Typology of axioms for a weighted modal logic

Bénédicte Legastelois [a,*], Marie-Jeanne Lesot [a], Adrien Revault d'Allonnes [b]

[a] *Sorbonne Universités, UPMC Univ. Paris 06, CNRS, LIP6 UMR 7606, 4 place Jussieu 75005 Paris, France*
[b] *Université Paris 8 – EA 4383 – LIASD, FR-93526, Saint-Denis, France*

A B S T R A C T

This paper introduces and studies extensions of modal logics by investigating the soundness of classical modal axioms in a weighted framework. It discusses the notion of relevant weight values, in a specific weighted Kripke semantics and exploits accessibility relation properties. Different generalisations of the classical axioms are constructed and, from these, a typology of weighted axioms is built, distinguishing between four types, depending on their relations to their classical counterparts and to the, possibly equivalent, frame conditions.

Keywords:
Modal logics
Kripke semantics
Graded modality
Weighted axioms

1. Introduction

Weighted extensions of modal logics aim at increasing their expressiveness by enriching the two classical modal operators, $\Box$ and $\Diamond$, with integer or real valued degrees. These extensions, described in more detail in Section 2, are usually based on infinitely many weighted modal operators $\Box_\alpha$ and $\Diamond_\alpha$, where α stands for the numerical weight. These extended modalities make it possible to introduce fine distinctions between the pieces of knowledge modeled in the formalism, which can then be used to infer nuanced new knowledge and thus, for example, allow reasoning on partial beliefs [1–3]. To do this, one needs to define adequate versions of the weighted axioms to express relevant partial belief manipulation rules, i.e. to define a weighted extension of KD45.

With this objective in mind, this paper studies weighted extensions of the classical modal axioms, seen as rules defining the combination of the modal operators $\Box$ and $\Diamond$, establishing relations between formulae in which they occur once, repeatedly or in combination. For instance the classical axiom (4), $\vdash \Box\varphi \to \Box\Box\varphi$, states that an implication holds between a single occurrence and repetitions of $\Box$. Similarly, axiom (D), $\vdash \Box\varphi \to \Diamond\varphi$, establishes a relation between the two modal operators.

This paper examines the transposition of these axioms to the case of a weighted modal logic, identifying rules for the combination of the weighted modal operators $\Box_\alpha$ and $\Diamond_\alpha$. Starting with candidate weighted axioms obtained by replacing each modality of a classical axiom with a weighted one, each with its own weight, the paper discusses how these weights depend on each other. This issue can be illustrated by axiom (D), whose associated weighted candidate takes the form $\vdash \Box_\alpha\varphi \to \Diamond_\beta\varphi$. The question is then to establish a relevant valuation for β depending on α, or vice versa.

* Corresponding author.
E-mail addresses: benedicte.legastelois@lip6.fr (B. Legastelois), marie-jeanne.lesot@lip6.fr (M.-J. Lesot), Adrien.Revault-d_Allonnes@paris8.fr

The paper proposes to address this task from a semantic point of view, interpreting the candidates in a particular weighted Kripke semantics: it first proposes a semantic interpretation for $\Box_\alpha$ and $\Diamond_\alpha$ based on a relative counting of accessible validating worlds which relaxes the conditions on the universal quantifier defining $\Box$ in Kripke's semantics. This semantics offers the advantage of being informative enough to serve as a basis for the definition of weighted axioms.

The applied approach identifies weight dependencies which hold either in any frame or under specific frame conditions. The paper also studies whether specific conditions are satisfied by the frames in which the obtained axioms hold. This should be considered as opening the way to the definition of a weighted correspondence theory. Note that the aim here is not to introduce an axiomatisation of the considered weighted modal logic semantics, but to study the transposition of classical axioms to the weighted case: the semantic approach provides a motivation and justification for the proposed weighted axioms and their weight values, they can then be used as candidates for partial belief manipulation rules, or any other form of modal, non-factual reasoning.

The paper then introduces a typology of weighted modal axioms which separates them in four types, depending on their relation to their classic, unweighted counterparts and their associated frame conditions: type I groups axioms which cannot be relaxed using the degrees of freedom offered by the proposed weights. Type II is made of weighted axioms that preserve the frame conditions of their usual versions. Types III and IV contain weighted axioms that require a modification of the conditions imposed on the frame, respectively when correspondence cannot be proved and when it can.

The obtained typology allows to introduce, for instance, a weighted extension of KD45: the four axiom types will allow to identify axioms that are required by compatibility to the classic case, e.g. with the fact that the associated frames should satisfy some specific properties as well as axioms that may be additionally considered, depending on the desired behaviour of the manipulation rules.

The paper is organised as follows: Section 2 presents an informal comparative study of existing weighted modal logics. Section 3 introduces the semantics used to build weighted axioms with the method described in Section 4. Section 5 presents an overview of the resulting typology of weighted modal axioms, whereas Sections 6 to 9 are dedicated to each type in turn.

2. Existing weighted modal logics

After presenting the notations used in this paper, this section briefly describes existing weighted extensions of modal logics, first with approaches which modify the definition of Kripke frames, integrating weights either in the accessibility relation or in the worlds. It then describes the counting models, which preserve the classical frame definition but alter the quantification used in the modal operator definitions.

Specific weighted modal systems, dedicated to particular applications, are not detailed in this section. These include, for instance, fuzzy temporal logic [4] or multi-agent modal logic [5].

2.1. Notations

We adopt the standard notation (e.g. see [6,7]): a frame $F = \langle W, R\rangle$ is a pair composed of a non-empty set W of worlds and a binary accessibility relation R on W. A model $\mathcal{M} = \langle F, s\rangle$ is a couple formed by a frame F and a valuation s which assigns truth values to each atomic formula, in each world in W.

For any formula φ and any world $w \in W$, the usual definition of the semantic consequence symbol $\vDash$ states that $\mathcal{M}, w \vDash \varphi$ if and only if φ is true in world w for the considered model $\mathcal{M}$ (the latter may be omitted when there is no ambiguity). The notation $F \vDash \varphi$ means that for any valuation s and for any world $w \in W$, $\langle F, s\rangle, w \vDash \varphi$. Finally, $\vDash \varphi$ means that for any frame $F = \langle W, R\rangle$, any valuation s and any world $w \in W$, $\langle F, s\rangle, w \vDash \varphi$.

For a given model $\mathcal{M}$ and any world w in W, we denote by R_w the set of worlds accessible from w:

$$R_w = \left\{ w' \in W \mid wRw' \right\} \tag{1}$$

We also define, for any formula φ, the set $R_w(\varphi)$:

$$R_w(\varphi) = \left\{ w' \in R_w \mid \mathcal{M}, w' \vDash \varphi \right\} \tag{2}$$

For any formula φ, the classical interpretations of $\Box\varphi$ and $\Diamond\varphi = \neg\Box\neg\varphi$ are respectively based on the universal and existential quantification of accessible worlds which satisfy φ. To use the previous notations, one can write:

$$\begin{aligned} \mathcal{M}, w \vDash \Box\varphi \quad &\Leftrightarrow \quad \forall w' \in R_w,\ \ \mathcal{M}, w' \vDash \varphi && (3)\\ &\Leftrightarrow \quad R_w(\varphi) = R_w \\ \mathcal{M}, w \vDash \Diamond\varphi \quad &\Leftrightarrow \quad \exists w' \in R_w,\ \ \mathcal{M}, w' \vDash \varphi && (4)\\ &\Leftrightarrow \quad R_w(\varphi) \neq \emptyset \\ &\Leftrightarrow \quad |R_w(\varphi)| > 0 \end{aligned}$$

2.2. Weighted accessibility relation

A first category of weighted modal logics extends the classical Kripke model by replacing the accessibility relation R with a set of indexed relations R^α, usually with $\alpha \in [0,1]$. They then define weighted modalities $\Box_\alpha$, respectively associated with each relation R^α, in accordance with the definitions given in Eqs. (3) and (4). Three approaches can be distinguished depending on the interpretation of the weight, which can belong to different formal frameworks such as probability theory, fuzzy set theory [8] or possibility theory [9].

In the probabilistic case [10], the interpretation given to the accessibility weights can, for instance, depend on the conditional probability of transitions from one world to another. Combinations of weights are, therefore, led in the usual probabilistic way.

In the fuzzy case [11], the weights represent the strength of the relation, expressing that a world is *more or less* accessible. The weighted relations R^α then correspond to α-cuts of the fuzzy relation R; therefore they satisfy a nesting property: $\forall \alpha, \beta \in [0,1]$, if $\alpha \geq \beta$ then $w_1 R^\alpha w_2 \Rightarrow w_1 R^\beta w_2$. This in turn implies relations between modalities, expressed as a decreasing graduality property:

$$\forall \alpha, \beta \in [0,1]\,, \text{ if } \alpha \geq \beta \text{ then } \vDash \Box_\alpha \varphi \to \Box_\beta \varphi \tag{5}$$

The fuzzy interpretation thus leads to a multi-modal logic with dependent – or at least comparable – modalities.

In the possibilistic case [12], the relation weights represent the uncertainty on the accessibility between worlds: they allow to express doubt on the very existence of a link between worlds, where the fuzzy model delivers information about its intensity. The possibilistic approach leads to multiple independent modalities.

2.3. Weighted worlds

A second category of weighted modal logics considers that weights bear on worlds and not on the relation. Consequently, they have a global effect, insofar as for any reference world, they do not depend on the considered successor. Conversely, weighted relations, as considered in the previous subsection, exhibit a local effect, since weights are specific to each couple of worlds.

Among these world weightings, classical Kripke frames can be enriched with a distribution of qualitative possibilities [9] over W, written π [13]: worlds are then considered as more or less possible. π is then used to define the accessibility relation as:

$$R_w = \{w' \in W | \pi(w) \leq \pi(w')\}$$

The $\Box$ and $\Diamond$ semantics are then defined in the classical way, see Eqs. (3) and (4), using this relation. As a consequence, a formula $\Box\varphi$ holds in w if and only if φ is satisfied in all worlds that are at least as possible as w. Note that $\Box$ and $\Diamond$ remain unweighted: this integration of weights actually does not lead to weighted modalities.

The accessibility relation induced by π is necessarily antisymmetric, transitive and reflexive, restricting the expressivity of the ensuing modalities: since the semantics is based on a total order, it corresponds to a knowledge modal system [14] and the defined modalities can be interpreted in an epistemic meaning [15].

A second possibilistic model [3] proposes to generalise the distribution of possibilities π to formulae, by setting $\Pi(\varphi) = \max_{w \in W}\{\pi(w) | \mathcal{M}, w \vDash \varphi\}$. It allows to build a generalised possibilistic logic, interpreted, once again, in an epistemic framework.

The classical Kripke model can also be extended with weights on the worlds representing some semantic property, independently of any formal paradigm [16,17]: each world is associated to a so-called exceptionality degree that represents how different – or unrepresentative – it is. An exceptionality degree is then derived for each formula as:

$$except(\varphi) = \min_{w \in W}\{except(w) \mid \mathcal{M}, w \vDash \varphi\}$$

The proposed definition for the induced weighted modality does not preserve the classical definition of Eq. (3) but states:

$$\mathcal{M}, w \vDash \Box_\alpha \varphi \ \Leftrightarrow\ except(\neg\varphi) \geq \alpha \tag{6}$$

This definition means that the more exceptional, i.e. the rarer, a contradiction, the higher the weight.

Two properties of this exceptionality-based definition of weighted modalities stand out: first, the validity of a modal formula is global and does not depend on the reference world where it is interpreted. Indeed, $\mathcal{M}, w \vDash \Box_\alpha \varphi \Leftrightarrow \mathcal{M} \vDash \Box_\alpha \varphi$. Second, due to the inequality in their definition, a relation between degrees can be observed: the decreasing graduality property defined in Eq. (5) also applies for this model.

2.4. Counting approaches

The counting approach [18–22] does not change the Kripke definitions of frames to integrate weights, neither on worlds nor on the relation, but modifies the definition of the modality. Because, here, the degrees are integers, they are written n.

The counting approach modifies the quantification constraints on accessible validating worlds in Eqs. (3) and (4). More precisely, the definition of $\Diamond_n$ is based on a hardening of the existential quantifier in Eq. (4): it is no longer required that at least one accessible world satisfy the formula, but that at least n do. Formally, the counting approach defines $\Diamond_n$ and, by duality, $\Box_n$, as, $\forall n \in \mathbb{N}$:

$$\mathcal{M}, w \models \Diamond_n \varphi \quad \Leftrightarrow \quad |R_w(\varphi)| \geq n \tag{7}$$

$$\mathcal{M}, w \models \Box_n \varphi \quad \Leftrightarrow \quad |R_w(\neg\varphi)| < n \tag{8}$$

The $\Box_n$ modality is weighted by the number of invalidating accessible worlds: n can be interpreted as a measure of contradiction, applying a principle similar to the one of exceptionality degrees (see Eq. (6)).

Whereas this definition relies on an absolute counting, majority logic [23] considers a specific case of relative counting: it introduces a modal operator expressing that a formula is true in more than half of the accessible worlds. Note that in this case the modification of the quantification does not lead to weighted modalities.

Contrary to the approaches described in the previous subsections 2.2 and 2.3, which rely on a semantic definition, the counting approach has been axiomatised, in both the absolute and relative cases [24,25].

3. Proposed semantics

This section describes the semantics we propose for a weighted modal logic. It relies on a relative counting approach, along the same lines as majority logic [23] mentioned in the previous section, but leading to weighted modalities. If the set of worlds W is infinite, this relative counting can be replaced by a probability, understood as the frequency limit. Even if the presentation in this section and the following focuses on the relative counting approach, it would provide the same results in a probabilistic framework. In both cases, the imposed normalisation constraint offers the benefits of rich information that allow to establish weighted extensions of the modal axioms, as discussed in Sections 4 and 5.

Syntactically, for $p \in \mathbb{P}$ denoting a set of propositional variables and $\alpha \in [0, 1]$ a numerical coefficient, we consider the set of all well-formed formulae according to the language

$$F := p \mid \neg F \mid F \wedge F \mid F \vee F \mid F \rightarrow F \mid \Box_\alpha F \mid \Diamond_\alpha F$$

3.1. Definition

The semantics we propose follows the same principle as the counting approach described in Section 2.4, viz. based on counting proportions of validating worlds to relax the universal and harden the existential quantification constraints of Eqs. (3) and (4).

It is defined when W is finite, in a frequentist interpretation, as a normalised cardinality. This proportion has the added benefit of making the modality weight independent of frame connectivity: the evaluation of the truth value of a formula $\Box_\alpha \varphi$ in a world w is not obscured by the number $|R_w|$ of accessible worlds w has.

Formally, the proposed weighted modality $\Box_\alpha$ is defined as, $\forall \alpha \in [0, 1]$:

$$\begin{cases} \mathcal{M}, w \models \Box_\alpha \varphi \quad \Leftrightarrow \quad \dfrac{|R_w(\varphi)|}{|R_w|} \geq \alpha & \text{if } R_w \neq \emptyset \\ \mathcal{M}, w \models \Box_\alpha \varphi & \text{otherwise} \end{cases} \tag{9}$$

This definition thus relaxes the universal quantification in Eq. (3), by only requiring that a proportion of the accessible worlds satisfy the formula φ, instead of all of them.

By duality, the modality $\Diamond_\alpha$ is defined as, $\forall \alpha \in [0, 1]$

$$\begin{cases} \mathcal{M}, w \models \Diamond_\alpha \varphi \Leftrightarrow \dfrac{|R_w(\varphi)|}{|R_w|} > 1 - \alpha & \text{if } R_w \neq \emptyset \\ \mathcal{M}, w \not\models \Diamond_\alpha \varphi & \text{otherwise} \end{cases} \tag{10}$$

The modality $\Diamond_\alpha$ requires that at least a proportion $1 - \alpha$ of accessible worlds satisfy φ, instead of at least one accessible world: similar to the counting approach of Section 2.4, it thus hardens the existential quantifier, requiring more than just one accessible validating world. Note that, consequently, the higher the α, the less demanding the condition. Also, because $\Diamond_\alpha \varphi = \neg\Box_\alpha \neg\varphi$, the loose inequality in Eq. (9) becomes a strict one for $\Diamond_\alpha$, in Eq. (10).

3.2. Properties

This section introduces and discusses some properties satisfied by the proposed weighted modal operators.

3.2.1. Boundary cases

As stated in the following proposition, the boundary case $\alpha = 1$ corresponds to the classical modalities, whereas $\alpha = 0$ is a tautology for $\Box$ and a contradiction for $\Diamond$:

Proposition 1.

$$\Box_1\varphi = \Box\varphi \qquad \vDash \Box_0\varphi$$
$$\Diamond_1\varphi = \Diamond\varphi \qquad \vDash \neg\Diamond_0\varphi$$

The proofs of this proposition follow directly from the definitions given in Eqs. (9) and (10) and are, thus, omitted.

Because of the tautologies of the case $\alpha = 0$, it should be considered trivial and uninformative and should, generally, be ignored. However, in the case where it is the only value for which a weighted formula holds, it expresses rich knowledge: considering $\Box_0$ for instance, for $w \in W$ such that $R_w \neq \emptyset$ and $|R_w(\varphi)|/|R_w| = 0$, $\mathcal{M}, w \vDash \Box_1\neg\varphi$.

3.2.2. Decreasing graduality

Due to the transitivity of the inequality relation on which the proposed semantics relies, the decreasing graduality property, defined in Eq. (5) and recalled below, is satisfied:

Proposition 2. *The definition of $\Box_\alpha$ given in Eq. (9) satisfies the graduality property:*

$$\forall\alpha, \beta \in [0, 1] \text{ if } \alpha \geq \beta, \text{ then } \vDash \Box_\alpha\varphi \to \Box_\beta\varphi$$

The proof follows directly from the definitions given in Eq. (9) and is, therefore, omitted.

Proposition 2 implies that, up to a maximal degree, a formula holds for all lower weights. Notice that this property provides another justification for the uninformativeness of the $\Box_0$ modality underlined above.

More generally, as a result, the most informative weight for the $\Box_\alpha$ modality is the maximal admissible value, since all others can be derived from it. This property will be crucial to establishing weighted extensions of modal axioms, as discussed in Section 4, where we focus on identifying the inferable upper bound for α.

By duality, similar results hold for the $\Diamond_\alpha$ modality, with an increasing graduality property: for $\Diamond_\alpha$, the most informative weight is the minimal admissible value.

3.2.3. Relations between $\Box_\alpha$ and $\Diamond_\alpha$

Let us underline that the preserved duality constraint, according to which $\Box_\alpha\varphi = \neg\Diamond_\alpha\neg\varphi$, does not guarantee the equivalence between $\Box_\alpha$ and $\Diamond_{1-\alpha}$: since $1-\alpha$ can be considered as the dual value of α, these two weighted modalities are intuitively related. However, due to the fact that the $\Box_\alpha$ definition relies on a non-strict inequality whereas $\Diamond_\alpha$ relies on a strict one, it can be shown that one implication holds but the other does not:

Proposition 3.

$$\forall\alpha \in [0, 1] \quad \vDash \Diamond_\alpha\varphi \to \Box_{1-\alpha}\varphi$$
$$\nvDash \Box_\alpha\varphi \to \Diamond_{1-\alpha}\varphi$$

Proof. For the first implication, if $\alpha = 0$, $\Diamond_\alpha\varphi$ is a contradiction (see Proposition 1), which makes the implication true. For any $\alpha \neq 0$, any model $\mathcal{M} = \langle\langle W, R\rangle, s\rangle$ and $w \in W$, it holds that

$$\begin{aligned}\mathcal{M}, w \vDash \Diamond_\alpha\varphi \quad &\Leftrightarrow \quad R_w \neq \emptyset \text{ and } \frac{|R_w(\varphi)|}{|R_w|} > 1-\alpha \\ &\Rightarrow \quad R_w \neq \emptyset \text{ and } \frac{|R_w(\varphi)|}{|R_w|} \geq 1-\alpha \\ &\Rightarrow \quad \mathcal{M}, w \vDash \Box_{1-\alpha}\varphi\end{aligned}$$

The fact that the second implication $\Box_\alpha\varphi \to \Diamond_{1-\alpha}\varphi$ is not a tautology can be proved by constructing a counterexample for any α. The frame in Fig. 2 on p. 354, for instance, illustrates the case $\alpha = 2/3$: $w \vDash \Box_{2/3}\varphi$ but $w \nvDash \Diamond_{1/3}\varphi$, as $|R_w(\varphi)|/|R_w| = 2/3$ does not satisfy a strict inequality. $\Box$

Another relation gives the equivalence between the classical $\Diamond$ and a weighted $\Box_\alpha$:

Proposition 4. *For any model $\mathcal{M} = \langle\langle W, R\rangle, s\rangle$ and any $w \in W$,*

$$\mathcal{M}, w \vDash \Diamond_1\varphi \quad \Leftrightarrow \quad R_w \neq \emptyset \text{ and } \mathcal{M}, w \vDash \Box_{\frac{1}{|R_w|}}\varphi$$

Proof. Let $\mathcal{M} = \langle\langle W, R\rangle, s\rangle$ be any model and $w \in W$. It holds that

$$\begin{aligned}\mathcal{M}, w \models \Diamond_1\varphi \quad &\Leftrightarrow \exists w' \in R_w, \mathcal{M}, w' \models \varphi \\ &\Leftrightarrow R_w \neq \emptyset \text{ and } \frac{|R_w(\varphi)|}{|R_w|} \geq \frac{1}{|R_w|} \\ &\Leftrightarrow R_w \neq \emptyset \text{ and } \mathcal{M}, w \models \Box_{\frac{1}{|R_w|}}\varphi \qquad \Box\end{aligned}$$

Note that this proposition can be considered as an improvement on Proposition 3. Indeed, the only value Proposition 3 gave for α was the tautological 0, which is less informative than $w \models \Box_{\frac{1}{|R_w|}}\varphi$.

4. Principles for building weighted extensions of modal axioms

Axioms in classical modal logic [7] can be seen as rules defining the handling and combination of the modal operators $\Box$ and $\Diamond$, that is establishing relations between formulae in which they occur once, repeatedly or in combination.

We propose to study their weighted transposition, defined as the formulae obtained when replacing each modality of a classical axiom with its own weighted version, each with its own weight. For instance, the classical axiom (4), $\vdash \Box\varphi \to \Box\Box\varphi$, leads to a weighted extension noted $\Box_\alpha\varphi \to \Box_\beta\Box_\gamma\varphi$.

More precisely, we propose to examine how these weights depend on each other, in a semantic approach based on the interpretation of weighted modal logic presented in the previous section: the method we consider consists in identifying weight dependencies which hold either in any frame or under specific frame conditions. Moreover, we study whether the frames in which the obtained axioms hold satisfy specific conditions. This section presents the principles used to set the values for the introduced weights, the obtained results are discussed in the next sections.

4.1. Inequality constraints on candidate weights

Seen as elements of an inference system, axioms, which essentially revolve around implications, aim at producing informative new knowledge, through rich inferences. As such, modal axioms with easily satisfied premises and informative conclusions should be favoured, whenever possible.

This section discusses the influence of the axiom structure on these criteria, considering two characteristics of the modalities they contain: the first is related to the position of the modality, either in the premise or in the conclusion, which influences the definition of relevant weights. The second one focuses on the case of combined, successive, modalities, that raise the question of weight trade-offs.

4.1.1. Modality positions

The principle that aims at establishing implications with easy to satisfy premises and informative conclusions gives hints regarding relevant weight values exploiting the axiom structure: the position of the considered modal operator $\Box_\alpha$, in the premise or in the conclusion plays a major role, when combined with the crucial decreasing graduality property (see Proposition 2, p. 345).

More precisely, when $\Box_\alpha$ is in the conclusion of the implication, α should be maximal. Indeed, all lower values can be inferred from it and the most informative case is the highest value. For instance, if, for any φ and ψ, it can be proved that $F_1 = \varphi \to \Box_{0.8}\psi$ and $F_2 = \varphi \to \Box_{0.3}\psi$, then F_1 is preferred to F_2. Indeed, F_2 can be derived from F_1, due to the decreasing graduality property (Proposition 2, p. 345), which implies that $\Box_{0.8}\psi \to \Box_{0.3}\psi$ and the transitivity of implication, which holds by compatibility with the classic logic case.

Conversely, if $\Box_\alpha$ is in the premise of the implication, α should be minimal: it indicates the lowest value that still allows to infer the conclusion, using modus ponens. For instance, if, for any φ and ψ, it can be proved that $F_3 = \Box_{0.8}\varphi \to \psi$ and $F_4 = \Box_{0.3}\varphi \to \psi$, then F_4 is preferred to F_3. Indeed, due to the decreasing graduality property, any proved formula of the form $\Box_\beta\varphi$ with greater β induces the required $\Box_\alpha\varphi$, triggering the axiom inference.

By duality, for the $\Diamond_\alpha$ operator, that satisfies an increasing graduality property, the converse definition of relevant values applies: α should be minimal for $\Diamond_\alpha$ in the conclusion and maximal in the premise.

4.1.2. Modality combinations

Some of the classic modal axioms have combined modalities, $\Box\Box$, $\Box\Diamond$ and $\Diamond\Box$. In these cases, the previous maximisation/minimisation principle becomes more complex and requires the definition of trade-offs between their respective weights. Indeed, considering, for instance, the case of $\Box_\alpha\Box_\beta$, the informativeness is maximal for the combination $\Box_1\Box_1$, but not all combinations can be compared: e.g. among the two cases $\Box_{0.5}\Box_{0.8}$ and $\Box_{0.8}\Box_{0.5}$ none should be favoured over the other. More generally, decreasing the value of α can be considered as relevant only if it allows to increase that of β.

The case of $\Box_\alpha\Diamond_\beta$ and $\Diamond_\beta\Box_\alpha$ can be discussed along the same lines, with the difference that the $\Diamond$ weight, β, should be minimised. Again, however, decreasing α in order to increase β, or vice versa, is relevant: between $\Box_{0.9}\Diamond_{0.5}$ and $\Box_{0.5}\Diamond_{0.9}$, neither can be considered as more informative than the other.

As a consequence, in the case of successive modalities, the general principle of maximising the $\Box_\alpha$ and minimising the $\Diamond_\beta$ weights when they appear in conclusions becomes more complex. The same principle applies for minimising α and

Table 1
Most common properties of a relation R defined on $W \times W$, where $u, v, w \in W$.

Serial	$\forall u, \exists v\ uRv$
Reflexive	$\forall u,\ uRu$
Symmetric	$\forall u, v\ uRv \Rightarrow vRu$
Shift-reflexive	$\forall u, v\ uRv \Rightarrow vRv$
Transitive	$\forall u, v, w\ (uRv \wedge vRw) \Rightarrow uRw$
Euclidean	$\forall u, v, w\ (uRv \wedge uRw) \Rightarrow vRw$
Dense	$\forall u, v\ uRv \Rightarrow \exists w\ (uRw \wedge wRv)$
Convergent	$\forall u, v, w\ (uRv \wedge uRw) \Rightarrow \exists x\ (vRx \wedge wRx)$
Functional	$\forall u, v, w\ (uRv \wedge uRw) \Rightarrow v = w$

maximising β in the premises. Therefore, several variants of axioms containing successive modalities may be established, as detailed in Section 9.

4.2. *Using frame conditions*

A second tool to establish weight dependence for weighted extensions of modal axioms is provided by the frame conditions associated to classical modal axioms in correspondence theory [14]. Indeed, the semantic counterparts of modal axioms come with specific classes of frames, constrained by conditions on the accessibility relation which is, for instance, required to be reflexive or symmetric. Table 1 lists the definition of the most frequent relation properties.

Using these, when interpreting the weighted extension of a classical modal axiom from a semantic point of view, we only consider frames satisfying the corresponding conditions, to examine if specific relations are imposed on the weight values under these assumptions. This principle guarantees compatibility with the boundary case where all introduced weights are equal to 1.

When occurring in the premise, the modality $\Box_\alpha$, which presents a more expressive interpretation than $\Box$, is also less specific. Indeed, knowing that a proportion of accessible worlds is a model for a given formula does not give information on this formula's evaluation in all considered worlds: $\Box_\alpha$ gives a global indication and leads to uncertainty for any precisely specified world. As a consequence, it is expected that establishing weighted extensions of the axioms may require to impose more constraining frame conditions. More precisely, it can be the case that the obtained axiom does not exclude the case where the best guaranteed weight in the conclusion is of the form $\Box_0\varphi$, which, as discussed in Section 3.2, is uninformative. The approach we propose thus consists in looking for conditions that exclude such frames, hardening the classic condition, when necessary.

On the other hand, when occurring in the conclusion, the relaxed modality $\Box_\alpha$ with $\alpha < 1$ provides less information and may be compensated for by relaxing its associated frame condition, requiring less constraints: in some cases, the weighted versions of axioms can remove the classical corresponding frame condition, as detailed in Section 5.

Finally, when a relevant weighted axiom has been established, under possibly hardened or weakened frame conditions, we study whether a converse proposition holds, i.e. whether the frames in which the obtained weighted axiom holds necessarily satisfy the considered condition. This can be considered as opening the way to the definition of a weighted correspondence theory.

4.3. *Principle of the established theorems and their proofs*

Using correspondence theory we can determine the relation between weights in weighted versions of the axioms, thereby describing the extent to which they can be relaxed. These correspondence theorems, which link a frame topology to an axiom, are usually of the form:

$$\begin{aligned}&\forall \langle W, R\rangle \\ &R \text{ satisfies property } P \Rightarrow \forall \alpha \in [0,1], \forall s, \quad \langle\langle W, R\rangle, s\rangle \models \varphi_\alpha\end{aligned}$$

where P is a property of the frame relation, e.g. taken from Table 1, and φ_α is a formula in which weighted modalities appear.

In some cases, as illustrated in the next section, such a theorem does not apply, which means it can be shown that there are a frame $\langle W, R\rangle$, whose relation R satisfies the property P, an $\alpha \in [0,1]$, a valuation s and a world $w \in W$ in which the property does not hold, that is $\langle\langle W, R\rangle, s\rangle, w \not\models \varphi_\alpha$. In most cases, in fact, an even more general result, with a universal quantification of the weight, can be established, in the form

$$\begin{aligned}&\forall\, \alpha \in [0,1] \\ &\exists \langle W, R\rangle \text{ such that } R \text{ satisfies property } P, \exists s\ \exists w \in W \quad \langle\langle W, R\rangle, s\rangle, w \not\models \varphi_\alpha\end{aligned}$$

$\langle\langle W, R\rangle, s\rangle, w$ is a counter-example which is usually explicitly constructed to prove this result.

Table 2
Obtained weighted axioms with associated (not necessarily corresponding) frame conditions and type, as defined in Section 5. α, β are real numbers in $[0, 1]$ and $\varepsilon \in (0, \alpha]$.

(K_α)	$\Box_\alpha(\varphi \to \psi) \to (\Box_\beta\varphi \to \Box_{\max(0,\alpha+\beta-1)}\psi)$		(II)
$(NA1_\alpha)$	$(\Box_\alpha\varphi \wedge \Box_\beta\psi) \to \Box_{\max(0,\alpha+\beta-1)}(\varphi \wedge \psi)$		(II)
$(NA1_\alpha)$	$\Box_\alpha(\varphi \wedge \psi) \to (\Box_\alpha\varphi \wedge \Box_\alpha\psi)$		(II)
$(NA2_\alpha)$	$(\Box_\alpha\varphi \vee \Box_\beta) \to \Box_{\min(\alpha,\beta)}(\varphi \vee \psi)$		(II)
(CD_α)	$\Diamond_\alpha\varphi \to \Box_{1-\alpha}\varphi$		(IV)
(M_α)	$\Box_1\varphi \to \varphi$	reflexive	(I)
(B_α)	$\varphi \to \Box_1\Diamond_1\varphi$	symmetric	(I)
$(\Box M_\alpha)$	$\Box_1(\Box_1\varphi \to \varphi)$	shift-reflexive	(I)
(5_α)	$\Diamond_1\varphi \to \Box_1\Diamond_1\varphi$	euclidean	(I)
(D_α)	$\Box_\alpha\varphi \to \Diamond_{1-\alpha+\varepsilon}\varphi$	serial	(II)
(C_α)	$\Diamond_1\Box_\alpha\varphi \to \Box_{1-\alpha+\varepsilon}\Diamond_1\varphi$	convergent	(III)
(4_α)	$\Box_\alpha\varphi \to \Box_1\Box_\alpha\varphi$	transitive & euclidean	(III)
$(C4_\alpha)$	$\Box_\alpha\Box_\beta\varphi \to \Box_\beta\varphi$	transitive & euclidean	(IV)
$(C4_\alpha)$	$\Box_\alpha\Box_1\varphi \to \Box_\alpha\varphi$	shift-reflexive	(IV)

When the constraints on the relation suffice to prove the implication, its converse can also be looked at, so as to establish a result of the form

$$\forall \langle W, R\rangle$$
$$\forall\alpha \in [0,1], \forall s, \langle\langle W, R\rangle, s\rangle \models \varphi_\alpha \Rightarrow R \text{ satisfies property } P$$

Such a result is usually proved using its contrapositive form

$$\forall \langle W, R\rangle$$
$$R \text{ does not satisfies property } P \Rightarrow \exists\alpha \in [0,1], \exists s, \exists w \in W\ \langle\langle W, R\rangle, s\rangle, w \not\models \varphi_\alpha$$

This proof, in turn, is established by exhibiting, for a frame $\langle W, R\rangle$ where R does not satisfy the considered property, a value for α, a valuation s and a world $w \in W$ such that φ_α does not hold in w.

5. Typology of weighted axioms

This section gives an overview on the obtained results, i.e. the weighted extensions of the classical modal axioms when applying the principles presented in the previous section. Regarding the semantic interpretation, we consider the definition presented in Section 3, which allows to establish weight dependence.

The results, listed in Table 2, are organised in a typology, whose four types are defined below:

(i) Unweightable axioms
(ii) Weighted axioms with classical correspondence
(iii) Weighted axioms without correspondence
(iv) Weighted axioms with enriched correspondence

These types, respectively discussed and illustrated in the next sections, depend on the relation between the weighted axioms and their classical counterparts and the frame conditions the latter correspond to: type I groups axioms that cannot be relaxed using the degrees of freedom offered by the weights. Type II is composed of weighted axioms that preserve the frame conditions of their usual counterparts. Types III and IV contain the weighted axioms that require a modification of the conditions imposed on the frame, respectively when correspondence cannot be proved and when it can.

Note that a given classical axiom can have several weighted extensions, depending on the considered frame conditions.

6. Type I: unweightable axioms

The first type of axioms are those for which the only possible weight is 1.

There exist two kinds of argument that justify not weighting an axiom: some axioms cannot be weighted because there is no guarantee on a non-negative value for their assigned weights; some others do not need to be relaxed because it would affect their power of manipulation. These two cases are successively described below: axiom (M) belongs to the former, ($\Box$M), (B) and (5) to the latter.

In this section, as well as in the following three, for all theorems, we detail the proofs in the case of finite sets of worlds W, using an explicit relative counting semantics. In a general probabilistic approach, the proofs can be established along the same lines.

6.1. *Axiom (M)*

The general weighted form of axiom (M) is $\vdash \Box_\alpha \varphi \rightarrow \varphi$. By compatibility with the classic case this formula must be true within any frame with a reflexive relation. However, in the case where the maximal admissible weight is $\alpha < 1$, the reflexivity constraint cannot guarantee the reference world is not one of the worlds where φ does not hold:

Theorem 1 *(M_α). $\forall \alpha \in [0, 1)$, there exists a model $\mathcal{M} = \langle\langle W, R\rangle, s\rangle$ with reflexive R and $w \in W$ such that $\mathcal{M}, w \models \Box_\alpha \varphi$ but $\mathcal{M}, w \models \neg\varphi$.*

Proof. Let $\alpha \in [0, 1)$. The proof consists in building such a model $\mathcal{M}$.

Let W be a set of n worlds, where n is such that $(n-1)/n \geq \alpha$ (such an n exists because $\alpha < 1$) and let w be a reference world in W. Let R be a reflexive relation such that $R_w = W$. Let s be the valuation such that

(i) $x \models \varphi$ for all $x \in W \setminus \{w\}$
(ii) $w \models \neg\varphi$

Setting $\mathcal{M} = \langle\langle W, R\rangle, s\rangle$, it holds that $\mathcal{M}, w \models \Box_\alpha \varphi$, as $|R_w| = n$, $|R_w(\varphi)| = n-1$ and $(n-1)/n \geq \alpha$, but $\mathcal{M}, w \not\models \varphi$. $\Box$

Other additional constraints (such as, e.g., transitivity, symmetry or euclideanity) would not give information about the valuation in the reference world w: guaranteeing that, for all frames and valuations, $w \in R_w(\varphi) \Leftrightarrow R_w(\varphi) = R_w$, which imposes that R is reflexive.

Therefore, the only relevant weight value is $\alpha = 1$, which corresponds to the classic case, making the axiom unweightable.

6.2. *Axiom ($\Box M$)*

The general weighted form of axiom ($\Box$M) is written $\vdash \Box_\alpha(\Box_\beta \varphi \rightarrow \varphi)$. As detailed below, it can be proved that the only relevant weight values are 1 for both α and β, when preserving the condition that the relation is shift-reflexive, by compatibility with the classical case.

Obviously, the case $\alpha = 0$ is a tautology, but it is not an interesting one, as discussed in Section 3. In the case $\beta = 1$, using classical correspondence theory, if the relation is shift-reflexive, then $\langle W, R\rangle \models \Box_1(\Box_1 \varphi \rightarrow \varphi)$, which induces all other α values due to Proposition 2. In the other cases, it holds that

Theorem 2 *($\Box\mathrm{M}_\alpha$). $\forall \alpha \in (0, 1]$, $\forall \beta \in [0, 1)$, there exists a model $\mathcal{M} = \langle\langle W, R\rangle, s\rangle$ with shift-reflexive R and $w \in W$ such that $\mathcal{M}, w \not\models \Box_\alpha(\Box_\beta \varphi \rightarrow \varphi)$*

Proof. Let $\alpha \in (0, 1]$ and $\beta \in [0, 1)$. The proof consists in building a counterexample model $\mathcal{M}$. It is very similar to the one used in the proof of Theorem 1, considering an additional world pointing to w: let W be a set of $n+1$ worlds, such that $(n-1)/n \geq \beta$, $u \in W$ and $w \in W$. Let R be such that $R_u = \{w\}$, $R_w = W \setminus \{u\}$ and $\forall w \in W \setminus \{u, w\}$, $R_x = \{x\}$. By construction R is shift-reflexive. Let s be the valuation such that (i) $x \models \varphi$ for all $x \in W \setminus \{w\}$ and (ii) $w \models \neg\varphi$.

Then, using the same proof as for Theorem 1, $w \models \neg(\Box_\beta \varphi \rightarrow \varphi)$, thus $u \models \Box_1 \neg(\Box_\beta \varphi \rightarrow \varphi)$ and so, for any $\alpha > 0$, $u \not\models \Box_\alpha(\Box_\beta \varphi \rightarrow \varphi)$. $\Box$

As above, additional constraints on the relation (e.g. transitivity, symmetry or euclideanity) would not give information about the valuation in the reference world w.

6.3. *Axiom (B)*

The general weighted form of axiom (B_α) is $\vdash \varphi \rightarrow \Box_\alpha \Diamond_\beta \varphi$. It presents two characteristics as opposed to the previously discussed axioms: the premise of the implication contains no modality and the conclusion contains a combination of modalities. The definition of relevant weights may, thus, consider decreasing the α value if it allows to increase β, as discussed in Section 4.1.2. By compatibility with the classic case, the semantic counterpart of the weighted axiom is studied under the hypothesis that the accessibility relation is symmetric: it can be shown that, except in the non-informative case where $\alpha = 0$, no other value than 1 can be determined for β:

Theorem 3 *(B_α). $\forall \alpha \in (0, 1]$, $\forall \beta \in [0, 1)$, there exists a model $\mathcal{M} = \langle\langle W, R\rangle, s\rangle$ with symmetric R and $w \in W$ such that $\mathcal{M}, w \models \varphi$ but $\mathcal{M}, w \not\models \Box_\alpha \Diamond_\beta \varphi$*

Proof. Let $\alpha \in (0, 1]$ and $\beta \in [0, 1)$. The proof consists in building such a model $\mathcal{M}$. Let $F = \langle W, R\rangle$ be a frame containing $n = \lfloor \frac{1}{1-\beta} \rfloor + 1$ worlds. Let $w, w' \in W$ be two reference worlds and R a symmetric accessibility relation such that $R_w = \{w'\}$ and $R_{w'} = W$. Let s be the valuation such that:

(i) $x \models \neg\varphi$ for all $x \in W \setminus \{w\}$
(ii) $w \models \varphi$

Then $\mathcal{M}, w' \models \neg\Diamond_\beta\varphi$: indeed, $\frac{|R_{w'}(\varphi)|}{|R_{w'}|} = \frac{1}{n}$. By definition of n, $n > \frac{1}{1-\beta}$ and thus $\frac{1}{n} < 1-\beta$. As a consequence, w' being the sole successor of w, $\mathcal{M}, w \not\models \Box_\alpha\Diamond_\beta\varphi$, whatever $\alpha \in (0,1]$. $\Box$

The case with $\beta = 1$ is obtained by compatibility with the classic case of non-weighted modal logics and the decreasing graduality property that holds for α.

6.4. Axiom (5)

The general form of the weighted axiom (5_α) is $\vdash \Diamond_\alpha\varphi \to \Box_\beta\Diamond_\gamma\varphi$: it contains a modality combination and has the specificity that its premise contains $\Diamond$.

If $\alpha = 1$, compatibility with the classic case implies setting $\beta = \gamma = 1$. The question is then whether it is possible to obtain a better result, for instance increasing the γ value, even if it possibly requires to decrease β or to increase α so as to have a more informative premise.

As for the previous theorems, we study this question from a semantic point of view, considering frames with a euclidean relation, again to conform to with the classic case. Another issue is also to consider more constrained relations, that satisfy an additional property, beyond euclideanity.

Theorem 4 below first shows that β necessarily equals 1; then Theorem 5 shows that the case where $\alpha < 1$ is not relevant either.

Theorem 4.

$$\forall\beta \in (0,1],\ \forall\gamma \in [0,1],$$
$$\forall\bigcirc \in \{\Box, \Diamond\},\ \forall\langle W, R\rangle,$$
$$R \textit{ euclidean } \Rightarrow \forall w \in W,\ \textit{if } w \models \Box_\beta \bigcirc_\gamma \varphi \textit{ then } w \models \Box_1 \bigcirc_\gamma \varphi$$

Proof. Let $\beta \in (0,1]$ and $\gamma \in [0,1]$ be two numerical values.

Let a model $\mathcal{M} = \langle\langle W, R\rangle, s\rangle$ be such that R is euclidean, let $w \in W$.

We now show that $\forall u, v \in R_w$, $R_u = R_v$. Indeed, as R is euclidean, wRu and wRv implies uRv. Then, for any $x \in R_u$, combining uRv with uRx implies vRx, i.e. $x \in R_v$, and therefore $R_u \subseteq R_v$. It can similarly be proved that $R_v \subseteq R_u$.

As a consequence, any modal formula satisfied in a given successor of w is satisfied in all its successors, as they all have the same set of successors. In this way, any considered proportion of w successors is representative of all its successors. Therefore, if $w \models \Box_\beta \bigcirc_\gamma \varphi$ then $w \models \Box_1 \bigcirc_\gamma \varphi$. $\Box$

This theorem shows that it is not relevant to consider a trade-off consisting in decreasing the value of β so as to increase that of γ: $\beta = 1$ follows from any lower β if the relation is euclidean.

The question is then whether decreasing the α value allows to establish a richer conclusion, of the form $\Box_1\Diamond_\gamma$ with $\gamma < 1$. The next theorem answers negatively:

Theorem 5 *(5_α). $\forall\alpha \in [0,1)$, $\forall\gamma \in [0,1)$, there exists a model $\mathcal{M} = \langle\langle W, R\rangle, s\rangle$ with euclidean R and $w \in W$ such that $\mathcal{M}, w \models \Diamond_\alpha\varphi$ but $\mathcal{M}, w \not\models \Box_1\Diamond_\gamma\varphi$.*

Proof. The theorem can be proved by building such a model. First, note that if R is euclidean, then for every $w, w' \in W$, if $w' \in R_w$, then $R_{w'} \subseteq R_w$, with no way of identifying validating worlds which belong to R_w and not to $R_{w'}$.

More precisely, for given α and $\gamma \in [0,1)$, let $n, m \in \mathbb{N}$ such that $\frac{m}{n+m} > 1-\gamma$ and $F = \langle W, R\rangle$ be the frame containing $n+m+1$ worlds and w a reference world in W. Let R and s be such that

(i) $|R_w| = n$ and $\forall w' \in R_w$, $w' \models \varphi$
(ii) $\forall w' \in R_w$, $R_{w'} = W \setminus \{w\}$
(iii) $\forall w' \notin R_w$, $R_{w'} = W \setminus (R_w \cup \{w\})$ and $w' \models \neg\varphi$

By construction, R is euclidean.

It holds that $\frac{|R_w(\varphi)|}{|R_w|} = 1 > 1-\alpha$ and thus $w \models \Diamond_\alpha\varphi$. Moreover $\forall w' \in R_w$, $\frac{|R_{w'}(\varphi)|}{|R_{w'}|} = \frac{n}{n+m} \leq 1-\gamma$. Therefore $w' \not\models \Diamond_\gamma\varphi$ and $w \not\models \Box_1\Diamond_\gamma\varphi$. $\Box$

Note that additionally imposing the relation R to be transitive can force the relation between a world and the successors of its own successors. In this way, by transitivity and euclideanity, the following property holds:

$$\forall\langle\langle W, R\rangle, s\rangle, \text{ such that } R \text{ is transitive and euclidean,}$$
$$\forall w \in W,\ \forall w' \in R_w,\ R_{w'} = R_w$$

Thus, if the world satisfies $\Diamond_\alpha\varphi$, all $w' \in R_w$ also satisfy it: so $w \models \Box_1\Diamond_\alpha\varphi$.

However, this approach imposes more constraints both on the frame relation and the premise of the implication, as its premise with a weight α is harder to satisfy than 1. Thus, even though it leads to an enriched conclusion, it reduces the axiom satisfiability; we choose to exclude this solution.

7. Type II: weighted axioms with classic correspondence

Type II axioms offer a relaxed version of their classic counterparts and can be established under the same frame conditions: the classically associated relation constraint is preserved and suffices to set relevant values. This principle in particular applies to the case of axioms that correspond to non-constrained relations, namely axiom (K) and normal axioms, but also to axiom (D), as successively discussed in the next subsections.

7.1. Axiom (K)

The weighted variant of axiom (K) takes the following form:

Theorem 6 *(K_α).* $\forall\alpha, \beta \in [0, 1]$

$$\models \Box_\alpha(\varphi \to \psi) \to (\Box_\beta\varphi \to \Box_\gamma\psi) \quad \textit{where } \gamma = \max(0, \alpha + \beta - 1)$$

Proof. Let $F = \langle W, R\rangle$ be a frame, s a valuation and $w \in W$. If $R_w = \emptyset$, w trivially satisfies all three modal formulae and thus the implication. If $|R_w| > 0$, the proof consists in applying the modus ponens in accessible worlds where both $\varphi \to \psi$ and φ are satisfied: $R_w(\varphi \to \psi) \cap R_w(\varphi) \subseteq R_w(\psi)$. Now by definition of the cardinal of set intersection:

$$|R_w(\varphi \to \psi) \cap R_w(\varphi)| = |R_w(\varphi \to \psi)| + |R_w(\varphi)| - |R_w(\varphi \to \psi) \cup R_w(\varphi)|$$

As $|R_w| \geq |R_w(\varphi \to \psi) \cup R_w(\varphi)|$, it holds that:

$$\begin{aligned}|R_w(\psi)| &\geq |R_w(\varphi \to \psi) \cap R_w(\varphi)| \\ &\geq |R_w(\varphi \to \psi)| + |R_w(\varphi)| - |R_w|\end{aligned}$$

Thus $\frac{|R_w(\psi)|}{|R_w|} \geq \alpha + \beta - 1$. $\Box$

7.2. Normal axioms

Normal axioms state how the modality can be factorised and distributed over conjunction and disjunction: for instance (NA1) states that $\vdash \Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$. Its weighted variant takes the following form

Theorem 7 *($\mathrm{NA1}_\alpha$).* $\forall\alpha, \beta, \gamma \in [0, 1]$

$$\models \Box_\alpha(\varphi \wedge \psi) \to (\Box_\alpha\varphi \wedge \Box_\alpha\psi)$$
$$\models (\Box_\beta\varphi \wedge \Box_\gamma\varphi) \to \Box_\alpha(\varphi \wedge \psi) \ \ \textit{with } \alpha = \max(\beta + \gamma - 1, 0)$$

Proof. Let $F = \langle W, R\rangle$ be a frame, s a valuation and $w \in W$. If $R_w = \emptyset$, w trivially satisfies all involved modal formulae and thus the two implications.

If $w \models \Box_\alpha(\varphi \wedge \psi)$, using the fact that $R_w(\varphi \wedge \psi) = R_w(\varphi) \cap R_w(\psi) \subseteq R_w(\varphi)$, one gets

$$\alpha \leq \frac{|R_w(\varphi \wedge \psi)|}{|R_w|} \leq \frac{|R_w(\varphi)|}{|R_w|}$$

which implies that $w \models \Box_\alpha\varphi$. As, similarly, $R_w(\varphi \wedge \psi) \subseteq R_w(\psi)$, it also holds that $w \models \Box_\alpha\psi$. Therefore $w \models \Box_\alpha\varphi \wedge \Box_\alpha\psi$.

Note that it may the case that $\Box_\beta\varphi$ (or similarly $\Box_\beta\psi$) holds for a β greater than α, but the general result that can be established is α.

For the factorisation direction, assuming that $w \models \Box_\beta\varphi \wedge \Box_\gamma\psi$, and using the definition of the intersection cardinal

$$\begin{aligned}\frac{|R_w(\varphi \wedge \psi)|}{|R_w|} = \frac{|R_w(\varphi) \cap R_w(\psi)|}{|R_w|} &= \frac{|R_w(\varphi)| + |R_w(\psi)| - |R_w(\varphi) \cup R_w(\psi)|}{|R_w|} \\ &\geq \frac{|R_w(\varphi)|}{|R_w|} + \frac{|R_w(\psi)|}{|R_w|} - \frac{|R_w|}{|R_w|} \\ &\geq \beta + \gamma - 1\end{aligned}$$

which implies that $w \models \Box_\alpha(\varphi \wedge \psi)$ with $\alpha = \max(\beta + \gamma - 1, 0)$. $\Box$

Normal axiom (NA2) considers the disjunction case, stating that factorisation holds: $\vdash (\Box\varphi \vee \Box\psi) \rightarrow \Box(\varphi \vee \psi)$. Its weighted extension takes the following form:

Theorem 8 *(NA2$_\alpha$f).*

$$\vDash (\Box_\alpha\varphi \vee \Box_\beta\psi) \rightarrow \Box_\gamma(\varphi \vee \psi) \quad \text{with } \gamma = \min(\alpha, \beta)$$

Proof. Let $F = \langle W, R\rangle$ be a frame, s a valuation and $w \in W$. If $R_w = \emptyset$, w trivially satisfies all three modal formulae and thus the implication.

If $|R_w| > 0$, the proof follows from the facts that $R_w(\varphi \vee \psi) = R_w(\varphi) \cup R_w(\psi)$, $R_w(\varphi) \subseteq R_w(\varphi) \cup R_w(\psi)$ and $R_w(\psi) \subseteq R_w(\varphi) \cup R_w(\psi)$.

If $w \vDash \Box_\alpha\varphi \vee \Box_\beta\psi$, two cases can occur: either $w \vDash \Box_\alpha\varphi$ and then $\frac{|R_w(\varphi)\cup R_w(\psi)|}{|R_w|} \geq \frac{|R_w(\varphi)|}{|R_w|} \geq \alpha$; or $w \vDash \Box_\beta\psi$, and, similarly, $\frac{|R_w(\varphi)\cup R_w(\psi)|}{|R_w|} \geq \beta$.

As a consequence, in both cases, $\frac{|R_w(\varphi)\cup R_w(\psi)|}{|R_w|} \geq \min(\alpha, \beta)$. $\Box$

The reverse implication, corresponding to $\vee$ distribution, does not hold in classical modal logic: $\nvdash \Box(\varphi \vee \psi) \rightarrow (\Box\varphi \vee \Box\psi)$. However, a weighted form, generally written $\Box_\alpha(\varphi \vee \psi) \rightarrow (\Box_\beta\varphi \vee \Box_\gamma\psi)$, can be discussed so as to characterise the triplet (α, β, γ). Only a general form can be established

Theorem 9 *(NA2$_\alpha$d).*

$$\vDash \Box_\alpha(\varphi \vee \psi) \rightarrow (\Box_\beta\varphi \vee \Box_\gamma\psi) \quad \text{with } \alpha \leq \beta + \gamma$$

Proof. Let $F = \langle W, R\rangle$ be a frame, s a valuation and $w \in W$. If $R_w = \emptyset$, w trivially satisfies all modal formulae and thus the implication.

Otherwise, the proof follows from the set and cardinality properties:

$$|R_w(\varphi \vee \psi)| = |R_w(\varphi) \cup R_w(\psi)| = |R_w(\varphi)| + |R_w(\psi)| - |R_w(\varphi) \cap R_w(\psi)|$$

which leads to $\alpha \leq \beta + \gamma$ when dividing both sides by $|R_w|$. $\Box$

However, no more precise result can be established: disjunction distribution requires to share γ among the values of α and β, there is no way to establish which quantity is assigned to each of them.

7.3. Axiom (D)

The weighted extension of axiom (D), which is classically in correspondence with serial frames, relates the two weighted modal operators, $\Box_\alpha$ and $\Diamond_\alpha$, completing the properties stated in Proposition 3 considering the converse implication:

Theorem 10 *(D$_\alpha$). $\forall F = \langle W, R\rangle$,*

R is serial

$$\Leftrightarrow \quad \forall\alpha \in (0, 1] \text{ and } \varepsilon \in (0, \alpha]\ F \vDash \Box_\alpha\varphi \rightarrow \Diamond_{1-\alpha+\varepsilon}\varphi$$

Proof. Let $F = \langle W, R\rangle$ be a frame where R is serial, s a valuation, $w \in W$ and $\alpha \in (0, 1]$. If $w \vDash \Box_\alpha\varphi$, as R is serial, $\frac{|R_w(\varphi)|}{|R_w|} \geq \alpha$ and therefore $\frac{|R_w(\varphi)|}{|R_w|} > \alpha - \varepsilon$ for any non-negative ε. This implies $w \vDash \Diamond_{1-\alpha+\varepsilon}\varphi$.

Reciprocally, if (D$_\alpha$) holds for all α, the relation is necessarily serial. Indeed, applying the classical correspondence theory, if R is not serial, the axiom does not hold for $\alpha = 1 = \varepsilon$. $\Box$

Note that the case $\alpha = 0$ can actually be included in the theorem, but it constitutes a trivial case, for which no ε value can be considered: as it is non-informative, we do not consider it.

In this case, the weighted extension of the classic axiom can be interpreted as providing enriched information: even in the case where less information is provided, i.e. for a world w such that $w \vDash \Box_\alpha\varphi$ with $\alpha < 1$, a richer conclusion than in the classic case can be provided, namely $w \vDash \Diamond_{1-\alpha+\varepsilon}\varphi$. This comes from the fact that the classic modalities have a lower expressivity than the weighted ones and do not allow to represent all available information.

8. Type III: weighted axiom without correspondence

This section groups axioms for which the classical frame conditions are not sufficient to establish weighted variants and proposes relevant additional constraints. However, in this category, correspondence cannot be proved, the weighted axioms do not allow to prove properties of the accessibility relation. Axioms (C) and (4) are successively discussed.

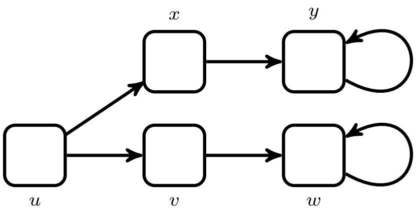

Fig. 1. Frame showing the converse of Theorem 11 does not hold, as stated by Theorem 12.

8.1. Axiom (C)

Axiom (C) $\vdash \Diamond\Box\varphi \to \Box\Diamond\varphi$ is in correspondence with the convergence of the accessibility relation. Its weighted version involves four weights and can be written $\Diamond_\alpha\Box_\beta\varphi \to \Box_{\beta'}\Diamond_{\alpha'}\varphi$.

First, in order to reduce constraints on the premise, the case where $\alpha = 1$ is considered. The other modalities can be weighted, enriching the axiom as compared to the classic case, while keeping the same frame condition. This makes a difference with axiom (4) discussed in the next section, where the frame condition needs to be modified.

Theorem 11 *(C_α).* $\forall F = \langle W, R\rangle$,

R is convergent

$$\Rightarrow \quad \forall s, \forall w \in W \text{ with } w \text{ not blind}, \forall \beta \in (0, 1, \forall \varepsilon \in (0, \beta]$$
$$\langle\langle W, R\rangle, s\rangle, w \vDash \Diamond_1\Box_\beta\varphi \to \Box_{\frac{1}{|R_w|}}\Diamond_{1-\beta+\varepsilon}\varphi$$

Note that if w is a blind world, it satisfies $w \vDash \Box_\alpha\psi$ for any α and any ψ, in particular $\psi = \Diamond_{1-\beta+\varepsilon}\varphi$ and thus it satisfies the implication in the theorem, except that the coefficient $\frac{1}{|R_w|}$ is not defined.

Proof. Let $\mathcal{M} = \langle\langle W, R\rangle, s\rangle$ such that R is convergent. Let $w \in W$ non-blind, $\beta \in (0, 1]$, $\varepsilon \in (0, \beta]$.

If $w \vDash \Diamond_1\Box_\beta\varphi$, then $R_w \neq \emptyset$ and $\exists w' \in R_w$ such that $w' \vDash \Box_\beta\varphi$.

As R is convergent, $R_{w'} \neq \emptyset$: considering twice wRw' and wRw', $\exists x$ such that $w'Rx$. As $\beta > 0$ and $\varepsilon \in (0, \beta]$, $w' \vDash \Box_\beta\varphi$ implies that $w' \vDash \Diamond_{1-\beta+\varepsilon}\varphi$. This can be established using a proof similar to that of Theorem 10 (D_α), using a local seriality property.

Therefore, $R_w(\Diamond_{1-\beta+\varepsilon}\varphi) \neq \emptyset$, so $w \vDash \Box_{\frac{1}{|R_w|}}\Diamond_{1-\beta+\varepsilon}\varphi$. □

However, the converse does not hold: if axiom (C_α) holds, the relation is not necessarily convergent, as the following theorem shows.

Theorem 12 *(4_α). There exists a frame $F = \langle W, R\rangle$ with non-convergent accessibility relation R such that $\forall\beta \in [0, 1], \forall\varepsilon \in (0, \beta]\ F \vDash \Diamond_1\Box_\beta\varphi \to \Box_{\frac{1}{|R_w|}}\Diamond_{1-\beta+\varepsilon}\varphi$.*

Proof. Fig. 1 shows such a frame with $W = \{u, v, z, x, y\}$: the accessibility relation is non-convergent, because there is no common successor to the worlds x and v.

Let s be any valuation, β and ε be two numerical coefficients with $\beta \in (0, 1]$, $\varepsilon \in (0, \beta]$ and let $\psi = \Diamond_1\Box_\beta\varphi \to \Box_{\frac{1}{|R_w|}}\Diamond_{1-\beta+\varepsilon}\varphi$ denote the considered formula. The proof considers each world successively.

If $u \nvDash \Diamond_1\Box_\beta\varphi$, $u \vDash \psi$. Otherwise, as $R_u = \{x, v\}$, either $x \vDash \Box_\beta\varphi$ or $v \vDash \Box_\beta\varphi$. In the first case, $x \vDash \Diamond_{1-\beta+\varepsilon}\varphi$, using the same proof as in Theorem 10 (D_α) with a local seriality property of the accessibility relation. Then $u \vDash \Box_{\frac{1}{|R_u|}}\Diamond_{1-\beta+\varepsilon}\varphi$. The proof is identical in the second case, for world v.

If $x \nvDash \Diamond_1\Box_\beta\varphi$, then $x \vDash \psi$. Otherwise, as y is x's only successor, $y \vDash \Box_\beta\varphi$, thus $y \vDash \Diamond_{1-\beta+\varepsilon}\varphi$, using the same proof as in Theorem 10 (D_α) with a local seriality property of the accessibility relation. Then $x \vDash \Box_{\frac{1}{|R_x|}}\Diamond_{1-\beta+\varepsilon}\varphi$. The same proof holds for world v, which also has a single successor w, as well as for worlds y and w which are their own single successors. □

8.2. Axiom (4)

Axiom (4), written $\vdash \Box\varphi \to \Box\Box\varphi$, is classically associated with transitivity. A weighted variant is of the form $\vdash \Box_\alpha\varphi \to \Box_\beta\Box_\gamma\varphi$ and the issue is to determine the appropriate values for β and γ for a given α. Now the sole condition that R is transitive does not allow to establish such a result:

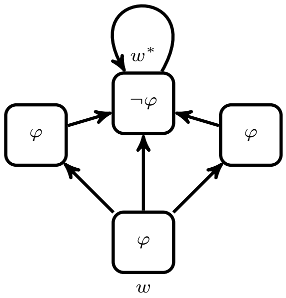

Fig. 2. Model illustrating Theorem 13 for $\alpha = \frac{2}{3}$.

Theorem 13. $\forall \alpha \in [0, 1)$, *there exists a model* $\mathcal{M} = \langle\langle W, R\rangle, s\rangle$ *with* R *transitive and* $w \in W$ *such that* $\mathcal{M}, w \models \Box_\alpha \varphi$ *and* $\mathcal{M}, w \models \Box_1 \Box_1 \neg\varphi$.

Proof. The proof consists in building such a model $\mathcal{M}$. For a given $\alpha < 1$, let $m, q \in \mathbb{N}^*$ such that $\alpha \leq m/(m+q)$. Let W be a set of $1+q+m$ worlds and $w \in W$ a reference world. Let then R be the binary relation on $W \times W$ and s the valuation such that

(i) $R_w = W \setminus \{w\}$
(ii) $|R_w(\varphi)| = m$
(iii) $|R_w(\neg\varphi)| = q$, let w_n be one world in $R_w(\neg\varphi)$.
(iv) $\forall u \in R_w,\ R_u = \{w_n\}$

By construction, R is transitive. Such a model is illustrated in Fig. 2 for $\alpha = 2/3$, with $m = 2$ and $q = 1$.

Denoting $\mathcal{M} = \langle\langle W, R\rangle, s\rangle$, it holds that $\mathcal{M}, w \models \Box_{m/(m+q)} \varphi$, therefore, using the graduality property, $\mathcal{M}, w \models \Box_\alpha \varphi$.

Moreover, as $\forall u \in R_w$, $\mathcal{M}, u \models \Box_1 \neg\varphi$, it holds that $\mathcal{M}, w \models \Box_1 \Box_1 \neg\varphi$. $\Box$

The theorem implies that for $\gamma > 0$ there is no $\beta > 0$ such that $\mathcal{M}, w \models \Box_\beta \Box_\gamma \varphi$.

Therefore, transitivity is not a sufficient condition to have guarantees on the values of β and γ. It is thus necessary to harden the frame conditions by adding another constraint, euclideanity in Theorem 14 below. Indeed, it can prevent the existence of sinkhole worlds, the ones in $R_w(\neg\varphi)$ in the previous proof. Transitivity is kept to preserve the compatibility with the classic case obtained when the weights equal 1, leading to the theorem:

Theorem 14 *(4_α).* $\forall F = \langle W, R\rangle$,

R is transitive and euclidean

$$\Rightarrow \quad \forall \alpha \in [0, 1],\ F \models \Box_\alpha \varphi \to \Box_1 \Box_\alpha \varphi$$

Proof. The proof relies on the fact that, for a transitive and euclidean relation, $\forall w' \in R_w$, $R_{w'} = R_w$ (see p. 351). As a consequence, for all valuations s and for all $\alpha \in [0, 1]$, if $w \models \Box_\alpha \varphi$, then all accessible worlds $w' \in R_w$ also satisfy $w' \models \Box_\alpha \varphi$, that is $w \models \Box_1 \Box_\alpha \varphi$ $\Box$

This axiom is powerful as the first modality in its conclusion is weighted by the maximal possible degree and the second one precisely by the degree α appearing in the premise of the implication. Therefore, a weighted axiom with greater degree cannot be considered, meaning this axiom cannot be further "improved". Note that, as opposed to the case of Axiom (5), discussed in Section 6.4, p. 350, the constraint is augmented, but the premise of the implication is relaxed, only requiring $\Box_\alpha \varphi$ instead of $\Box_1 \varphi$.

However, the converse does not hold:

Theorem 15 *(4_α). There exists a frame* $F = \langle W, R\rangle$ *with non-euclidean accessibility relation* R *such that for any valuation* s *and* $\forall \alpha \in [0, 1]$, $\langle F, s\rangle \models \Box_\alpha \varphi \to \Box_1 \Box_\alpha \varphi$

Proof. Fig. 3 shows such a model with $W = \{u, v, w\}$: the accessibility relation is transitive, but not euclidean, because the relation wRv, which would be required because uRw and uRv, is missing.

Let us denote $\psi = \Box_\alpha \varphi \to \Box_1 \Box_\alpha \varphi$ the considered formula. Let s be any valuation and $\alpha \in [0, 1]$. The proof considers each world successively, starting with v: if $v \not\models \Box_\alpha \varphi$, $v \models \psi$. Otherwise, as $R_v = \{v, w\}$ and $w \models \Box_\alpha \varphi$ because w is blind, all successors of v are models of $\Box_\alpha \varphi$. Therefore, $v \models \Box_1 \Box_\alpha \varphi$ and thus v is model of ψ. The proof for world u is identical:

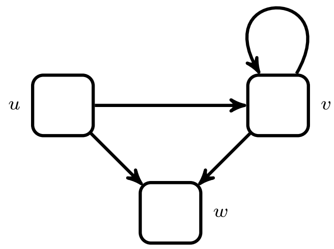

Fig. 3. Frame showing the converse of Theorem 14 does not hold, as stated by Theorem 15.

if $u \nvDash \Box_\alpha \varphi$, $u \vDash \psi$. Otherwise, as $R_u = R_v$, $u \vDash \Box_\alpha \varphi$ implies $v \vDash \Box_\alpha \varphi$. Again, $w \vDash \Box_\alpha \varphi$ because w is blind. Therefore all successors of u are models of $\Box_\alpha \varphi$, thus $u \vDash \psi$. Finally, as w is blind, it is a model of all modal formulae involved in the implication. $\Box$

9. Type IV: weighted axioms with enriched correspondence

Weighted axioms of type IV are defined as extensions for which modified frame conditions can be considered. The difference with type III comes from the fact that, in their case, correspondence can be proved. Two types of modifications can be considered: the classical frame condition can be removed, leading to a less constrained relation, or additional conditions can be introduced.

9.1. Axiom (CD)

Axiom (CD), classically in correspondence with frames with functional relations, states that $\vdash \Diamond\varphi \to \Box\varphi$.

Weighted versions of this axiom are established by Propositions 3 and 4, in Section 3.2.3, p. 345. Indeed, according to the proposed semantics, Proposition 3 states that $\vDash \Diamond_\alpha \varphi \to \Box_{1-\alpha}\varphi$. As it imposes no constraint on the relation, the correspondence is trivially established.

Proposition 4 states that $\vDash \Diamond_\alpha \varphi \to \Box_{\frac{1}{|R_w|}} \varphi$ for any relation and any world w (if w is blind, the coefficient $1/|R_w|$ is undefined, but w is a model of any modal formula $\Box_\alpha \psi$). As discussed in Section 3.2.3, in the case of $\alpha = 1$, it proposes a more informative result, since the previous axiom states that $\Diamond_1 \varphi \to \Box_0 \varphi$.

In both cases, augmenting the premise strength, requiring $\Diamond_\alpha \varphi$ instead of $\Diamond\varphi$, allows to remove the frame functionality condition.

9.2. Axiom (C4)

Axiom ($C4_\alpha$) illustrates the case of a more constrained frame condition: its classic counterpart states $\vdash \Box\Box\varphi \to \Box\varphi$ and is associated to the density frame condition. The general weighted version takes the form $\Box_\alpha \Box_\beta \varphi \to \Box_\gamma \varphi$ but, as stated in the following theorem, density alone is not sufficient to guarantee such a property: for any α and β value, a model can be built for which $\gamma = 0$.

Theorem 16. $\forall \alpha \in [0, 1), \forall \beta \in [0, 1]$, *there exists a model* $\mathcal{M} = \langle \langle W, R \rangle, s \rangle$ *with* R *dense and* $w \in W$ *such that* $\mathcal{M}, w \vDash \Box_\alpha \Box_\beta \varphi$ *and* $\mathcal{M}, w \vDash \Box_1 \neg\varphi$.

Proof. The proof consists in building such a model. For a given $\alpha \in [0, 1)$ and $\beta \in [0, 1]$, let $n \in \mathbb{N}$ be such that $n \geq \alpha/(1-\alpha)$. Let W be a set of $n+2$ worlds, w_1 and w_2 two distinct worlds from W, R the relation and s the valuation such that

(i) $\mathcal{M}, w_2 \vDash \varphi$
(ii) $R_{w_1} = W \setminus \{w_2\}$
(iii) $\forall w \in R_{w_1}, \mathcal{M}, w \vDash \neg\varphi$
(iv) $\forall w \in W \setminus \{w_1\}, R_w = \{w_2\}$

By construction, R is dense. Such a counter-example frame is illustrated on Fig. 4 for $\alpha = 0.75$.

Then $\forall w \in W \setminus \{w_1\}$, $\mathcal{M}, w \vDash \Box_1 \varphi$, which implies by decreasing graduality, $\mathcal{M}, w \vDash \Box_\beta \varphi$. Therefore

$$\begin{aligned} |R_{w_1}(\Box_\beta \varphi)| = n \quad &\Rightarrow \quad \frac{|R_{w_1}(\Box_\beta \varphi)|}{|R_{w_1}|} = \frac{n}{n+1} \geq \alpha \\ &\Rightarrow \quad \mathcal{M}, w_1 \vDash \Box_\alpha \Box_\beta \varphi \end{aligned}$$

But $\mathcal{M}, w_1 \vDash \Box_1 \neg\varphi$ so $\nexists \gamma > 0$ such that $\mathcal{M}, w_1 \vDash \Box_\gamma \varphi$. $\Box$

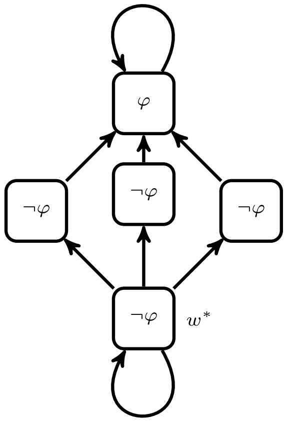

Fig. 4. Counterexample model proving Theorem 16 for $\alpha = \frac{3}{4}$.

As a consequence, to guarantee a strictly positive γ value, additional constraints on the relation are considered. As listed in Table 2 two distinct sets of constraints can be added, leading to two weighted extensions of (C4_α). They differ by the informativeness of their conclusion and the correlated level of constraint their premise imposes, due to the succession of modalities, as discussed in Section 4.1.2, p. 346.

The strongest version is stated in the next theorem. Note that the compatibility with the classic case is preserved, because euclideanity implies density.

Theorem 17 *(C4_α).* $\forall F = \langle W, R\rangle$,

R is transitive and euclidean

$$\Leftrightarrow \quad \forall\alpha \in (0,1], \forall\beta \in [0,1],\ F \vDash \Box_\alpha\Box_\beta\varphi \to \Box_\beta\varphi$$

Proof. Let $\langle W, R\rangle$ be such that R is transitive and euclidean, s be a valuation $\alpha, \beta \in (0, 1]$ (if $\beta = 0$ then $\Box_\beta\varphi$ is true, and thus the implication is) and $w \in W$. If $R_w = \emptyset$, w trivially satisfies all involved formulae and thus the implication. In the following, $|R_w| > 0$.

If $w \nvDash \Box_\alpha\Box_\beta\varphi$, then w is a model of the implication.

If $w \vDash \Box_\alpha\Box_\beta\varphi$, then a proportion $\alpha > 0$ of worlds accessible from w satisfy $\Box_\beta\varphi$, let u be such a world: $\frac{|R_u(\varphi)|}{|R_u|} \geq \beta$. As R is transitive and euclidean, it holds that $\forall w' \in R_w,\ R_{w'} = R_w$ (see p. 351). In particular, $R_u = R_w$ and thus $w \vDash \Box_\beta\varphi$.

The converse can be proved by contraposition: for any frame $\langle W, R\rangle$, if the relation R is not transitive or not euclidean, there exist α, β and a valuation s and $\exists w \in W$ such that $w \nvDash$ (C4_α). The case of non-transitivity is detailed below.

Let W be a finite set of worlds and R a non-transitive relation: there exist $u, v, w \in W$ such that $uRv \wedge vRw \wedge \neg uRw$. Let $\alpha = \frac{1}{|R_u|}$ and $\beta = \frac{1}{|R_v|}$. Let s be the valuation such that

(i) $w \vDash \varphi$
(ii) $\forall x \in W \setminus \{w\},\ x \nvDash \varphi$.

Fig. 5 illustrates an example of a such model. It holds that:

- $u \vDash \Box_1\neg\varphi$ because the only world satisfying φ, w, is not accessible from u. Therefore $u \nvDash \Box_\beta\varphi$.
- $v \vDash \Box_\beta\varphi$ because $w \in R_v$ and $w \vDash \varphi$
- $u \vDash \Box_\alpha\Box_\beta\varphi$ because $v \in R_u$ and $v \vDash \Box_\beta\varphi$

Following the same principle, it can be shown that if the relation is not euclidean then there exists a model which does not satisfy (C4_α). $\Box$

With frame conditions weaker than transitivity and euclideanity, other relevant values can be established for the weights, provided a richer premise is considered in the axiom: the next theorem considers the case of shift-reflexive relations (which are thus dense). The implication premise is changed to $\Box_\alpha\Box_1\varphi$ which is richer than the general case $\Box_\alpha\Box_\beta\varphi$ considered in Theorem 17.

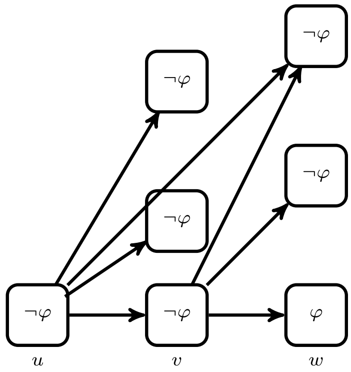

Fig. 5. Frame illustrating the proof of Theorem 17.

Theorem 18 *($C4_\alpha$).* $\forall F = \langle W, R\rangle$,

$$R \text{ is shift-reflexive} \Leftrightarrow \forall \alpha \in (0,1],\ F \vDash \Box_\alpha \Box_1 \varphi \to \Box_\alpha \varphi$$

Proof. Let $\langle W, R\rangle$ be such that R is shift-reflexive. Let s be a valuation and $\alpha \in (0,1]$. Let a world $w \in W$. If $R_w = \emptyset$, w trivially satisfies all involved formulae and thus the implication. In the following, $|R_w| > 0$.

If $w \nvDash \Box_\alpha \Box_1 \varphi$, then $w \vDash \Box_\alpha \Box_1 \varphi \to \Box_\alpha \varphi$.

If $w \vDash \Box_\alpha \Box_1 \varphi$, as R is shift-reflexive, it holds that $\forall w' \in R_w,\ w' \vDash \Box_1 \varphi$ implies $w' \vDash \varphi$, i.e. $R_w(\Box_1\varphi) \subseteq R_w(\varphi)$. Therefore

$$\alpha \le \frac{|R_w(\Box_1\varphi)|}{|R_w|} \le \frac{|R_w(\varphi)|}{|R_w|}$$

i.e. $w \vDash \Box_\alpha \varphi$.

The converse can be proved by contraposition: if the relation is not shift-reflexive, then axiom ($C4_\alpha$) does not hold. Let W be a finite set of worlds and R a non-shift-reflexive relation: let $w \in W$ and $w' \in R_w$ such that $w' \notin R_{w'}$. Let s be the valuation such that $w' \nvDash \varphi$ and $\forall x \in W \setminus \{w'\}, x \vDash \varphi$.

Then $w \nvDash \Box_1 \varphi$ because of w', but $w \vDash \Box_1 \Box_1 \varphi$. Therefore, setting $\alpha = 1$, $w \vDash \Box_\alpha \Box_1 \varphi$ but $w \nvDash \Box_\alpha \varphi$, which implies $w \nvDash (C4_\alpha)$. $\Box$

10. Conclusion and future works

This paper studied rules for the combination of weighted modal operators, through the extension of classical axioms. In doing so, it offered a typology of weighted axioms with respect to their relation to their classical counterparts and to the frame conditions the latter correspond to. It discussed the expressiveness increase allowed by the weighting of axioms and how the hardened relation properties allow to balance the induced lack of information.

The perspectives of this work include the study of the axiomatisation of the proposed semantics, e.g. using an approach similar to that of Rijke [26]. They also include the study of more general frame conditions: in this paper, weighted modal axioms have been established considering binary classical relation properties. It would be interesting to consider relaxed versions, e.g. in the spirit of some α-symmetry, to examine what other extended versions of the axioms can be established.

Future works also aim at specifying the proposed weighted modal logic to the doxastic framework, so as to study a belief-based adaptation. From a semantic point of view, the interpretation of the weights as belief degrees will be studied; from an axiomatic point of view, the weighted axioms of the modal logic KD45 and their properties will be considered from the set of established axioms.

References

[1] F. Bacchus, A.J. Grove, J.Y. Halpern, D. Koller, From statistical knowledge bases to degrees of belief, Artif. Intell. 87 (1) (1996) 75–143.
[2] M. Banerjee, D. Dubois, A simple modal logic for reasoning about revealed beliefs, in: Symbolic and Quantitative Approaches to Reasoning with Uncertainty, Springer, 2009, pp. 805–816.
[3] D. Dubois, H. Prade, S. Schockaert, Stable models in generalized possibilistic logic, in: 13th Int. Conf. on the Principles of Knowledge Representation and Reasoning (KR), 2012, pp. 519–529.
[4] H. Thiele, S. Kalenka, On fuzzy temporal logic, in: Second IEEE Int. Conf. on Fuzzy Systems, IEEE, 1993, pp. 1027–1032.
[5] E. Lorini, R. Demolombe, From binary trust to graded trust in information sources: a logical perspective, in: Trust in Agent Societies, Springer, 2008, pp. 205–225.
[6] B.F. Chellas, Modal Logic: An Introduction, vol. 316, Cambridge Univ. Press, 1980.
[7] P. Blackburn, M. De Rijke, Y. Venema, Modal Logic, Cambridge University Press, 2001.

[8] L. Zadeh, Fuzzy sets, Inf. Control 8 (1965) 338–353, http://dx.doi.org/10.1016/S0019-9958(65)90241-X.
[9] L. Zadeh, Fuzzy sets as the basis for a theory of possibility, Fuzzy Sets Syst. 1 (1978) 3–328, http://dx.doi.org/10.1016/0165-0114(78)90029-5.
[10] A. Shirazi, E. Amir, Probabilistic modal logic, in: Proc. of the 22nd AAAI Conf. on Artificial Intelligence, vol. 7, 2007, pp. 489–495.
[11] F. Bou, F. Esteva, L. Godo, R. Rodrıguez, Characterizing fuzzy modal semantics by fuzzy multimodal systems with crisp accessibility relations, in: Proc. of the Joint 2009 IFSA World Congress and EUSFLAT Conference, 2009, pp. 1541–1546.
[12] L. Fariñas del Cerro, A. Herzig, A modal analysis of possibility theory, in: Fundamentals of Artificial Intelligence Research, Springer, 1991, pp. 11–18.
[13] C. Boutilier, Modal logics for qualitative possibility theory, Int. J. Approx. Reason. 10 (2) (1994) 173–201, http://dx.doi.org/10.1016/0888-613X(94)90015-9.
[14] J. Van Benthem, Correspondence Theory, Springer, 1984.
[15] J. Hintikka, Knowledge and Belief: An Introduction to the Logic of the Two Notions, vol. 181, Cornell University Press, Ithaca, 1962.
[16] N. Laverny, J. Lang, From knowledge-based programs to graded belief-based programs. Part I: on-line reasoning, in: Proc. of the European Conf. on Artificial Intelligence (ECAI), IOS Press, 2004, pp. 368–372.
[17] N. Laverny, J. Lang, From knowledge-based programs to graded belief-based programs, Part II: off-line reasoning, in: Proc. of the 9th Int. Joint Conf. on Artificial Intelligence (IJCAI'05), Gallus, 2005, pp. 497–502.
[18] K. Fine, In so many possible worlds, Notre Dame J. Form. Log. 13 (4) (1972) 516–520, http://dx.doi.org/10.1305/ndjfl/1093890715.
[19] M. Fattorosi-Barnaba, F. De Caro, Graded modalities I, Stud. Log. 44 (2) (1985) 197–221, http://dx.doi.org/10.1007/BF00379767.
[20] F. De Caro, Graded modalities II, Stud. Log. 47 (2) (1988) 1–10.
[21] M. Fattorosi-Barnaba, C. Cerrato, Graded modalities III, Stud. Log. 47 (1988) 99–110, http://dx.doi.org/10.1007/BF00379767.
[22] W. van der Hoek, J.-J. Meyer, Graded Modalities in Epistemic Logic, Springer, 1992.
[23] E. Pacuit, S. Salame, Majority logic, in: Int. Conf. on the Principles of Knowledge Representation and Reasoning (KR), vol. 4, 2004, pp. 598–605.
[24] W. van der Hoek, M. De Rijke, Counting objects, J. Log. Comput. 5 (3) (1995) 325–345, http://dx.doi.org/10.1093/logcom/5.3.325.
[25] E. Pacuit, S. Salame, Majority logic: axiomatization and completeness, Stud. Log.
[26] M. de Rijke, A note on graded modal logic, Stud. Log. 64 (2000) 271–283.