An elliptic triptych

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ABSTRACT: We clarify three aspects of non-compact elliptic genera. Firstly, we give a path integral derivation of the elliptic genus of the cigar conformal field theory from its non-linear sigma-model description. The result is a manifestly modular sum over a lattice. Secondly, we discuss supersymmetric quantum mechanics with a continuous spectrum. We regulate the theory and analyze the dependence on the temperature of the trace weighted by the fermion number. The dependence is dictated by the regulator. From a detailed analysis of the dependence on the infrared boundary conditions, we argue that in non-compact elliptic genera right-moving supersymmetry combined with modular covariance is anomalous. Thirdly, we further clarify the relation between the flat space elliptic genus and the infinite level limit of the cigar elliptic genus.

Contents

1 Introduction

Mock modular forms have an illustrious history in mathematics [1]. However, a systematic understanding of mock modular forms is recent [2] and evolving. Mock modular forms also appeared in physics in various guises [3–5]. A natural habitat for mock modular forms and their non-holomorphic modular completion was provided by the demonstration that they arise as elliptic genera of two-dimensional superconformal field theories with continuous spectrum [6]. As such the completed forms appear also as duality covariant counterparts to black hole entropy counting functions [7].

In this paper, we wish to clarify three aspects of non-compact elliptic genera. The first comment we make is on the compact form of the elliptic genus of the cigar derived

by Eguchi and Sugawara in [8]. It is a modular covariant sum over lattice points which is an exponentially regulated Eisenstein series. Since it is manifestly modular covariant, one can wonder whether it has a simple direct path integral derivation. We demonstrate that a path integration of the non-linear sigma-model description of the cigar provides such a derivation. The second remark, in section 3, is based on an analysis of the temperature dependence of the weighted trace $Tr(-1)^F e^{-\beta H}$ in supersymmetric quantum mechanics with a continuous spectrum. Upon regularization, the trace becomes β-dependent in a manner that hinges upon the choice of regulator. We demonstrate this in detail, analyze the supersymmetric regulator and its path integral incarnation, and the role of infrared boundary conditions. We use it to lay bare the unresolvable tension between right-moving supersymmetry and modularity in the non-compact elliptic genus. In a third and final part, we clarify the relation between the flat space superconformal field theory and the infinite level limit of the cigar conformal field theory using their elliptic genera.

2 The path integral lattice sum

In this section, we wish to obtain a simpler path integral understanding of the compact formula for the elliptic genus of the cigar in terms of a lattice sum, derived in [8]. To that end, we provide a new derivation of the elliptic genus of the cigar, through its supersymmetric non-linear sigma-model description. The latter has the advantage of being parameterized in terms of the physical degrees of freedom only.

2.1 The guises of the genus

The cigar elliptic genus

$$\chi_{\text{cig}}(\tau, \alpha) = Tr_{RR}(-1)^{F_L + F_R} e^{2\pi i \alpha Q} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \tag{2.1}$$

is a partition sum in the Ramond-Ramond sector, weighted by left- and right-moving fermion numbers $F_{L,R}$, as well as twisted by the left-moving R-charge Q. It was computed manifestly covariantly through a path integral over maps from the torus into the coset $\text{SL}(2,\mathbb{R})/\text{U}(1)$ target space [6]. The result obtained in [6, 9, 10] was

$$\chi_{\text{cig}}(\tau, \alpha) = k \int_0^1 ds_{1,2} \sum_{m,w \in \mathbb{Z}} \frac{\theta_1(s_1\tau + s_2 - \alpha\frac{k+1}{k}, \tau)}{\theta_1(s_1\tau + s_2 - \frac{\alpha}{k}, \tau)} e^{2\pi i \alpha w} e^{-\frac{k\pi}{\tau_2}|(m+s_2)+(w+s_1)\tau|^2}, \tag{2.2}$$

where the θ_1 functions arise from partition functions of fermions and bosons with twisted boundary conditions on the torus, the integers m, w are winding numbers for the maps from the torus onto the target space angular direction, and the angles $s_{1,2}$ are holonomies on the torus for the U(1) gauge field used to gauge an elliptic isometry of $\text{SL}(2,\mathbb{R})$. The twist with respect to the left-moving R-charge is given by α. This modular Lagrangian result was put into a Hamiltonian form in which the elliptic genus could be read directly as a sum over right-moving ground states plus an integral over the differences of spectral densities for the continuous spectrum of bosonic and fermionic right-movers [6, 10]. The difference of spectral densities is determined by the asymptotic supercharge [6, 11, 12].

In [8], a rewriting of the result (2.2) in terms of a lattice sum was obtained. The resulting expression for the cigar elliptic genus is

$$\chi_{\text{cig}}(\tau, \alpha) = \frac{\theta_1(\alpha, \tau)}{2\pi\eta^3} \sum_{m,w\in\mathbb{Z}} \frac{e^{-\frac{\pi}{k\tau_2}(\alpha^2 + |m-w\tau|^2 + 2\alpha(m-w\bar\tau))}}{\alpha + m - w\tau} \,. \tag{2.3}$$

This expression is also manifestly modular covariant, because it is written as a sum over a lattice $\mathbb{Z} + \mathbb{Z}\tau$. Our goal in this section is to understand the formula (2.3) in a more direct manner than through the route laid out in [6, 8–10]. We recall that a key step in the derivation of the lattice sum (2.3) was to first compute the elliptic genus of the infinite cover of the $\mathbb{Z}_k$ orbifold of the trumpet geometry [8, 13].

2.2 The infinite cover of the orbifolded trumpet

We start our calculation from the cigar geometry [14–16]

$$ds^2 = \alpha' k(d\rho^2 + \tanh^2\rho\, d\theta^2)$$
$$e^\Phi = e^{\Phi_0}/\cosh\rho\,, \tag{2.4}$$

where the angle θ is identified modulo 2π. The range of the radial coordinate ρ is from 0 to ∞. The metric and dilaton determine the couplings of a conformal two-dimensional non-linear sigma-model. The T-dual geometry is the $\mathbb{Z}_k$ orbifold of the trumpet:

$$ds^2 = \alpha'\left(kd\rho^2 + \frac{1}{k}\coth^2\rho\, d\theta^2\right)$$
$$e^\Phi = e^{\Phi_0}/\sinh\rho \tag{2.5}$$

where the angle θ is again identified modulo 2π. The trumpet geometry is singular at the rim of the horn, at $\rho = 0$. The infinite cover of the orbifold of the trumpet is the geometry in which we no longer impose any equivalence relation on the variable θ.

We perform the path integral on the cover as follows. Firstly, we consider the integral over the zero modes and the oscillator modes separately. We suppose that the oscillator contribution on the left is proportional to the free field result

$$Z_{\text{osc}}^\infty = \frac{1}{4\pi^2\tau_2}\frac{\theta_1(\alpha, \tau)}{\eta^3}\,, \tag{2.6}$$

for a left-moving fermion of R-charge 1 and two uncharged bosonic fields. The factor $1/(4\pi^2\tau_2)$ has two sources. One can be viewed as the result of the space-time covariant integral over the radial momentum (at $\alpha' = 1$) while the second is the proper normalization of the θ zero mode volume integral (to be performed shortly). The right-moving oscillators cancel among each other.

We want to focus on the remaining integral over zero modes, which contains the crucial information on the modularly completed Appell-Lerch sum [2]. The left-moving fermionic zero modes have been lifted by the R-charge twist. Thus, we can concentrate on the integration over the bosonic zero modes as well as the right-moving fermionic zero modes, with measure

$$d\rho\, d\theta\, d\tilde\psi^\rho d\tilde\psi^\theta\,. \tag{2.7}$$

The square root of the determinant in the diffeomorphism invariant measures has canceled between the bosons and the fermions. The relevant action is the $N = (1, 1)$ supersymmetric extension of the non-linear sigma-model on the curved target space.[1] The term in the action that lifts the right moving fermion zero modes is [17]

$$S_{\text{lift}} = \frac{1}{4\pi} \int d^2 z \, G_{\mu\nu} \tilde{\psi}^\mu \Gamma^\nu_{\rho\sigma} \partial X^\rho \tilde{\psi}^\sigma \tag{2.8}$$

and more specifically, the term proportional to the Christoffel connection symbols

$$\Gamma_{\theta\theta\rho} = -\Gamma_{\rho\theta\theta} = \frac{1}{2} \partial_\rho G_{\theta\theta} . \tag{2.9}$$

This leads to a term in the action equal to

$$S_{\text{lift}} = \frac{1}{4\pi} \int d^2 z \, \tilde{\psi}^\theta \tilde{\psi}^\rho \partial_\rho G_{\theta\theta} \partial\theta . \tag{2.10}$$

We can descend this term once from the exponential in order to absorb the right-moving zero modes and obtain a non-zero result.

We wish to introduce a twist in the worldsheet time direction for the target space angular direction θ because we insert a R-charge twist operator in the elliptic genus, and the field θ is charged under the R-symmetry [6, 8–10]. We thus must twist

$$\theta(\sigma_1 + 2\pi\tau_1, \sigma_2 + 2\pi\tau_2) = \theta(\sigma_1, \sigma_2) + 2\pi\alpha , \tag{2.11}$$

and we still have $\theta(\sigma_1 + 2\pi) = \theta(\sigma_1)$. Since we study the infinite cover of the $\mathbb{Z}_k$ orbifold of the trumpet, there are no winding sectors. We thus obtain the classical configuration

$$\theta_{cl} = \sigma_2 \alpha / \tau_2 . \tag{2.12}$$

We plug this classical solution (2.12) into the action for the infinite order orbifold of the trumpet, and descend a single insertion of (2.10) to lift the right-moving zero mode, use the Christoffel connection (2.9) and then find the zero mode integral

$$\begin{aligned} Z_0^\infty &= 2\pi N_\infty \int_0^\infty d\rho \, \alpha \partial_\rho \left(-\frac{\pi}{k} \coth^2 \rho \right) e^{-\frac{\pi\alpha^2}{k\tau_2} \coth^2 \rho} \\ &= 2\pi N_\infty \frac{\tau_2}{\alpha} e^{-\frac{\pi\alpha^2}{k\tau_2}} . \end{aligned} \tag{2.13}$$

We have represented the integral over the variable θ by a factor of $2\pi N_\infty$ where we think of N_∞ as the order of the cover, which goes to infinity. The singular geometry at $\rho = 0$ makes for a vanishing contribution from this region. Putting this together with the oscillator factor (2.6) we proposed previously, we find

$$Z^\infty = N_\infty \frac{\theta_1(\alpha, \tau)}{\eta^3} \frac{1}{2\pi\alpha} e^{-\frac{\pi\alpha^2}{k\tau_2}} . \tag{2.14}$$

This precisely agrees with the elliptic genus of the infinite cover of the orbifolded trumpet calculated in [8].[2]

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[1] See e.g. formula (12.3.27) in [17].
[2] The factor N_∞ is absorbed in the definition of Z^∞ in [8, 13].

2.3 The lattice sum

Our next step is the path integral incarnation of the procedure of the derivation of the lattice sum formula in [8]. We undo the infinite order orbifold of the cigar, i.e. we undo the infinite order cover of the orbifolded trumpet. This will reproduce the lattice sum elliptic genus formula.

There are two changes that we need to carefully track. The first one is that since the field θ becomes an angular variable with period 2π, we must sum over the world sheet winding sectors. Thus, we introduce the identifications

$$\theta(\sigma_1 + 2\pi\tau_1, \sigma_2 + 2\pi\tau_2) = \theta(\sigma_1, \sigma_2) + 2\pi(\alpha + m)$$
$$\theta(\sigma_1 + 2\pi, \sigma_2) = \theta(\sigma_1, \sigma_2) + 2\pi w\,, \tag{2.15}$$

which lead to the classical solutions

$$\theta_{cl} = \sigma^1 w + \sigma^2(m + \alpha - w\tau_1)/\tau_2$$
$$= \frac{-i}{2\tau_2}(z(m + \alpha - w\bar{\tau}) - \bar{z}(m + \alpha - w\tau))\,. \tag{2.16}$$

We then have the classical contribution to the action

$$\partial\theta_{cl}\bar{\partial}\theta_{cl} = \frac{1}{4\tau_2^2}(m + \alpha - w\bar{\tau})(m + \alpha - w\tau)$$
$$= \frac{1}{4\tau_2^2}(|\lambda|^2 + \alpha(\lambda + \bar{\lambda} + \alpha)) \tag{2.17}$$

where $\lambda = m - w\tau$. After tracking normalization factors, one finds that the action acquires another overall factor of $4\pi\tau_2/k$ (see e.g. [27]).

The second effect we must take into account is that the left-moving R-charge corresponds to the left-moving momentum of the angle field. When we introduce a winding number w, we must properly take into account the contribution of the winding number to the left-moving momentum. This amounts to adding a factor of $e^{-2\pi i\alpha w/k}$ to a contribution arising from winding number w. (Recall that the radius is $R^2/\alpha' = 1/k$.) We rewrite

$$e^{-2\pi i\alpha w/k} = e^{\alpha(\lambda - \bar{\lambda})\frac{\pi}{k\tau_2}} \tag{2.18}$$

which leads to a total contribution to the exponent equal to

$$-\frac{\pi}{k\tau_2}(|\lambda|^2 + \alpha(\lambda + \bar{\lambda}) + \alpha^2 + \alpha(-\lambda + \bar{\lambda})) = -\frac{\pi}{k\tau_2}(|\lambda|^2 + 2\alpha\bar{\lambda} + \alpha^2)\,. \tag{2.19}$$

The denominator in the final expression is obtained from a factor $(\lambda + \alpha)(\bar{\lambda} + \alpha)$ in the denominator that arises from the exponent (2.17) in the generalized zero mode integral (2.13) on the one hand, and a factor of $\bar{\lambda} + \alpha$ in the numerator from the z-derivative of the angular variable θ on the other hand (arising from the zero mode lifting term (2.10)). Multiplying these, we find the final formula

$$\chi_{\text{cig}}(\tau, \alpha) = \frac{\theta_1(\alpha, \tau)}{2\pi\eta^3} \sum_{m,w\in\mathbb{Z}} \frac{e^{-\frac{\pi}{k\tau_2}(\alpha^2 + |m - w\tau|^2 + 2\alpha(m - w\bar{\tau}))}}{\alpha + m - w\tau}\,, \tag{2.20}$$

which is the compact lattice sum form [8] of the cigar elliptic genus. We have given a direct derivation of the lattice sum form, using the non-linear sigma model description. This concludes the first panel of our triptych.

3 Supersymmetric quantum mechanics on a half line

In this section, we wish to render the fact that the non-holomorphic term in non-compact elliptic genera arises from a contribution due to the continuum of the right-moving supersymmetric quantum mechanics [6] even more manifest. For that purpose, we discuss to what extent the right-moving supersymmetric quantum mechanics can be regularized in a supersymmetric invariant way, or a modular covariant manner, but not both. That fact leads to the holomorphic anomaly [6]. The plan of this section is to first review how boundary conditions in ordinary quantum mechanics show up in its path integral formulation. We then extend this insight to supersymmetric quantum mechanics. We illustrate the essence of the phenomenon in the simplest of systems. We end with a discussion of how the regulator of the non-compact elliptic genus cannot be both modular and supersymmetric, which leads to an anomaly.

3.1 Quantum mechanics on a half line

We are used to path integrals that map spaces with boundaries into closed manifolds. Less frequently, we are confronted with path integrals from closed spaces to spaces with boundaries. It is the latter case that we study in the following in the very simple setting of quantum mechanics.

In particular, we discuss quantum mechanics on a half line, its path integral formulation, and pay particular attention to the path integral incarnation of the boundary conditions. The easiest way to proceed will be to relate the problem to quantum mechanics on the whole real line. What follows is a review of the results derived in e.g. [18–20], albeit from an original perspective.

3.1.1 Quantum mechanics on the line

Firstly, we rapidly review quantum mechanics on the real line. We work with a Hilbert space which consists of quadratically integrable functions on the line parameterized by a coordinate x. We have a Hamiltonian operator H of the form

$$H = -\frac{1}{2}\partial_x^2 + V(x) \,, \tag{3.1}$$

where $V(x)$ is a potential. We can define a Feynman amplitude to go from an initial position x_i to a final position x_f in time t through the path integral

$$A(x_i, x_f, t) = \int_{x(0)=x_i}^{x(t)=x_f} dx \, e^{iS[x]} \,, \tag{3.2}$$

where the action is equal to

$$S = \int_0^t dt' \left(\frac{\dot{x}^2}{2} - V(x) \right) \,. \tag{3.3}$$

The Schrödinger equation for the wave-function of the particle reads

$$i\partial_t \Psi = H\Psi \,, \tag{3.4}$$

and we work with normalized wave-functions Ψ. We can also write the amplitude in terms of an integral over energy eigenstates Ψ_E:

$$A(x_i, x_f, t) = \int dE\, e^{-iEt} \Psi_E(x_i) \Psi_E(x_f) \,, \tag{3.5}$$

and the amplitude satisfies the δ-function completeness relation at $t = 0$, as well as the Schrödinger equation (3.4) in the initial and final position variables x_i and x_f.

3.1.2 Quantum mechanics on the half line

The subtleties of quantum mechanics on the open real half line $x \geq 0$ have been understood for a long time [21]. Boundary conditions compatible with unitarity have been classified. The path integral formulation for quantum mechanics on the half line has resurfaced several times over the last decades [18–20], and is also well-understood. We review what is known.

The half-line has a boundary, and we must have that the probability current vanishes at the boundary. This is guaranteed by the Robin boundary conditions

$$\partial_x \Psi(0) = c\, \Psi(0) \,. \tag{3.6}$$

When the constant c is zero, we have a Neumann boundary condition and when it is infinite, the boundary condition is in effect Dirichlet, $\Psi(0) = 0$. Suppose we are given a Hamiltonian H of the form (3.1) with a potential $V(x)$ on the half line $x > 0$. We can extend the quantum mechanics on the half line to the whole real line by extending the potential in an even fashion, declaring that $V(-x) = V(x)$. It is important to note that this constraint leaves the potential to take any value at the origin $x = 0$. We can then think of the quantum mechanics on the half line as a folded version of the quantum mechanics on the real line.[3] The even quantum mechanics that we constructed on the real line has a global symmetry group $\mathbb{Z}_2$. We can divide the quantum mechanics problem on the real line, including its Hilbert space, by the $\mathbb{Z}_2$ operation, and find a well-defined quantum mechanics problem on the half line, which is the original problem we wished to discuss.

An advantage of this way of thinking is that the measure for quantum mechanics on the whole line is canonical. It leads to the Green's function (3.5). Since the quantum mechanics that we constructed has a global $\mathbb{Z}_2$ symmetry, we can classify eigenfunctions in terms of the representation they form under the $\mathbb{Z}_2$ symmetry, namely, we can classify them into even and odd eigenfunctions of the Hamiltonian. We then obtain the whole line Green's function in the form that separates the even and odd energy eigenfunction contributions

$$A(x_i, x_f, t) = \int dE\, e^{-iEt} (\Psi_{E,e}(x_i) \Psi_{E,e}(x_f) + \Psi_{E,o}(x_i) \Psi_{E,o}(x_f)) \,. \tag{3.7}$$

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[3]In string theory, one would say that we think of the half line as an orbifold of the real line.

The Green's function

$$A^{\frac{1}{2},D}(x_i, x_f, t) = \frac{1}{2}(A(x_i, x_f, t) - A(x_i, -x_f, t)) = \int dE e^{-iEt} \Psi_{E,o}(x_i) \Psi_{E,o}(x_f), \quad (3.8)$$

is well-defined on the half-line and satisfies Dirichlet boundary conditions. We divide by a factor of two since we are projecting onto $\mathbb{Z}_2$ invariant states. From the path integral perspective, the subtraction corresponds to a difference over paths that go from x_i to x_f and that go from x_i to $-x_f$, on the whole real line, with the canonical measure (divided by two). This prescription generates a measure on the half line which avoids the origin, since we subtract all paths that cross to the other side [18, 19].[4] If we represent the $\mathbb{Z}_2$ action oppositely on the odd wave-functions, we arrive at the Green's function that satisfies Neumann boundary conditions:

$$A^{\frac{1}{2},N}(x_i, x_f, t) = \frac{1}{2}(A(x_i, x_f, t) + A(x_i, -x_f, t)) = \int dE e^{-iEt} \Psi_{E,e}(x_i) \Psi_{E,e}(x_f). \quad (3.9)$$

In this second option, we add paths to the final positions x_f and $-x_f$ with their whole line weights (divided by two). This path integral represents a sum over paths that reflect an even or an odd number of times off the origin $x = 0$, and in particular, allows the particle to reach the end of the half line.

We clearly see that the naive folding operation projects the states of the quantum mechanics onto those states that are even, or those that are odd.[5] However, concentrating on these two possibilities only fails to fully exploit the loop hole that the even potential $V(x)$ allows, which is an arbitrary value $V(0)$ at the fixed point $x = 0$ of the folding operation.[6] We can make use of this freedom by taking as the total potential an even potential $V(x)$, zero at $x = 0$, complemented with a δ-function:

$$H^c(x) = -\partial_x^2/2 + V(x) + c\,\delta(x). \quad (3.10)$$

We take the wave-function on the whole line to be even and continuous, with a discontinuous first derivative at the origin. When we consider the one-sided derivative at zero, we find that the wave-function satisfies the Robin boundary condition [19]

$$\partial_x \Psi(0^+) = c\Psi(0). \quad (3.11)$$

We have gone from a purely even continuous and differentiable wave-function on the real line that satisfies the Neumann boundary condition (at $c = 0$) to an even wave-function that satisfies mixed Robin boundary conditions, by influencing the wave-function near zero with a delta-function interaction.[7] It is intuitively clear, and argued in detail in [19] that

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[4]This is a common manipulation in probability theory.

[5]These are states in the untwisted sector of an orbifold, projected onto invariants under the gauged discrete symmetry.

[6]In string theory orbifolds, the fixed point hosts extra degrees of freedom which in that case are very strongly constrained by consistency.

[7]The even wave-function on the side $x > 0$ corresponds to the linear combination $\Psi(x) \propto (\Phi_{E,e}(x) + c\,\Phi_{E,o}(x))$ in terms of even and odd solutions to the problem on the real line without the delta-function interaction [19]. It is an invariant under the $\mathbb{Z}_2$ action with discontinuous derivative at the origin.

it is harder to push an initial problem with Dirichlet boundary conditions at the origin towards a mixed boundary condition problem. In order to achieve this, one needs a very deep well [19]. For later purposes, we note in particular that an ordinary delta-function insertion at the origin will not influence an initial Dirichlet boundary value problem.

As an intuitive picture, we can imagine that the delta-function is generated by possible extra degrees of freedom that are localized at the origin, and whose interaction with the quantum mechanical degree of freedom we concentrate on induces the delta-function potential localized at the origin.

Thus far, we briefly reviewed the results of [18, 19] on path integrals on the half line and discussed how they are consistent with folding. Next, we render these techniques compatible with supersymmetry.

3.2 Supersymmetric quantum mechanics on the half line

In this section, we extend our perspective on quantum mechanics on the half line to a quantum mechanical model with supersymmetry. We again start from a quantum mechanics on the whole of the real line, with extra fermionic degrees of freedom and supersymmetry. In a second stage, we fold the quantum mechanics onto the half line in a manner consistent with supersymmetry.

3.2.1 Supersymmetric quantum mechanics on the line

We discuss the supersymmetric system with Euclidean action (see e.g. [22])

$$S_E = \int_0^t d\tau \left(\frac{1}{2} \partial_\tau x^2 + \frac{1}{2} W^2 - \psi^*(\partial_\tau - W')\psi \right), \tag{3.12}$$

where $W'(x) = \partial_x W(x)$. The action permits two supersymmetries with infinitesimal variations

$$\begin{aligned}
\delta x &= \epsilon^* \psi + \psi^* \epsilon \\
\delta \psi^* &= -\epsilon^*(\partial_\tau x + W) \\
\delta \psi &= \epsilon(\partial_\tau x - W).
\end{aligned} \tag{3.13}$$

When we quantize the fermionic degrees of freedom, we tensor the space of quadratically integrable functions with a two component system. We call one component bosonic and the other fermionic. The two components have the Hamiltonians [22][8]

$$H_\pm = p^2 + W^2 \mp W'. \tag{3.14}$$

We introduced the operator

$$p = -i\partial_x \tag{3.15}$$

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[8]We follow standard conventions for supersymmetric quantum mechanics in this section. These differ by a factor of two from the standard conventions for quantum mechanics used in section 3.1.

and can represent the supercharges by

$$Q = (p + iW) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$Q^\dagger = (p - iW) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} . \tag{3.16}$$

When we trace over the fermionic degrees of freedom, we need to compute the fermionic determinant with anti-periodic boundary conditions. It evaluates to [22]

$$Z_f^{\text{anti−per}}(x) = \int d\psi d\psi_{\text{anti−per}}^* \exp(\psi^*(\partial_\tau - W')\psi) = \cosh\left(\int_0^t d\tau \frac{W'(x)}{2}\right), \tag{3.17}$$

after regularization. This is the path integral counterpart to the calculation of the Hamiltonians (3.14).

3.2.2 Supersymmetric quantum mechanics on the half line

We study the supersymmetric quantum mechanics on the half line by folding the supersymmetric quantum mechanics on the whole line. We wish for the folding $\mathbb{Z}_2$ symmetry to preserve supersymmetry. Since the particle position x is odd under the $\mathbb{Z}_2$ action (as is its derivative with respect to time, since we choose world line time to be invariant), we demand that the superpotential $W(x)$ is odd under parity, and that the fermionic variables ψ and ψ^* are odd as well. See equation (3.13). Thus, we have the $\mathbb{Z}_2$ action

$$(x, \psi, \psi^*) \to (-x, -\psi, -\psi^*) , \tag{3.18}$$

and the superpotential W is odd. For the moment, we consider the superpotential to be continuous, and therefore zero at zero.

We project onto states invariant under the $\mathbb{Z}_2$ action (3.18). Thus, in any path integral, we will insert a projection operator $P_{\mathbb{Z}_2}$ that consists of

$$P_{\mathbb{Z}_2} = \frac{1}{2}(1 + P(-1)^F) \tag{3.19}$$

where P is the parity operator that maps $P : x \to -x$ and $(-1)^F$ maps fermions to minus themselves. When we trace over the fermionic degrees of freedom with a $(-1)^F$ insertion, we must impose periodic boundary conditions on the fermions. The fermionic determinant in this case evaluates to [22]

$$Z_f^{\text{per}}(x) = \int d\psi d\psi_{\text{per}}^* \exp(\psi^*(\partial_\tau - W')\psi) = \sinh\left(\int_0^T d\tau \frac{W'(x)}{2}\right), \tag{3.20}$$

which leads to the same Hamiltonians (3.14) for the two component system, and when we compare to equation (3.17) we find a minus sign up front in the path integral over the second component. As a consequence, for the first component of the two component

system, from the insertion of the projection operator $P_{\mathbb{Z}_2}$ in equation (3.19), we will obtain a path integral measure

$$\frac{1}{2}\left(\int_{x_i}^{x_f} dx + \int_{x_i}^{-x_f} dx\right),\qquad(3.21)$$

while for the second component, we obtain a path integral measure

$$\frac{1}{2}\left(\int_{x_i}^{x_f} dx - \int_{x_i}^{-x_f} dx\right).\qquad(3.22)$$

Thus, from the discussion in subsection 3.1, the upper component, which we will call fermionic and indicate with a minus sign, will satisfy a Neumannn boundary condition at zero, while the bosonic component will satisfy the Dirichlet boundary condition. We carefully crafted our set-up to be consistent with supersymmetry, and must therefore expect the boundary conditions we obtain to be consistent with supersymmetry as well. Indeed, the operator Q maps the derivative of the fermionic wave-function to the bosonic wave-function (when evaluated at the boundary, and using $W(0) = 0$). Thus, the operator Q maps the boundary conditions into one another.[9]

The next case we wish to study is when the superpotential is well-defined on the half-line for $x > 0$, and approximates a non-zero constant as we tend towards $x = 0$. Since the superpotential is odd on the line, the distributional derivative of the superpotential will be a delta-function with coefficient twice the limit of the superpotential as it tends towards zero. If we call the latter value W_0, then we have the equation

$$W'(0) = 2W_0\,\delta(x)\,.\qquad(3.23)$$

The derivative of the superpotential arises as a term in the component Hamiltonians (3.14). The δ-function interaction at the origin will result in a change in the Neumann (but not the Dirichlet) boundary conditions, as we saw in subsection 3.1. If we follow through the consequences, we find that the supersymmetric quantum mechanics on the half line that we obtain by folding now satisfies the boundary conditions

$$\begin{aligned}
\Psi_+(0) &= 0\\
\partial_x\Psi_-(0) &= W_0\Psi_-(0)\,.
\end{aligned}\qquad(3.24)$$

These boundary conditions are consistent with supersymmetry.

3.2.3 An interval

We have used the folding technique to obtain a supersymmetric or ordinary quantum mechanics problem on a half line. We can use the same technique to generate quantum mechanics problems on an interval. We perform a second folding by the reflection symmetry $x \to 2L - x$ where L is the length of the desired interval. The fermions also transform with

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[9]Note that the choice of action of $(-1)^F$ on the two components (assigning to one component a plus sign) broke the symmetry between Q and $Q^\dagger$ in this discussion. In other words, the opposite assignment would have resulted in the operator $Q^\dagger$ mapping one boundary condition into the other.

a minus sign under the second $\mathbb{Z}_2$ generator. Again, we can render the superpotential odd under the second flip, take into account a possible delta-function potential on the second end of the interval, and find boundary conditions consistent with supersymmetry on both ends. Our application of these ideas lies in regulating a weighted trace, and we proceed immediately to apply them in that particular context.

3.3 Infrared regulators and the weighted trace

We wish to discuss the trace

$$Z(\beta) = Tr(-1)^F e^{-\beta H} \tag{3.25}$$

over the Hilbert space of states, weighted with a sign $(-1)^F$ corresponding to their fermion number F. It is well-known that this weighted trace is equal to the supersymmetric (Witten) index when the spectrum of the supersymmetric quantum mechanics is discrete [23]. It then reduces to the index which equals the number of bosonic minus the number of fermionic ground states.[10]

When the spectrum of the supersymmetric quantum mechanics is continuous, the situation is considerably more complicated (see e.g. [11, 24, 25]), and the debate in the literature on this quantity may not have culminated in a clear pedagogical summary. We attempt to improve the state of affairs in this subsection. The origin of the difficulties is that the trace over a continuum of states is an ill-defined concept. An infinite set of states contributing a finite amount gives rise to a divergent sum. A proper definition requires a regulator. An infrared regulator will reduce the continuum to a discretuum and render the trace finite. The alternating sum can remain finite in the limit where we remove the regulator. There has been a discussion on whether and how the resulting weighted trace $Z(\beta)$ depends on the inverse temperature β, and on the infrared regulator. To understand the main issues at stake, and to draw firm conclusions, it is sufficient to consider the example of a free supersymmetric particle on the half line.

3.3.1 The free supersymmetric particle on the half line

Let us consider a supersymmetric quantum mechanics, based on the superpotential which is equal to a constant for $x > 0$, namely $W(x > 0) = W_0$. We obtain the half line supersymmetric quantum mechanics by folding the problem on the whole line, and induce supersymmetric boundary conditions at the end of the half line. We recall the Hamiltonians

$$H_\pm = p^2 + W_0^2 \mp 2\,W_0\,\delta(x)\,, \tag{3.26}$$

with boundary conditions

$$\begin{aligned} \partial_x \Psi_- &= W_0 \Psi_- \\ \Psi_+(0) &= 0\,. \end{aligned} \tag{3.27}$$

We can then solve for the wave-functions on the half line. The solutions for energy $E = p^2 + W_0^2$ are given by reflecting waves. The phase shift is set by the boundary condition.

---

[10]We use the name weighted trace because we will soon encounter contexts in which it is not an index.

We have the wave-functions on the half line $x \geq 0$

$$\Psi_+(x) = c_+(e^{ipx} - e^{-ipx}),$$

$$\Psi_-(x) = c_-\left(e^{ipx} + \frac{ip - W_0}{ip + W_0}e^{-ipx}\right). \tag{3.28}$$

We find that the supercharge Q maps the wave-function Ψ_- into Ψ_+ if we identify $c_-(p + iW_0) = c_+$. Thus, we have computed the space of eigenfunctions for bosons and fermions and how they are related.

3.3.2 The weighted trace

Our intermediate goal is to evaluate the weighted trace $Z(\beta)$ in this model. To evaluate the trace, we need an infrared regulator. Moreover, the weighted trace depends on the infrared regulator, as we will demonstrate. In any case, we need to introduce an infrared regulator to make the trace well-defined. We cut off the space at large $x = x_{\mathrm{IR}}$. We need to impose boundary conditions at this second end, at x_{IR}. As a result, the spectrum becomes discrete, and we will be able to perform the trace over states weighted by the corresponding fermion number. We consider two regulators in detail.

In a first regularization, we construct the supersymmetric quantum mechanics on the interval as we described previously. The result will be a Hamiltonian

$$H_{\mp} = p^2 + W_0^2 \pm 2W_0\,\delta(x) \mp 2W_0\,\delta(x - x_{\mathrm{IR}}), \tag{3.29}$$

and boundary conditions

$$\partial_x \Psi_f(0^+) = W_0 \Psi_f$$
$$\Psi_b(0) = 0$$
$$\partial_x \Psi_f(x_{\mathrm{IR}}^-) = W_0 \Psi_f$$
$$\Psi_b(x_{\mathrm{IR}}) = 0. \tag{3.30}$$

The reason that the boundary condition on both sides is the same despite the sign flip in the δ function coefficient in (3.29) is because we are evaluating either the derivative with a left or a right approach to the singular point. Because the $\mathbb{Z}_2 \times \mathbb{Z}_2$ folding procedures commute with supersymmetry, the infrared regulated model preserves supersymmetry. Explicitly, we have a spectrum determined by the infrared boundary condition

$$e^{ip_n x_{\mathrm{IR}}} - e^{-ip_n x_{\mathrm{IR}}} = 0, \tag{3.31}$$

which implies

$$p_n = \frac{\pi n}{x_{\mathrm{IR}}} \tag{3.32}$$

where n is an integer. All states are two-fold degenerate. The state with the lowest energy has energy equal to $E = W_0^2$. The weighted trace reduces to a supersymmetric index and the Witten index is equal to zero.

A second regularization of the weighted trace proceeds as follows. We rather put Dirichlet boundary conditions at the infrared cut-off x_{IR} for both component wave-functions. We can intuitively argue that we expect a normalizable wave-function to drop off at infinity, and that the Dirichlet boundary condition is a good approximation to this expectation. It has the added advantage of not introducing extra degrees of freedom at the end point which we imagine to be responsible for a delta-function potential. The disadvantage is that this infrared regulator breaks supersymmetry. The regulated weighted trace will now sum over bosonic and fermionic states determined by the respective conditions (see (3.28))

$$
e^{ip_n^b x_{\text{IR}}} - e^{ip_n^b x_{\text{IR}}} = 0 \,,
$$
$$
e^{ip_{n'}^f x_{\text{IR}}} + \frac{ip_{n'}^f - W_0}{ip_{n'}^f + W_0} e^{-ip_{n'}^f x_{\text{IR}}} = 0 \,. \tag{3.33}
$$

We define the phase shift

$$
e^{i\delta(p)} = \frac{ip + W_0}{ip - W_0} \tag{3.34}
$$

of the fermionic wave-function. Then the solutions to the bosonic and fermionic boundary conditions are

$$
p_n^b = \frac{\pi n}{x_{\text{IR}}}
$$
$$
2p_{n'}^f x_{\text{IR}} + \delta(p_{n'}^f) = 2\pi \left(n' + \frac{1}{2} \right) \,. \tag{3.35}
$$

As the infrared cut-off is taken larger, the number of states per small dp interval will grow, to finally reach the continuum we started out with. To measure this growth, we can compute the bosonic and fermionic densities of states

$$
\rho^b(p) = \frac{dn}{dp} = \frac{x_{\text{IR}}}{\pi}
$$
$$
\rho^f(p) = \frac{dn'}{dp} = \frac{1}{2\pi} \left(2x_{\text{IR}} + \frac{d\delta(p)}{dp} \right) \,. \tag{3.36}
$$

Thus, when we approximate the weighted trace at large infrared cut-off by the appropriate integral formula, we find [11]

$$
Tr(-1)^F e^{-\beta H} = \int_0^\infty dp(\rho^b(p) - \rho^f(p))e^{-\beta E(p)} \tag{3.37}
$$

where the difference of densities of states is given by

$$
\Delta\rho = \rho^b(p) - \rho^f(p) = \frac{1}{2\pi}\delta'(p)
$$
$$
= \frac{1}{2\pi i}\frac{d}{dp}\log\frac{ip + W_0}{ip - W_0} = \frac{1}{2\pi}\left(\frac{1}{ip + W_0} - \frac{1}{ip - W_0} \right) \,. \tag{3.38}
$$

This second way of regularizing shows that the boundary condition we impose at the infrared end of our interval is crucial in determining the end result. When we put, as

we did in the first case, a boundary condition consistent with supersymmetry, then the difference of spectral densities is zero for all values of the cut-off, and therefore also in the limit of infinite cut-off. When we put identical boundary conditions for fermions and bosons at the infrared endpoint, then the spectral densities differ by the phase shift in the continuum problem. It should now be clear that one can choose another mix of boundary conditions that will lead to yet another outcome for the spectral measure. Before a choice of regulator, the weighted trace is ill-defined. The final result depends on the regulator choice, even after we remove the regulator. We have illustrated this effect in two cases, but there is an infinite number of choices, and the β-dependence of the final result $Z(\beta)$ is determined by the choice of regulator. We should rather think of the weighted trace $Z(\beta, \text{regulator})$ as a function of both the inverse temperature β and the regulator.

The first regulator is interesting, since it preserves supersymmetry. The second regulator, with identical boundary conditions for bosons and fermions is also interesting, it turns out. Although we computed the spectral density in our particular model of the free particle on a half line, the final result is universal in an appropriate sense. The relative phase shift of bosons and fermions at large x_{IR} is determined by the asymptotic form of the supercharge Q alone. This can be seen from the fact that the fermionic wave function in the infrared is determined by the bosonic wave function in the infrared and the asymptotic supercharge. Thus, only the asymptotic value of the superpotential $\lim_{x\to\infty} W(x) = W_0$, which we assume to be constant, will enter the phase shift and spectral density formula [11]. Thus, the result for the β-dependent weighted trace is universal, *given* the regularization procedure. Both the universality and the caveat are crucial.

The final result for our free particle on the half line with Dirichlet infrared regulator becomes [11]

$$\begin{aligned} Z(\beta, \text{Dirichlet}) &= \int_0^\infty dp \frac{1}{2\pi} \left(\frac{1}{ip + W_0} - \frac{1}{ip - W_0} \right) e^{-\beta(p^2 + W_0^2)} \\ &= \int_{-\infty}^{+\infty} dp \frac{1}{2\pi} \frac{1}{ip + W_0} e^{-\beta(p^2 + W_0^2)} . \end{aligned} \tag{3.39}$$

Conclusion. Of course, we recuperated the standard wisdom that any supersymmetric regulator makes the weighted trace into a supersymmetric Witten index which is β-independent. However, another choice of infrared regulator can give rise to a β-dependent weighted trace, and the β-dependence is dictated by the regulator.

It is quite striking that there are applications of supersymmetric quantum mechanics on a half line in which the infrared regulator is dictated by another symmetry of an overarching, higher dimensional model. In such circumstances, the weighted trace and its β-dependence become well-defined and useful concepts.

3.4 The application to the elliptic genus

In the calculation of the cigar elliptic genus (2.1), there is a weighted trace over the right-moving supersymmetric quantum mechanics. For each sector labeled by the right-moving momentum $\bar{m}$ on the asymptotic circle of the cigar, there is a supersymmetric quantum mechanics with superpotential W that asymptotes to $W_0 = \bar{m}$ [12]. The point is now that,

as we saw, each of the right-moving supersymmetric quantum mechanics labeled by the right-moving momentum can be cut-off supersymmetrically using a δ-function potential with coefficient depending on the right-moving momentum $\bar{m}$. The resulting elliptic genus would be equal to the mock modular Appell-Lerch sum. The cut-off depending on the right-moving momentum is not modular covariant though. The right-moving momentum is a combination of a winding number of torus maps, and the Poisson dual of the other winding number of torus maps, and as a result does not transform modular covariantly. The second alternative (and the one generically preferred in the context of a two-dimensional theory of gravity in which we wish to preserve large diffeomorphisms as a symmetry group) is to have a Dirichlet cut-off for all these supersymmetric quantum mechanics labeled by the right-moving momentum. This choice *is* covariant under modular transformations, but is not supersymmetric, as we have shown. The result of the second regularization is a modular completion of the mock modular form. We have thus shown that an anomaly arises in the combination of right-moving supersymmetry and modular covariance.

Let us comment on how generic the anomaly is. In cases where the asymptotic supercharge relates a radial direction to a circle (as is the case in conical geometries), we expect the supercharge to depend on radial and angular momentum. This momentum dependence entails a difference in the density of states as well. In such a rather generic set-up, there is an anomaly.[11]

Our analysis of supersymmetric quantum mechanics is interesting in itself. It also provides the technical details of the reasoning in [6, 10], and thus produces a second panel in our elliptic triptych. Moreover, our technical tinkering paints the background to continuum contributions to indices, or rather their continuous counterparts in two-dimensional theories [28] as well as in four-dimensional theories with eight supercharges [29, 30]. In particular, it clarifies both the regulator dependence as well as the universality of the results on weighted traces in the presence of supersymmetry and a continuum.

4 A flat space limit conformal field theory

In [26], we studied the infinite level limit of the cigar elliptic genus. In this limit, the target space is flattened. One is tempted to interpret the resulting conformal field theory as a flat space supersymmetric conformal field theory at central charge $c = 3$. Still, the theory has features that distinguish it from a mundane flat space theory. In this third panel, we add remarks to the discussion provided in [26], to which we also refer for further context.

4.1 Flat space regulated

Firstly, we consider a flat space conformal field theory on $\mathbb{R}^2$, with two free bosonic scalar fields, and two free Majorana fermions, for a total central charge of $c = 3$, and with $N = (2, 2)$ supersymmetry. We consider the Ramond-Ramond sector of the left- and right-moving fermions.

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[11]Exceptionally, the *integral* over the density of states can still be zero. An example is provided by the cigar conformal field theory at level one.

The ordinary bosonic partition function is divergent. There is an overall volume factor arising from the integral over bosonic zero modes which makes the partition function ill-defined. We can regulate the divergence in various ways. One regulator would be to compactify the target space on a torus of volume V, and then take the radii of the torus to infinity. The result is that the partition function approximates (see e.g. [27])

$$Z_V = \frac{V}{\alpha'}(4\pi^2\tau_2)^{-1}|\eta|^{-4}\,, \tag{4.1}$$

where V/α' represents the volume divergence. Alternatively, we can compute the partition function through zeta-function regularization and the first Kronecker limit formula. See e.g. [31]. The result is identical. If we regulate the bosons in this manner, and leave the finite fermionic partition function unaltered, both the right-moving fermions and the left-moving fermions will provide a zero mode in the Ramond-Ramond sector partition sum. Thus, we will find that the regulated supersymmetric Witten index is zero for all finite values of the volume regulator V. The limit of the supersymmetric index will be zero under these circumstances.

A different way of regularizing is to twist the phase of the complex boson $Z = X^1 + iX^2$. In the path integral calculation of the complex boson partition function, this is implemented in a modular covariant way by demanding that the field configurations we integrate over pick up a phase as we go around a cycle of the torus. The phase is a character of the $\mathbb{Z}^2$ homotopy group of the torus. If we parameterize the phases by $e^{2\pi ium + 2\pi ivw}$ (for winding numbers m, w on the two cycles of the torus), the result can be obtained either as the Ray-Singer analytic torsion [32] (to the power minus two) or by using the second Kronecker limit formula. The modular invariant result is

$$Z_{\text{twist}} = \left| e^{-\frac{\pi(\text{Im}(\beta))^2}{\tau_2}}\frac{\theta_1(\beta,\tau)}{\eta} \right|^{-2}\,, \tag{4.2}$$

where $\beta = u - v\tau$ is the complexified twist. Near zero twist, there is a second order divergence that is proportional to $|\beta|^{-2}|\eta|^{-4}$ in accord with equation (4.1). The twist regulator breaks the translation invariance in space-time and preserves the rotational invariance. In fact, it uses the rotation invariance to twist the angular direction and to remove all bosonic zero modes. (The idea is generic in that one can use twists by global symmetries to lift divergences in numerous contexts.) If we leave the fermions undisturbed, we again have the fermionic zero modes that give rise to a zero elliptic genus for the full conformal field theory.

The twist regulator suggests an interesting alternative. We can twist the bosons and preserve world sheet supersymmetry at the same time. The (tangent indexed) fermions naturally transform under the SO(2) rotating the two space-time directions, and if we twist with respect to the complete action of the space-time rotations, we twist the fermions as well. In that case, we find a partition function that equals one

$$Z_{\text{twist}} = \left| e^{-\frac{\pi(\text{Im}(\beta))^2}{\tau_2}}\frac{\theta_1(\beta,\tau)}{\eta} \right|^2 \times \left| e^{-\frac{\pi(\text{Im}(\beta))^2}{\tau_2}}\frac{\theta_1(\beta,\tau)}{\eta} \right|^{-2} = 1\,. \tag{4.3}$$

The two fermionic zero modes have canceled the quadratic volume divergence. The supersymmetric partition function (or Witten index) is now equal to one for all values of the twist, and therefore equals one in the limit where we remove the twist.

Again, as in section 3, we see that the final result is regulator dependent (as is infinity times zero). We have two regulators that preserve world sheet supersymmetry as well modular invariance, and they give rise to index equal to zero, or to one.

4.2 Twist two

We analyze how the above remarks influence our reading of the infinite level limit of the cigar elliptic genus [26]. First off, we further twist the left-moving fermions (only) by their left-moving R-charge, and wind up with the modular invariant flat space partition sum

$$Z_{\text{twist two}} = \left| \frac{e^{-\frac{\pi(\text{Im}(\alpha+\beta))^2}{\tau_2}} \theta_1(\alpha+\beta,\tau)}{e^{-\frac{\pi(\text{Im}(\beta))^2}{\tau_2}} \theta_1(\beta,\tau)} \right| . \tag{4.4}$$

This chiral partition function suffers from a chiral anomaly. We have again decided (for now) on a modular invariant choice of phase. The regulating twist β has canceled the right-moving zero mode against the anti-holomorphic pole due to the infinite volume. The left-moving R-charge twist α (when non-equivalent to zero) has reintroduced the holomorphic pole in β, also associated to the divergent volume. When we take the limit $\beta \to 0$, we therefore again find an infinite result.

Once more, there are various ways to regularize the expression. One straightforward way to obtain the result in [26] is to perform a modular covariant minimal subtraction. We expand the expression (4.4) near $\beta = 0$, and subtract the pole. Given the dictum of a modular covariant transformation rule for the constant term (e.g. the desired modular covariant transformation rule for the elliptic genus [33]) one then obtains the result [26]

$$Z_{\text{ms,cov}} = -\frac{1}{2\pi} \frac{\partial_\alpha \theta_{11}(\alpha,\tau)}{\eta^3} - \frac{\alpha}{2\tau_2} \frac{\theta_{11}(\alpha,\tau)}{\eta^3} . \tag{4.5}$$

We note that the elliptic properties are lost in the large level limit since the periodicity grows with the level. This contrasts sharply with the (alternatively regulated) zero result for the flat space elliptic genus which exhibits a very large symmetry group. [12]

The cigar elliptic genus manages to regulate the pole at $\beta = 0$ in a more subtle manner than the covariant minimal subtraction advocated above [6]. It goes as follows. One introduces an extra circle. Then, one couples the circle to the angular direction of the plane (or the cigar), and gauges a U(1) such as to identify the two circular directions. The net effect on the toroidal partition function is to incorporate the twist β into a modular covariant holonomy integral. The integral over the angle of the twist kills the divergent holomorphic pole, and renders the final result finite. The result is identical to the one obtained by covariant minimal subtraction (see [26] for the detailed derivation of this statement).

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[12]The lack of periodicity makes the partition function hard to interpret as an infinite set of Dirac indices. To recuperate such an interpretation as an infinite set of Dirac indices, one reverts to the holomorphically regulated expression, i.e. the mock modular part of the partition function.

4.3 A miniature

Finally, we wish to assemble a miniature triptych. Firstly, we revisit the path integral approach of section 2 and apply it to flat space. We T-dualize flat space, consider the infinite covering, and find instead of the zero mode factor (2.13)

$$Z_0^{\infty,\text{flat}} = 2\pi N_\infty \int_0^R dr \partial_r(-\pi r^{-2})\alpha e^{-r^{-2}\frac{\pi}{\tau_2}\alpha^2}$$
$$= 2\pi N_\infty \frac{\tau_2}{\alpha} e^{-\frac{\pi}{R^2\tau_2}\alpha^2}, \tag{4.6}$$

where we have introduced an infrared cut-off R on the radial integral. Thus, we find for the infinite cover of the T-dual of flat space the infrared regulated elliptic genus

$$Z^{\infty,\text{flat}}(R) = N_\infty \frac{\theta_1(\alpha,\tau)}{\eta^3} \frac{1}{2\pi\alpha} e^{-\frac{\pi\alpha^2}{R^2\tau_2}}. \tag{4.7}$$

For flat space then, we find the same lattice sum (see equation (2.20)) as for the cigar elliptic genus, with the level k replaced by the infrared cut-off R^2.

Our second panel, in section 3, makes it manifest that we have implicitly used the same boundary conditions for bosons and fermions, since we considered a single measure, a hard infrared cut-off R, and no delta-function insertion. Hence we find the anti-holomorphic $\bar\tau$ dependence in our result (4.7). Furthermore, our discussion in this section agrees with the fact that if we take the limit $R \to \infty$ term by term, neglecting the exponential factor in (4.7), then we find a divergent result. Indeed, the lattice sum will be divergent.

Finally, we note that (at $R = \infty$) the genus can be regulated in the manner of the Weierstrass ζ-function (which is the regulated lattice sum of $1/\alpha$). If we take that ad hoc route, the result can be made holomorphic and non-modular, and equal to only the first term in (4.5), using the formula

$$\zeta(\alpha,\tau) - G_2(\tau)\alpha = \frac{\partial_\alpha \theta_1(\alpha,\tau)}{\theta_1(\alpha,\tau)}, \tag{4.8}$$

where G_2 is the second Eisenstein series (and multiplying in the prefactor $\theta_1(\alpha,\tau)/\eta^3$). On the other hand, if we infrared regulate with a radial cut-off as in (4.7), or using the cigar model in the large level limit, we obtain the modular covariant, non-holomorphic result (4.5) which equals the exponentially regulated Eisenstein series as proven in [8, 26]. This final miniature illustrates how our conceptual triptych folds together seamlessly.

5 Conclusion

Our aim in this paper was to further explain conceptual features of completed mock modular non-compact elliptic genera [6] with elementary means. Using the supersymmetric cigar conformal field theory as an example, we provided a simple path integral derivation of the lattice sum formula [8] for the completed mock modular form. We derived the elliptic genus from the non-linear sigma-model.[13] We also laid bare the unresolvable tension between right-moving supersymmetry and modular covariance in defining the weighted trace

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[13]Other derivations are based on the coset conformal field theory or the gauged linear sigma-model [34, 35] descriptions.

with an infrared regulator, and we analyzed the quirks of the identification of the large level limit of the cigar model [26] with a flat space conformal field theory.

We believe these conceptual pointers provide a looking glass with which to revisit higher dimensional elliptic genera, including the $K3$, the ALE [36] and the higher dimensional linear dilaton space genera [37]. The ubiquitous possibility to factor the appropriate powers of θ_1/η^3 bodes well for this enterprise. For four-dimensional examples, for instance, we expect the doubling of the number of right-moving zero modes to be correlated to an elliptic Weierstrass $\wp$ factor in the result, et cetera. It will be interesting to study these generalizations.

Acknowledgments

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