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# ON THE CO-ORBITAL MOTION IN THE PLANAR RESTRICTED THREE-BODY PROBLEM: THE QUASI-SATELLITE MOTION REVISITED

ALEXANDRE POUSSE, PHILIPPE ROBUTEL, AND ALAIN VIENNE

ABSTRACT. In the framework of the planar and circular restricted three-body problem, we consider an asteroid that orbits the Sun in quasi-satellite motion with a planet. A quasi-satellite trajectory is a heliocentric orbit in co-orbital resonance with the planet, characterized by a non zero eccentricity and a resonant angle that librates around zero. Likewise, in the rotating frame with the planet it describes the same trajectory as the one of a retrograde satellite even though the planet acts as a perturbator.

In the last few years, the discoveries of asteroids in this type of motion made the term “quasi-satellite” more and more present in the literature. However, some authors rather use the term “retrograde satellite” when referring to this kind of motion in the studies of the restricted problem in the rotating frame.

In this paper we intend to clarify the terminology to use, in order to bridge the gap between the perturbative co-orbital point of view and the more general approach in the rotating frame. Through a numerical exploration of the co-orbital phase space, we describe the quasi-satellite domain and highlight that it is not reachable by low eccentricities by averaging process. We will show that the quasi-satellite domain is effectively included in the domain of the retrograde satellites and neatly defined in terms of frequencies.

Eventually, we highlight a remarkable high eccentric quasi-satellite orbit corresponding to a frozen ellipse in the heliocentric frame. We extend this result to the eccentric case (planet on an eccentric motion) and show that two families of frozen ellipses originate from this remarkable orbit.

Restricted Three-Body Problem and Co-orbital motion and Quasi-satellite and Averaged Hamiltonian

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**Abbreviations.**

RF: Rotating frame with the planet  
 AP: Averaged problem  
 RAP: Reduced averaged problem  
 RS: Retrograde-satellite  
 TP: Tadpole  
 HS: Horseshoe  
 QS: Quasi-satellite  
 sRS : “Satellized” retrograde satellite  
 QS<sub>b</sub> : Binary quasi-satellite  
 QS<sub>h</sub> : Heliocentric quasi-satellite

**List of symbols.**

$L_1, L_2, L_3$ : Circular Eulerian aligned configurations

$L_4, L_5$ : Circular Lagranian equilateral configurations

$\mathcal{L}_4^l, \mathcal{L}_5^l$ : In the RF, long period families that originate from  $L_4$  and  $L_5$ .

$\mathcal{L}_3, \mathcal{L}_4^s, \mathcal{L}_5^s$ : In the RF, short period families that originate from  $L_3, L_4$  and  $L_5$ .

Family  $f$ : In the RF, one-parameter family of simple-periodic symmetrical retrograde satellite orbits that extends from an infinitesimal neighbourhood of the planet to the collision with the Sun. For  $\varepsilon < 0.0477$ , it is stable but contains two particular orbits where the frequencies  $\nu$  and  $1 - g$  are in 1 : 3 resonance. These two orbits decomposed the neighbourhood of the family  $f$  in three domains: sRS, QS<sub>b</sub> and QS<sub>h</sub>.

1,  $\nu$ ,  $g$ : Frequencies respectively associated with the fast variations (the mean longitudes  $\lambda$  and  $\lambda'$ ), the semi-fast component of the dynamics (oscillation of the resonant angle  $\theta$ ) and the secular evolution of a trajectory (precession of the periastron argument  $\omega$ ).

$\mathcal{N}_{L_4}^u, \mathcal{N}_{L_5}^u$ : In the RAP, the AP and the RF, families of  $2\pi/\nu$ -periodic orbits parametrized by  $|u| \leq 0$  and that originates from  $L_4$  and  $L_5$ . Moreover, they correspond to  $\mathcal{L}_4^l$  and  $\mathcal{L}_5^l$  in the RF.

$\mathcal{G}_{L_3}^{e_0}, \mathcal{G}_{L_4}^{e_0}, \mathcal{G}_{L_5}^{e_0}$ : In the RAP, families of fixed points parametrized by  $e_0$  and that originate from  $L_3, L_4$  and  $L_5$ . In the AP and the RF, these fixed points correspond to periodic orbits of frequency respectively  $g$  and  $1 - g$ . Moreover, they correspond to  $\mathcal{L}_3, \mathcal{L}_4^s$  and  $\mathcal{L}_5^s$  in the RF.

$\mathcal{G}_{QS}^{e_0}$ : In the RAP, family of fixed points parametrized by  $e_0$ . In the AP and the RF, these fixed points correspond to periodic orbits of frequency respectively  $g$  and  $1 - g$ . Moreover, this family corresponds to a part of the family  $f$  that belongs to the QS<sub>h</sub> domain.

$G_{L_3}, G_{L_4}, G_{L_5}, G_{QS}$ : In the RAP, fixed points that belong to  $\mathcal{G}_{L_3}^{e_0}, \mathcal{G}_{L_4}^{e_0}, \mathcal{G}_{L_5}^{e_0}$  and  $\mathcal{G}_{QS}^{e_0}$  and characterized by  $g = 0$ . In the AP, sets of fixed points (also denoted as “circles of fixed points”) parametrized by  $\omega(t = 0)$ . In the RF, sets of  $2\pi$ -periodic orbits parametrized by  $(\lambda' - \omega)_{t=0}$ .

$G_{L_3,1}^{e'}, G_{L_3,2}^{e'}, G_{L_4,1}^{e'}, G_{L_4,2}^{e'}, G_{L_5,1}^{e'}, G_{L_5,2}^{e'}, G_{QS,1}^{e'}, G_{QS,2}^{e'}$ : In the AP with  $e' \geq 0$ , families of fixed points that originate from the circles of fixed points  $G_{L_3}, G_{L_4}, G_{L_5}, G_{QS}$  when  $e' = 0$ .

## 1. INTRODUCTION

Following the discoveries, in 1899 and 1908, of the retrograde moons Phoebe and Pasiphea moving at great distances from their respective primaries Saturn and Jupiter, Jackson (1913) published the first study dedicated to the motion of the retrograde satellites (RS). Seeking to understand how a moon could still be satellized at this remote distance (close to the limit of the planet Hill’s sphere), he highlighted in the Sun-Jupiter system that where “[...] *the solar forces would prohibited direct motion, [...] the solar and the Jovian forces would go hand in hand to maintain a retrograde satellite*”. Thus, by this remark the author was the first to confirm the existence and stability of remote retrograde satellite objects in the solar system.

Afterwards, the existence and stability of some retrograde satellite orbits far from the secondary body have also been established in the planar restricted three-body problem with two equal masses (Strömngren, 1933; Moeller, 1935; Henon, 1965a,b)<sup>1</sup> and in the Earth-Moon system (Broucke, 1968).

In the framework of the Hill’s approximation, Henon (1969) extended Jackson’s study and highlighted that there exists a one-parameter family of simple-periodic symmetrical retrograde satellite orbits (denoted family  $f$ ) that could exist beyond the Eulerian configurations  $L_1$  and  $L_2$ . This has been confirmed in Henon and Guyot (1970) in the restricted three-body problem. The authors showed in the rotating frame with the planet (RF), that the family  $f$  extends from the retrograde satellite orbits in an infinitesimal neighbourhood of the secondary to the collision orbit with the primary. Besides, they pointed out that if  $\varepsilon$ , the ratio of the secondary mass over the sum of the system masses, is less than 0.0477, the whole family is stable. Benest (1974, 1975, 1976) extended these results by studying the stability of the neighbourhood of the family  $f$  in the configuration space for  $0 \leq \varepsilon \leq 1$ .

After these theoretical works, the study of the retrograde satellite orbits was addressed in a more practical point of view, with the project to inject a spacecraft in a circum-Phobos orbit. Remark that as the Phobos Hill’s sphere is too close to its surface, remote retrograde satellites are particularly adapted trajectories. Hence, at the end of the eighties, the terminology “quasi-satellite”<sup>2</sup> (QS) appeared in the USSR astrodynamist community to define trajectories in the restricted three-body problem in rotating frame that correspond to retrograde satellite orbits outside the Hill’s sphere of the secondary body (see Fig.1a). The Phobos mission study led to the works of Kogan (1990) and Lidov and Vashkov’yak (1993, 1994a,b).

At the end of the nineties, the quasi-satellite motion appeared in the celestial mechanics community in the view of asteroid trajectories in the solar system.

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<sup>1</sup>The two firsts are works of the Copenhagen group that extensively explored periodic orbit solutions in the planar restricted three-body problem with two equal masses. The two lasts are the first numerical explorations of all the solutions of the restricted three-body problem that recovered and completed the precedent works.

<sup>2</sup>Let us still mention that the “quasi-satellite” terminology has already been used in the paper of Danielson and Ip (1972) but this was to describe the resonant behaviour of the near-Earth Object 1685 Toro and therefore was completely disconnected to retrograde satellite motion.

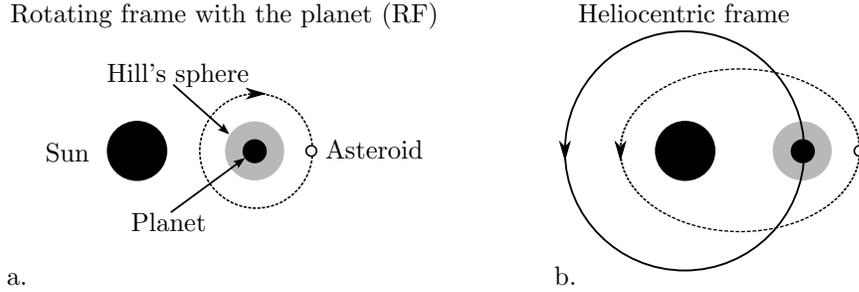


FIGURE 1. Asteroid on a quasi-satellite orbit (QS). In the rotating frame with the planet (RF) (a.), the trajectory is those of a retrograde satellite (RS) outside the planet Hill's sphere. In the heliocentric frame (b.), the trajectory is represented by heliocentric osculating ellipses with a non zero eccentricity (in the circular case) and a resonant angle  $\theta = \lambda - \lambda'$  that librates around zero.

Let us suppose a QS-type asteroid far enough from the planet so that the influence of the Sun dominates its movement and therefore that the planet acts as a perturbator. Then, its trajectory could be represented by heliocentric osculating ellipses whose variations are governed by the influence of the planet. In this context, Mikkola and Innanen (1997) remarked that the asteroid and the planet are in 1 : 1 mean motion resonance and therefore that the quasi-satellite orbits correspond to a particular kind of configurations in the co-orbital resonance. Unlike the tadpole (TP) orbits that librate around the Lagrangian equilibria  $L_4$  and  $L_5$  or the horseshoes (HS) that encompass  $L_3$ ,  $L_4$  and  $L_5$ , the quasi-satellite orbits are characterized by a resonant angle  $\theta = \lambda - \lambda'$  that librates around zero (where  $\lambda$  and  $\lambda'$  are the mean longitudes of the asteroid and the planet) and a non zero eccentricity if the planet gravitates on a circle (see Fig.1b). In their paper, these authors also described a first perturbative treatment to study the long term stability of quasi-satellites in the solar system.

At that time no natural object was known to orbit this configuration. However, they suggested that, at least, the Earth and Venus could have quasi-satellite companions. Following this work, Wiegert et al. (2000) also predicted, via a numerical investigation of the stability around the giant planets, that Uranus and Neptune could harbour QS-type asteroids whereas they did not found stable solutions for Jupiter and Saturn.

Subsequently, Namouni (1999) and Namouni et al. (1999) became the reference in term of co-orbital dynamics with close encounters. Using Hill's approximation, these authors highlighted that in the spatial case, transitions between horseshoe and quasi-satellite trajectories could occurred. They exhibited new kinds of compound trajectories denoted HS-QS, TP-QS or TP-QS-TP which means that there exists stable transitions exit between quasi-satellite, tadpole and horseshoe orbits. Later, Nesvorný et al. (2002) recovered these new co-orbital structures in a global study of the co-orbital resonance phase space. By developing a perturbative scheme using numerical averaging techniques, they showed how the

tadpole, horseshoe, quasi-satellite and compound orbits vary with the asteroid eccentricity and inclination in the planar-circular, planar-eccentric and spatial-circular models. Particularly, they showed that the higher the asteroid’s eccentricity is, the larger the domain occupied by the quasi-satellite orbits in the phase space is.

Eventually, the quasi-satellite long-term stability has been studied using perturbation theory in Mikkola et al. (2006) and Sidorenko et al. (2014). The first ones developed a practical algorithm to detect QS-type asteroids on temporary or perpetual regime, while the last ones established conditions of existence of quasi-satellite motion and also explore its different possible regimes.

Following these theoretical works, many objects susceptible to be at least temporary quasi-satellites have been found in the solar system. The first confirmed minor body was 2002 VE68 in co-orbital motion with Venus in Mikkola et al. (2004). The Earth (Brasser et al., 2004; Connors et al., 2002, 2004; de la Fuente Marcos and de la Fuente Marcos, 2014; Wajer, 2009, 2010) and Jupiter (Kinoshita and Nakai, 2007; Wajer and Królikowska, 2012) are the two planets with the largest number of documented QS-type objects. Likewise, Saturn (Gallardo, 2006), Uranus (Gallardo, 2006; de la Fuente Marcos and de la Fuente Marcos, 2014), Neptune (de la Fuente Marcos and de la Fuente Marcos, 2012) possess at least one of this type.

At last, let us mention that quasi-satellite motion could play a role in other celestial problems: according to Kortenkamp (2005) and (2013), planetesimals could be trapped in quasi-satellite motion around the protoplanet as well as interplanetary dust particles around Earth. Eventually, although no co-orbital exoplanet system has been found, several studies on the planar planetary three-body problem showed the existence and the stability of two co-orbital planets in quasi-satellite motion (Hadjidemetriou et al., 2009; Hadjidemetriou and Voyatzis, 2011; Giuppone et al., 2010).

During these last twenty years, even though the “quasi-satellite” terminology becomes dominant in the literature, some studies use rather “retrograde satellite” (Namouni, 1999; Nesvorný et al., 2002) in reference to the neighbourhood of the family  $f$  in the restricted problem in rotating frame with the planet. Hence, there exists an ambiguity in terms of terminology that is a consequence of the several approaches to describe these orbits, depending on the distance between the two co-orbitals. One of our purposes is thus to clarify the terminology to use between “quasi-satellite” and “retrograde satellite”. Then, we chose to revisit the classical works on the family  $f$  (Henon and Guyot, 1970; Benest, 1974) in the section 4 and through a study on its frequencies, we show that the neighbourhood of the family is split in three different domains connected by an orbit; one corresponding to the “satellized” retrograde satellite orbits while the two others to the quasi-satellites. Among these two quasi-satellite domains, we identify one that is associated with asteroid trajectories in the solar system. This is on this last one that the paper is focussed.

An usual approach for these co-orbital trajectories in the restricted (Mikkola et al., 2006; Nesvorný et al., 2002; Sidorenko et al., 2014) and planetary (Robutel and Pousse, 2013) problems consists on averaging the Hamiltonian over the fast angle of the system (the planet mean longitude) to reduce the study of the problem to its semi-fast and secular components. This approach is generally denoted as the “averaged problem” (AP). However,

as mentioned in Robutel and Pousse (2013) and Robutel et al. (2016), this one has the important drawback to reflect poorly the dynamics close to the singularity associated with the collision with the planet. Some quasi-satellite trajectories having close encounter with the planet, these are located close to the singularity in the averaged problem which implies that this approach would not be appropriate for them. Thus, in order to estimate a validity limit of the averaged problem for the study of quasi-satellite motion, we also revisit the co-orbital resonance via the averaged problem.

Firstly, in the section 2, we develop the Hamiltonian formalism of the problem and introduce the averaged problem. Subsequently, in the section 3, we focus on the circular case (i.e. planet on a circular orbit) that allows possible reduction. We introduce the reduced averaged problem (RAP) that seems to be the most adapted approach to understand the dynamics in the co-orbital resonance. Focussing on quasi-satellite motion, we exhibit a family of fixed points in the reduced averaged problem representing the family  $f$  that allows us to estimate the validity limit of the averaged problem.

Next, to bridge a gap between the averaged problem and the works of Henon and Guyot (1970) and Benest (1975), we devote the section 4 to revisit the motion in the rotating frame in the circular case in order to describe the family  $f$  as well as its reachable part in the averaged problem and characterize its neighbourhood. Through this study, we show how the quasi-satellite domain reachable in the averaged problem shrinks by increasing  $\varepsilon$ .

At last, in the section 5, we come back to the averaged problem with the aim to extend in the eccentric case (i.e. planet on an eccentric orbit) a result on co-orbital frozen ellipses that has been highlighted in section 3.4.

## 2. THE AVERAGED PROBLEM

In the framework of the planar restricted three-body problem, we consider a primary with a mass  $1 - \varepsilon$  (the Sun or a star), a secondary (a planet) with a mass  $\varepsilon$  small with respect to 1 and a massless third body (particle or asteroid). We assume that the planet is in elliptic Keplerian motion whose eccentricity is denoted  $e'$ . Without loss of generality, we set that its semi-major axis is equal to 1 and that the argument of the periaster is equal to zero. Likewise, we fix its orbital period to  $2\pi$  (and therefore its mean motion to 1) which imposes the gravitational constant to be equal to 1.

In an heliocentric frame, the Hamiltonian of the problem reads

$$(1) \quad \mathcal{H}(\mathbf{r}, \dot{\mathbf{r}}, t) = \mathcal{H}_K(\mathbf{r}, \dot{\mathbf{r}}) + \mathcal{H}_P(\mathbf{r}, t)$$

with

$$\mathcal{H}_K(\mathbf{r}, \dot{\mathbf{r}}) := \frac{1}{2} \|\dot{\mathbf{r}}\|^2 - \frac{1}{\|\mathbf{r}\|}$$

and

$$\mathcal{H}_P(\mathbf{r}, t) := \varepsilon \left( -\frac{1}{\|\mathbf{r} - \mathbf{r}'(t)\|} + \frac{1}{\|\mathbf{r}\|} + \frac{\mathbf{r} \cdot \mathbf{r}'(t)}{\|\mathbf{r}'(t)\|^3} \right).$$

In this expression,  $\mathbf{r}$  is the heliocentric position of the particle,  $\dot{\mathbf{r}}$  its conjugated variable and  $\mathbf{r}'(t)$  is the position of the planet at the time  $t$ .

In order to work with an autonomous Hamiltonian, we extend the phase space by introducing  $\Lambda'$ , the conjugated variable of  $\lambda' := t$  that corresponds to the mean longitude of the planet. As a consequence the Hamiltonian becomes, on the extended phase space, equal to  $\Lambda' + \mathcal{H}$ .

In order to define a canonical coordinate system related to the elliptic elements  $(a, e, \lambda, \omega)$  (respectively semi-major axis, eccentricity, mean longitude and argument of the periastron) and adapted to the co-orbital resonance, we introduce the canonical coordinates  $(\theta, u, -i\bar{x}, x, \lambda', \tilde{\Lambda}')$  where

$$(2) \quad \theta := \lambda - \lambda' \quad \text{and} \quad u := \sqrt{a} - 1$$

are the resonant variables,

$$(3) \quad x := \sqrt{\Gamma} \exp(i\omega) \quad \text{with} \quad \Gamma := \sqrt{a}(1 - \sqrt{1 - e^2})$$

is the Poincaré's variable associated with the eccentricity  $e$ , and  $\tilde{\Lambda}'$  that is the conjugated variable of  $\lambda'$  such as

$$(4) \quad \Lambda' = \tilde{\Lambda}' - u.$$

If we denote  $\Phi$ , the canonical transformation such that

$$\Phi : \begin{cases} \mathbb{T} \times \mathbb{R} \times \mathbb{C}^2 \times \mathbb{T} \times \mathbb{R} & \longrightarrow & \mathbb{R}^4 \times \mathbb{T} \times \mathbb{R} \\ (\theta, u, -i\bar{x}, x, \lambda', \tilde{\Lambda}') & \longmapsto & (\mathbf{r}, \dot{\mathbf{r}}, \lambda', \Lambda') \end{cases},$$

the Hamiltonian of the problem reads  $\tilde{\Lambda}' + H$  with

$$(5) \quad H := (\Lambda' + \mathcal{H}) \circ \Phi - \tilde{\Lambda}' = H_K - u + H_P$$

where

$$H_K := -\frac{1}{2(1+u)^2} \quad \text{and} \quad H_P := \mathcal{H}_P \circ \Phi.$$

In these variables, the Hamiltonian possesses 3 degrees of freedom, each one corresponding to a particular component of the dynamics inside the co-orbital resonance. Indeed, the resonant angle  $\theta$  varies slowly with respect to the fast angle  $\lambda'$ . Thus the degree of freedom  $(\theta, u)$  is generally known as the “semi-fast” component of the dynamics while the degree of freedom  $(-i\bar{x}, x)$  is associated with the “secular” variations of the trajectory. As a consequence, a natural way to reduce the dimension of the problem in order to study the “semi-fast” and “secular” dynamics of the co-orbital motion is to average the Hamiltonian over  $\lambda'$ . In the following, this averaged Hamiltonian will be denoted  $\bar{H}$ .

**2.1. The averaged Hamiltonian.** According to the perturbation theory, there exists a canonical transformation

$$\mathcal{C} : \begin{cases} \mathbb{T} \times \mathbb{R} \times \mathbb{C}^2 \times \mathbb{T} \times \mathbb{R} & \longrightarrow & \mathbb{T} \times \mathbb{R} \times \mathbb{C}^2 \times \mathbb{T} \times \mathbb{R} \\ (\theta, u, -i\bar{x}, x, \lambda', \tilde{\Lambda}') & \longmapsto & (\theta, u, -i\bar{x}, x, \lambda', \tilde{\Lambda}'), \end{cases}$$

such as, in the averaged variables  $(\theta, u, -i\bar{x}, x, \lambda', \tilde{\Lambda}')$ , the Hamiltonian reads

$$(6) \quad \tilde{\Lambda}' + \mathbf{H} = (\tilde{\Lambda}' + H) \circ \mathcal{C} \quad \text{with} \quad \mathbf{H} := \bar{H} + H_*$$

where

$$\overline{H} := H_K - u + \overline{H}_P$$

with

$$\overline{H}_P(\theta, u, -i\overline{x}, x) := \frac{1}{2\pi} \int_0^{2\pi} H_P(\theta, u, -i\overline{x}, x, \lambda') d\lambda'.$$

$H_*$  is a remainder that is supposed to be small with respect to  $\overline{H}_P$ . More precisely, the transformation  $\mathcal{C}$  is close to the identity and could be constructed with the time-one map of the Hamiltonian flow generated by some auxiliary function  $\chi$  (for further details, see Robutel et al., 2016). As a consequence, if  $\{f, g\}$  represents the Poisson bracket of the two functions  $f$  and  $g$  and if  $y$  stands for one of the variables  $(\theta, u, -i\overline{x}, x, \lambda', \tilde{\Lambda}')$ , then the two coordinate systems are related by

$$(7) \quad y = y + \{\chi, y\} + \mathcal{O}(\varepsilon^2)$$

with

$$\chi(\theta, u, -i\overline{x}, x, \lambda') = \int_0^{\lambda'} \left[ \overline{H}_P(\theta, u, -i\overline{x}, x) - H_P(\theta, u, -i\overline{x}, x, \tau) \right] d\tau.$$

In this paper, we only consider the restriction at first order in  $\varepsilon$  of the Hamiltonian in the equation (6). This approximation of the initial problem that is described by  $\overline{H}$  is generally known as the ‘‘averaged problem’’ (AP). Thus, the averaged problem possesses two degrees of freedom and two parameters,  $\varepsilon$  and  $e'$ , respectively the planetary mass ratio and eccentricity of the planet..

For the sake of clarity, the ‘‘underdot’’ used to denote the averaged coordinates will be omitted below.

**2.2. Numerical averaging.** There exists at least two classical averaging techniques adapted to the co-orbital resonance: an analytical one based on an expansion of the Hamiltonian in power series of the eccentricity (e.g. Morais, 2001; Robutel and Pousse, 2013), and a numerical one consisting on a numerical evaluation of  $\overline{H}$  and its derivatives (e.g. Nesvorný et al., 2002; Giuppone et al., 2010; Beaugé and Roig, 2001; Mikkola et al., 2006; Sidorenko et al., 2014). Whereas for low eccentricities the analytical technique is very efficient, reaching higher values of eccentricity requires high order expansions which generate very heavy expressions. Thus, in this case, the use of numerical methods may be more convenient. Then in order to explore the phase space of the co-orbital resonance for all eccentricities lower than one, we use the numerical averaging method developed by Nesvorný et al. (2002). This method consists on a numerical evaluation of the integral (7). More generally, let  $F$  be a generic function depending on  $(\theta, u, -i\overline{x}, x, E, E')$  where  $E$  and  $E'$  are the eccentric anomaly of the particle and the planet. As its average over the mean longitude  $\lambda'$  is computed for a given fixed value of  $\theta$ , we have  $d\lambda' = d\lambda = (1 - e(x) \cos E) dE$ . As

$$(8) \quad \theta = \lambda - \lambda' = E + \omega(x) - E' - e(x) \sin E + e' \sin E',$$

the eccentric anomalies  $E'$  can be expressed in terms of  $(\theta, E, x, e')$ . Eventually, the integrals reads

$$(9) \quad \overline{F}(\theta, u, -i\bar{x}, x) = \frac{1}{2\pi} \int_0^{2\pi} F(\theta, u, -i\bar{x}, x, E, E'(\theta, E, x, e')) (1 - e(x) \cos E) dE,$$

which can be computed by discretizing the variable  $E$  as  $E_k = \frac{k2\pi}{N}$  and  $100 \leq N \leq 300$  (see Nesvorný et al., 2002, for more details).

### 3. THE CO-ORBITAL RESONANCE IN THE CIRCULAR CASE ( $e' = 0$ )

In the circular case – that is the case where the planet gravitates on a circle –, the averaged problem defined by  $\overline{H}$  is invariant under the action of the symmetry group  $SO(2)$  associated with the rotations around the vertical axis. Thereby, in the vicinity of the quasi-circular orbits ( $|x| \ll 1$ ), the expansion of  $\overline{H}$  in power series of  $x$  and  $\bar{x}$  reads

$$(10) \quad \sum_{(p, \bar{p}) \in \mathbb{N}^2} \Psi_{p, \bar{p}}(\theta, u) x^p \bar{x}^{\bar{p}}$$

where the integers occurring in these summations satisfy the relation

$$(11) \quad p - \bar{p} = 0$$

that results from the d'Alembert rule. Hence, we have

$$(12) \quad \frac{\partial \overline{H}}{\partial \omega}(\theta, u, -i\bar{x}, x) = 0 = \dot{x}\bar{x} + x\dot{\bar{x}} = \dot{\Gamma},$$

which imposes  $\Gamma$  to be a first integral. As a consequence, in the averaged problem, the two degrees of freedom of the problem are separable and a reduction is possible.

By fixing the value of the parameter  $\Gamma = |x|^2$  and eliminating the cyclic variable  $\omega = \arg(x)$ , we remove one degree of freedom. We call this new problem the “reduced averaged problem” (RAP). However, instead of using  $\Gamma$  as a parameter, we introduce  $e_0$  such as

$$(13) \quad \Gamma = (1 + u)(1 - \sqrt{1 - e^2}) = 1 - \sqrt{1 - e_0^2}.$$

Then, if  $u \ll 1$ , the parameter  $e_0$  that is equal to  $e + \mathcal{O}(u)$  provides an approximation of the eccentricity value  $e$  of the trajectory.

**3.1. The reduced Hamiltonian.** For a given value  $e_0 = a$  such that  $0 \leq a < 1$ , let us define  $\overline{\mathcal{M}}_{e_0} \subset \mathbb{T} \times \mathbb{R} \times \mathbb{C}^2$  the intersection of the phase space of the averaged problem (denoted  $\overline{\mathcal{M}} \subset \mathbb{T} \times \mathbb{R} \times \mathbb{C}^2$ ) with the hyperplane  $\{e_0 = a\}$ , and  $\overline{\mathcal{M}}_{e_0}/SO(2)$ , the quotient space of this section by the symmetry group  $SO(2)$ . Under the action of the application

$$(14) \quad \psi_{e_0} : \begin{cases} \overline{\mathcal{M}}_{e_0} & \longrightarrow \overline{\mathcal{M}}_{e_0}/SO(2) \\ (\theta, u, -i\bar{x}, x) & \longmapsto (\theta, u) \end{cases},$$

the problem is reduced to one degree of freedom and is associated with the reduced Hamiltonian

$$(15) \quad \overline{H}_{e_0} := \overline{H}(\cdot, \cdot, -i\bar{x}(e_0), x(e_0)).$$

Thus, for a fixed  $e_0$ , a trajectory in the RAP is generally a periodic orbit, but can also be a fixed point. As a consequence, the description of the RAP's phase portrait obtained for various values of  $e_0$  allows to understand the global dynamics of the co-orbital resonance in the circular case.

The AP being more usual to illustrate the semi-fast and secular variations of the orbital elements and the rotating frame (RF) more classic to understand the dynamics of the restricted three-body problem, we will see in the next section how a given orbit is represented in these three different points of view.

**3.2. Correspondence between the RAP, the AP and the RF.** For a given value of  $e_0$ , let us consider a periodic trajectory of frequency  $\nu$  in the RAP. The correspondence between the RAP and the AP consists in the pullback of a trajectory belonging to  $\overline{\mathcal{M}}_{e_0}/SO(2)$  by the application  $\psi_{e_0}^{-1}$ . However,  $\omega = \arg(x)$  being ignorable in the RAP,  $\psi_{e_0}^{-1}$  is not an injection, which implies that a set of orbits in the AP parametrized by  $\omega_0 := \omega(t=0) \in \mathbb{T}$  is mapped by  $\psi_{e_0}$  to the initial trajectory. Furthermore, as

$$\dot{\omega}(t) = -\frac{\partial}{\partial \Gamma} \overline{H}(\theta(t), u(t)),$$

then  $\dot{\omega}(t)$  is  $2\pi/\nu$ -periodic and could be decomposed such as

$$(16) \quad \dot{\omega}(t) = g - \left[ \frac{\partial}{\partial \Gamma} \overline{H}(\theta(t), u(t)) + g \right]$$

where

$$g := \frac{\nu}{2\pi} \int_0^{2\pi/\nu} -\frac{\partial}{\partial \Gamma} \overline{H}_{e_0}(\theta(t), u(t)) dt$$

is the secular precession frequency of  $\omega$ . Thus, for each orbits of the family, the temporal evolution of the argument of its periastrer is given by

$$(17) \quad \omega(t) = \omega_0 + gt - \int_0^t \left[ \frac{\partial}{\partial \Gamma} \overline{H}_{e_0}((\theta(\tau), u(\tau)) + g \right] d\tau.$$

As a consequence, a given periodic trajectory in the RAP generally corresponds, in the AP, to a set of quasi-periodic orbits of frequencies  $\nu$  and  $g$ . Nevertheless,  $\omega$  being ignorable when the osculating ellipses are circles (i.e.  $e_0 = 0$ ), the trajectories are fixed points or periodic orbits of frequency  $\nu$  in both approaches. When  $e_0 > 0$  and  $g = 0$ , a periodic trajectory of the RAP provides a set of periodic orbits of frequency  $\nu$  in the AP. Likewise a fixed point corresponds to a set of degenerated fixed points. These fixed points being distributed along a circle in the phase space represented by the variables  $(x, -i\bar{x})$ . Their set will be describe as a ‘‘circle of fixed points’’ in what follows.

Next, to connect the AP with the RF, we firstly have to apply  $\mathcal{C}$  to the trajectory which adds the fast frequency in the variations of the orbital elements, i.e. the planet mean motion. In the circular case, the d'Alembert rule implies that  $\tilde{\Lambda}' + H$  only depends on the angles  $\lambda' - \omega$  and  $\theta$ . Consequently, by defining the canonical transformation

$$\widehat{\psi} : \begin{cases} \mathcal{M} & \longrightarrow & \widehat{\psi}(\mathcal{M}) \\ (\theta, u, -i\bar{x}, x, \lambda', \tilde{\Lambda}') & \longmapsto & (\theta, u, -i\bar{\xi}, \xi, \lambda', \tilde{\Lambda}' - \Gamma) \end{cases}$$

Approach	$e_0 = 0$		$e_0 > 0$			
			$g \neq 0$		$g = 0$	
	FP	PO ( $\nu$ )	FP	PO ( $\nu$ )	FP	PO ( $\nu$ )
<b>RAP</b>	FP	PO ( $\nu$ )	FP	PO ( $\nu$ )	FP	PO ( $\nu$ )
↓ <b>AP</b>	FP	PO ( $\nu$ )	$S_{\omega_0}$ PO ( $g$ )	$S_{\omega_0}$ QPO ( $\nu, g$ )	$S_{\omega_0}$ FP	$S_{\omega_0}$ PO ( $\nu$ )
↓ <b>RF</b>	FP	PO ( $\nu$ )	$S_{\varphi_0}$ PO ( $1 - g$ )	$S_{\varphi_0}$ QPO ( $\nu, 1 - g$ )	$S_{\varphi_0}$ PO (1)	$S_{\varphi_0}$ QPO ( $\nu, 1$ )

TABLE 1. Correspondence between the three approaches for a given trajectory in the RAP.  $S_{\omega_0}, S_{\varphi_0}$ : set of solutions parametrized by  $\omega_0$  and  $\varphi_0 \in \mathbb{T}$ . **FP**: Fixed point. **PO**: Periodic orbit. **QPO**: Quasi-periodic orbit. Parenthesis: associated frequencies.

with  $\mathcal{M}$  that corresponds to the non-averaged phase space<sup>3</sup>,  $\xi = \sqrt{\Gamma} \exp(i\varphi)$  and  $\varphi = \lambda' - \omega$ , the Hamiltonian  $(\tilde{\Lambda}' + H) \circ \hat{\psi}^{-1}$  becomes autonomous with two degrees of freedom associated with the frequencies  $\nu$  and  $1 - g$ . Moreover, this Hamiltonian is related to those in the RF by the pullback by  $\Phi^{-1}$ , that is the canonical transformation in Cartesian coordinates. Thus, a trajectory in the RF is generally quasi-periodic with two frequencies. As a consequence, a given trajectory of the RAP generally corresponds to a set of orbits in the RF parametrized by  $\varphi_0 := \varphi(t = 0) \in \mathbb{T}$  with one more frequency.

For the sake of clarity, we summarize the status of the remarkable orbits in the three different approaches in the table 1.

**3.3. Phase portraits of the RAP.** The figure 2 displays the phase portraits of the RAP associated with six different values of the parameter  $e_0$  for a Sun-Jupiter like system ( $\varepsilon = 0.001$ ).

In Fig.2a,  $e_0$  is equal to zero: the osculating ellipses of all the orbits are circles. The singular point located at  $\theta = u = 0$  corresponds to the collision between the asteroid and the planet, where  $\overline{H}$  is not defined (the integral (7) is divergent). The two elliptic fixed points, in  $(\theta, u) = (\pm 60^\circ, 0)$ , correspond to the Lagrangian equilateral configurations  $L_4$  and  $L_5$  whereas the hyperbolic fixed point, close to  $(\theta, u) = (180^\circ, 0)$ , is associated with the Eulerian aligned configuration  $L_3$ .

On the phase portraits described by Nesvorný et al. (2002) two additional equilibria appears located at  $\theta = 0^\circ$ : the Eulerian aligned configurations  $L_1$  and  $L_2$ . But as it has been shown in Robutel and Pousse (2013), there exists a neighbourhood of the collision singularity inside which the averaged Hamiltonian does not reflect properly the dynamics of the “initial” problem. Indeed, a remainder which depends on the fast variable and that is supposed to be small with respect to  $\overline{H}_P$  is generated by the averaging process; we denoted it  $H_*$  in the expression (6). Although  $H_*$  is equal to  $\mathcal{O}(\varepsilon^2)$  in the major part of

<sup>3</sup>As we have to take into account the degree of freedom  $(\lambda', \tilde{\Lambda}')$ , we have  $\mathcal{M} \subset \mathbb{T} \times \mathbb{R} \times \mathbb{C}^2 \times \mathbb{T} \times \mathbb{R}$ .

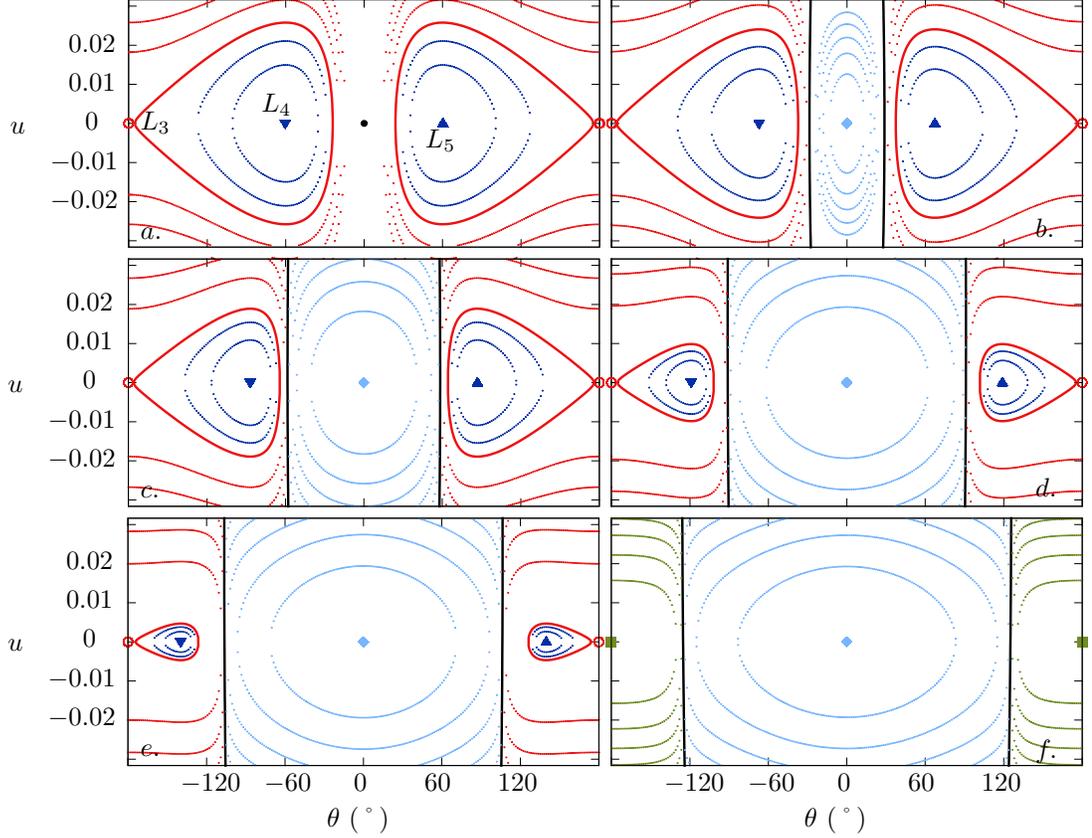


FIGURE 2. Phase portraits of a Sun-Jupiter like system in the circular case. For a, b, c, d, e and f,  $e_0$  is equal to 0, 0.25, 0.5, 0.75, 0.85 and 0.95 . The black dot (a.) and curves represent the collision with the planet. The blue, sky blue and red dots are level curves of TP, QS and HS orbits. For  $e_0 = 0$ , the blue triangles and red circles represents  $L_4$ ,  $L_5$  and  $L_3$ , while for  $e_0 > 0$  they form the families  $\mathcal{G}_{L_4}^{e_0}$ ,  $\mathcal{G}_{L_5}^{e_0}$  and  $\mathcal{G}_{L_3}^{e_0}$ . From  $L_3$  and the unstable part of  $\mathcal{G}_{L_3}^{e_0}$  originates a separatrix that is represented by a red curve. The sky blue diamonds form the family  $\mathcal{G}_{QS}^{e_0}$ . Eventually the green squares represents the stable part of  $\mathcal{G}_{L_3}^{e_0}$  around which trajectories represented by green dots librate.

the phase space, when the distance to the collision is of order  $\varepsilon^{1/3}$  and less,  $H_*$  is at least of the same order than the perturbation  $\overline{H}_P$  (Robutel et al., 2016). Thus, this define an “exclusion zone” inside which the trajectories, and especially the equilibria  $L_1$  and  $L_2$ , fall outside the scope of the averaged Hamiltonian.

The orbits that librate around  $L_4$  or  $L_5$  lying inside the separatrix originated from  $L_3$  correspond to the tadpoles (TP) orbits. For  $e_0 = 0$ , these two domains form two families

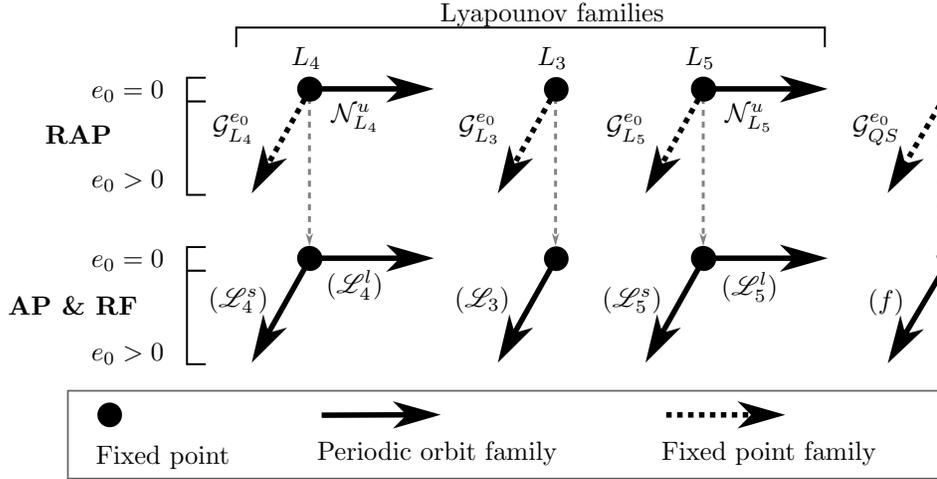


FIGURE 3. Representation of the co-orbital families of periodic orbits from the three different points of view. From each Lagrangian triangular equilibrium originates two Lyapounov families that correspond to a periodic orbit family and a fixed point family in the RAP. These families are associated with the long and short period families in the AP and the RF.  $L_3$  being a saddle center type in the RAP, only one Lyapounov family emanates from this equilibrium that is a fixed point family in the RAP and a periodic orbit family in the AP and RF. Eventually, for  $e_0 > 0$ , there exists a family of fixed points in the RAP that is not a Lyapounov family:  $\mathcal{G}_{QS}^{e_0}$ . This family is associated with a periodic orbit family in the RF: the family  $f$ .

of  $2\pi/\nu$ -periodic orbits originating in  $L_4$  and  $L_5$  and that are parametrized by  $u \geq 0$ . We denote them  $\mathcal{N}_{L_4}^u$  and  $\mathcal{N}_{L_5}^u$ . More precisely, they are the Lyapounov families of the Lagrangian equilateral configurations associated with the libration and generally known as the long period families  $\mathcal{L}_4^l$  and  $\mathcal{L}_5^l$  in the RF (see Meyer and Hall, 1992). Eventually, outside the separatrix lies the horseshoe (HS) domain: the orbits that encompass the three equilibria  $L_3$ ,  $L_4$  and  $L_5$ .

If, when  $e_0 = 0$ , the domain of definition of  $\overline{H}_{e_0}$  excludes the origin  $\theta = u = 0$ , the location of its singularities (associated with the collision with the planet) evolves with the parameter  $e_0$ . Indeed, as soon as  $e_0 > 0$ , the origin becomes a regular point while the set of singular points describes a curve that surrounds the origin. The phase space is now divided in two different domains.

For small  $e_0$  (for example  $e_0 = 0.25$  represented in Fig.2b), the domain outside the collision curve has the same topology as for  $e_0 = 0$ : two stable equilibria close to the  $L_4$  and  $L_5$ 's location and a separatrix emerging from an hyperbolic fixed point close to  $L_3$  that bounds the TP and the HS domains. However, contrarily to  $e_0 = 0$ , the fixed points do not correspond to equilibria in the AP and the RF but to periodic orbits of frequency

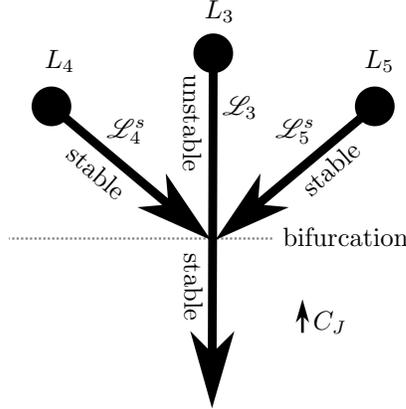


FIGURE 4. Representation of the result of Deprit et al. (1967) in the RF: the merge of the short period families  $\mathcal{L}_4^s$  and  $\mathcal{L}_5^s$  with  $\mathcal{L}_3$  and bifurcation of the latter that becomes stable.

respectively  $g$  and  $1 - g$ . Consequently, orbits in their vicinity correspond to quasi-periodic orbits. Thus, by varying  $e_0$ , these fixed points form three one-parameter families that we denote  $\mathcal{G}_{L_3}^{e_0}$ ,  $\mathcal{G}_{L_4}^{e_0}$  and  $\mathcal{G}_{L_5}^{e_0}$ . In the RF, these ones are known as the short period families  $\mathcal{L}_4^s$ ,  $\mathcal{L}_5^s$  and  $\mathcal{L}_3$ , the Lyapounov families associated with the precession, that emanate from  $L_4$ ,  $L_5$  and  $L_3$  (see Meyer and Hall, 1992).

Inside the collision curve appears a new domain containing orbits that librate around a fixed point of coordinates close to the origin: the QS domain. By varying  $e_0$ , the fixed points form a one-parameter family characterized by  $\theta = 0^\circ$  and that originates from the singular point for  $e_0 = 0$ ; we denote it  $\mathcal{G}_{QS}^{e_0}$ . In the RF, these fixed points correspond<sup>4</sup> to periodic retrograde satellite orbits of frequency  $1 - g$ . As a consequence, the family  $\mathcal{G}_{QS}^{e_0}$  is related to the family  $f$  that is<sup>5</sup> the one-parameter family of simple-periodic symmetrical retrograde satellite orbits.

Thus, for small eccentricities, TP, HS and QS domains are structured around two periodic orbit families ( $\mathcal{N}_{L_4}^u$  and  $\mathcal{N}_{L_5}^u$ ) and four fixed point families ( $\mathcal{G}_{L_3}^{e_0}$ ,  $\mathcal{G}_{L_4}^{e_0}$ ,  $\mathcal{G}_{L_5}^{e_0}$  and  $\mathcal{G}_{QS}^{e_0}$ ) that we outline in Fig.3 to clarify their representations in the different approaches.

For higher values of  $e_0$  (see Fig.2c, d, e and f), the topology of the phase portraits does not change inside the collision curve: the QS domain is always present, but its size increases until it dominates the phase portrait for high eccentricity values. Outside the collision curve, the situation is different. As  $e_0$  increases, the two stable equilibria get closer to the hyperbolic fixed point, which implies that the TP domains shrink and vanish when the three merge. This bifurcation generates a new domain new domain inside of which the orbits librate around the fixed point close to  $(\theta, u) = (180^\circ, 0)$  (see Fig.2f). A similar

<sup>4</sup>See the section 3.2.

<sup>5</sup>See the section 4 for further details on the family  $f$ .

result was found by Deprit et al. (1967) for an Earth-Moon like system in the circular case ( $\varepsilon = 1/81$ ). In the RF, the authors showed that the short period families  $\mathcal{L}_4^s$  and  $\mathcal{L}_5^s$  terminate on a periodic orbit of  $\mathcal{L}_3$  (see the outline in Fig.4).

Now, let us focus on the QS domain. As mentioned above, there exists an exclusion zone in the vicinity of the collision curve such that the QS orbits does not represent “real” trajectories of the initial problem. For high eccentricities, the QS dominates the phase portraits; the size of the intersection between the QS domain and the exclusion zone is small relatively to the whole domain. However by decreasing  $e_0$ , the QS domain shrinks with the collision curve. As a consequence, the relative size of the intersection increases until a critical value of  $e_0$  where the exclusion zone contains all the QS orbits. In this case, the AP and a fortiori the RAP are not relevant to study the QS motion.

A simple way to estimate a validity limit of these two approaches is to consider that the whole QS domain is excluded if and only if  $\mathcal{G}_{QS}^{e_0}$  is inside the exclusion zone. Thus the study of the fixed points family  $\mathcal{G}_{QS}^{e_0}$  allows to determinate the eccentricity value under which the averaging method cannot be applied to QS motion.

**3.4. Fixed point families of the RAP.** For a given value of  $e_0$ , let us consider a fixed point of the RAP, denoted  $(\theta_0, u_0)$ , such as

$$(18) \quad \frac{\partial}{\partial \theta} \overline{H}_{e_0}(\theta_0, u_0) = 0 \quad \text{and} \quad \frac{\partial}{\partial u} \overline{H}_{e_0}(\theta_0, u_0) = 0.$$

The linear stability of this fixed point, is deduced from the eigenvalues of the matrix<sup>6</sup>

$$\mathcal{M} := \begin{pmatrix} \frac{\partial^2 \overline{H}_{e_0}}{\partial \theta \partial u} & \frac{\partial^2 \overline{H}_{e_0}}{\partial u^2} \\ -\frac{\partial^2 \overline{H}_{e_0}}{\partial \theta^2} & -\frac{\partial^2 \overline{H}_{e_0}}{\partial \theta \partial u} \end{pmatrix}$$

that comes from the variational equations

$$\begin{pmatrix} \dot{\theta} \\ \dot{u} \end{pmatrix} = \mathcal{M}(\theta_0, u_0) \begin{pmatrix} \theta \\ u \end{pmatrix}$$

associated with the linearization of the equations of motion in the vicinity of  $(\theta_0, u_0)$ . When this fixed point is elliptic, its eigenvalues are equal to  $\pm i\nu$ , where the real number  $\nu$  is the rotation frequency around the equilibrium. Moreover, the secular precession frequency of its corresponding orbits in the AP is equal to

$$(19) \quad g = -\frac{\partial}{\partial \Gamma} \overline{H}_{e_0}(\theta_0, u_0).$$

The evolution of the location and of the frequencies of the orbits associated with the families  $\mathcal{G}_{QS}^{e_0}$ ,  $\mathcal{G}_{L_3}^{e_0}$ ,  $\mathcal{G}_{L_4}^{e_0}$  and  $\mathcal{G}_{L_5}^{e_0}$  versus  $e_0$  are described in Fig.5 for a mass ratio equal to  $\varepsilon = 10^{-3}$  (a Sun-Jupiter like system).

The red curve close to  $(\theta, u) = (180^\circ, 0)$  represents the family  $\mathcal{G}_{L_3}^{e_0}$  while the two blue curves that start in  $L_4$  and  $L_5$  correspond to  $\mathcal{G}_{L_4}^{e_0}$  and  $\mathcal{G}_{L_5}^{e_0}$ . As described in section 3.3, by

<sup>6</sup>In practice, the matrix  $\mathcal{M}(\theta_0, u_0)$  is provided by a numerical differentiation of the equations of motion at the fixed point  $(\theta_0, u_0)$ .

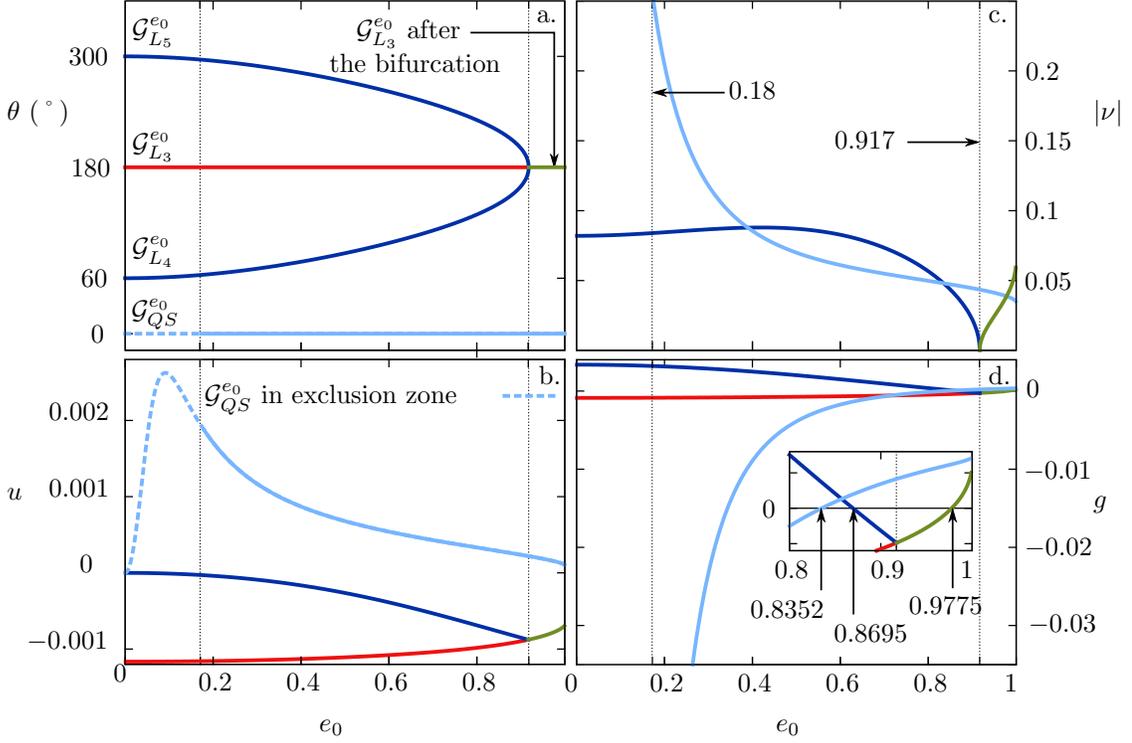


FIGURE 5. Location in  $\theta$  (a.) and  $u$  (b.) and frequencies  $|\nu|$  (c.) and  $g$  (d.) of the fixed points families for a Sun-Jupiter like system ( $\varepsilon = 10^{-3}$ ).  $\mathcal{F}_{L_4}$  and  $\mathcal{F}_{L_5}$  (blue curves) merge with  $\mathcal{G}_{L_3}^{e_0}$  (red curve) which gives rise to a stable family of fixed points (green curve). The AP is relevant for QS motion when  $\mathcal{G}_{QS}^{e_0}$  (sky blue curve) is a continuous curve. There are particular orbits without precession on each family which correspond to degenerated fixed points of the AP.

increasing  $e_0$  these two last families merge with  $\mathcal{G}_{L_3}^{e_0}$  for  $e_0 \simeq 0.917$  (vertical dashed line). Above this critical value, the last family becomes stable (green curves in Fig.5). The sky blue curve located nearby  $(\theta, u) = (0^\circ, 0)$  represents the family  $\mathcal{G}_{QS}^{e_0}$ . Along this family, for  $0.4 \leq e_0 < 1$ , the frequencies  $|\nu|$  and  $|g|$  are of the same order as those of the TP equilibria, but the sign of  $g$  is different. Then, by decreasing  $e_0$ , the moduli of the frequencies increase and tend to infinity. When the frequencies reach values of the same order or higher than the fast frequency,  $\mathcal{G}_{QS}^{e_0}$  enters the exclusion zone and the averaged problem does not describe accurately the quasi-satellite's motion. In order to estimate an eccentricity range where the averaged problem is adapted to QS motion, we consider that  $\mathcal{G}_{QS}^{e_0}$  is outside the exclusion zone when  $|g|$  and  $|\nu|$  are lower than  $1/4$ . Fig.5 shows that this quantity is given by  $e_0 = 0.18$  (vertical dashed line). Therefore,

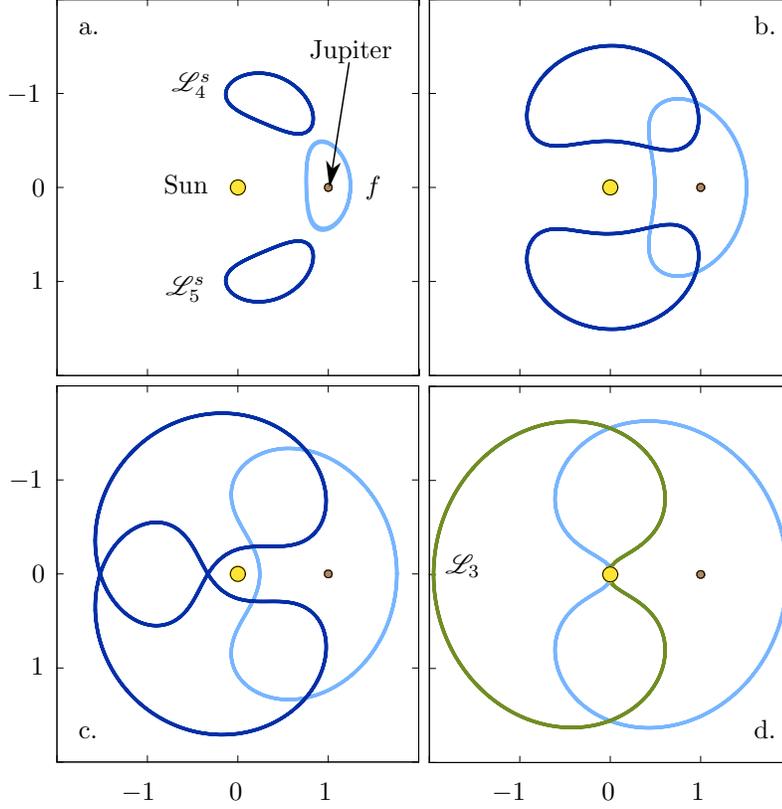


FIGURE 6. Periodic orbits in the rotating frame associated with stable orbits of  $\mathcal{G}_{L_4}^{e_0}$ ,  $\mathcal{G}_{L_5}^{e_0}$ ,  $\mathcal{G}_{L_3}^{e_0}$  and  $\mathcal{G}_{QS}^{e_0}$  for  $e_0 = 0.25$  (a.),  $0.5$  (b.),  $0.75$  (c.) and  $0.95$  (d.) (see Fig.2 b, c, d and f). The blue curves are associated with  $\mathcal{L}_4^s$ ; the sky blue curve with the family  $f$  and the green curve corresponds to  $\mathcal{L}_3$  after the bifurcation.

the AP and RAP are relevant to study  $\mathcal{G}_{QS}^{e_0}$  and thus the QS motion for  $e_0 \geq 0.18$  in the Sun-Jupiter system.

Now, we focus on the variations of  $g$  along each families of fixed points. For each of them, the frequency is monotonous and crosses zero for a critical value of eccentricity:  $e_0 \simeq 0.8352$  for  $\mathcal{G}_{QS}^{e_0}$ ,  $e_0 \simeq 0.8695$  for  $\mathcal{G}_{L_4}^{e_0}$  and  $\mathcal{G}_{L_5}^{e_0}$ , and  $e_0 \simeq 0.9775$  for  $\mathcal{G}_{L_3}^{e_0}$ . According to the section 3.2, these particular trajectories in the RAP correspond to circles of fixed points in the AP, and  $2\pi$ -periodic orbits in the RF, i.e. frozen ellipses in the heliocentric frame. We denote them  $G_{QS}$ ,  $G_{L_4}$ ,  $G_{L_5}$  and  $G_{L_3}$ .

To conclude this section, we connect the fixed points families in the RAP to the corresponding trajectory in the RF. Outside the exclusion zone, the transformation of these ones by  $\Phi \circ \hat{\psi} \circ \mathcal{C} \circ \psi_{e_0}^{-1}$  provides us a first order approximation of their initial conditions in the RF. Therefore, by improving them with an iterative algorithm that removes the frequency  $\nu$  (Couetdic et al., 2010), we integrated the corresponding trajectories in the

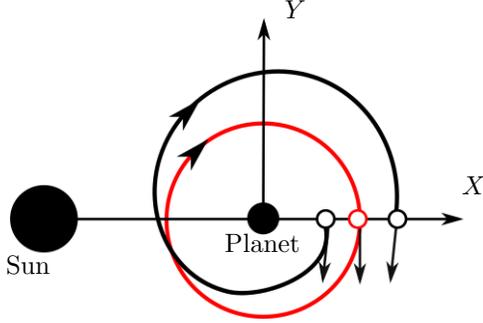


FIGURE 7. Representation of retrograde satellite trajectories in the RF. Red curve: simple-periodic symmetrical retrograde satellite trajectory that belongs to the family  $f$ . Black curve: trajectory in the neighbourhood of the previous one and that intersects the Poincaré section (black circles).

RF. An example of stable trajectories is represented on the figure 6 for several values of  $e_0$ .

For a Sun-Jupiter like system, the families  $\mathcal{G}_{L_4}^{e_0}$ ,  $\mathcal{G}_{L_5}^{e_0}$  provide the entire short period families, from their respective equilibrium to their merge with  $\mathcal{G}_{L_3}^{e_0}$  and its collision orbit with the Sun. On the contrary,  $\mathcal{G}_{Q_S}^{e_0}$  provide only a part of the family  $f$ , from the collision with the Sun to the orbit with an eccentricity  $e \simeq 0.18$ . The figure 6 shows that by increasing  $e_0$ , the size of the periodic trajectories in the RF increases. As expected, the libration center of the family  $f$  is located close to the planet, while those of  $\mathcal{L}_4^s$  and  $\mathcal{L}_5^s$  shift from  $L_4$  and  $L_5$  towards  $L_3$  where they merge with those of  $\mathcal{L}_3$ . After the bifurcation, only trajectories of the  $f$  and  $\mathcal{L}_3$  families remain.

#### 4. QUASI-SATELLITE'S DOMAINS IN THE ROTATING FRAME WITH THE PLANET

**4.1. The family  $f$  in the RF.** The RAP seems to be the most adapted approach to understand the co-orbital motion in the circular case. However, the averaged approaches have the drawback to be poorly significant in the exclusion zone that surround the singularity associated with the collision with the planet. For the QS motion, we showed in the section 3 that the whole domain could not be reachable by low eccentric orbits, that is when the trajectories get closer to the planet. As a consequence, to understand the QS nearby the planet and connect our results in the averaged approaches, we chose to revisit the classical works in the RF (Henon and Guyot, 1970; Benest, 1975) on the simple-periodic symmetrical family of retrograde satellite orbits, generally known as the family  $f$ .

In the RF with the planet on a circular orbit, the problem has two degrees of freedom that we represent by the position  $\mathbf{r} = (X, Y)$  and the velocity  $\dot{\mathbf{r}} = (\dot{X}, \dot{Y})$  in the frame whose origin is the planet position, the horizontal axis is the Sun-planet alignment and the vertical axis, its perpendicular (see Fig.7). This problem is autonomous and possesses a

first integral  $C_J$  generally known as the Jacobi constant.

For a given value of  $C_J$ , a simple-periodic symmetrical retrograde satellite orbit crosses the axes  $\{Y = 0\}$  with  $\dot{Y} < 0$  and  $\dot{X} = 0$  when  $X > 0$ . By defining the Poincaré map  $\Pi_T$  associated with the section  $\{Y = 0; \dot{Y} < 0\}$  where  $T$  is the time between two consecutive crossings, the problem could be reduced to one degree of freedom represented by  $(X, \dot{X})$  and  $\dot{Y} = \dot{Y}(X, Y, \dot{X}, C_J)$ . As a consequence, an orbit of the family  $f$  corresponds to a fixed point in this Poincaré section whose coordinates in the RF are  $(X, 0, 0, \dot{Y})$  with

$$(20) \quad T = 2\pi/(1 - g)$$

where  $g$  is the precession frequency of the periaster argument  $\omega$ .

Moreover the stability of the fixed point is deduced from the trace of the monodromy matrix  $d\Pi_T(X, 0)$  evaluated at the fixed point. When the fixed point is stable, the frequency  $\nu$  that characterized the oscillation of the resonant angle  $\theta$  is obtained<sup>7</sup> from its two conjugated eigenvalues  $(\kappa, \bar{\kappa})$  such as

$$(21) \quad \kappa = \exp(i\nu T).$$

**4.2. Application to a Sun-Jupiter like system.** The figure 8 and 9a represent the family  $f$  in the  $(X, \dot{Y})$  plane (red curve) and its reachable part in the averaged approaches (sky blue curve).

Fig.8 shows that the family  $f$  extends from the orbits in an infinitesimal neighbourhood of the planet ( $X \simeq 0$ ) to the collision orbit with the Sun. Although, the whole family is linearly stable, we cannot predict the size of the stable region surrounding it. Indeed, this domain depends strongly on the position of the resonances between the fundamental frequencies  $1 - g$  and  $\nu$ , which are themselves conditioned by the value of  $X$ . This is what occurs in particular orbits of the family  $f$  (blue crosses and dashed lines) where the stability domain's diameter tends to zero. Consequently, these two orbits divide the neighbourhood of the family  $f$  in three connected domains that we outlined in grey in Fig.8 and Fig.9a.

The figure 9b exhibits the variations of the frequencies<sup>8</sup>  $\nu$  and  $1 - g$ . Comparing to Fig.5b, we remark that  $\nu$  does not tend to infinity when the periodic orbits get closer to the planet, but increases and tends to 1. Likewise, Fig.9b highlights that the resonance between the frequencies of the system is  $\nu/(1 - g) = 1/3$  and that the three domains are neatly defined in terms of frequencies as follows:

$$\text{sRS} : \begin{cases} 3\nu < 1 - g \\ |g| > 1 \end{cases}, \quad \text{QS}_b : \quad 3\nu > 1 - g \quad \text{and} \quad \text{QS}_h : \begin{cases} 3\nu < 1 - g \\ |g| < 1 \end{cases}.$$

The closest domain to the planet corresponds to the ‘‘satellized’’ retrograde satellite orbits (sRS). Indeed, as the upper bound of this domain matches with the  $L_2$  position, we recovered the notion of Hill’s sphere in the context of the retrograde satellite trajectories. Hence, this domain consists of trajectories dominated by the gravitational influence of the planet

<sup>7</sup>Floquet theory; for further details, see Meyer and Hall (1992).

<sup>8</sup>In practice, the numerical algorithm of the Poincaré map provides  $g$  as in the equation (20) while the frequency  $\nu$  is obtained via the monodromy matrix  $d\Pi_T$  (see equation (21)) that is calculated with a numerical differentiation algorithm on the Poincaré map.

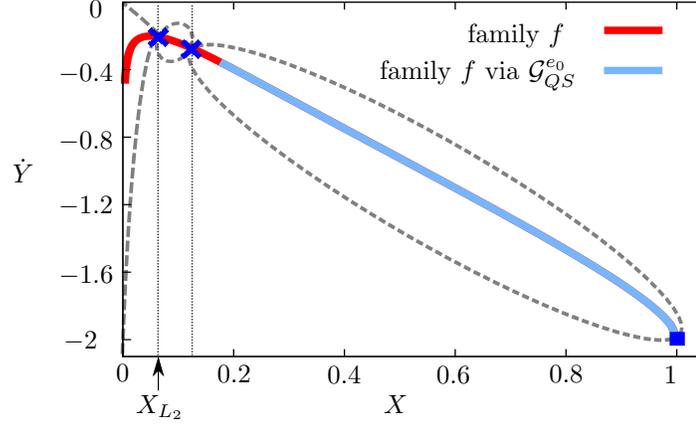


FIGURE 8. The family  $f$  in the  $(X, \dot{Y})$  plane (red curve) and its reachable part in the AP via  $\mathcal{G}_{QS}^{eo}$  (sky blue curve). The two blue crosses indicate the particular orbits (whose fundamental frequencies are in  $1 : 3$  resonance) that split the neighbourhood of the family  $f$ . The blue square indicates the collision orbit with the Sun while  $\{X = 0\}$  corresponds to the collision with the planet. The grey outline schematizes the three connected domains of the family  $f$  neighbourhood.

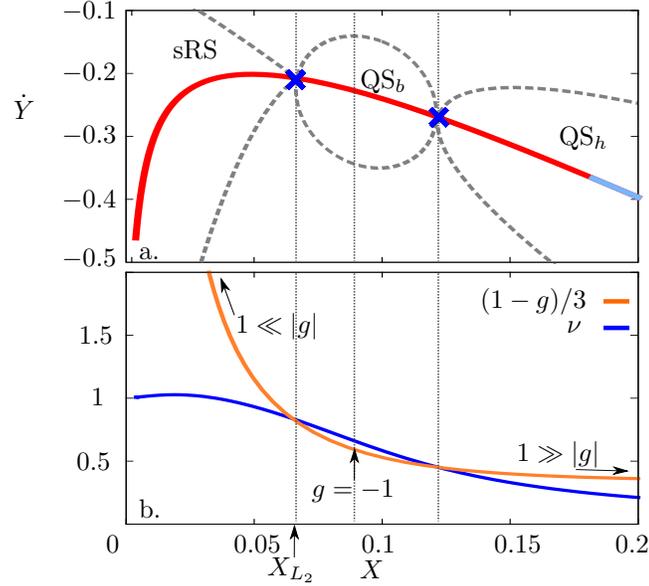


FIGURE 9. (a.) Zoom in of Fig.8 on the two particular orbits whose fundamental frequencies are in  $1 : 3$  resonance. (b.) Variation of the frequencies of the system along the family  $f$ . Comparing to Fig.5b,  $\nu$  does not tend to infinity when the periodic orbits get closer to the planet ( $\{X = 0\}$ ), but increases and tends to 1. The  $1 : 3$  resonance splits the neighbourhood in three domains neatly defined in terms of frequencies: “satellized” retrograde satellite (sRS), binary quasi-satellite ( $QS_b$ ) and heliocentric quasi-satellite ( $QS_h$ )

whereas the star acts as a perturbator. Therefore the planetocentric osculating ellipses are the most relevant variables to represent the motion and perturbative treatments are possible.

The domain outside the Hill's sphere corresponds to the QS that is divided in two others domains.

The domain of  $QS_h$  orbits, that is the heliocentric QS, corresponds to the farthest domain to the planet, which implies that this body acts as a perturbator whereas the influence of the star dominates the dynamics. Therefore, the heliocentric orbital elements are well suited to the problem, and the perturbative treatment as well as the averaging over the fast angle are natural. As a consequence, it is the  $QS_h$  trajectories that are reachable in the averaged problem. As the orbits of the family  $f$  included in the  $QS_h$  domain cross the Poincaré section at their aphelion, the  $X$  coordinates is related to  $e_0$  by the expression

$$(22) \quad X = e = e_0 + \mathcal{O}(\varepsilon).$$

The third domain, that we called the binary QS domain ( $QS_b$ ), is intermediate between the sRS and  $QS_h$  ones. In this region, none of the two massive bodies has a dominant influence on the massless one. As a consequence, the frequencies  $g$  could be of the same order or even equal to 1, making inappropriate any method of averaging.

Remark that in the planetary problem, Hadjidemetriou and Voyatzis (2011) highlight a family of periodic orbits that corresponds to the family  $f$ . Indeed, along this family that ranges from orbits for which the two planets collide with the star to the orbits where the two planets are mutually satellized, all trajectories are stable and satisfy  $\theta = 0^\circ$ . These authors also decomposed the family in three domains, denoted  $A$ ,  $B$  and planetary, which seem to correspond to our sRS,  $QS_b$  and  $QS_h$  domains.

**4.3. Extension to arbitrary mass ratio.** By varying the mass ratio  $\varepsilon$ , we follow the evolution of the boundaries of the three domains along the family  $f$  as well as the validity limit of the averaged problem. In Fig.10, the parameter  $\varepsilon$  ranges from  $10^{-7}$  to 0.0477 which is the critical mass ratio where a part of the family  $f$  becomes unstable (see Henon and Guyot, 1970). For Sun-terrestrial planet systems, the size of the  $QS_b$  and sRS domains is negligible with respect to the  $QS_h$  one. As a consequence, for these systems, the AP and RAP are fully adapted to describe the main part of the family  $f$  and its neighbourhood (except for very small eccentricities). For Sun-giant planet systems as well as the Earth-Moon system, the gravitational influence of the planet being stronger, the size of the  $QS_b$  domain  $f$  increases until to be of the same order than those of the sRS one while the size of the  $QS_h$  decreases. By the equation (22), we established that for the Sun-Uranus, Sun-Saturn, Sun-Jupiter and Earth-Moon systems, the  $QS_h$  orbits are reachable in the averaged problem by  $e_0$  greater than 0.08, 0.13, 0.18 and 0.5. Then, by increasing  $\varepsilon$ , the  $QS_b$  domain becomes dominant while the  $QS_h$  one is reduced so much that the averaged problem becomes useless for all values of  $e_0$  ( $\varepsilon \simeq 0.04$ ). Consequently, for the Pluto-Charon system ( $\varepsilon \simeq 1/10$ ) none  $QS_h$  trajectory could be described in the averaged approaches. Moreover, according to the stability map of the family  $f$  in Benest (1975),

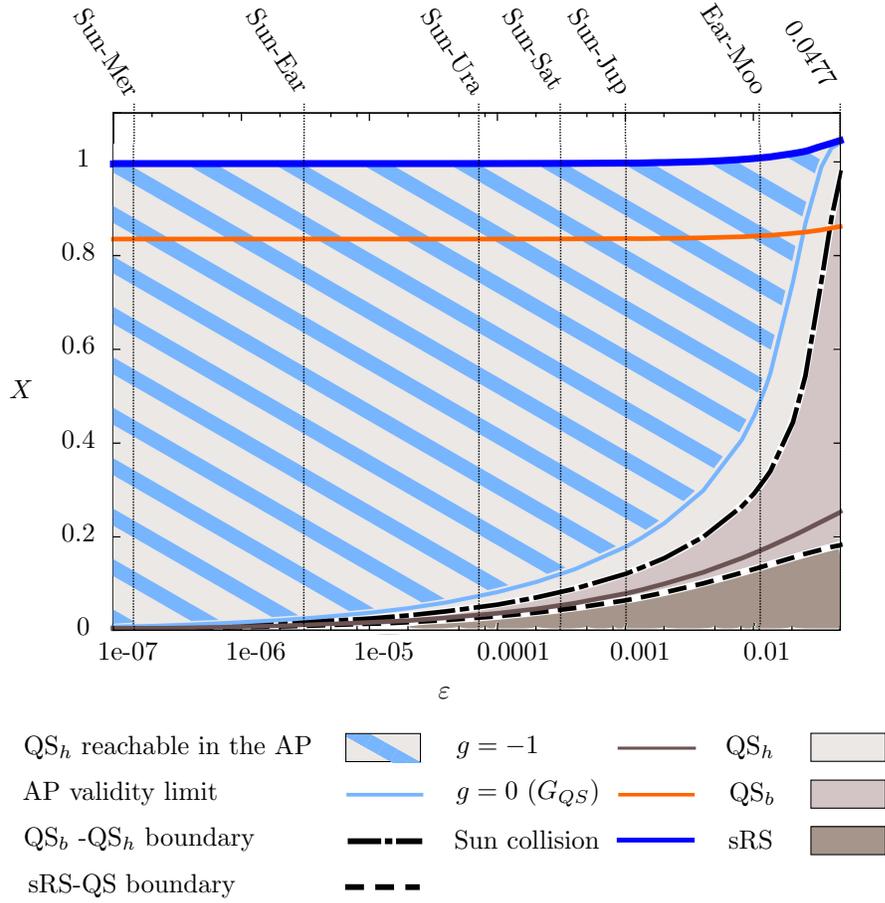


FIGURE 10. Evolution of the sRS,  $QS_b$  and  $QS_h$  boundaries along the family  $f$  by varying the mass ratio  $\varepsilon$ . For small  $\varepsilon$ , the  $QS_h$  domain dominates the family  $f$  implying that the AP and the RAP are fully adapted to study the QS motion. By increasing  $\varepsilon$ , the size of the part associated with the sRS and the  $QS_b$  trajectories increases making not reachable the orbits with small eccentricities in the averaged problem. Eventually, for  $\varepsilon > 0.01$ , the sRS and the  $QS_b$  domains become dominant while the  $QS_h$  one is reduced so much that the averaged problem becomes useless for all values of  $e_0$ .

this system could not harbour a  $QS_h$  companion: only  $QS_b$  and sRS trajectories exist for this value of mass ratio.

5. ON THE FROZEN ELLIPSES: AN EXTENSION TO THE ECCENTRIC CASE ( $e' \geq 0$ )

An important result of our study in the circular case has been to highlight the particular orbits  $G_{QS}$ ,  $G_{L_3}$ ,  $G_{L_4}$  and  $G_{L_5}$ , that correspond<sup>9</sup> to circles of fixed points in the averaged problem and therefore frozen ellipses in the heliocentric frame.

A natural question is to know if these structures are preserved when a small eccentricity is given to the planetary orbit. This question can be addressed in a perturbative way. Indeed, for sufficiently low values of planet's eccentricity, the Hamiltonian of the problem reads  $\bar{H}|_{e'=0} + e'R$ , i.e. the perturbation of the Hamiltonian in the circular case by the first order term in planetary eccentricity. However, as  $\omega = \arg(x)$  is no longer an ignorable variable in this Hamiltonian, the dimension of the phase space could not be reduced as in the section 3 and the persistence of the set of degenerated fixed points is not necessary guaranteed.

In the present paper, we limit our approach to numerical explorations of the phase space associated with  $\bar{H}|_{e' \geq 0}$ . For a very low value of  $e'$  in a Sun-Jupiter like system, the (numerical) solving of the equations of motion in the averaged problem,

$$(23) \quad \begin{cases} \frac{\partial}{\partial \theta} \bar{H}(\theta, u, -i\bar{x}, x) = 0 \\ \frac{\partial}{\partial u} \bar{H}(\theta, u, -i\bar{x}, x) = 0, \\ \frac{\partial}{\partial x} \bar{H}(\theta, u, -i\bar{x}, x) = 0 \end{cases}$$

shows that although each circle of fixed points is destroyed, two fixed points survived to the perturbation. One is stable and the other unstable. We denote these fixed points  $G_{X,1}^{e'}$  and  $G_{X,2}^{e'}$  with  $X$  corresponding to QS,  $L_4$ ,  $L_5$  and  $L_3$ . By varying  $e'$ , we followed them and show families of fixed points of the averaged problem that originate from  $G_{L_3}$ ,  $G_{L_4}$ ,  $G_{L_5}$  and  $G_{QS}$ .

For a given  $e'$  in the AP, the linear dynamics in the vicinity of a fixed point is given by two couples of eigenvalues:  $\pm\mu$  or  $\pm i\nu$  and  $\pm f$  or  $\pm ig$  where  $\mu$ ,  $f$ ,  $\nu$  and  $g$  are real. If these eigenvalues are all imaginary then they characterized an elliptic fixed point with libration and secular precession frequencies  $\nu$  and  $g$ . Otherwise, the fixed point is unstable. Thus, we also characterized the stability variations of these families of fixed points by varying  $e'$ . Their initial conditions and the moduli of the real and imaginary part of the eigenvalues versus  $e'$  are plotted on Fig.11, 12 and 13.

According to the figures 11, 12 and 13, we find eight families of fixed points in the averaged problem that correspond to frozen ellipses in the heliocentric frame. For  $e' = 0$ , these equilibria of the averaged problem belong to the set of degenerated fixed points or "circles of fixed points" that exist for  $\omega \in \mathbb{T}$  and that we denoted  $G_{L_3}$ ,  $G_{L_4}$ ,  $G_{L_5}$  and  $G_{QS}$ .

Among these eight families of fixed points, two are more relevant:  $G_{QS,1}^{e'}$  and  $G_{L_3,1}^{e'}$ . The fixed points of  $G_{QS,1}^{e'}$  originates from  $G_{QS}$  and are stable until  $e' \simeq 0.8$ . It corresponds to a

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<sup>9</sup>See the figure 5.

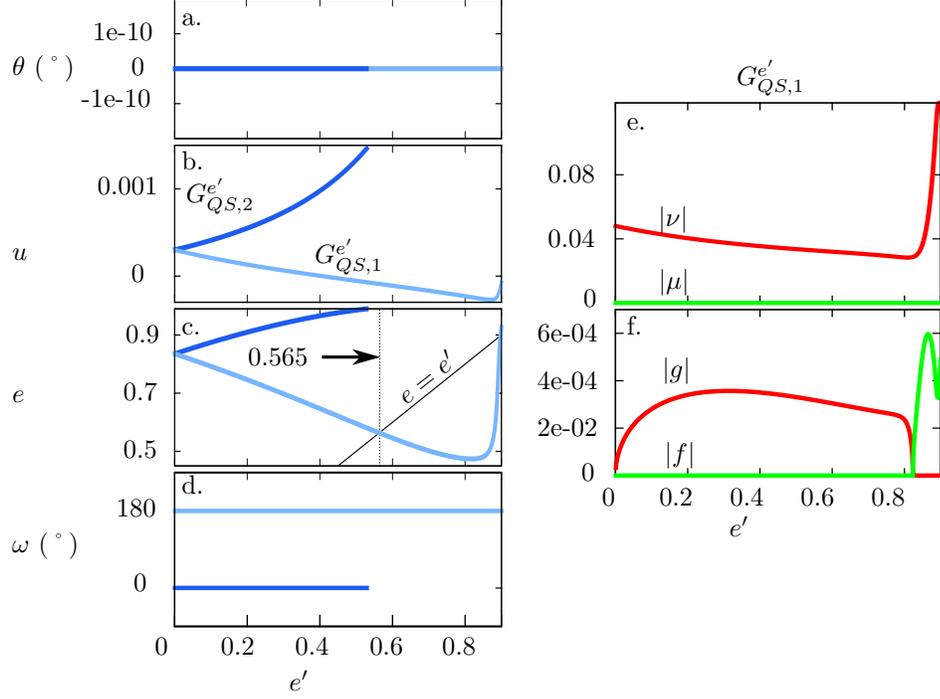


FIGURE 11. a, b, c and d: orbital elements of the families of fixed points  $G_{QS,2}^{e'}$  and  $G_{QS,1}^{e'}$  versus  $e'$ . e and f: variations of the moduli of the real and imaginary part of the eigenvalues of the Hessian matrix along  $G_{QS,1}^{e'}$ .  $G_{QS,1}^{e'}$  describe a configuration of two ellipses with opposite periaster and  $\theta = 0^\circ$ . The fixed points are stable until  $e' < 0.8$ . Moreover, this family possesses a particular orbit where  $e = e' \simeq 0.565$ . On the contrary, the family  $G_{QS,2}^{e'}$  is unstable and describe a configuration of two ellipses with aligned periaster and  $\theta = 0^\circ$ .

configuration of two ellipses with two opposite periaster ( $\omega = 180^\circ$ ),  $\theta = 0^\circ$  and a very high eccentricity that decreases when  $e'$  increases (the slope being close to  $\frac{de}{de'} = -1/2$ ). On the contrary, the fixed points of  $G_{L3,1}^{e'}$  originates from  $G_{L3}^{e_0}$  and is only stable for  $0 \leq e' \leq 0.15$ . It describes a configuration of two ellipses with aligned periaster ( $\omega = 0^\circ$ ),  $\theta = 180^\circ$  and a very high eccentricity that decreases when  $e'$  increases.

Along these two families, there exists a critical value of  $e'$  where the planet and the asteroid ellipses have the same eccentricities. The dashed lines of the figures 11 and 12 show that these particular orbits exist for  $e' = e \simeq 0.565$  along  $G_{QS,1}^{e'}$  and  $e' = e \simeq 0.73$  along  $G_{L3,1}^{e'}$ .

Let us notice that these two families of configurations have been highlighted in the planetary problem. Indeed, these two families have certainly to do with the stable and unstable families of periodic orbits described in Hadjidemetriou et al. (2009) and Hadjidemetriou and Voyatzis (2011). As regard  $G_{QS,1}^{e'}$ , it could also be associated with the QS fixed point

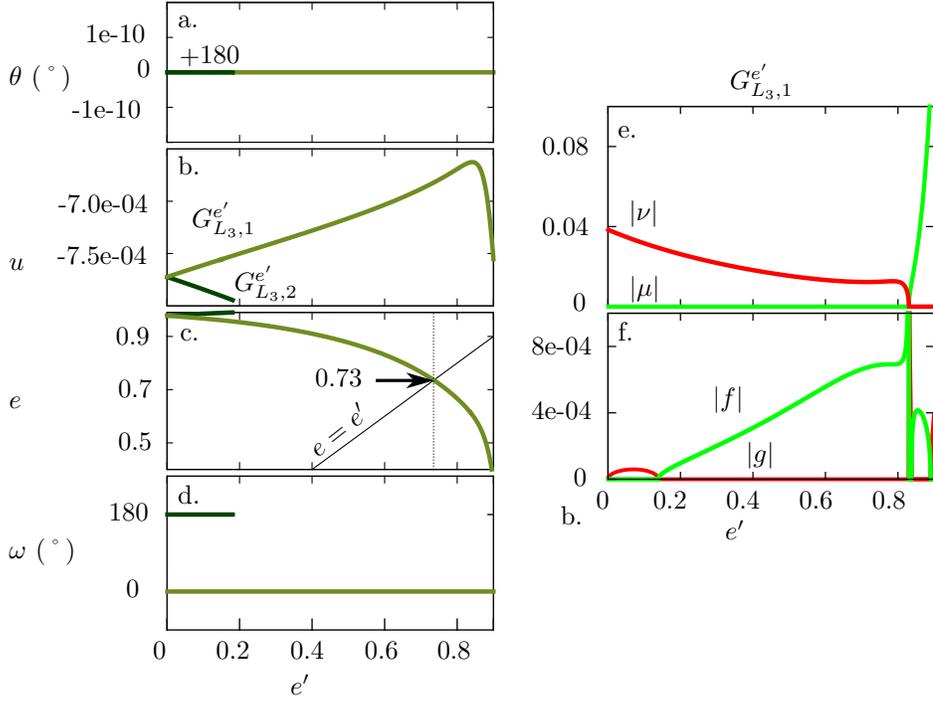


FIGURE 12. a, b, c and d: orbital elements of the families of fixed points  $G_{L_{3,2}}^{e'}$  and  $G_{L_{3,1}}^{e'}$  versus  $e'$ . e and f: variations of the moduli of the real and imaginary part of the eigenvalues of the Hessian matrix along  $G_{L_{3,1}}^{e'}$ .  $G_{L_{3,1}}^{e'}$  describe a configuration of two ellipses with aligned periastron and  $\theta = 180^\circ$ . The fixed points are stable for  $0 \leq e' \leq 0.15$ . Moreover, this family possesses a particular orbit with  $e' = e \simeq 0.73$  where the planet and the particle share the same ellipses. On the contrary,  $G_{L_{3,2}}^{e'}$  is unstable and describe a configuration of two ellipses with opposite periastron and  $\theta = 180^\circ$ .

family in Giuppone et al. (2010). In Giuppone et al. (2010) as well as in Hadjidemetriou and Voyatzis (2011), these authors remarked that the configuration described by  $G_{QS,1}^{e'}$  with two equal eccentricities exists with an eccentricity value close to 0.565 for several planetary mass ratio. In our study, we establish that this particular orbit also exists in the restricted three-body problem for  $e = e' \simeq 0.565$  (see Fig.11). Likewise, according to Hadjidemetriou et al. (2009), the configuration described by  $G_{L_{3,1}}^{e'}$  with two equal eccentricities seems to exist in the planetary problem for an eccentricity value close to 0.73. Consequently, this suggests that these two particular configurations are weakly dependent on the ratio of the planetary masses.

Eventually, we remark that the existence of some of these eight configurations has already been showed. Indeed, in the range  $0.01 \leq e' \leq 0.5$ , Nesvorný et al. (2002) exhibit QS stable and unstable fixed points. In addition, these authors also shown very high eccentric fixed

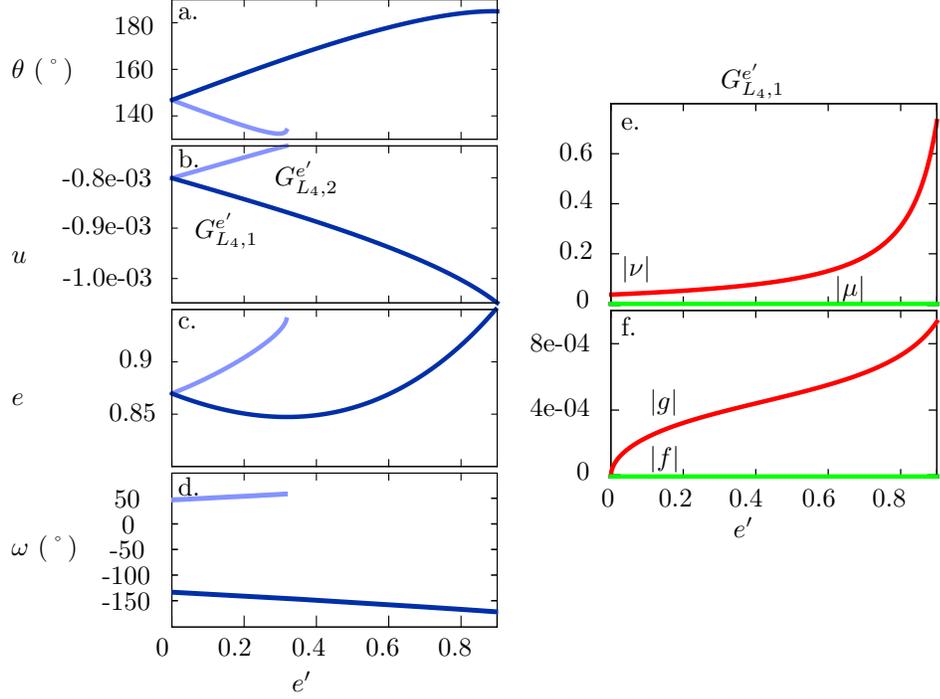


FIGURE 13. a, b, c and d: orbital elements of the families of fixed points  $G_{L4,2}^{e'}$  and  $G_{L4,1}^{e'}$  versus  $e'$ . e and f: variations of the moduli of the real and imaginary part of the eigenvalues of the Hessian matrix along  $G_{L4,1}^{e'}$ . The whole family  $G_{L4,1}^{e'}$  is stable whereas  $G_{L4,2}^{e'}$  is unstable.

points that correspond to the configuration of  $G_{L4,1}^{e'}$  and  $G_{L4,2}^{e'}$ .

Likewise, Bien (1978) and Edelman (1985) highlighted some frozen ellipses in co-orbital motion in the Sun-Jupiter system with  $e' = e'_{Jupiter} \simeq 0.048$ . The first author found six very high eccentric fixed points denoted  $P_1, Q_1, P_2, Q_2, P_3, Q_3$  that correspond to  $G_{L4,1}^{e'}, G_{L4,2}^{e'}, G_{L5,1}^{e'}, G_{L5,2}^{e'}, G_{QS,1}^{e'}$  and  $G_{QS,2}^{e'}$ . The other found a frozen ellipse in co-orbital resonance with  $e = 0.975$  and  $\theta = 180^\circ$ , that is an orbit of  $G_{L3,1}^{e'}$ .

## 6. CONCLUSIONS

In this paper, we clarify the definition of quasi-satellite motion and estimate a validity limit of the averaged approach by revisiting the planar and circular restricted three-body problem .

First of all, we focussed on the co-orbital resonance via the averaged problem and showed that the studies of the phase portraits of the reduced averaged problem parametrized by  $e_0$  allow to understand its global dynamics. Indeed, they reveal that tadpole, horseshoe

and quasi-satellite domains are structured around four families of fixed points originating from  $L_4$ ,  $L_5$  ( $\mathcal{G}_{L_4}^{e_0}$  and  $\mathcal{G}_{L_5}^{e_0}$ ),  $L_3$  ( $\mathcal{G}_{L_3}^{e_0}$ ) and the singularity point for  $e_0 = 0$  ( $\mathcal{G}_{QS}^{e_0}$ ). By increasing  $e_0$ , the quasi-satellite orbits appear inside the domain opened by the collision curve for  $e_0 > 0$  and becomes dominant for high eccentricities. On the contrary, tadpole and horseshoe domains shrink and vanish when  $\mathcal{G}_{L_4}^{e_0}$  and  $\mathcal{G}_{L_5}^{e_0}$  get closer and merge  $\mathcal{G}_{L_3}^{e_0}$ . Moreover, we showed that this remaining family bifurcates and generates a new domain of high eccentric orbits librating around  $(\theta, u) = (180^\circ, 0)$ .

However, the averaged approaches having the drawback to be poorly significant in the exclusion zone, we highlighted that for sufficiently small eccentricities, the whole quasi-satellite domain is contained inside it which makes this type of motion not reachable by averaging process. The study of the evolution of the libration and secular precession frequencies along  $\mathcal{G}_{QS}^{e_0}$ , allowed us to show that the family  $f$  and a fortiori the quasi-satellite domain are not reachable by  $0 \leq e_0 < 0.18$  in a Sun-Jupiter like system.

In order to clarify the terminology to use between “retrograde satellite” or “quasi-satellite” when these orbits are close encountering trajectories with the planet, we revisited the works in the rotating frame on the family of simple-periodic symmetrical retrograde satellite orbits, or family  $f$ .

We highlighted that the family  $f$  possesses two particular orbits that divide its neighbourhood in three connected areas: “satellized” retrograde satellite, binary quasi-satellite and heliocentric quasi-satellite domains. We established that the last one is the only one reachable in the averaged approaches.

The study of the frequencies of the fixed point families of the reduced averaged problem has also shown some frozen ellipses in the heliocentric frame which are equivalent to sets of degenerated fixed points (also denoted “circles of fixed points”) in the averaged problem. In order to exhibit fixed points when the planet’s orbit is eccentric, we highlighted numerically that from each circles of fixed points originates at least two families of fixed points parametrized by the planet eccentricity. Among them,  $G_{QS,1}^{e'}$  is the most interesting as it is in quasi-satellite motion with a configuration of two ellipses with opposite perihelia and connected to the stable family described in Hadjidemetriou et al. (2009) in the planetary problem. Moreover,  $G_{QS,1}^{e'}$  as well as the family in the planetary problem possess a configuration with equal eccentricities for any mass ratio with an eccentricity value close to 0.565. As a consequence, this suggests that this remarkable configuration is weakly dependent to the ratio between the planetary masses. Likewise, let us mentioned that this configuration is similar to those of the family “A.1/1” described in Broucke (1975) in the general three-body problem with three equal masses which suggests a connection between them.

When  $e_0 > 0.4$ , we denoted that the moduli of the libration frequency  $\nu$  and of the secular precession frequency  $g$  along the family  $f$  are of the same order than those of the two tadpole periodic orbit families. Thus, in the framework of long-term dynamics of the Jovian quasi-satellite asteroids in the solar system, we can assume that a study of the global dynamics by means of the frequency map analysis will reveal resonant structures close to those of the trojans identified in Robutel and Gabern (2006). However, by remarking

that the direction of the perihelion precession being the opposite of those of the planets in the solar system, resonances with these secular frequencies should be of higher order in comparison to the tadpoles orbits. On the contrary, resonances with their node precession should be of lower order. These questions will be addressed in a forthcoming work.

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