For every $k \in \mathbb{N}$, we intend to find the best subspace W of H of dimension smaller than or equal to k that minimizes the mean error between $u(\gamma)$ and $u_W(\gamma)$. That is, W solves

$$\min_{Z\in\mathcal{S}_k} \bar{a}(u-u_Z,u-u_Z), \tag{8}$$

where $\mathcal{S}_k$ is the family of subspaces of H of dimension smaller than or equal to k. This problem will be proved to have a solution in Sections 3 and 4. We will then use this result to approximate the solution u of problem (3) by a deflation algorithm.

To finish this section we provide some equivalent formulations of the problem. First we observe that

Proposition 2.1 *For every closed subspace $Z \subset H$, the function u_Z defined by (7) is also the unique solution of*

$$\min_{z\in L^2(\Gamma,Z;d\mu)} \bar{a}(u-z,u-z). \tag{9}$$

Moreover, for $d\mu$-a.e. $\gamma \in \Gamma$, the vector $u_Z(\gamma)$ is the solution of

$$\min_{z\in Z} a(u(\gamma)-z,u(\gamma)-z;\gamma). \tag{10}$$

Proof: It is a classical property of the Galerkin approximation of the variational formulation of linear elliptic problems that u_Z satisfies (9). Indeed, the symmetry of $\bar{a}$ gives

$$\bar{a}(u-z,u-z) = \bar{a}(u-u_Z,u-u_Z) + 2\bar{a}(u-u_Z,u_Z-z) + \bar{a}(u_Z-z,u_Z-z),$$

for every $z \in L^2(\Gamma,H;d\mu)$, where by (4), (5) and (7) the second term on the right-hand side vanishes, while the third one is nonnegative. This proves (9).

The proof of (10) is the same by taking into account (3) and (6) instead of (4) and (7). ∎

As a consequence of Proposition 2.1 and definition (2) of $\bar{a}$, we have

Corollary 2.2 *A space $W \in S_k$ is a solution of (8) if and only if it is a solution of*

$$\min_{Z\in S_k}\min_{z\in L^2(\Gamma,Z;d\mu)} \bar{a}(u-z,u-z). \tag{11}$$

Moreover

$$\min_{Z\in S_k}\min_{z\in L^2(\Gamma,Z;d\mu)} \bar{a}(u-z,u-z) = \min_{Z\in S_k}\int_\Gamma \min_{z\in Z} a(u(\gamma)-z,u(\gamma)-z;\gamma)d\mu(\gamma). \tag{12}$$

Remark 2.3 *Optimization problem* (11) *is reminiscent of the Kolmogorov k-width related to the best approximation of the manifold $(u(\gamma))_{\gamma\in\Gamma}$ by subspaces in H with dimension k as presented in* [19]. *In the present minimization problem, we use the norm of $L^2(\Gamma,H,d\mu)$ instead of the norm of $L^\infty(\Gamma,H,d\mu)$ as used there. The minimization problem in* [19] *can indeed be written as*

$$\min_{Z\in\mathcal{S}_k}\operatorname*{ess\,sup}_{\gamma\in\Gamma}\min_{z\in Z} a(u(\gamma)-z,u(\gamma)-z;\gamma),$$

if one uses $a(\cdot,\cdot;\gamma)$ as the inner product in H.

For a function $v \in L^2(\Gamma,V;d\mu)$, we denote by $R(v)$ the vectorial space spanned by $v(\gamma)$ when γ belongs to Γ; more exactly, taking into account that v is only defined up to sets of zero measure, the correct definition of $R(v)$ is given by

$$R(v) = \bigcap_{\mu(N)=0} \text{Span}\,\big\{v(\gamma):\ \gamma\in\Gamma\setminus N\big\}. \tag{13}$$

Taking into account (11), a new formulation of (8) is given by

Proposition 2.4 *If W is a solution of (8), then u_W is a solution of*

$$\min_{\substack{v\in L^2(\Gamma,H;d\mu)\\ \dim R(v)\leq k}} \bar{a}(u-v,u-v). \tag{14}$$

Reciprocally, if $\hat{u}$ is a solution of (14), then $R(\hat{u})$ is a solution of (8) and $\hat{u}=u_{R(\hat{u})}$.

The next proposition provides another formulation for (8) which depends on f and not on the solution u of (3).

Proposition 2.5 *The subspace $W\in\mathcal{S}_k$ solves problem* (8) *if and only if it is a solution of the problem*

$$\max_{Z\in\mathcal{S}_k}\int_\Gamma \langle f(\gamma),u_Z(\gamma)\rangle\,d\mu(\gamma). \tag{15}$$

Proof: As in the proof of the first part of Proposition 2.1, one deduces from (4), (5) and (7) that

$$\bar{a}(u-u_Z,z)=0,\quad \forall z\in L^2(\Gamma,Z;d\mu).$$

Using the symmetry of $\bar{a}$, we then have

$$\bar{a}(u-u_Z,u-u_Z)=\bar{a}(u,u)-a(u_Z,u)=\bar{a}(u,u)-\bar{a}(u_Z,u_Z)$$
$$=\bar{a}(u,u)-\int_\Gamma\langle f(\gamma),u_Z(\gamma)\rangle\,d\mu(\gamma).$$

Thus W solves (8) if and only if it solves (15). ∎

Remark 2.6 *In* [7] *a problem similar to* (8) *has been studied, namely*

$$(P_k)'\qquad \min_{Z\in\mathcal{S}_k}\int_\Gamma (u(\gamma)-u_Z(\gamma),u(\gamma)-u_Z(\gamma))_H\,d\mu(\gamma), \tag{16}$$

where $(\cdot,\cdot)_H$ is an inner product on H. In this case a solution of $(P_k)'$ is the space generated by the first k eigenfunctions of the POD operator $\mathcal{P}:H\mapsto H$, which is given by

$$\mathcal{P}(v)=\int_\Gamma (v,u(\gamma))_H\,u(\gamma)\,d\mu(\gamma),\quad \forall v\in H.$$

In the present case, due the dependence of a with respect to γ, it does not seem that the problem can be reduced to a spectral problem. As an example, we consider the case $k=1$. Then problem (14) can be written as

$$\min_{v\in H,\,\varphi\in L^2(\Gamma;d\mu)}\int_\Gamma a(u(\gamma)-\varphi(\gamma)v,u(\gamma)-\varphi(\gamma)v;\gamma)d\mu(\gamma). \tag{17}$$

So, taking the derivative of the functional

$$(v,\varphi)\in H\times L^2(\Gamma;d\mu)\mapsto\int_\Gamma a(u(\gamma)-\varphi(\gamma)v,u(\gamma)-\varphi(\gamma)v;\gamma)d\mu(\gamma),$$

we deduce that if $(w,\psi)\in H\times L^2(\Gamma;d\mu)$ is a solution of (17), with $w\neq 0$, then

$$\psi(\gamma)=\frac{a(u(\gamma),w;\gamma)}{a(w,w;\gamma)},\quad d\mu\text{-a.e. }\gamma\in\Gamma, \tag{18}$$

and w is a solution of the non-linear variational problem

$$\int_\Gamma \frac{a(u(\gamma),w;\gamma)}{a(w,w;\gamma)} a(u(\gamma),v;\gamma)d\mu(\gamma) = \int_\Gamma \frac{a(u(\gamma),w;\gamma)^2}{a(w,w;\gamma)^2} a(w,v;\gamma)d\mu(\gamma), \quad \forall\, v \in H. \tag{19}$$

Note that if $w=0$, then $u=0$ and therefore $f=0$.

If a does not depend on γ, statement (19) can be written as

$$a\left(\int_\Gamma a(u(\gamma),w)\, u(\gamma)d\mu(\gamma), v\right) = a\left(\frac{\displaystyle\int_\Gamma a(u(\gamma),w)^2 d\mu(\gamma)}{a(w,w)}\, w, v\right), \quad \forall\, v \in H,$$

which implies that

$$\int_\Gamma a(u(\gamma),w)\, u(\gamma)d\mu(\gamma) = \frac{\displaystyle\int_\Gamma a(u(\gamma),w)^2 d\mu(\gamma)}{a(w,w)}\, w,$$

i.e. w is an eigenvector of the operator

$$v \in H \mapsto \int_\Gamma a(u(\gamma),v)u(\gamma)d\mu(\gamma)$$

for the eigenvalue

$$\frac{\displaystyle\int_\Gamma a(u(\gamma),w)^2 d\mu(\gamma)}{a(w,w)}.$$

In contrast, when a depends on γ problem (19) does not correspond to an eigenvalue equation.

3 One-dimensional approximations

In Section 4 we shall show the existence of the solution of problem (8) for any arbitrary k. However a particularly interesting case from the point of view of the applications is $k = 1$. We dedicate this section to this special case. Observe that for $Z \in \mathcal{S}_1$, there exists $z \in H$ such that $Z = Span\{z\}$. The problem to solve can be reformulated as follows.

Lemma 3.1 *Assume $f \not\equiv 0$. Then, the subspace $W \in \mathcal{S}_1$ solves problem* (15) *if and only if $W = \mathrm{span}\{w\}$, where w is a solution of*

$$\max_{\substack{z \in H \\ z \neq 0}} \int_\Gamma \frac{\langle f(\gamma), z\rangle^2}{a(z, z; \gamma)}\, d\mu(\gamma). \tag{20}$$

Proof: Let $Z \in \mathcal{S}_1$. Then $Z = \mathrm{span}\{z\}$, for some $z \in H$, and there exists a function $\varphi : \Gamma \mapsto \mathbb{R}$ such that

$$u_Z(\gamma) = \varphi(\gamma)\, z, \quad d\mu\text{-a.e. } \gamma \in \Gamma.$$

If $z \neq 0$, then, as $u_Z(\gamma)$ is the solution to the variational equation (7), we derive that

$$\varphi(\gamma) = \frac{\langle f(\gamma), z\rangle}{a(z, z; \gamma)}, \quad d\mu\text{-a.e. } \gamma \in \Gamma.$$

Using this formula we obtain that

$$\int_\Gamma \langle f, u_Z(\gamma)\rangle\, d\gamma = \int_\Gamma \frac{\langle f(\gamma), z\rangle^2}{a(z, z; \gamma)}\, d\mu(\gamma). \tag{21}$$

If the maximum in (15) is obtained by a space of dimension one, then formula (21) proves the desired result.

In contrast, if the maximum in (15) is obtained by the null space, then the maximum in $\mathcal{S}_1$ is equal to zero. Therefore the right-hand side of (21) is zero for every $z \in H$, which implies that $f = 0$ $d\mu$-a.e. in Γ, in contradiction with the assumption $f \not\equiv 0$. ∎

Remark 3.2 *Since the integrand which appears in (20) is homogenous of degree zero in z, problem* (20) *is equivalent to*

$$\max_{\substack{z \in H \\ \|z\|=1}} \int_\Gamma \frac{\langle f(\gamma), z\rangle^2}{a(z, z; \gamma)}\, d\mu(\gamma).$$

We now prove the existence of a solution to problem (20).

Theorem 3.3 *Assume $f \not\equiv 0$. Problem* (20) *admits at least a solution.*

Note that if $f \equiv 0$, then, every vector $w \in H \setminus \{0\}$ is a solution of (20).

Proof: Define

$$M^* := \sup_{\substack{z \in H \\ \|z\|=1}} \int_\Gamma \frac{\langle f(\gamma), z\rangle^2}{a(z, z; \gamma)}\, d\mu(\gamma), \tag{22}$$

and consider a sequence $w_n \subset H$, with $\|w_n\| = 1$ such that

$$\lim_{n\to\infty} \int_\Gamma \frac{\langle f(\gamma), w_n\rangle^2}{a(w_n, w_n; \gamma)}\, d\mu(\gamma) = M^*. \tag{23}$$

Up to a subsequence, we can assume the existence of $w \in H$, such that w_n converges weakly in H to w. Taking into account that $f(\gamma) \in H'$, $a(\cdot,\cdot,\gamma) \in B_s(H)$ $d\mu$-a.e. $\gamma \in \Gamma$ and (1) is satisfied, we get

$$\lim_{n\to\infty} \langle f(\gamma), w_n\rangle = \langle f(\gamma), w\rangle, \quad d\mu\text{-a.e. } \gamma \in \Gamma, \tag{24}$$

$$\liminf_{n\to\infty} a(w_n, w_n; \gamma) \geq a(w, w; \gamma), \quad d\mu\text{-a.e. } \gamma \in \Gamma. \tag{25}$$

On the other hand, we observe that (1) and $\|w_n\| = 1$ imply

$$|\langle f(\gamma), w_n\rangle| \leq \|f(\gamma)\|_{H'}, \quad \frac{1}{a(w_n, w_n; \gamma)} \leq \frac{1}{\alpha} \qquad d\mu\text{-a.e. } \gamma \in \Gamma. \tag{26}$$

If $w = 0$, then (24), (26) and Lebesgue's dominated convergence theorem imply

$$\lim_{n\to\infty} \int_\Gamma \frac{\langle f(\gamma), w_n\rangle^2}{a(w_n, w_n; \gamma)}\, d\mu(\gamma) = 0,$$

which by (23) is equivalent to $M^* = 0$. Taking into account (1) and the definition (22) of M^*, this is only possible if $f \equiv 0$ is the null function. As we are assuming $f \not\equiv 0$, we conclude that w is different of zero. Then, (26) proves

$$0 \leq \frac{\|f(\gamma)\|_{H'}^2}{\alpha} - \frac{\langle f(\gamma), w_n\rangle^2}{a(w_n, w_n; \gamma)}, \quad d\mu\text{-a.e. } \gamma \in \Gamma,$$

while (24) and (25) prove

$$\liminf_{n\to\infty} \left(\frac{\|f(\gamma)\|_{H'}^2}{\alpha} - \frac{\langle f(\gamma), w_n\rangle^2}{a(w_n, w_n; \gamma)} \right) \geq \frac{\|f(\gamma)\|_{H'}^2}{\alpha} - \frac{\langle f(\gamma), w\rangle^2}{a(w, w; \gamma)}, \quad d\mu\text{-a.e. } \gamma \in \Gamma. \tag{27}$$

Using (23), Fatou's lemma implies

$$\int_\Gamma \left(\frac{\|f(\gamma)\|_{H'}^2}{\alpha} - \frac{\langle f(\gamma), w\rangle^2}{a(w, w; \gamma)} \right) d\mu(\gamma) \leq \liminf_{n\to\infty} \int_\Gamma \left(\frac{\|f(\gamma)\|_{H'}^2}{\alpha} - \frac{\langle f(\gamma), w_n\rangle^2}{a(w_n, w_n; \gamma)} \right) d\mu(\gamma)$$
$$= \int_\Gamma \frac{\|f(\gamma)\|_{H'}^2}{\alpha}\, d\mu(\gamma) - M^*,$$

or equivalently

$$M^* \leq \int_\Gamma \frac{\langle f(\gamma), w\rangle^2}{a(w, w; \gamma)} d\mu(\gamma). \tag{28}$$

By definition (22) of M^*, this proves that the above inequality is an equality and that w is a solution of (20). ∎

Remark 3.4 *Actually, in place of (27), one has the stronger result*

$$\liminf_{n\to\infty} \left(\frac{\|f(\gamma)\|_{H'}^2}{\alpha} - \frac{\langle f(\gamma), w_n\rangle^2}{a(w_n, w_n; \gamma)} \right) = \frac{\|f(\gamma)\|_{H'}^2}{\alpha} - \frac{\langle f(\gamma), w\rangle^2}{\liminf\limits_{n\to\infty} a(w_n, w_n; \gamma))}, \quad d\mu\text{-a.e. } \gamma \in \Gamma,$$

which by the proof used to prove (28) shows

$$M^* \leq \int_\Gamma \frac{\langle f(\gamma), w\rangle^2}{\liminf\limits_{n\to\infty} a(w_n, w_n; \gamma)} d\mu(\gamma).$$

Combined with

$$M^* = \int_\Gamma \frac{\langle f(\gamma), w\rangle^2}{a(w,w;\gamma)} d\mu(\gamma)$$

and (25), this implies

$$a(w,w;\gamma) = \liminf_{n\to\infty} a(w_n,w_n;\gamma) \quad d\mu\text{-a.e. } \gamma\in\Gamma \quad \textit{such that } \langle f(\gamma), w\rangle \neq 0.$$

By (1) and $f \not\equiv 0$, this proves the existence of a subsequence of w_n which converges strongly to w.

Since this proof can be carried out by replacing w_n by any subsequence of w_n, we conclude that the whole sequence w_n (which we extracted just after (23) assuming that it converges weakly to some w) actually converges strongly to w.

The above result may be used to build a computable approximation of a solution of (20)*. Indeed, for $f \not\equiv 0$, let $\{H_n\}_{n\geq 1}$ be an internal approximation of H, that is a sequence of subspaces of finite dimension of H such that*

$$\lim_{n\to\infty} \inf_{\psi\in H_n} \|z-\psi\| = 0, \quad \forall z \in H.$$

and consider a solution w_n of

$$\max_{\substack{z\in H_n\\ \|z\|=1}} \int_\Gamma \frac{\langle f(\gamma), z\rangle^2}{a(z,z;\gamma)}\, d\mu(\gamma).$$

The existence of such a w_n can be obtained by reasoning as in the proof of Theorem 3.3 or just using Weierstrass theorem because the dimension of H_n is finite.

Taking $\tilde w$ a solution of (20) and a sequence $\tilde w_n \in H_n$ converging to $\tilde w$ in H, we have

$$\int_\Gamma \frac{\langle f(\gamma), \tilde w\rangle^2}{a(\tilde w,\tilde w;\gamma)}\, d\mu(\gamma) = \lim_{n\to\infty} \int_\Gamma \frac{\langle f(\gamma), \tilde w_n\rangle^2}{a(\tilde w_n,\tilde w_n;\gamma)}\, d\mu(\gamma)$$
$$\leq \liminf_{n\to\infty} \int_\Gamma \frac{\langle f(\gamma), w_n\rangle^2}{a(w_n,w_n;\gamma)}\, d\mu(\gamma) \leq \limsup_{n\to\infty} \int_\Gamma \frac{\langle f(\gamma), w_n\rangle^2}{a(w_n,w_n;\gamma)}\, d\mu(\gamma) \leq \int_\Gamma \frac{\langle f(\gamma), \tilde w\rangle^2}{a(\tilde w,\tilde w;\gamma)}\, d\mu(\gamma),$$

and thus

$$\lim_{n\to\infty} \int_\Gamma \frac{\langle f(\gamma), w_n\rangle^2}{a(w_n,w_n;\gamma)}\, d\mu(\gamma) = \int_\Gamma \frac{\langle f(\gamma), \tilde w\rangle^2}{a(\tilde w,\tilde w;\gamma)}\, d\mu(\gamma) = M^*.$$

This proves that the sequence w_n satisfies (23). Therefore any subsequence of w_n which converges weakly to some w converges strongly to w which is a solution of (20)*.*

4 Higher-dimensional approximations

This section is devoted to the proof of the existence of an optimal subspace which is solution of (8) when $k \geq 1$ is any given number.

Theorem 4.1 *For any given $k \geq 1$, problem (8) admits at least one solution.*

Proof: As in the proof of Theorem 3.3, we use the direct method of the Calculus of Variations. Denoting by m_k

$$m_k = \inf_{Z \in \mathcal{S}_k} \bar{a}(u - u_Z, u - u_Z), \tag{29}$$

we consider a sequence of spaces $W_n \in \mathcal{S}_k$ such that $w_n := u_{W_n}$ satisfies

$$\lim_{n \to \infty} \bar{a}(u - w_n, u - w_n) = m_k. \tag{30}$$

Taking into account that by Proposition 2.1

$$Z \subset \tilde{Z} \Longrightarrow \bar{a}(u - u_{\tilde{Z}}, u - u_{\tilde{Z}}) \leq \bar{a}(u - u_Z, u - u_Z), \tag{31}$$

we can assume that the dimension of W_n is equal to k. Moreover, we observe that (30) implies that w_n is bounded in $L^2(\Gamma, H; d\mu)$.

Let $(z_n^1, \cdots, z_n^k)$ be an orthonormal basis of W_n. It holds

$$w_n(\gamma) = \sum_{j=1}^{k} (w_n(\gamma), z_n^j)\, z_n^j, \quad d\mu\text{-a.e. } \gamma \in \Gamma. \tag{32}$$

Since the norm of the vectors z_n^j is one, there exists a subsequence of n and k vectors $z^j \in H$ such that

$$z_n^j \rightharpoonup z^j \ \text{ in } H, \quad \forall\, j \in \{1, \cdots, k\}. \tag{33}$$

Using also

$$|(w_n(\gamma), z_n^j)| \leq \|w_n(\gamma)\|, \qquad d\mu\text{-a.e } \gamma \in \Gamma,$$

we get that (w_n, z_n^j) is bounded in $L^2(\Gamma, H; d\mu)$ for every j and thus, there exists a subsequence of n and k functions $p^j \in L^2(\Gamma; d\mu)$ such that

$$(w_n, z_n^j) \rightharpoonup p^j \ \text{ in } L^2(\Gamma, H; d\mu), \ \forall\, j \in \{1, \cdots, k\}. \tag{34}$$

We claim that

$$w_n \rightharpoonup w := \sum_{j=1}^{n} p^j z^j \ \text{ in } L^2(\Gamma; d\mu). \tag{35}$$

Indeed, taking into account that w_n is bounded in $L^2(\Gamma, H; d\mu)$ and (32), it is enough to show

$$\lim_{n \to \infty} \int_\Gamma \big((w_n, z_n^j) z_n^j, \varphi\, v\big)\ d\mu(\gamma) = \int_\Gamma (p^j z^j, \varphi\, v)\, d\mu(\gamma), \quad \forall\, \varphi \in L^2(\Gamma; d\mu),\ \forall\, v \in H. \tag{36}$$

This is a simple consequence of

$$\int_\Gamma \big((w_n, z_n^j) z_n^j, \varphi\, v\big)\ d\mu(\gamma) = (z_n^j, v) \int_\Gamma (w_n, z_n^j)\, \varphi\, d\mu(\gamma),$$

combined with (33) and (34).

From the continuity and convexity of the quadratic form associated to $\bar{a}$, as well as from (35) and (30), we have

$$\bar{a}(u-w,u-w) \leq \lim_{n\to\infty} \bar{a}(u-w_n,u-w_n) = m_k. \tag{37}$$

Using that $W = \mathrm{Span}\{z^1, \cdots, z^k\} \in \mathcal{S}_k$, and that (see Proposition 2.1)

$$\bar{a}(u-u_W,u-u_W) \leq \bar{a}(u-w,u-w), \tag{38}$$

we conclude that W is a solution of (8). ∎

Remark 4.2 *From (37), (38), definition (29) of m_k and Proposition 2.1, we have that $w = u_W$ in the proof of Theorem 4.1. Moreover,*

$$\bar{a}(u-w,u-w) = m_k = \lim_{n\to\infty} \bar{a}(u-w_n,u-w_n),$$

which combined with (35) proves that w_n converges strongly to w in $L^2(\Gamma, H; d\mu)$. As in Remark 3.4, this can be used to build a strong approximation of a solution of (8) by using an internal approximation of H.

5 An iterative algorithm by deflation

In the previous section, for any given $k \geq 1$, we have proved the existence of an optimal subspace for problem (8). We use here this fact to build an iterative approximation of the solution of (3) by a deflation approach. Let us denote

$$\Pi_k(v) = \left\{ v_W \mid \quad W \text{ solves } \min_{Z \in \mathcal{S}_k} \bar{a}(v - v_Z, v - v_Z) \right\}, \quad \forall\, v \in L^2(\Gamma, H; d\mu). \tag{39}$$

The deflation algorithm is as follows

- Initialization:

$$u_0 = 0 \tag{40}$$

- Iteration: Assuming $u_{i-1} \in H$ known for $i = 1, 2, \cdots$, set

$$u_i = u_{i-1} + s_i, \quad \text{ with } s_i \in \Pi_k(e_{i-1}), \text{ where } e_{i-1} = u - u_{i-1} \tag{41}$$

Remark 5.1 *Since one has $e_{i-1} = u - u_{i-1}$ by (41) and since by (5), u is the solution of (4), the function e_{i-1} satisfies*

$$\begin{cases} e_{i-1} \in L^2(\Gamma, H; d\mu), \\ \bar{a}(e_{i-1}, v) = \displaystyle\int_\Gamma \langle f(\gamma), v(\gamma) \rangle \, d\mu(\gamma) - \bar{a}(u_{i-1}, v), \quad \forall\, v \in L^2(\Gamma, H; d\mu). \end{cases}$$

Therefore Proposition 2.5 applied to the case where f is replaced by the function $\hat{f}_i$ defined by

$$\int_\Gamma \langle \hat{f}_i(\gamma), v(\gamma) \rangle d\mu(\gamma) = \int_\Gamma \langle f(\gamma), v(\gamma) \rangle \, d\mu(\gamma) - \bar{a}(u_{i-1}, v), \quad \forall\, v \in L^2(\Gamma, H; d\mu),$$

proves that $s_i \in \Pi_k(e_{i-1})$ is equivalent to $s_i = (e_{i-1})_W$, where W is a solution of

$$\max_{Z \in \mathcal{S}_k} \left\{ \int_\Gamma \langle f(\gamma), (e_{i-1})_Z(\gamma) \rangle \, d\mu(\gamma) - \bar{a}(u_{i-1}, (e_{i-1})_Z) \right\},$$

where, in accordance to (7), $(e_{i-1})_Z$ denotes the solution of

$$\begin{cases} (e_{i-1})_Z \in L^2(\Gamma, Z; d\mu), \\ \bar{a}\big((e_{i-1})_Z, z\big) = \displaystyle\int_\Gamma \langle f(\gamma), z(\gamma) \rangle \, d\mu(\gamma) - \bar{a}(u_{i-1}, z), \quad \forall\, z \in L^2(\Gamma, Z; d\mu). \end{cases}$$

This observation allows one to carry out the iterative process without knowing the function u (compare with (41)).

The convergence of the algorithm is given by the following theorem. Its proof follows the ideas of [7].

Theorem 5.2 *The sequence u_i provided by the least-squares PGD algorithm (40)-(41) strongly converges in $L^2(\Gamma, H; d\mu)$ to the parameterized solution $\gamma \in \Gamma \mapsto u(\gamma) \in H$ of problem (3).*

Proof: By (41) and Proposition 2.4 applied to the case where u is replaced by e_{i-1}, we have that s_i is a solution of

$$\min_{\substack{v\in L^2(\Gamma,H;d\mu)\\ \dim R(v)\le k}} \bar{a}(e_{i-1}-v, e_{i-1}-v). \tag{42}$$

This proves in particular that s_i is a solution of

$$\min_{\substack{v\in L^2(\Gamma,H;d\mu)\\ R(v)\subset R(s_i)}} \bar{a}(e_{i-1}-v, e_{i-1}-v),$$

and therefore

$$\bar{a}(e_{i-1}-s_i, v) = 0, \quad \forall\, v\in L^2(\Gamma,H;d\mu) \ \text{ with } R(v)\subset R(s_i).$$

But (41) implies that

$$e_{i-1}-s_i = e_i, \tag{43}$$

which gives

$$\bar{a}(e_i, v) = 0, \quad \forall\, v\in L^2(\Gamma,H;d\mu) \ \text{ with } R(v)\subset R(s_i). \tag{44}$$

Taking $v=s_i$ and using again (43) we get

$$\bar{a}(e_{i-1}, e_{i-1}) = \bar{a}(s_i, s_i) + \bar{a}(e_i, e_i), \quad \forall\, i\ge 1, \tag{45}$$

and therefore

$$\bar{a}(e_i, e_i) + \sum_{j=1}^{i} \bar{a}(s_j, s_j) = \bar{a}(e_0, e_0), \quad \forall\, i\ge 1. \tag{46}$$

Thus, we have

$$e_i \ \text{ is bounded in } L^2(\Gamma,H;d\mu), \tag{47}$$

$$\sum_{j=1}^{\infty} \bar{a}(s_j, s_j) \le \bar{a}(e_0, e_0). \tag{48}$$

By (47), there exists a subsequence e_{i_n} of e_i and $e\in L^2(\Gamma,H;d\mu)$, such that

$$e_{i_n} \rightharpoonup e \ \text{ in } L^2(\Gamma,H;d\mu). \tag{49}$$

On the other hand, since s_{i_n+1} is a solution of (42) with $i-1$ replaced by i_n, we get

$$\begin{gathered}\bar{a}(e_{i_n}-s_{i_n+1}, e_{i_n}-s_{i_n+1}) \le \bar{a}(e_{i_n}-v, e_{i_n}-v) = \bar{a}(e_{i_n}, e_{i_n}) - 2\bar{a}(e_{i_n}, v) + \bar{a}(v,v),\\ \forall\, v\in L^2(\Gamma,H;d\mu),\ \dim R(v)\le k,\end{gathered} \tag{50}$$

and then

$$\bar{a}(e_{i_n}-s_{i_n+1}, e_{i_n}-s_{i_n+1}) - \bar{a}(e_{i_n}, e_{i_n}) \le -2\bar{a}(e_{i_n}, v) + \bar{a}(v,v), \ \ \forall\, v\in L^2(\Gamma,H;d\mu),\ \dim R(v)\le k,$$

or in other terms

$$-2\bar{a}(e_{i_n}, s_{i_n+1}) + \bar{a}(s_{i_n+1}, s_{i_n+1}) \le -2\bar{a}(e_{i_n}, v) + \bar{a}(v,v), \ \ \forall\, v\in L^2(\Gamma,H;d\mu),\ \dim R(v)\le k.$$

Thanks to (47) and (48), the left-hand side tends to zero when n tends to infinity, while in the right-hand side we can pass to the limit by (49). Thus, we have

$$2\bar{a}(e,v) \le \bar{a}(v,v), \quad \forall\, v\in L^2(\Gamma,H;d\mu), \ \ \dim \mathrm{R}(v)\le k.$$

Replacing in this equality v by tv with $t > 0$, dividing by t, letting t tend to zero and writing the resulting inequality for v and $-v$, we get

$$\bar{a}(e, v) = 0, \quad \forall\, v \in L^2(\Gamma, H; d\mu), \quad \dim \mathrm{R}(v) \le k.$$

Taking $v = w\varphi$, with $w \in H$, $\varphi \in L^2(\Gamma; d\mu)$, and recalling definition (2) of $\bar{a}$ we deduce

$$\int_\Gamma a(e(\gamma), w; \gamma)\, \varphi(\gamma)\, d\mu(\gamma) = 0, \quad \forall\, z \in H,\ \forall\, \varphi \in L^2(\Gamma; d\mu),$$

and then for any $w \in H$, there exists a subset $N_w \in \mathcal{B}$ with $\mu(N_w) = 0$ such that

$$a(e(\gamma), w; \gamma) = 0, \quad \forall\, \gamma \in \Gamma \setminus N_w.$$

The separability of H implies that N_w can be chosen independent of w, and then we have

$$a(e(\gamma), w; \gamma) = 0, \quad \forall\, w \in H,\ d\mu\text{-a.e. } \gamma \in \Gamma,$$

and therefore

$$e(\gamma) = 0 \quad d\mu\text{-a.e. } \gamma \in \Gamma. \tag{51}$$

This proves that e does not depend on the subsequence in (49) and that

$$e_i \rightharpoonup 0 \ \text{ in } L^2(\Gamma, H; d\mu). \tag{52}$$

Let us now prove that in (52) the convergence is strong in $L^2(\Gamma, H; d\mu)$. We use that thanks to (43), we have

$$e_i = -\sum_{j=1}^{i} s_j + e_0, \quad \forall\, i \ge 1,$$

and so,

$$\bar{a}(e_i, e_i) = -\sum_{j=1}^{i} \bar{a}(e_i, s_j) + \bar{a}(e_i, e_0), \quad \forall\, i \ge 1. \tag{53}$$

In order to estimate the right-hand side of the latest equality, we introduce, for $i, j \ge 1$, the function $z_{i,j}$ as the solution of

$$z_{i,j} \in L^2(\Gamma, \mathrm{R}(s_j); d\mu), \qquad \bar{a}(z_{i,j}, v) = \bar{a}(e_{i-1}, v), \ \ \forall\, v \in L^2(\Gamma, \mathrm{R}(s_j); d\mu). \tag{54}$$

We have

$$\big|\bar{a}(e_{i-1}, s_j)\big| = \big|\bar{a}(z_{i,j}, s_j)\big| \le \bar{a}(z_{i,j}, z_{i,j})^{\frac{1}{2}}\, \bar{a}(s_j, s_j)^{\frac{1}{2}}. \tag{55}$$

Using (45), (43), the fact that s_i is a solution of (42) and $\dim \mathrm{R}(s_j) \le k$

$$\bar{a}(e_{i-1}, e_{i-1}) - \bar{a}(s_i, s_i) = \bar{a}(e_{i-1} - s_i, e_{i-1} - s_i) \le \bar{a}(e_{i-1} - z_{i,j}, e_{i-1} - z_{i,j}).$$

Expending the right-hand side and using $v = z_{i,j}$ in (54) this gives

$$\bar{a}(z_{i,j}, z_{i,j}) \le \bar{a}(s_i, s_i),$$

which combined with (55) provides the estimate

$$\big|\bar{a}(e_{i-1}, s_j)\big| \le \bar{a}(s_i, s_i)^{\frac{1}{2}}\, \bar{a}(s_j, s_j)^{\frac{1}{2}}, \quad \forall\, i, j \ge 1.$$

Using the latest estimate in (53) and then Cauchy-Schwarz's inequality, we get

$$
\left\{
\begin{aligned}
&\bar{a}(e_i, e_i) \le \bar{a}(s_{i+1}, s_{i+1})^{\frac{1}{2}} \sum_{j=1}^{i} \bar{a}(s_j, s_j)^{\frac{1}{2}} + \bar{a}(e_i, e_0) \\
&\le \bar{a}(s_{i+1}, s_{i+1})^{\frac{1}{2}}\, i^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} \bar{a}(s_j, s_j) \right)^{\frac{1}{2}} + \bar{a}(e_i, e_0), \quad \forall\, i \ge 1.
\end{aligned}
\right. \tag{56}
$$

But the criterion of comparison of two series with nonnegative terms and the facts that (see (48))

$$
\sum_{i=1}^{\infty} \frac{1}{i} = \infty, \qquad \sum_{i=1}^{\infty} \bar{a}(s_i, s_i) < \infty,
$$

prove that

$$
\liminf_{i \to \infty}\ \bar{a}(s_{i+1}, s_{i+1})\, i = \liminf_{i \to \infty} \frac{\bar{a}(s_{i+1}, s_{i+1})}{\frac{1}{i}} = 0.
$$

Since $\bar{a}(e_i, e_i)$ is a decreasing sequence by (45) and since (52) asserts that e_i converges weakly to zero, we can pass to the limit in (56), to deduce

$$
\begin{aligned}
&\lim_{i \to \infty} \bar{a}(e_i, e_i) = \liminf_{i \to \infty}\ \bar{a}(e_i, e_i) \\
&\le \liminf_{i \to \infty} \left(\bar{a}(s_{i+1}, s_{i+1})^{\frac{1}{2}}\, i^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} \bar{a}(s_j, s_j) \right)^{\frac{1}{2}} + \bar{a}(e_i, e_0) \right) = 0.
\end{aligned}
$$

This proves that e_i converges strongly to zero in $L^2(\Gamma, H; d\mu)$. Since $e_i = u - u_i$ this finishes the proof of Theorem 5.2. ∎

Remark 5.3 *In many cases the corrections s_i decrease exponentially in the sense that:*

$$
\|s_i\| = O(\rho^{-i}) \quad \textit{as } i \to +\infty, \quad \textit{for some } \rho > 1.
$$

This occurs in particular for the standard POD expansion when Γ is an open set of $\mathbb{R}^N$, μ is the Lebesgue measure and the function $f = f(\gamma)$ is analytic with respect to γ (see [5]). Then $\|s_i\|$ is a good estimator for the error $\|u - u_i\|$.

6 Conclusion

In this paper we have introduced an iterative deflation algorithm to solve parametric symmetric elliptic equations. It is a Proper Generalized Decomposition algorithm as it builds a tensorized representation of the parameterized solutions, by means of optimal subspaces that minimize the residual in mean quadratic norm. It is intrinsic in the sense that in each deflation step the residual is minimized in the "natural" mean quadratic norm generated by the parametric elliptic operator. It is conceptually close to the Proper Orthogonal Decomposition with the difference that in the POD the residual is minimized with respect to a fixed mean quadratic norm. Due to this difference, spectral theory cannot be applied.

We have proved the existence of the optimal subspaces of dimension less than or equal to a fixed number, as required in each iteration of the deflation algorithm, with a specific analysis for the one-dimensional case. Also, we have proved the strong convergence in the natural mean quadratic norm of the deflation algorithm for quite general parametric elliptic operators.

We will next focus our research on the analysis of convergence rates of the deflation algorithm that we introduced. We will compare the convergence rates with those of the POD expansion, to determine whether the use of the "natural" mean quadratic norm provides improved convergence rates. We will also work on the numerical approximation of the algorithm, based upon a "trust" solution on high-fidelity finite-dimensional subspaces of the given Hilbert space, to construct a feasible Reduced Order Modeling algorithm.

All the results obtained in the present paper refer to a symmetric. In a future work we will consider the non-symmetric case.

Acknowledgements

The work of Mejdi Azaiez and Tomás Chacón has been partially supported by the Spanish Government - Feder EU grant MTM2015-64577-C2-1-R.

The work of Juan Casado-Díaz and François Murat has been partially supported by the Spanish Government - Feder EU grant MTM2014-53309-P.

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