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Junctions between two plates and a family of beams

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The aim of this paper was to study the junction between a periodic family of beams and two thin plates. This structure depends on 3 small parameters. We use the decompositions of the displacement fields in every beam and plate to obtain a priori estimates. Then in the case for which the displacements of both plates match, we derive the asymptotic behavior of this structure.

KEYWORDS

beam, junction, Korn inequality, linear elasticity, plate

1 | INTRODUCTION

This paper concerns the asymptotic behavior of an elastic multistructure composed by a family of elastic beams having the same length of order δ and as cross-section a disk of radius r. The beams are ε-periodically distributed between two plates of different thicknesses of order δ. We assume $r < \varepsilon/2$ and $r \leq \delta$ so as to deal with a family of distinct beams. The lateral boundary of the lower plate is clamped, and the other parts of the boundary are free of forces. The mechanical model is the linear isotropic elasticity. In this paper, the main novelty is to propose a way to obtain sharp estimates.

The aim of this paper was to introduce a simplified model of the skin. So the top layer stands for the epidermis, while the lower one is the hypoderm. The beams periodically distributed between these two layers stand for the collagen fibers in the dermis (for more details, see Blasselle and Griso[1]).

When we want to study an elastic multistructure the first difficulty is as follows: how can we obtain sharp estimates of the displacements? The Korn inequalities for a plate, a beam, or a bounded 3D regular domain are unfortunately not sufficient. To overcome this difficulty, here we use the decomposition of plates and beams displacements introduced by Griso[2,3] for straight and curved rods, shells, or plates. These decompositions have been extended to the structures made of beams or plates by Griso.[4,5] A beam displacement is written as the sum of an elementary displacement and a warping. The elementary displacement is affine in the cross-sections; it depends on two vector fields define on the centerline of the beam (see (3.2). The warping stands for the deformation of the cross-sections. Similarly, a plate displacement is also written as the sum of an elementary displacement and a warping. Here, the elementary displacement is affine in the fibers (see (3.13). In Sections 3.1 and 3.2, for both decompositions, we recall the full estimates of the warpings and the estimates of the strain tensors of the elementary displacements with respect to the strain tensors of the displacements. We use these decompositions for the set of beams and for the two plates. Then it remains to obtain the full estimates of the elementary displacements. To do that we compare the terms of the elementary displacements in the small portions of beams included in the plates. In particular, we prove that the estimates of the displacements in both plates differ by the

factor $1+\frac{\varepsilon\delta^2}{r^2}+\frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}$. Hence, if we want to deal with displacements of the same order in both plates, we assume that the above factor is uniformly bounded. We link the small parameters δ, ε, and r:

$$\varepsilon=\kappa_0\delta^\rho,\quad r=\kappa_1\delta^\eta,\quad \eta>0,\quad \rho>0.$$

Under the previous assumptions on these small parameters, we show that the couple (ρ,η) must belong to a convex polygon. We introduce two unfolding operators Π_δ and Π_r; they make possible both reductions of dimension $\delta\to 0$ and $r\to 0$.

For the mathematical modeling of beams and plates, we refer to previous studies.[6-8] Concerning the multistructures in linear elasticity and the junctions between beams and 3D domains or beams and plates, we refer to previous studies.[2,5,9-18] The junction between a beam and a plate in nonlinear elasticity is treated by Blanchard and Griso.[19,20] Concerning the periodic homogenization, we refer to Blanchard and Griso.[21] The following two recent papers[22,23] concern problems in domains with rough boundaries.

The paper is organized as follows: In Section 2 we describe the structure, and we introduce some notations; we also present the elasticity problem. Section 3 concerns the estimates of the admissible displacements of the structure. In Section 4 we link the small parameters δ, ε, and r. We opt to devote the next sections to the general case that corresponds to the interior of a polygon. Section 5 is dedicated to the applied forces. For the sake of simplicity we do not choose surface forces. In Section 6 we introduce the unfolding operators Π_δ and Π_r, and we give their first properties. In Section 7, Theorem 1 gives the weak limits of the different terms involved in the decompositions. We show that the limit displacements are of Kichhoff-Love type in both plates and also in the set of beams. In Sections 8.1 to 8.3 we obtain the limits of the strain and stress tensors. Section 8.4 is concerned by the limit problem, which links the bending and the membrane displacements of the plates. The convergence of the total elastic energy is given in Section 9. Finally, Section 10 is dedicated to the proof of a Poincaré-Wirtinger type inequality.

Throughout this paper

- *the Greek indexes α and β belong to $\{1,2\}$, while the Latin indexes i,j,k,l belong to $\{1,2,3\}$,*
- *the constants, which are denoted by C, do not depend on δ,ε, and r, and*
- *we use the Einstein convention of summation over repeated indices.*

2 | THE GEOMETRY OF THE STRUCTURE AND THE ELASTICITY PROBLEM

The structure is composed of two plates Ω^a_δ and Ω^b_δ whose thicknesses are $2\kappa_a\delta$, $2\kappa_b\delta$; they are connected by a family of beams (see Figure 1), regularly spaced, whose thicknesses are of order r and lengths $2\delta-\delta(\kappa_a+\kappa_b)$, which also represents the distance between the two plates, where $\kappa_a+\kappa_b<1$.

Set

- $\omega=(-L/2,L/2)^2$ the reference midsurface of the plates,
- $I^a=(1-\kappa_a,1+\kappa_a)$, $I^b=(-1-\kappa_b,-1+\kappa_b)$,
- $I^a_\delta=\delta I^a=\big(\delta(1-\kappa_a),\delta(1+\kappa_a)\big)$, $I^b_\delta=\delta I^b=\big(-\delta(1+\kappa_b),-\delta(1-\kappa_b)\big)$,
- $I=(-1-\kappa_b,1+\kappa_a)$, $I_\delta=\delta I=(-\delta(1+\kappa_b),\delta(1+\kappa_a))$,
- $I^{\mathrm{be}}=(-1+\kappa_b,1-\kappa_a)$, $I^{\mathrm{be}}_\delta=\delta I^{\mathrm{be}}=\big(\delta(-1+\kappa_b),\delta(1-\kappa_a)\big)$,
- $\Omega^a_\delta=\omega\times I^a_\delta$ the upper plate, $\Omega^a=\omega\times I^a$,
- $\Omega^b_\delta=\omega\times I^b_\delta$ the lower plate, $\Omega^b=\omega\times I^b$,
- $Y=\left(-\frac{1}{2},\frac{1}{2}\right)^2$, $\varepsilon=\frac{L}{N}$, $N\in\mathbb{N}^*$ and $Y_\varepsilon=\varepsilon Y=\left(-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right)^2$,
- $\Xi_\varepsilon=\{1,2,\ldots,N\}^2$, one has $\omega=\mathrm{int}\Big(\bigcup_{\xi\in\Xi_\varepsilon}\big(\varepsilon\xi+\overline{Y}_\varepsilon\big)\Big)$,
- $D_r=\{(x_1,x_2)\in\mathbb{R}^2\ \mid\ x_1^2+x_2^2<r^2\}$, $r>0$,
- $\mathbf{P}_\delta=D_r\times I_\delta$ the beam whose cross-section is the disc D_r and length $2\delta+\delta(\kappa_a+\kappa_b)$,
- the family of beams $B_{\delta,\varepsilon,r}=\cup_{\xi\in\Xi_\varepsilon}\big(\varepsilon\xi+D_r\big)\times I^{\mathrm{be}}_\delta$,
- the whole structure $\Omega_{\delta,\varepsilon,r}=\mathrm{int}\big(\overline{\Omega}^a_\delta\cup\overline{\Omega}^b_\delta\cup\overline{B_{\delta,\varepsilon,r}}\big)$,
- $B=D_1\times I$, $B^{\mathrm{be}}=D_1\times I^{\mathrm{be}}$ the reference beams.

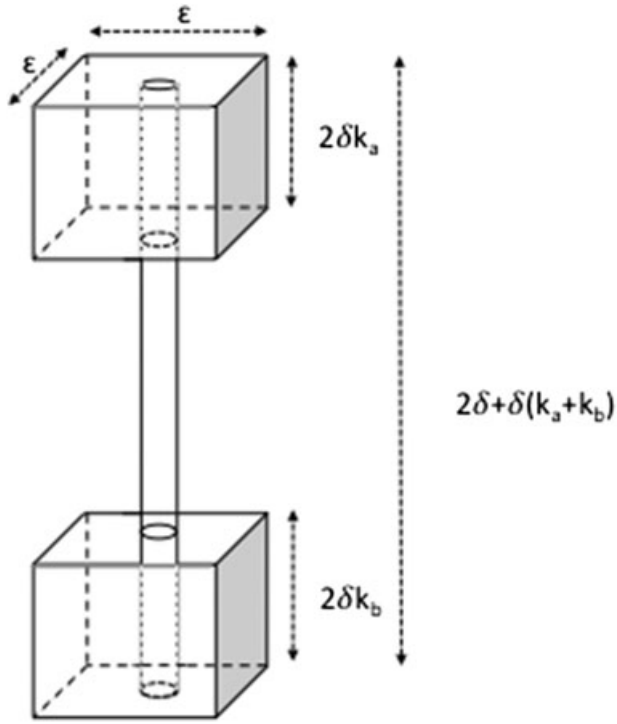

FIGURE 1 The beam P_δ

Throughout this paper, we denote $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the standard basis of $\mathbb{R}^3$.

Let u be a displacement belonging to $H^1(\mathcal{O}; \mathbb{R}^m)$, $m \in \{2, 3\}$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^m$. The linearized strain tensor field or the symmetric gradient field of u is defined by

$$\big(\nabla u\big)_S = \frac{1}{2}\big(\nabla u + (\nabla u)^T\big),$$

or equivalently by its components

$$\gamma_{kl}(u) = \frac{1}{2}\Big(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\Big), \quad (k, l) \in \{1, \dots, m\}^2.$$

The plates and the beams are made of homogeneous and isotropic elastic materials; for the sake of simplicity, one chooses the same Lamé constants for the plates. Set

$$\begin{aligned} &\lambda(x) = \lambda^{pl}, \qquad \mu(x) = \mu^{pl} \qquad \text{a.e. in } \Omega_\delta^a \cup \Omega_\delta^b, \\ &\lambda(x) = \lambda^{be}, \qquad \mu(x) = \mu^{be} \qquad \text{a.e. in } B_{\delta,\varepsilon,r}, \end{aligned} \tag{2.1}$$

where λ^{pl}, μ^{pl}, λ^{be}, and μ^{be} are the Lamé's constants of the materials. They are strictly positive constants.

Let $\{u_\delta\}_\delta$ be a sequence of displacements belonging to $H^1(\Omega_{\delta,\varepsilon,r}; \mathbb{R}^3)$.* The Cauchy stress tensor σ_δ in $\Omega_{\delta,\varepsilon,r}$ is linked to the symmetric gradient $(\nabla u_\delta)_S$ through the standard Hooke's law:

$$\sigma_{ij,\delta} = \lambda\big(\gamma_{kk}(u_\delta)\big)\delta_{ij} + 2\mu\gamma_{ij}(u_\delta), \qquad \text{a.e. in } \Omega_{\delta,\varepsilon,r}, \tag{2.2}$$

where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.

In the domain $\Omega_{\delta,\varepsilon,r}$, consider the standard problem of elasticity, and the equations of equilibrium in $\Omega_{\delta,\varepsilon,r}$ are

$$-\frac{\partial \sigma_{ij,\delta}}{\partial x_j} = f_{i,\delta} \text{ in } \Omega_{\delta,\varepsilon,r}, \tag{2.3}$$

where $f^\delta \in L^2(\Omega_{\delta,\varepsilon,r}; \mathbb{R}^3)$ denotes the applied forces.

*For the sake of clarity, in this section, we decide to omit the dependence of the fields with respect to the parameters ε and r. In Section 7, we will link the parameters δ, ε, and r, and then we will only use the parameter δ for any fields.

To specify the boundary conditions on $\partial\Omega_{\delta,\varepsilon,r}$, one assumes that the $3d$ plate Ω_δ^b is clamped on its lateral boundary $\partial\omega \times I_\delta^b = \Gamma_\delta^b$:

$$u_\delta = 0 \text{ on } \Gamma_\delta^b \tag{2.4}$$

and that the boundary $\partial\Omega_{\delta,\varepsilon,r} \setminus \Gamma_\delta^b$ is free of forces:

$$\sigma_\delta \nu = 0 \text{ on } \partial\Omega_{\delta,\varepsilon,r} \setminus \Gamma_\delta^b, \tag{2.5}$$

where ν denotes the exterior unit normal vector to $\Omega_{\delta,\varepsilon,r}$.

Remark 1. The boundary condition (2.5) means that the applied surface forces on the boundary $\partial\Omega_{\delta,\varepsilon,r} \setminus \Gamma_\delta^b$ are null. This assumption is not necessary to carry on the analysis, but it is natural as far as the family of beams is concerned.

The variational formulation of (2.3) to (2.5) is standard. If $V_{\delta,\varepsilon,r}$ denotes the space of admissible displacements

$$V_{\delta,\varepsilon,r} = \left\{ v \in H^1(\Omega_{\delta,\varepsilon,r}; \mathbb{R}^3) \mid v = 0 \text{ on } \Gamma_\delta^b \right\},$$

the variational formulation is

$$\begin{cases} \text{Find } u_\delta \in V_{\delta,\varepsilon,r} \text{ such that,} \\ \displaystyle\int_{\Omega_{\delta,\varepsilon,r}} \sigma_{ij,\delta}\gamma_{ij}(v)dx = \int_{\Omega_{\delta,\varepsilon,r}} f_{i,\delta} v_i dx, \qquad \forall v \in V_{\delta,\varepsilon,r}. \end{cases} \tag{2.6}$$

We equip $H^1(\Omega_{\delta,\varepsilon,r}; \mathbb{R}^3)$ with the seminorm:

$$\|v\|_V = \|(\nabla v)_S\|_{[L^2(\Omega_{\delta,\varepsilon,r})]^9}$$

and throughout the paper and for every $v \in V_{\delta,\varepsilon,r}$ we denote by

$$\mathcal{E}(v) = \int_{\Omega_{\delta,\varepsilon,r}} \left[\lambda\big(\gamma_{kk}(v)\big)^2 + 2\mu\gamma_{ij}(v)\gamma_{ij}(v) \right] dx$$

the total elastic energy of the displacement v. Indeed, choosing $v = u_\delta$ in (2.6) leads to the usual energy relation:

$$\mathcal{E}(u_\delta) = \int_{\Omega_{\delta,\varepsilon,r}} f_{i,\delta} u_{i,\delta} dx. \tag{2.7}$$

For a.e. $z \in \mathbb{R}^2$, we denote $[z] \in \mathbb{Z}^2$ the integer part of z and $\{z\} \in Y$ its fractional part, hence

$$z = [z] + \{z\}.$$

3 | ESTIMATES FOR THE STRUCTURE DISPLACEMENTS

To obtain a priori estimates on the displacements of the whole structure, one needs a Korn's inequality for this kind of domain. Here, we are concerned with a multistructure, and it is not convenient to estimate the constant in Korn's type inequality with respect to ε, δ, and r. To overcome this difficulty, we use decompositions of the beams displacements and of the plates displacements.

3.1 | Estimates for the set of beams

In this paper, one considers the following assumptions:

$$r < \varepsilon/2, \qquad r \le \delta. \tag{3.1}$$

With the first assumption, one claims that the structure is made of distinct beams, with the second one, one only wants to deal with a set of beams between the two plates.[†]

The space $H^1(\mathbf{P}_\delta; \mathbb{R}^3)$ is equipped with the seminorm:

$$\forall v \in H^1(\mathbf{P}_\delta; \mathbb{R}^3), \qquad \|v\|_{\mathrm{P}_\delta} = \|(\nabla v)_S\|_{[L^2(\mathrm{P}_\delta)]^9}.$$

Let u be an element belonging to $H^1(\Omega_{\delta,\varepsilon,r}; \mathbb{R}^3)$, theorem 3.1 in Griso[3] gives a decomposition of the restrictions u_ξ of u to the beam $\varepsilon\xi + \mathbf{P}_\delta$, $\xi \in \Xi_\varepsilon$.

[†]If we assume that $\delta/r \to 0$, then between the two plates we get small plates. In this case, to obtain the full estimates, the strategy must be different.

For a.e. $x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \in \varepsilon\xi + \mathbf{P}_\delta$, we write ($x' = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$):

$$\begin{aligned} u_\xi(x) = u(\varepsilon\xi + x) &= U_\xi(x_3) + R_\xi(x_3) \wedge x' + \ddot{u}_\xi(x) \\ &= U^e_\xi(x) + \ddot{u}_\xi(x), \end{aligned} \tag{3.2}$$

where $U_\xi \in H^1(I_\delta;\mathbb{R}^3)$, $R_\xi \in H^1(I_\delta;\mathbb{R}^3)$ and $\ddot{u}_\xi \in H^1(\mathbf{P}_\delta;\mathbb{R}^3)$. The residual displacement $\ddot{u}_\xi$ (named the warping) satisfies for a. e. $x_3 \in I_\delta$,

$$\begin{aligned} &\int_{D_r} \ddot{u}_\xi(x_1,x_2,x_3)dx_1dx_2 = 0, \\ &\int_{D_r} x_1\ddot{u}_{3,\xi}(x_1,x_2,x_3)dx_1dx_2 = \int_{D_r} x_2\ddot{u}_3(x_1,x_2,x_3)dx_1dx_2 = 0, \\ &\int_{D_r} \big(x_1\ddot{u}_{2,\xi}(x_1,x_2,x_3) - x_2\ddot{u}_{1,\xi}(x_1,x_2,x_3)\big)dx_1dx_2 = 0. \end{aligned} \tag{3.3}$$

The following estimates of the terms of the decomposition (3.2) are proven in theorem 3.1 in Griso[3]:

$$\begin{aligned} &\|\ddot{u}_\xi\|_{[L^2(\mathrm{P}_\delta)]^3} \le Cr\|u_\xi\|_{\mathrm{P}_\delta}, \qquad \|\nabla\ddot{u}_\xi\|_{[L^2(\mathrm{P}_\delta)]^9} \le C\|u_\xi\|_{\mathrm{P}_\delta}, \\ &\Big|\Big|\frac{dR_\xi}{dx_3}\Big|\Big|_{[L^2(I_\delta)]^3} \le \frac{C}{r^2}\|u_\xi\|_{\mathrm{P}_\delta}, \qquad \Big|\Big|\frac{dU_\xi}{dx_3} - R_\xi \wedge \mathbf{e}_3\Big|\Big|_{[L^2(I_\delta)]^3} \le \frac{C}{r}\|u_\xi\|_{\mathrm{P}_\delta}. \end{aligned} \tag{3.4}$$

The strain tensor field of u_ξ is

$$\big(\nabla u_\xi\big)_S = \begin{pmatrix} \gamma_{11}(\ddot{u}_\xi) & \gamma_{12}(\ddot{u}_\xi) & \frac{1}{2}\Big[\Big(\frac{dU_{1,\xi}}{dx_3} - R_{2,\xi}\Big) - x_2\frac{dR_{3,\xi}}{dx_3}\Big] + \gamma_{13}(\ddot{u}_\xi) \\ * & \gamma_{22}(\ddot{u}_\xi) & \frac{1}{2}\Big[\Big(\frac{dU_{2,\xi}}{dx_3} + R_{1,\xi}\Big) + x_1\frac{dR_3}{dx_3}\Big] + \gamma_{23}(\ddot{u}_\xi) \\ * & * & \frac{dU_{3,\xi}}{dx_3} + x_2\frac{dR_{1,\xi}}{dx_3} - x_1\frac{dR_{2,\xi}}{dx_3} + \gamma_{33}(\ddot{u}_\xi) \end{pmatrix}. \tag{3.5}$$

Set

$$\mathbf{R}^{\mathrm{be}}_\xi = \begin{pmatrix} 0 & -R_{3,\xi} & R_{2,\xi} \\ R_{3,\xi} & 0 & -R_{1,\xi} \\ -R_{2,\xi} & R_{1,\xi} & 0 \end{pmatrix}$$

From the expression (3.2) of u_ξ and after a straightforward calculation one derives

$$\|\nabla u_\xi - \mathbf{R}^{\mathrm{be}}_\xi\|_{[L^2(\mathrm{P}_\delta)]^9} \le C\left(\|\nabla\ddot{u}_\xi\|_{[L^2(\mathrm{P}_\delta)]^9} + r\Big|\Big|\frac{dU_\xi}{dx_3} - R_\xi \wedge \mathbf{e}_3\Big|\Big|_{[L^2(I_\delta)]^3} + r^2\Big|\Big|\frac{dR_\xi}{dx_3}\Big|\Big|_{[L^2(I_\delta)]^3}\right).$$

The estimates (3.4) and the above one lead to

$$\big|\big|\nabla u_\xi - \mathbf{R}^{\mathrm{be}}_\xi\big|\big|_{[L^2(\mathrm{P}_\delta)]^9} \le C\|u_\xi\|_{\mathrm{P}_\delta}. \tag{3.6}$$

Denote

$$\mathbf{s} = \begin{cases} +1 \text{ if } d = a, \\ -1 \text{ if } d = b. \end{cases}$$

For every $\phi \in L^1(I^d_\delta)$, $d \in \{a,b\}$, set

$$\mathcal{M}_{I^d_\delta}(\phi) = \frac{1}{2\kappa_d\delta}\int_{I^d_\delta}\phi(x_3)dx_3.$$

Recall the following consequences of the Poincaré-Wirtinger inequality ($d \in \{a,b\}$):

$$\forall\phi \in H^1(I), \qquad \begin{aligned} &\|\phi - \mathcal{M}_{I^d_\delta}(\phi)\|_{L^2(I_\delta)} \le C\delta\|\phi'\|_{L^2(I_\delta)}, \\ &|\mathcal{M}_{I^a_\delta}(\phi) - \mathcal{M}_{I^b_\delta}(\phi)| \le C\delta^{1/2}\|\phi'\|_{L^2(I_\delta)}. \end{aligned} \tag{3.7}$$

Lemma 1. *One has* ($d \in \{a,b\}$)

$$\sum_{\xi\in\Xi_\varepsilon}\big|\big|\nabla u(\varepsilon\xi + \cdot) - \mathcal{M}_{I^d_\delta}\big(\mathbf{R}^{be}_\xi\big)\big|\big|^2_{[L^2(\mathrm{P}_\delta)]^9} \le \frac{C\delta^2}{r^2}\|u\|^2_V \tag{3.8}$$

and

$$\sum_{\xi\in\Xi_\varepsilon} \left|\left|u_3(\varepsilon\xi+\cdot) - \mathcal{M}_{I_\delta^d}\left(U_{3,\xi}\right) + x_1\mathcal{M}_{I_\delta^d}\left(R_{2,\xi}\right) - x_2\mathcal{M}_{I_\delta^d}\left(R_{1,\xi}\right)\right|\right|^2_{L^2(\mathrm{P}_\delta)} \le C\delta^2\|u\|_V^2,$$
$$\sum_{\xi\in\Xi_\varepsilon} \left|\left|u(\varepsilon\xi+\cdot) - \mathcal{M}_{I_\delta^d}\left(U_\xi\right) - \mathcal{M}_{I_\delta^d}\left(R_\xi\right)\wedge(x-\mathbf{s}\delta\mathbf{e}_3)\right|\right|^2_{[L^2(\mathrm{P}_\delta)]^3} \le \frac{C\delta^4}{r^2}\|u\|_V^2. \tag{3.9}$$

The constants do not depend on δ, ε, and r.

Proof. From $(3.4)_3$ and $(3.7)_1$ one obtains $d\in\{a,b\}$:

$$\forall\xi\in\Xi_\varepsilon,\qquad \|\mathbf{R}_\xi^{\mathrm{be}} - \mathcal{M}_{I_\delta^d}\left(\mathbf{R}_\xi^{\mathrm{be}}\right)\|_{[L^2(I_\delta)]^9} \le C\delta\left|\left|\frac{dR_\xi}{dx_3}\right|\right|_{[L^2(I_\delta)]^3} \le \frac{C\delta}{r^2}\|u_\xi\|_{\mathrm{P}_\delta}. \tag{3.10}$$

Then (3.8) follows. Now, $(3.4)_4$ and (3.10) give

$$\forall\xi\in\Xi_\varepsilon,\qquad \begin{aligned} &\left|\left|\frac{dU_{3,\xi}}{dx_3}\right|\right|_{L^2(I_\delta)} \le \frac{C}{r}\|u_\xi\|_{\mathrm{P}_\delta}\\ &\left|\left|\frac{dU_{1,\xi}}{dx_3} - \mathcal{M}_{I_\delta^d}\left(R_{2,\xi}\right)\right|\right|_{L^2(I_\delta)} + \left|\left|\frac{dU_{2,\xi}}{dx_3} + \mathcal{M}_{I_\delta^d}\left(R_{1,\xi}\right)\right|\right|_{L^2(I_\delta)} \le \frac{C\delta}{r^2}\|u_\xi\|_{\mathrm{P}_\delta}. \end{aligned}$$

Hence

$$\begin{aligned} &\|U_{3,\xi} - \mathcal{M}_{I_\delta^d}\left(U_{3,\xi}\right)\|_{L^2(I_\delta)} \le \frac{C\delta}{r}\|u_\xi\|_{\mathrm{P}_\delta},\\ &\|U_{1,\xi} - \mathcal{M}_{I_\delta^d}\left(U_{1,\xi}\right) - (x_3-\mathbf{s}\delta)\mathcal{M}_{I_\delta^d}\left(R_{2,\xi}\right)\|_{L^2(I_\delta)} \le \frac{C\delta^2}{r^2}\|u_\xi\|_{\mathrm{P}_\delta},\\ &\|U_{2,\xi} - \mathcal{M}_{I_\delta^d}\left(U_{2,\xi}\right) + (x_3-\mathbf{s}\delta)\mathcal{M}_{I_\delta^d}\left(R_{1,\xi}\right)\|_{L^2(I_\delta)} \le \frac{C\delta^2}{r^2}\|u_\xi\|_{\mathrm{P}_\delta}. \end{aligned}$$

The above estimates together with $(3.4)_1$ and (3.10) yield (3.9). □

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^l$, $l\in\mathrm{N}^*$. For every measurable function ϕ on $\varepsilon\Xi_\varepsilon\times\mathcal{O}$, denote $\widetilde{\phi}$ the piecewise constant function defined on $\omega\times\mathcal{O}$ by

$$\widetilde{\phi}(z_1,z_2,x) = \phi_\xi(x)\quad \text{for all } (z_1,z_2)\in\varepsilon\xi+\varepsilon Y,\ \ \xi\in\Xi_\varepsilon \text{ and for a.e. } x\in\mathcal{O}. \tag{3.11}$$

The fields associated to the decomposition (3.2) of u_ξ are denoted:

$$\widetilde{U},\ \ \widetilde{R}\in L^2(\omega;H^1(I_\delta;\mathbb{R}^3)),\ \widetilde{\mathbf{R}}^{\mathrm{be}}\in L^2(\omega;H^1(I_\delta;\mathbb{R}^9))\ \text{ and }\ \widetilde{u}\in L^2(\omega;H^1(\mathrm{P}_\delta;\mathbb{R}^3)).$$

As a consequence of (3.4), one has

$$\begin{aligned} &\|\widetilde{u}\|_{[L^2(\omega\times\mathrm{P}_\delta)]^3} \le Cr\varepsilon\|u\|_V, \qquad \|\nabla_x\widetilde{u}\|_{[L^2(\omega\times\mathrm{P}_\delta)]^9} \le C\varepsilon\|u\|_V,\\ &\left|\left|\frac{\partial\widetilde{R}}{\partial x_3}\right|\right|_{[L^2(\omega\times I_\delta)]^3} \le C\frac{\varepsilon}{r^2}\|u\|_V, \qquad \left|\left|\frac{\partial\widetilde{U}}{\partial x_3} - \widetilde{R}\wedge\mathbf{e}_3\right|\right|_{[L^2(\omega\times I_\delta)]^3} \le C\frac{\varepsilon}{r}\|u\|_V. \end{aligned} \tag{3.12}$$

3.2 | Decomposition of the plate displacements

Let u be in $H^1(\Omega_{\delta,\varepsilon,r};\mathbb{R}^3)$. In the plates Ω_δ^a and Ω_δ^b, the displacement u is decomposed as (see the decomposition of the plate displacements introduced in Griso[3,5])

$$u(x) = \mathcal{U}^d(x') + (x_3-\mathbf{s}\delta)\mathcal{R}^d(x') + \overline{u}^d(x),\quad \text{for a.e. } x=(x_1,x_2,x_3)\in\Omega_\delta^d \tag{3.13}$$

where $x' = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, $\mathcal{U}^d = \mathcal{U}_1^d\mathbf{e}_1 + \mathcal{U}_2^d\mathbf{e}_2 + \mathcal{U}_3^d\mathbf{e}_3 \in H^1(\omega;\mathbb{R}^3)$, $\mathcal{R}^d = \mathcal{R}_1^d\mathbf{e}_1 + \mathcal{R}_2^d\mathbf{e}_2 \in H^1(\omega;\mathbb{R}^2)$, and $\overline{u}^d\in H^1(\Omega_\delta^d;\mathbb{R}^3)$, $d\in\{a,b\}$. The residual displacement $\overline{u}^d$—named the warping—satisfies

$$\int_{I_\delta^d}\overline{u}^a(x',x_3)dx_3 = 0,\qquad \int_{I_\delta^d}\left(x_3-\mathbf{s}\delta\right)\overline{u}_\alpha^a(x',x_3)dx_3 = 0\quad \text{for a.e. } x'\in\omega. \tag{3.14}$$

Moreover, the following estimates hold, $d\in\{a,b\}$, (see thoerem 4.1 of Griso[3]):

$$\sum_{\alpha,\ \beta=1}^{2}\left(\delta\left|\left|\gamma_{\alpha\beta}\left(\mathcal{R}^d\right)\right|\right|_{L^2(\omega)}+\|\gamma_{\alpha\beta}(\mathcal{U}_m^d)\|_{L^2(\omega)}\right)+\left|\left|\frac{\partial\mathcal{U}_3^d}{\partial x_1}+\mathcal{R}_1^d\right|\right|_{L^2(\omega)}+\left|\left|\frac{\partial\mathcal{U}_3^d}{\partial x_2}+\mathcal{R}_2^d\right|\right|_{L^2(\omega)}$$
$$\leq\frac{C}{\delta^{1/2}}\|u\|_V,\quad \|\overline{u}^d\|_{[L^2(\Omega_\delta^d)]^3}\leq C\delta\|u\|_V,\quad \|\nabla\overline{u}^d\|_{[L^2(\Omega_\delta^d)]^9}\leq C\|u\|_V, \tag{3.15}$$

where $\mathcal{U}_m^d=\mathcal{U}_1^d\mathbf{e}_1+\mathcal{U}_2^d\mathbf{e}_2$.

The strain tensor field of the displacement u is given by ($d\in\{a,b\}$):

$$\left(\nabla u\right)_S=\begin{pmatrix}\Gamma_{11}^d & \Gamma_{12}^d & \frac{1}{2}\left(\mathcal{R}_1^d+\frac{\partial\mathcal{U}_3^d}{\partial x_1}\right)+\gamma_{13}(\overline{u}^d)\\ * & \Gamma_{22}^d & \frac{1}{2}\left(\mathcal{R}_2^d+\frac{\partial\mathcal{U}_3^d}{\partial x_2}\right)+\gamma_{23}(\overline{u}^d)\\ * & * & \gamma_{33}(\overline{u}^d)\end{pmatrix}\qquad \text{a.e. in } \Omega_\delta^d \tag{3.16}$$

where

$$\Gamma_{\alpha\beta}^d=\gamma_{\alpha\beta}(\mathcal{U}_m^d)+(x_3-\mathbf{s}\delta)\gamma_{\alpha\beta}(\mathcal{R}^d)+\gamma_{\alpha\beta}(\overline{u}^d).$$

From now on, one assumes that the displacements belong to $V_{\delta,\varepsilon,r}$.

3.2.1 | Estimates for the plate Ω_δ^b

Observe that if $u\in V_{\delta,\varepsilon,r}$, then all the terms of the decomposition of u vanish on Γ_δ^b. In particular, one has

$$\mathcal{R}^b\in H_0^1(\omega;\mathbb{R}^2),\qquad \mathcal{U}^b\in H_0^1(\omega;\mathbb{R}^3).$$

Hence, because of the $2D$-Korn inequality and (3.15), one obtains

$$\|\mathcal{R}^b\|_{[H^1(\omega)]^2}\leq\frac{C}{\delta^{3/2}}\|u\|_V,\qquad \|\mathcal{U}_m^b\|_{[H^1(\omega)]^2}\leq\frac{C}{\delta^{1/2}}\|u\|_V. \tag{3.17}$$

Again, (3.15) together with $(3.17)_1$ leads to

$$\left|\left|\nabla\mathcal{U}_3^b\right|\right|_{[L^2(\omega)]^2}\leq\|\mathcal{R}^b\|_{[L^2(\omega)]^2}+\frac{C}{\delta^{1/2}}\|u\|_V\leq\frac{C}{\delta^{3/2}}\|u\|_V. \tag{3.18}$$

Apply the Poincaré inequality that gives

$$\|\mathcal{U}_3^b\|_{H_0^1(\omega)}\leq\frac{C}{\delta^{3/2}}\|u\|_V. \tag{3.19}$$

Hence, from (3.15), (3.17), (3.18), and (3.19) one derives the classical estimates for a plate clamped on its lateral boundary:

$$\|u_\alpha\|_{L^2(\Omega_\delta^b)}\leq C\|u\|_V,\qquad \|\nabla u\|_{[L^2(\Omega_\delta^b)]^9}+\|u_3\|_{L^2(\Omega_\delta^b)}\leq\frac{C}{\delta}\|u\|_V. \tag{3.20}$$

3.2.2 | Estimates for the plate Ω_δ^a

Consider the membrane displacement $\mathcal{U}_m^a=\mathcal{U}_1^a\mathbf{e}_1+\mathcal{U}_2^a\mathbf{e}_2$. We know that there exists a rigid displacement $\mathbf{r}^a$ such that

$$\mathbf{r}_1^a(x')=a_1^a-b_3^ax_2,\qquad \mathbf{r}_2^a(x')=a_2^a+b_3^ax_1,\qquad x'\in\omega,$$
$$\|\mathcal{U}_m^a-\mathbf{r}^a\|_{[H^1(\omega)]^2}\leq C\sum_{\alpha,\beta=1}^{2}\|\gamma_{\alpha\beta}(\mathcal{U}_m^a)\|_{L^2(\omega)}\leq\frac{C}{\delta^{1/2}}\|u\|_V. \tag{3.21}$$

There also exists a second rigid displacement $\mathbf{R}^a$ such that

$$\mathbf{R}_1^a(x')=b_1^a-cx_2,\qquad \mathbf{R}_2^a(x')=b_2^a+cx_1,\qquad x'\in\omega,$$
$$\|\mathcal{R}^a-\mathbf{R}^a\|_{[H^1(\omega)]^2}\leq C\sum_{\alpha,\beta=1}^{2}\|\gamma_{\alpha\beta}(\mathcal{R}^a)\|_{L^2(\omega)}\leq\frac{C}{\delta^{3/2}}\|u\|_V. \tag{3.22}$$

Then (3.15) and the above estimate (3.22) yield

$$\left|\left|\frac{\partial\mathcal{U}_3^a}{\partial x_1}+\mathbf{R}_1^a\right|\right|_{L^2(\omega)}+\left|\left|\frac{\partial\mathcal{U}_3^a}{\partial x_2}+\mathbf{R}_1^a\right|\right|_{L^2(\omega)}\leq\frac{C}{\delta^{3/2}}\|u\|_V. \tag{3.23}$$

One has

$$\frac{\partial}{\partial x_2}\Big(\frac{\partial \mathcal{U}_3^a}{\partial x_1}+\mathbf{R}_1^a\Big)-\frac{\partial}{\partial x_1}\Big(\frac{\partial \mathcal{U}_3^a}{\partial x_2}+\mathbf{R}_2^a\Big)=-2c \qquad \text{in } H^{-1}(\omega).$$

This equality and (3.23) lead to $|c|\le \frac{C}{\delta^{3/2}}\|u\|_V$, which, in turn, with (3.22) give

$$\|\nabla\mathcal{R}^a\|_{[L^2(\omega)]^4}\le \frac{C}{\delta^{3/2}}\|u\|_V. \tag{3.24}$$

3.2.3 | Comparison of the terms of the plate decompositions

Set

$$\mathbf{R}_{\rm pl}^a=\begin{pmatrix}0 & -b_3^a & \mathcal{R}_1^a\\ b_3^a & 0 & \mathcal{R}_2^a\\ -\mathcal{R}_1^a & -\mathcal{R}_2^a & 0\end{pmatrix},\qquad \mathbf{R}_{\rm pl}^b=\begin{pmatrix}0 & 0 & \mathcal{R}_1^b\\ 0 & 0 & \mathcal{R}_2^b\\ -\mathcal{R}_1^b & -\mathcal{R}_2^b & 0\end{pmatrix}.$$

Lemma 2. *One has*

$$\|\nabla u-\mathbf{R}_{pl}^a\|_{[L^2(\Omega_\delta^b)]^9}\le C\|u\|_V,\qquad \|\nabla u-\mathbf{R}_{pl}^b\|_{[L^2(\Omega_\delta^b)]^9}\le C\|u\|_V. \tag{3.25}$$

The constants do not depend on δ, ε, and r.

Proof. From the expression (3.13) of u in the plate Ω_δ^a, one expresses $\nabla u-\mathbf{R}_{\rm pl}^a$. Then the estimates (3.15), (3.21), and (3.24) give

$$\begin{aligned}\|\nabla u-\mathbf{R}_{\rm pl}^a\|_{[L^2(\Omega_\delta^a)]^9}\le C\Big(&\|\nabla\overline{u}^a\|_{[L^2(\Omega_\delta^a)]^9}+\delta^{3/2}\|\nabla\mathcal{R}^a\|_{[L^2(\omega)]^4}+\delta^{1/2}\Big|\Big|\frac{\partial\mathcal{U}_3^a}{\partial x_1}+\mathcal{R}_1^a\Big|\Big|_{L^2(\omega)}\\ &+\delta^{1/2}\Big|\Big|\frac{\partial\mathcal{U}_3^a}{\partial x_2}+\mathcal{R}_2^a\Big|\Big|_{L^2(\omega)}+\delta^{1/2}\|\nabla(\mathcal{U}_m^a-\mathbf{r}^a)\|_{[L^2(\omega)]^{2\times2}}\Big)\le C\|u\|_V.\end{aligned}$$

In the same way one shows $(3.25)_2$. □

For $\phi\in L^1(\omega)$, define the piecewise constant function $\mathcal{M}_r(\phi)$ belonging to $L^\infty(\omega)$ by

$$\mathcal{M}_r(\phi)(x')\doteq\mathcal{M}_r(\phi)(\varepsilon\xi)=\frac{1}{|D_r|}\int_{D_r}\phi(\varepsilon\xi+z)dz,\quad \text{for a.e. } x'\in\varepsilon\xi+Y_\varepsilon,\quad \xi\in\Xi_\varepsilon.$$

Recall that for every $\phi\in H^1(\omega)$[‡]

$$\begin{aligned}\|\phi(\varepsilon\xi+\cdot)-\mathcal{M}_r(\phi)(\varepsilon\xi)\|_{L^2(Y_\varepsilon)}&\le C\varepsilon\sqrt{\ln\Big(\frac{\varepsilon}{r}\Big)}\|\nabla\phi\|_{[L^2(Y_\varepsilon)]^2},\quad \forall\xi\in\Xi_\varepsilon,\\ \|\phi-\mathcal{M}_r(\phi)\|_{L^2(\omega)}&\le C\varepsilon\sqrt{\ln\Big(\frac{\varepsilon}{r}\Big)}\|\nabla\phi\|_{[L^2(\omega)]^2}.\end{aligned} \tag{3.26}$$

Lemma 3. *One has* ($d\in\{a,b\}$)

$$\begin{aligned}\|\mathcal{R}^a-\mathcal{R}^b\|_{[L^2(\omega)]^2}&\le C\left[\frac{\varepsilon\delta^2}{r^2}+\frac{\varepsilon^2}{r}\sqrt{\ln\Big(\frac{\varepsilon}{r}\Big)}\right]\frac{\|u\|_V}{\delta^{3/2}},\\ \|\mathcal{R}^a\|_{[L^2(\omega)]^2}&\le C\left[1+\frac{\varepsilon\delta^2}{r^2}+\frac{\varepsilon^2}{r}\sqrt{\ln\Big(\frac{\varepsilon}{r}\Big)}\right]\frac{\|u\|_V}{\delta^{3/2}},\\ \big|\big|\mathbf{R}_{pl}^d-\widetilde{\mathbf{R}}^{be}\big|\big|_{[L^2(\omega\times I_\delta)]^9}&\le C\left[\frac{\varepsilon\delta^2}{r^2}+\frac{\varepsilon^2}{r}\sqrt{\ln\Big(\frac{\varepsilon}{r}\Big)}\right]\frac{\|u\|_V}{\delta},\\ \big|\big|\widetilde{\mathbf{R}}^{be}\big|\big|_{[L^2(\omega\times I_\delta)]^9}&\le C\left[1+\frac{\varepsilon\delta^2}{r^2}+\frac{\varepsilon^2}{r}\sqrt{\ln\Big(\frac{\varepsilon}{r}\Big)}\right]\frac{\|u\|_V}{\delta}.\end{aligned} \tag{3.27}$$

The constants do not depend on δ, ε, and r.

[‡]In Appendix we give a short proof of these classical results.

Proof. Using (3.17), $(3.25)_1$ (resp. (3.24), $(3.25)_2$), and $(3.26)_2$ one deduces ($d \in \{a, b\}$):

$$\begin{aligned} \left|\left|\mathbf{R}^d_{\mathrm{pl}} - \mathcal{M}_r(\mathbf{R}^d_{\mathrm{pl}})\right|\right|_{[L^2(\omega)]^9} &\leq C\varepsilon\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\frac{\|u\|_V}{\delta^{3/2}}, \\ \left|\left|\nabla u - \mathcal{M}_r\left(\mathbf{R}^d_{\mathrm{pl}}\right)\right|\right|_{[L^2(\Omega^d_\delta)]^9} &\leq C\left(1 + \frac{\varepsilon}{\delta}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right)\|u\|_V. \end{aligned} \tag{3.28}$$

The above estimate $(3.28)_2$ and (3.8) allow to obtain

$$\sum_{\xi\in\Xi_\varepsilon} \delta r^2 \left|\mathcal{M}_r\left(\mathbf{R}^d_{\mathrm{pl}}\right)(\varepsilon\xi) - \mathcal{M}_{I^d_\delta}\left(\mathbf{R}^{\mathrm{be}}_\xi\right)\right|^2 \leq C\left(\frac{\delta^2}{r^2} + \frac{\varepsilon^2}{\delta^2}\ln\left(\frac{\varepsilon}{r}\right)\right)\|u\|^2_V, \tag{3.29}$$

which, in turn, with $(3.28)_1$ lead to

$$\left|\left|\mathbf{R}^d_{\mathrm{pl}} - \mathcal{M}_{I^d_\delta}\left(\widetilde{\mathbf{R}}^{\mathrm{be}}\right)\right|\right|^2_{[L^2(\Omega^d_\delta)]^9} \leq C\left(\frac{\varepsilon^2\delta^2}{r^4} + \frac{\varepsilon^4}{\delta^2 r^2}\ln\left(\frac{\varepsilon}{r}\right)\right)\|u\|^2_V. \tag{3.30}$$

Besides, from $(3.7)_2$ and $(3.12)_3$ we get

$$\begin{aligned} \|\widetilde{\mathbf{R}}^{\mathrm{be}} - \mathcal{M}_{I^d_\delta}\left(\widetilde{\mathbf{R}}^{\mathrm{be}}\right)\|^2_{[L^2(\omega\times I_\delta)]^9} &\leq C\frac{\delta^2\varepsilon^2}{r^4}\|u\|^2_V, \\ \|\mathcal{M}_{I^a_\delta}\left(\widetilde{\mathbf{R}}^{\mathrm{be}}\right) - \mathcal{M}_{I^b_\delta}\left(\widetilde{\mathbf{R}}^{\mathrm{be}}\right)\|^2_{[L^2(\omega)]^9} &\leq C\frac{\delta\varepsilon^2}{r^4}\|u\|^2_V. \end{aligned} \tag{3.31}$$

Hence, (3.30) and $(3.31)_1$ yield $(3.27)_3$, while (3.30) and $(3.31)_2$ give

$$\|\mathbf{R}^a_{\mathrm{pl}} - \mathbf{R}^b_{\mathrm{pl}}\|^2_{[L^2(\omega)]^9} \leq C\frac{\varepsilon^2}{r^2\delta}\left(\frac{\delta^2}{r^2} + \frac{\varepsilon^2}{\delta^2}\ln\left(\frac{\varepsilon}{r}\right)\right)\|u\|^2_V.$$

Thus, $(3.27)_1$ is proven. The above estimate together with $(3.17)_1$ leads to $(3.27)_2$. Finally, $(3.31)_1$ and $(3.17)_1$ yield $(3.27)_4$. □

As a consequence of (3.15) and $(3.27)_2$, one gets

$$\|\nabla\mathcal{U}^a_3\|_{[L^2(\omega)]^2} \leq C\left[1 + \frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]\frac{\|u\|_V}{\delta^{3/2}}. \tag{3.32}$$

In the following lemma, one estimates the L^2 norm of the midsurface displacement $\mathcal{U}^a$.

Lemma 4. *There exists a constant C (independent of δ, ε, and r), such that*

$$\begin{aligned} \|\mathcal{U}^a_3 - \mathcal{U}^b_3\|_{L^2(\omega)} &\leq C\varepsilon\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\left[1 + \frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]\frac{\|u\|_V}{\delta^{3/2}}, \\ \|\mathcal{U}^a_3\|_{L^2(\omega)} &\leq C\left[1 + \frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]^2\frac{\|u\|_V}{\delta^{3/2}}, \\ \|\mathcal{U}^d_3 - \widetilde{U}_3\|_{L^2(\omega\times I_\delta)} &\leq C\varepsilon\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\left[1 + \frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]\frac{\|u\|_V}{\delta}, \\ \|\widetilde{U}_3\|_{L^2(\omega\times I_\delta)} &\leq C\left[1 + \frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]^2\frac{\|u\|_V}{\delta}, \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} \|\mathcal{U}^a_\alpha - \mathcal{U}^b_\alpha\|_{L^2(\omega)} + \|\mathcal{U}^a_\alpha\|_{L^2(\omega)} &\leq C\left[1 + \frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]^2\frac{\|u\|_V}{\delta^{1/2}}, \\ \|\nabla\mathcal{U}^a_\alpha\|_{[L^2(\omega)]^2} &\leq C\left[1 + \frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]\frac{\|u\|_V}{\delta^{1/2}}, \\ \|\widetilde{U}_\alpha\|_{L^2(\omega\times I_\delta)} &\leq C\left[1 + \frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]\|u\|_V. \end{aligned} \tag{3.34}$$

Proof.

Step 1. One proves (3.33). Because of (3.15) and $(3.28)_1$ one first obtains

$$\Big|\Big|\frac{\partial \mathcal{U}_3^d}{\partial x_1} + \mathcal{M}_r\big(\mathcal{R}_1^d\big)\Big|\Big|_{L^2(\omega)} + \Big|\Big|\frac{\partial \mathcal{U}_3^d}{\partial x_2} + \mathcal{M}_r\big(\mathcal{R}_2^d\big)\Big|\Big|_{L^2(\omega)} \le C\left(1 + \frac{\varepsilon}{\delta}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right)\frac{\|u\|_V}{\delta^{1/2}}.$$

Using $(3.26)_1$ and the above inequality yield

$$\begin{aligned}\sum_{\varepsilon\in\Xi_\varepsilon}\int_{Y_\varepsilon}&\big|\mathcal{U}_3^d(\varepsilon\xi + x') - \mathcal{M}_r\big(\mathcal{U}_3^d\big)(\varepsilon\xi) + x_1\mathcal{M}_r\big(\mathcal{R}_1^d\big)(\varepsilon\xi) + x_2\mathcal{M}_r\big(\mathcal{R}_2^d\big)(\varepsilon\xi)\big|^2 dx_1dx_2\\ &\le C\varepsilon^2\ln\left(\frac{\varepsilon}{r}\right)\left(1 + \frac{\varepsilon^2}{\delta^2}\ln\left(\frac{\varepsilon}{r}\right)\right)\frac{\|u\|_V^2}{\delta}.\end{aligned}$$

This estimate together with $(3.15)_3$ and $(3.28)_1$ leads to

$$\begin{aligned}\sum_{\varepsilon\in\Xi_\varepsilon}&\big|\big|u_3(\varepsilon\xi + \cdot) - \mathcal{M}_r\big(\mathcal{U}_3^d\big)(\varepsilon\xi) + x_1\mathcal{M}_r\big(\mathcal{R}_1^d\big)(\varepsilon\xi) + x_2\mathcal{M}_r\big(\mathcal{R}_2^d\big)(\varepsilon\xi)\big|\big|^2_{L^2(Y_\varepsilon\times I_\delta^d)}\\ &\le C\Big[\delta^2 + \varepsilon^2\ln\left(\frac{\varepsilon}{r}\right) + \frac{\varepsilon^4}{\delta^2}\Big(\ln\left(\frac{\varepsilon}{r}\right)\Big)^2\Big]\|u\|_V^2.\end{aligned}$$

This with $(3.9)_1$ gives

$$\begin{aligned}&\sum_{\varepsilon\in\Xi_\varepsilon} r^2\delta\big|\mathcal{M}_r\big(\mathcal{U}_3^d\big)(\varepsilon\xi) - \mathcal{M}_{I_\delta^d}\big(U_{3,\xi}\big)\big|^2 \le C\Big[\delta + \frac{\varepsilon^2}{\delta}\ln\left(\frac{\varepsilon}{r}\right)\Big]^2\|u\|_V^2,\\ &\implies \big|\big|\mathcal{M}_r\big(\mathcal{U}_3^d\big) - \mathcal{M}_{I_\delta^d}\big(\widetilde{U}_3\big)\big|\big|^2_{L^2(\omega)} \le \frac{C\varepsilon^2}{r^2\delta}\Big[\delta + \frac{\varepsilon^2}{\delta}\ln\left(\frac{\varepsilon}{r}\right)\Big]^2\|u\|_V^2.\end{aligned}\tag{3.35}$$

Besides, from $(3.7)_2$ and $(3.12)_4$ we get

$$\begin{aligned}&\|\widetilde{U}_3 - \mathcal{M}_{I_\delta^d}\big(\widetilde{U}_3\big)\|^2_{[L^2(\omega\times I_\delta)]^9} \le C\frac{\delta^2\varepsilon^2}{r^2}\|u\|_V^2,\\ &\|\mathcal{M}_{I_\delta^a}\big(\widetilde{U}_3\big) - \mathcal{M}_{I_\delta^b}\big(\widetilde{U}_3\big)\|^2_{[L^2(\omega)]^9} \le C\frac{\delta\varepsilon^2}{r^2}\|u\|_V^2\end{aligned}\tag{3.36}$$

then with $(3.35)_2$

$$\big|\big|\mathcal{M}_r\big(\mathcal{U}_3^a\big) - \mathcal{M}_r\big(\mathcal{U}_3^b\big)\big|\big|^2_{L^2(\omega)} \le C\frac{\varepsilon^2}{r^2\delta}\Big[\delta + \frac{\varepsilon^2}{\delta}\ln\left(\frac{\varepsilon}{r}\right)\Big]^2\|u\|_V^2$$

which, in turn, with $(3.26)_2$, (3.18), and (3.32) give $(3.33)_1$ and $(3.33)_2$ (observe that due to assumption (3.1) one has $\frac{\varepsilon}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)} \ge 1$), while $(3.35)_2$, $(3.36)_1$, and again $(3.26)_2$, (3.18), (3.32) lead to $(3.33)_3$ and $(3.33)_4$.

Step 2. One proves (3.34). Consider $(3.17)_2$, (3.21), and $(3.26)_1$. One gets

$$\begin{aligned}&\sum_{\varepsilon\in\Xi_\varepsilon}\big|\big|\mathcal{U}_1^b(\varepsilon\xi + \cdot) - \mathcal{M}_r\big(\mathcal{U}_1^b\big)(\varepsilon\xi)\big|\big|^2_{L^2(Y_\varepsilon)} \le C\varepsilon^2\ln\left(\frac{\varepsilon}{r}\right)\frac{\|u\|_V^2}{\delta},\\ &\sum_{\varepsilon\in\Xi_\varepsilon}\big|\big|\mathcal{U}_1^a(\varepsilon\xi + \cdot) - \mathcal{M}_r\big(\mathcal{U}_1^a\big)(\varepsilon\xi)\big|\big|^2_{L^2(Y_\varepsilon)} \le C\varepsilon^2\ln\left(\frac{\varepsilon}{r}\right)\|\nabla\mathcal{U}_1^a\|^2_{[L^2(\omega)]^2},\\ \implies &\sum_{\varepsilon\in\Xi_\varepsilon}\big|\big|\mathcal{U}_1^a(\varepsilon\xi + \cdot) - \mathcal{M}_r\big(\mathcal{U}_1^a\big)(\varepsilon\xi)\big|\big|^2_{L^2(Y_\varepsilon)} \le C\varepsilon^2\ln\left(\frac{\varepsilon}{r}\right)\Big(\frac{\|u\|_V^2}{\delta} + |b_3^a|^2\Big).\end{aligned}$$

Because of (3.15), $(3.28)_1$, and the previous estimates

$$\begin{aligned}&\sum_{\varepsilon\in\Xi_\varepsilon}\big|\big|u_1(\varepsilon\xi + \cdot) - \mathcal{M}_r\big(\mathcal{U}_1^b\big)(\varepsilon\xi) + (x_3 + \delta)\mathcal{M}_r\big(\mathcal{R}_2^b\big)(\varepsilon\xi)\big|\big|^2_{L^2(Y_\varepsilon\times I_\delta^b)} \le C\Big[\delta^2 + \varepsilon^2\ln\left(\frac{\varepsilon}{r}\right)\Big]\|u\|_V^2,\\ &\sum_{\varepsilon\in\Xi_\varepsilon}\big|\big|u_1(\varepsilon\xi + \cdot) - \mathcal{M}_r\big(\mathcal{U}_1^a\big)(\varepsilon\xi) + (x_3 - \delta)\mathcal{M}_r\big(\mathcal{R}_2^a\big)(\varepsilon\xi)\big|\big|^2_{L^2(Y_\varepsilon\times I_\delta^a)} \le C\Big[\delta^2 + \varepsilon^2\ln\left(\frac{\varepsilon}{r}\right)\Big]\|u\|_V^2 + C\varepsilon^2\ln\left(\frac{\varepsilon}{r}\right)\delta|b_3^a|^2.\end{aligned}$$

The above estimates together with $(3.9)_2$ leads to

$$\begin{aligned}
&\left\|\mathcal{M}_r(\mathcal{U}_1^b) - \mathcal{M}_{I_\delta^b}(\widetilde{U}_1)\right\|_{L^2(\omega)}^2 \le \frac{C\varepsilon^2}{r^2\delta}\left[\frac{\delta^4}{r^2} + \varepsilon^2 \ln\left(\frac{\varepsilon}{r}\right)\right]\|u\|_V^2,\\
&\left\|\mathcal{M}_r(\mathcal{U}_1^a) - \mathcal{M}_{I_\delta^a}(\widetilde{U}_1)\right\|_{L^2(\omega)}^2 \le \frac{C\varepsilon^2}{r^2\delta}\left[\frac{\delta^4}{r^2} + \varepsilon^2 \ln\left(\frac{\varepsilon}{r}\right)\right]\|u\|_V^2 + \frac{C\varepsilon^4}{r^2}\ln\left(\frac{\varepsilon}{r}\right)|b_3^a|^2.
\end{aligned} \tag{3.37}$$

Besides, from (3.7), $(3.12)_4$, and $(3.27)_4$, one derives

$$\begin{aligned}
\|\widetilde{U}_1 - \mathcal{M}_{I_\delta^d}(\widetilde{U}_1)\|_{L^2(\omega\times I_\delta)}^2 &\le C\delta^2\left(\frac{\varepsilon^2}{r^2}\|u\|_V^2 + \|\widetilde{\mathbf{R}}^{\mathrm{be}}\|_{[L^2(\omega\times I_\delta)]^9}^2\right)\\
&\le C\left[1 + \frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]^2\|u\|_V^2,\\
\|\mathcal{M}_{I_\delta^a}(\widetilde{U}_1) - \mathcal{M}_{I_\delta^b}(\widetilde{U}_1)\|_{L^2(\omega)}^2 &\le C\left[1 + \frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]^2\frac{\|u\|_V^2}{\delta}.
\end{aligned} \tag{3.38}$$

Hence with (3.37)

$$\|\mathcal{M}_r(\mathcal{U}_1^a) - \mathcal{M}_r(\mathcal{U}_1^b)\|_{L^2(\omega)}^2 \le C\left[1 + \frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]^2\frac{\|u\|_V^2}{\delta} + C\frac{\varepsilon^4}{r^2}\ln\left(\frac{\varepsilon}{r}\right)|b_3^a|^2.$$

The above estimate and $(3.26)_2$ yield

$$\begin{aligned}
\|\mathcal{U}_1^a - \mathcal{U}_1^b\|_{L^2(\omega)}^2 &\le C\varepsilon^2\ln\left(\frac{\varepsilon}{r}\right)\left(\|\nabla\mathcal{U}_1^a\|_{[L^2(\omega)]^2}^2 + \|\nabla\mathcal{U}_1^b\|_{[L^2(\omega)]^2}^2\right)\\
&\quad + C\left[1 + \frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]^2\frac{\|u\|_V^2}{\delta} + C\frac{\varepsilon^4}{r^2}\ln\left(\frac{\varepsilon}{r}\right)|b_3^a|^2\\
&\le C\left[1 + \frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]^2\frac{\|u\|_V^2}{\delta} + C\frac{\varepsilon^4}{r^2}\ln\left(\frac{\varepsilon}{r}\right)|b_3^a|^2.
\end{aligned}$$

An upper bound of $\|\mathcal{U}_2^a - \mathcal{U}_2^b\|_{L^2(\omega)}^2$ is obtained in the same way, the estimate is the same. As a consequence, using $(3.17)_2$, the following estimate holds ($\alpha \in \{1,2\}$):

$$\|\mathcal{U}_\alpha^a\|_{L^2(\omega)}^2 \le C\left[1 + \frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]^2\frac{\|u\|_V^2}{\delta} + C\frac{\varepsilon^4}{r^2}\ln\left(\frac{\varepsilon}{r}\right)|b_3^a|^2. \tag{3.39}$$

Now, observe that

$$\int_\omega \left(x_1\ \mathcal{U}_2^a(x') - x_2\ \mathcal{U}_1^a(x')\right)dx' = \int_\omega \left(x_1(\mathcal{U}_2^a(x') - \mathbf{r}_2^a(x')) - x_2(\mathcal{U}_1^a(x') - \mathbf{r}_1^a(x'))\right)dx' + 2b_3^a\int_\omega (x_1^2 + x_2^2)dx'.$$

Then (3.21) and (3.39) give

$$|b_3^a|^2 \le C\left[1 + \frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]^2\frac{\|u\|_V^2}{\delta} + C\frac{\varepsilon^4}{r^2}\ln\left(\frac{\varepsilon}{r}\right)|b_3^a|^2.$$

If $\frac{\varepsilon^4}{r^2}\ln\left(\frac{\varepsilon}{r}\right)$ is small enough, that gives an estimate of b_3^a. Then summarizing the estimates of this step and using (3.21) lead to $(3.34)_1$ and $(3.34)_2$, then $(3.37)_1$, $(3.38)_1$, $(3.34)_2$, and (3.17) give $(3.34)_3$. □

As a consequence of the above two lemmas, one obtains the estimates of the restriction of u to the plate Ω^a_δ:

$$\begin{aligned}
\|\nabla u\|_{[L^2(\Omega^a_\delta)]^9} &\le C\left[1+\frac{\varepsilon\delta^2}{r^2}+\frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]\frac{\|u\|_V}{\delta},\\
\|u_\alpha\|_{L^2(\Omega^a_\delta)} &\le C\left[1+\frac{\varepsilon\delta^2}{r^2}+\frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]^2\|u\|_V,\\
\|u_3\|_{L^2(\Omega^a_\delta)} &\le C\left[1+\frac{\varepsilon\delta^2}{r^2}+\frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]^2\frac{\|u\|_V}{\delta}.
\end{aligned}\tag{3.40}$$

Furthermore, from (3.6) to $(3.27)_4$, (3.4), $(3.33)_4$, and $(3.34)_3$, one derives the following estimates of the restriction of u to the set of beams $B_{\delta,\varepsilon,r}$:

$$\begin{aligned}
\|\nabla u\|_{[L^2(B_{\delta,\varepsilon,r})]^9} &\le C\frac{r}{\varepsilon\delta}\left[1+\frac{\varepsilon\delta^2}{r^2}+\frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]\|u\|_V,\\
\|u_\alpha\|_{L^2(B_{\delta,\varepsilon,r})} &\le C\frac{r}{\varepsilon}\left[1+\frac{\varepsilon\delta^2}{r^2}+\frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]|u\|_V,\\
\|u_3\|_{L^2(B_{\delta,\varepsilon,r})} &\le C\frac{r}{\varepsilon\delta}\left[1+\frac{\varepsilon\delta^2}{r^2}+\frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right]^2\|u\|_V.
\end{aligned}\tag{3.41}$$

4 | MAIN CASES

In view of (3.33), (3.34), and in order that both midsurface displacements $\mathcal{U}^a$ and $\mathcal{U}^b$ match, one assumes that

$$\frac{\varepsilon\delta^2}{r^2}\quad\text{is uniformly bounded from above and}\quad\frac{\varepsilon^4}{r^2}\ln\left(\frac{\varepsilon}{r}\right)\text{ is small.}\tag{4.1}$$

Now, the 3 small parameters δ, r, and ε are linked. Set

$$\varepsilon=\kappa_0\delta^\rho,\quad r=\kappa_1\delta^\eta,\quad \eta>0,\quad \rho>0.\tag{4.2}$$

Conditions (4.1) and assumptions (3.1) lead to

$$1\le\eta<2\rho\qquad\text{and}\qquad 2\eta\le\rho+2,\qquad \begin{array}{ll}\kappa_1\le 1, & \text{if } \eta=1,\\ \kappa_1<\kappa_0/2, & \text{if } \rho=\eta.\end{array}$$

The couple (η,ρ) must belong to the convex polygon (without the edge $\eta=2\rho$) whose vertexes are

$$(1,1),\qquad(1,1/2),\qquad(4/3,2/3),\qquad(2,2).$$

Thus, there are 6 cases to analyze. They correspond to 2 vertexes, 3 edges, and the interior of the polygon. The interior of this convex polygon corresponds to the most general situation. We will analyze this case in the next sections.

From now on, one assumes (3.1) *and* (4.1).

Now, we rewrite the estimates (3.40) and (3.41) obtained in the previous section:

$$\begin{aligned}
&\|u_\alpha\|_{L^2(\Omega^a_\delta)}\le C\|u\|_V, && \|\nabla u\|_{[L^2(\Omega^a_\delta)]^9}+\|u_3\|_{L^2(\Omega^a_\delta)}\le\frac{C}{\delta}\|u\|_V,\\
&\|u_\alpha\|_{L^2(B_{\delta,\varepsilon,r})}\le C\frac{r}{\varepsilon}\|u\|_V, && \|\nabla u\|_{[L^2(B_{\delta,\varepsilon,r})]^9}+\|u_3\|_{L^2(B_{\delta,\varepsilon,r})}\le C\frac{r}{\varepsilon\delta}\|u\|_V.
\end{aligned}\tag{4.3}$$

5 | ASSUMPTIONS ON THE APPLIED FORCES

In view of the energy relation (2.7) and the estimates (3.20) and (4.3), one can scale the applied forces:

- in the plate Ω^b_δ and Ω^a_δ, the applied forces are given by ($d\in\{a,b\}$):

$$\begin{aligned}
f_{\alpha,\delta}(x)&=\delta f^d_\alpha\left(x_1,x_2,\frac{x_3}{\delta}\right)\text{ for a.e. } x \text{ in } \Omega^d_\delta,\\
f_{3,\delta}(x)&=\delta^2 f^d_3\left(x_1,x_2,\frac{x_3}{\delta}\right)\text{ for a.e. } x \text{ in } \Omega^d_\delta,
\end{aligned}\tag{5.1}$$

  where f^d belongs to $L^2(\Omega^d;\mathbb{R}^3)$,

- in the set of beams $B_{\delta,\varepsilon,r}$, the applied forces are given by

$$\begin{aligned} f_{\alpha,\delta}(x) &= \frac{\varepsilon^2\delta}{r^2} f_\alpha^{\mathrm{be}}\Big(\varepsilon\xi, \frac{x_1-\varepsilon\xi_1}{r}, \frac{x_2-\varepsilon\xi_2}{r}, \frac{x_3}{\delta}\Big), \\ f_{3,\delta}(x) &= \frac{\varepsilon^2\delta^2}{r^2} f_3^{\mathrm{be}}\Big(\varepsilon\xi, \frac{x_1-\varepsilon\xi_1}{r}, \frac{x_2-\varepsilon\xi_2}{r}, \frac{x_3}{\delta}\Big), \\ &\text{for a.e. } x \text{ in } \big(\varepsilon\xi + D_r\big)\times I_\delta^{\mathrm{be}}, \quad \xi\in\Xi_\varepsilon, \end{aligned} \tag{5.2}$$

where $f^{\mathrm{be}} \in C(\overline{\omega}; L^2(B^{\mathrm{be}};\mathbb{R}^3))$.

As a consequence of (3.20) to (4.3), one obtains the following bound of the total elastic energy:

$$\mathcal{E}(u_\delta) = \int_{\Omega_{\delta,\varepsilon,r}} f_{i,\delta} u_{i,\delta} dx \le C\delta^{3/2}\|u_\delta\|_V, \tag{5.3}$$

where C is a constant independent of δ, ε, and r. Taking into account to (2.1) and (2.2), there exists a constant $c > 0$ independent of δ, ε, and r such that

$$c\|u_\delta\|_V^2 \le \mathcal{E}(u_\delta) \le C\delta^{3/2}\|u_\delta\|_V.$$

Hence

$$\|u_\delta\|_V \le C\delta^{3/2} \quad \text{and} \quad \mathcal{E}(u_\delta) \le C\delta^3. \tag{5.4}$$

6 | THE RESCALING OPERATORS

6.1 | First rescaling operator

Let φ be a measurable function on Ω_δ^d, $d \in \{a, b\}$. Define the measurable function $\Pi_\delta(\varphi)$ by

$$\Pi_\delta(\varphi)(x', X_3) = \varphi(x', \delta X_3) \qquad \text{for a.e. } (x', X_3) \in \Omega^d.$$

The linear operator Π_δ also satisfies for every $(\phi, \psi) \in [L^2(\Omega_\delta^d)]^2$: $\Pi_\delta(\phi)\Pi_\delta(\psi) = \Pi_\delta(\phi\psi)$. Moreover, for every $\phi \in L^2(\Omega_\delta^d)$ one has

$$\int_{\Omega^d} \Pi_\delta(\phi) dx' dX_3 = \frac{1}{\delta}\int_{\Omega_\delta^d} \phi\, dx' dx_3 \qquad \|\Pi_\delta(\phi)\|_{L^2(\Omega^d)} = \frac{1}{\delta^{1/2}}\|\phi\|_{L^2(\Omega_\delta^d)}. \tag{6.1}$$

For every $\phi \in L^2(\omega; H^1(I_\delta^d))$

$$\frac{\partial \Pi_\delta(\phi)}{\partial X_3} = \delta\Pi_\delta\Big(\frac{\partial\phi}{\partial x_3}\Big).$$

As a consequence of the above equality one gets

$$\Big\|\frac{\partial \Pi_\delta(\phi)}{\partial X_3}\Big\|_{L^2(\Omega^d)} = \delta^{1/2}\Big\|\frac{\partial\phi}{\partial x_3}\Big\|_{L^2(\Omega_\delta^d)}. \tag{6.2}$$

6.2 | Second rescaling operator

For ψ measurable on $\omega \times \mathbf{P}_\delta$, define the measurable function $\Pi_r(\psi)$ by

$$\Pi_r(\psi)(x', X_1, X_2, X_3) = \psi(x', rX_1, rX_2, \delta X_3) \qquad \text{for a.e. } (x', X) \in \omega \times B.$$

For every $\Phi \in L^2(\omega \times \mathbf{P}_\delta)$, one has

$$r^2\delta \int_{\omega\times B} \Pi_r(\Phi) dx' dX = \int_{\omega\times \mathrm{P}_\delta} \Phi(x', x)\, dx' dx.$$

Hence,

$$\|\Pi_r(\Phi)\|_{L^2(\omega\times B)} = \frac{1}{\delta^{1/2} r}\|\Phi\|_{L^2(\omega\times \mathrm{P}_\delta)}. \tag{6.3}$$

For every $\phi \in L^2(\omega; H^1(\mathbf{P}_\delta))$

$$\frac{\partial \Pi_r(\Phi)}{\partial X_\alpha} = r\Pi_r\Big(\frac{\partial \Phi}{\partial x_\alpha}\Big) \qquad \text{in } L^2(\omega\times B), \quad \alpha = 1,2,$$
$$\frac{\partial \Pi_r(\Phi)}{\partial X_3} = \delta\Pi_r\Big(\frac{\partial \Phi}{\partial x_3}\Big) \qquad \text{in } L^2(\omega\times B).$$

From these equalities one derives

$$\begin{aligned}
&\Big|\Big|\frac{\partial \Pi_r(\Phi)}{\partial X_\alpha}\Big|\Big|_{L^2(\omega\times B)} = \frac{1}{\delta^{1/2}}\Big|\Big|\frac{\partial \Phi}{\partial x_\alpha}\Big|\Big|_{L^2(\omega\times \mathbf{P}_\delta)}, \quad \alpha = 1,2,\\
&\Big|\Big|\frac{\partial \Pi_r(\Phi)}{\partial X_3}\Big|\Big|_{L^2(\omega\times B)} = \frac{\delta^{1/2}}{r}\Big|\Big|\frac{\partial \Phi}{\partial x_3}\Big|\Big|_{L^2(\omega\times \mathbf{P}_\delta)}.
\end{aligned} \tag{6.4}$$

From now on, the couple(ρ,η)belongs to the interior of the polygon. In this case observe that

$$\lim_{\delta\to 0}\frac{r}{\varepsilon} = \lim_{\delta\to 0}\frac{r}{\delta} = 0 \qquad \textit{and} \qquad \lim_{\delta\to 0}\frac{\varepsilon\delta^2}{r^2} = \lim_{\delta\to 0}\frac{\varepsilon^4}{r^2}\ln\Big(\frac{\varepsilon}{r}\Big) = 0. \tag{6.5}$$

7 | THE LIMIT FIELDS

Let $\{u_\delta\}_\delta$ be a sequence of displacements, $u_\delta \in V_{\delta,\varepsilon,r}$, satisfying

$$\|u_\delta\|_V \le C\delta^{3/2}. \tag{7.1}$$

Using (6.3) and (6.4), now the estimates (3.12) become

$$\begin{aligned}
&\|\Pi_r\big(\widetilde{\widetilde{u}}_\delta\big)\|_{[L^2(\omega\times B)]^3} + \Big|\Big|\frac{\partial \Pi_r\big(\widetilde{\widetilde{u}}_\delta\big)}{\partial X_\alpha}\Big|\Big|_{[L^2(\omega\times B)]^3} \le C\varepsilon\delta,\\
&\Big|\Big|\frac{\partial \Pi_r\big(\widetilde{\widetilde{u}}_\delta\big)}{\partial X_3}\Big|\Big|_{[L^2(\omega\times B)]^3} \le C\frac{\varepsilon\delta^2}{r}, \quad \Big|\Big|\frac{\partial \Pi_r\big(\widetilde{R}_\delta\big)}{\partial X_3}\Big|\Big|_{[L^2(\omega\times I)]^3} \le C\frac{\varepsilon\delta^2}{r^2},\\
&\Big|\Big|\frac{\partial \Pi_r\big(\widetilde{U}_\delta\big)}{\partial X_3} - \delta\Pi_r\big(\widetilde{R}_\delta\big)\wedge \mathbf{e}_3\Big|\Big|_{[L^2(\omega\times I)]^3} \le C\frac{\varepsilon\delta^2}{r}.
\end{aligned} \tag{7.2}$$

and estimates $(3.27)_3$, $(3.27)_4$, $(3.33)_4$, $(3.34)_3$, and $(7.2)_3$ $(d\in\{a,b\})$

$$\begin{aligned}
&\|\Pi_r\big(\widetilde{\mathbf{R}}^{\mathrm{be}}_\delta\big)\|_{[L^2(\omega\times I)]^3} \le C, \quad \Big|\Big|\mathbf{R}^d_{\mathrm{pl},\delta} - \Pi_r\big(\widetilde{\mathbf{R}}^{\mathrm{be}}_\delta\big)\Big|\Big|_{[L^2(\omega\times I)]^9} \le C\left[\frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\Big(\frac{\varepsilon}{r}\Big)}\right],\\
&\Big|\Big|\frac{\partial \Pi_r\big(\widetilde{U}_{\alpha,\delta}\big)}{\partial X_3}\Big|\Big|_{L^2(\omega\times I)} + \|\Pi_r\big(\widetilde{U}_{\alpha,\delta}\big)\|_{L^2(\omega\times I)} \le C\delta, \qquad \|\Pi_r\big(\widetilde{U}_{3,\delta}\big)\|_{L^2(\omega;H^1(I))} \le C.
\end{aligned} \tag{7.3}$$

Estimates (3.15), (3.17) to (3.19), $(3.27)_1$, $(3.27)_2$, $(3.33)_1$, $(3.33)_2$, and $(3.34)_1$ become

$$\begin{aligned}
&\|\mathcal{U}^d_{\alpha,\delta}\|_{H^1(\omega)} + \Big|\Big|\frac{\partial \mathcal{U}^d_{3,\delta}}{\partial x_\alpha} + \mathcal{R}_{\alpha,d}\Big|\Big|_{L^2(\omega)} \le C\delta, \quad \|\mathcal{U}^d_{3,\delta}\|_{H^1(\omega)} + \|\mathcal{R}^d_\delta\|_{[H^1(\omega)]^2} \le C,\\
&\|\mathcal{R}^a_{\alpha,\delta} - \mathcal{R}^b_{\alpha,\delta}\|_{L^2(\omega)} \le C\left[\frac{\varepsilon\delta^2}{r^2} + \frac{\varepsilon^2}{r}\sqrt{\ln\Big(\frac{\varepsilon}{r}\Big)}\right], \quad \|\mathcal{U}^a_{3,\delta} - \mathcal{U}^b_{3,\delta}\|_{L^2(\omega)} \le C\varepsilon\sqrt{\ln\Big(\frac{\varepsilon}{r}\Big)}\\
&\Big|\Big|\frac{\partial \Pi_\delta\big(\overline{u}^d_\delta\big)}{\partial x_\alpha}\Big|\Big|_{[L^2(\Omega^d)]^3} \le C\delta, \quad \Big|\Big|\frac{\partial \Pi_\delta\big(\overline{u}^d_\delta\big)}{\partial X_3}\Big|\Big|_{[L^2(\Omega^d)]^3} + \Big|\Big|\Pi_\delta\big(\overline{u}^d_\delta\big)\Big|\Big|_{[L^2(\Omega^d)]^3} \le C\delta^2.
\end{aligned} \tag{7.4}$$

As a consequence of the above estimates one obtains the following theorem.

Theorem 1. *Let $\{u_\delta\}_\delta$ be a sequence of displacements belonging to $V_{\delta,\varepsilon,r}$ and satisfying (7.1). There exist a subsequence of $\{\delta\}$, still denoted $\{\delta\}$ and $\widetilde{\widetilde{u}} \in L^2(\omega\times I; H^1(D;\mathbb{R}^3))$, $\widetilde{R}_\alpha$, $\widetilde{U}_\alpha \in L^2(\omega)$, $(\widetilde{U}_3, \widehat{R}_1, \widehat{R}_2, \widetilde{R}_3, \widehat{U}_3) \in [L^2(\omega; H^1(I))]^5$ satisfying $\widetilde{R}_3 = 0$ in $\omega\times(I^a\cup I^b)$, $\widehat{R}_\alpha(\cdot,0) = \widehat{U}_3(\cdot,0) = 0$ a.e. in ω and $\widetilde{Z}_\alpha \in L^2(\omega\times I)$ such that*

$$
\begin{aligned}
&\frac{1}{\varepsilon\delta}\Pi_r(\widetilde{\ddot{u}}_\delta) \rightharpoonup \widetilde{\ddot{u}} \qquad \textit{weakly in } L^2(\omega\times I; H^1(D;\mathbb{R}^3)),\\
&\frac{r}{\varepsilon\delta^2}\frac{\partial \Pi_r(\widetilde{\ddot{u}}_\delta)}{\partial X_3} \rightharpoonup 0 \qquad \textit{weakly in } L^2(\omega\times B;\mathbb{R}^3),\\
&\Pi_r(\widetilde{R}_\delta) \rightharpoonup \widetilde{R} \qquad \textit{weakly in } L^2(\omega; H^1(I;\mathbb{R}^3)),\\
&\frac{r^2}{\varepsilon\delta^2}\Pi_r(\widetilde{R}_{\delta,\alpha} - \widetilde{R}_{\delta,\alpha}(\cdot,0)) \rightharpoonup \widehat{R}_\alpha \qquad \textit{weakly in } L^2(\omega; H^1(I)),\\
&\frac{r^2}{\varepsilon\delta^2}\Pi_r(\widetilde{R}_{\delta,3}) \rightharpoonup \widetilde{R}_3 \qquad \textit{weakly in } L^2(\omega; H^1(I)),\\
&\frac{1}{\delta}\Pi_r(\widetilde{U}_{\alpha,\delta}) \rightharpoonup \widetilde{U}_\alpha \qquad \textit{weakly in } L^2(\omega; H^1(I)),\\
&\Pi_r(\widetilde{U}_{3,\delta}) \rightharpoonup \widetilde{U}_3 \qquad \textit{weakly in } L^2(\omega; H^1(I)),\\
&\frac{r}{\varepsilon\delta^2}\Pi_r\big(\widetilde{U}_{3,\delta} - \widetilde{U}_{3,\delta}(\cdot,0)\big) \rightharpoonup \widehat{U}_3 \qquad \textit{weakly in } L^2(\omega; H^1(I)),\\
&\frac{r}{\varepsilon\delta^2}\Big(\frac{\partial \Pi_r(\widetilde{U}_{1,\delta})}{\partial X_3} - \delta\Pi_r(\widetilde{R}_{2,\delta})\Big) \rightharpoonup \widetilde{Z}_1 \qquad \textit{weakly in } L^2(\omega\times I),\\
&\frac{r}{\varepsilon\delta^2}\Big(\frac{\partial \Pi_r(\widetilde{U}_{2,\delta})}{\partial X_3} + \delta\Pi_r(\widetilde{R}_{1,\delta})\Big) \rightharpoonup \widetilde{Z}_2 \qquad \textit{weakly in } L^2(\omega\times I).
\end{aligned}
\tag{7.5}
$$

Furthermore, there exist $\overline{u}^d \in L^2(\omega; H^1(I^d;\mathbb{R}^3))$, $\mathcal{R}_\alpha \in H_0^1(\omega)$, $\mathcal{U}_\alpha^d \in H_0^1(\omega)$, $\mathcal{U}_3 \in H_0^2(\omega)$ *and* $Z_\alpha^d \in L^2(\omega)$, $d \in \{a,b\}$, *such that*

$$
\begin{aligned}
&\frac{1}{\delta^2}\Pi_\delta(\overline{u}_\delta^d) \rightharpoonup \overline{u}^d \qquad \textit{weakly in } L^2(\omega; H^1(I^d;\mathbb{R}^3)),\\
&\frac{1}{\delta}\frac{\partial \Pi_\delta(\overline{u}_\delta^d)}{\partial x_\alpha} \rightharpoonup 0 \qquad \textit{weakly in } L^2(\Omega^d;\mathbb{R}^3),\\
&\mathcal{R}_{\alpha,\delta}^d \rightharpoonup \mathcal{R}_\alpha \qquad \textit{weakly in } H^1(\omega),\\
&\frac{1}{\delta}\mathcal{U}_{\alpha,\delta}^d \rightharpoonup \mathcal{U}_\alpha^d \qquad \textit{weakly in } H^1(\omega),\\
&\mathcal{U}_{3,\delta}^d \rightharpoonup \mathcal{U}_3 \qquad \textit{weakly in } H^1(\omega),\\
&\frac{1}{\delta}\Big(\frac{\partial \mathcal{U}_{3,\delta}^d}{\partial x_\alpha} + \mathcal{R}_{\alpha,\delta}^d\Big) \rightharpoonup Z_\alpha^d \qquad \textit{weakly in } L^2(\omega).
\end{aligned}
\tag{7.6}
$$

Moreover,

$$
\frac{\partial \mathcal{U}_3}{\partial x_\alpha} = -\mathcal{R}_\alpha, \quad \textit{a.e. in } \omega, \qquad \widetilde{U}_3 = \mathcal{U}_3 \quad \textit{a. e. in } \omega\times I. \tag{7.7}
$$

Set

$$
\mathcal{U}_\alpha = \frac{1}{2}\big(\mathcal{U}_\alpha^a + \mathcal{U}_\alpha^b\big).
$$

The limit fields $\frac{\partial \widetilde{U}_\alpha}{\partial X_3}$, $\widetilde{U}_\alpha$ *and* $\mathcal{U}_\alpha$, $\frac{\partial \mathcal{U}_3}{\partial x_\alpha}$ *are linked by the following junction conditions:*

$$
\begin{aligned}
&\frac{\partial \widetilde{U}_1}{\partial X_3} = -\frac{\partial \mathcal{U}_3}{\partial x_1} = \widetilde{R}_2, \qquad \frac{\partial \widetilde{U}_2}{\partial X_3} = -\frac{\partial \mathcal{U}_3}{\partial x_2} = -\widetilde{R}_1,\\
&\widetilde{U}_\alpha(\cdot,X_3) = \mathcal{U}_\alpha(\cdot) - X_3\frac{\partial \mathcal{U}_3}{\partial x_\alpha}(\cdot), \qquad \mathcal{U}_\alpha^d = \mathcal{U}_\alpha - \mathbf{s}\frac{\partial \mathcal{U}_3}{\partial x_\alpha},
\end{aligned}
\qquad \textit{a.e. in } \omega\times I. \tag{7.8}
$$

Proof. The relations (6.5) are extensively used in the proof of this theorem even if this fact is not always specified.

Step 1. As a consequence of the estimates (7.2) and (7.3) there exist a subsequence of $\{\delta\}$, still denoted $\{\delta\}$ and functions such that the convergences $(7.5)_1$, $(7.5)_3$ to $(7.5)_{10}$, $(7.6)_1$, and $(7.6)_3$ to $(7.6)_6$ hold. From estimates (7.4) one deduces that the sequences $\{\mathcal{R}_{\alpha,\delta}^a\}_\delta$, $\{\mathcal{R}_{\alpha,\delta}^a\}_d$ (resp. $\{\mathcal{U}_{3,\delta}^a\}_\delta$, $\{\mathcal{U}_{3,\delta}^a\}_d$) converge to the same limit. Moreover, the boundary conditions on the functions $\mathcal{U}_{\alpha,\delta}^b$, $\mathcal{U}_{3,\delta}^b$, and $\mathcal{R}_{\alpha,\delta}^b$ yield

$$
\mathcal{U}_\alpha^b, \quad \mathcal{U}_3, \quad \mathcal{R}_\alpha \in H_0^1(\omega).
$$

Step 2. Because of (7.2), the sequence $\frac{r}{\varepsilon\delta^2}\left\{\Pi_r(\widetilde{\ddot{u}}_\delta)\right\}_\delta$ is bounded in $L^2(\omega;H^1(B;\mathbb{R}^3))$. Hence, up to a subsequence it weakly convergences to a limit belonging to $L^2(\omega;H^1(B;\mathbb{R}^3))$. Convergence $(7.5)_1$ implies

$$\frac{r}{\varepsilon\delta^2}\Pi_r(\widetilde{\ddot{u}}_\delta)=\frac{r}{\delta}\left(\frac{1}{\varepsilon\delta}\Pi_r(\widetilde{\ddot{u}}_\delta)\right)\to 0 \qquad \text{strongly in } L^2(\omega;H^1(B;\mathbb{R}^3)).$$

Hence $(7.5)_2$. Similarly, one proves $(7.6)_2$.

Step 3. The first equality of (7.7) is a consequence of $(7.4)_1$; thus, $\mathcal{U}_3$ belongs to $H_0^2(\omega)$. From $(3.33)_3$ one deduces that $\widetilde{U}_3=\mathcal{U}_3$ a.e. in $\omega\times I$, which proves $(7.7)_2$.

Step 4. Recall that $\left|\left|\frac{\partial\Pi_\delta(\widetilde{U}_\delta)}{\partial X_3}-\delta\Pi_\delta(\widetilde{R}_\delta)\wedge\mathbf{e}_3\right|\right|_{L^2(\omega\times I)}\le C\frac{\varepsilon\delta^2}{r}$, then the convergences $(7.5)_3$ and $(7.5)_5$ lead to

$$\frac{\partial\widetilde{U}_1}{\partial X_3}=\widetilde{R}_2,\quad \frac{\partial\widetilde{U}_2}{\partial X_3}=-\widetilde{R}_1 \quad \text{a.e. in } \omega\times I. \tag{7.9}$$

Therefore, $\widetilde{U}_\alpha$ belongs to $L^2(\omega;H^2(I))$. Because of the estimates (7.2) and (7.3), the field $\widetilde{R}$ does not depend on the variable X_3. Then $(7.3)_2$ implies $\widetilde{R}_3=0$ in $\omega\times(I^a\cup I^b)$.

Now, because $\lim\limits_{\varepsilon\to 0}\frac{\varepsilon\delta}{r}=\lim\limits_{\varepsilon\to 0}\left(\frac{\varepsilon\delta^2}{r^2}\cdot\frac{r}{\delta}\right)=0$, $(7.3)_2$ also yields

$$\mathcal{R}_1=\widetilde{R}_2,\qquad \mathcal{R}_2=-\widetilde{R}_1,\qquad \text{a.e. in } \omega. \tag{7.10}$$

Step 5. From (7.7), (7.9), and (7.10), there exists $C_\alpha\in L^2(\omega)$ such that

$$\widetilde{U}_\alpha(\cdot,X_3)=-X_3\frac{\partial\mathcal{U}_3}{\partial x_\alpha}(\cdot)+C_\alpha(\cdot),\qquad \text{in } L^2(\omega;H^1(I)). \tag{7.11}$$

Besides, from (3.37)

$$\left|\left|\mathcal{M}_r\left(\mathcal{U}^d_{\alpha,\delta}\right)-\mathcal{M}_{I^d_\delta}\left(\widetilde{U}_{\alpha,\delta}\right)\right|\right|_{L^2(\omega)}\le C\delta\left(\frac{\varepsilon\delta^2}{r^2}+\frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right). \tag{7.12}$$

Let ϕ be in $L^1(\Omega^d)$. A simple change of variable gives

$$\mathcal{M}_{I^d_\delta}(\phi)(x')=\mathcal{M}_{I^d}\left(\Pi_\delta(\phi)\right)(x') \quad \text{for a.e. } x'\in\omega \tag{7.13}$$

where

$$\forall\Phi\in L^1(\omega\times I^d),\qquad \mathcal{M}_{I^d}(\Phi)(x')=\frac{1}{2\kappa_d}\int_{-\kappa_d}^{\kappa_d}\Phi(x',X_3-\mathbf{s})dX_3,\quad \text{for a.e. } x'\in\omega.$$

Hence, from (3.26) to (7.4) and (7.12) together with the above equalities, one gets

$$\left|\left|\frac{1}{\delta}\mathcal{U}^d_{\alpha,\delta}-\frac{1}{\delta}\mathcal{M}_r\left(\mathcal{U}^d_{\alpha,\delta}\right)\right|\right|_{L^2(\omega)}\le C\varepsilon\sqrt{\ln\left(\frac{\varepsilon}{r}\right)},$$

$$\left|\left|\frac{1}{\delta}\mathcal{M}_r\left(\mathcal{U}^d_{\alpha,\delta}\right)-\mathcal{M}_{I^d}\left(\frac{1}{\delta}\Pi_\delta(\widetilde{U}_{\alpha,\delta})\right)\right|\right|_{L^2(\omega)}\le C\left(\frac{\varepsilon\delta^2}{r^2}+\frac{\varepsilon^2}{r}\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\right).$$

Passing to the limit gives $\mathcal{U}^d_\alpha=\mathcal{M}_{I^d}(\widetilde{U}_\alpha)$ and then with (7.11)

$$\mathcal{U}^d_\alpha=\mathcal{M}_{I^d}(\widetilde{U}_\alpha)=-\mathbf{s}\frac{\partial\mathcal{U}_3}{\partial x_\alpha}+C_\alpha.$$

Then one deduces the expression of $\widetilde{U}_\alpha(\cdot,X_3)$ in $\omega\times I$ and $\mathcal{U}^d_\alpha$ in ω (see (7.8). □

As a consequence of the above theorem and the decompositions (3.2) to (3.13), one has

$$\frac{1}{\delta}\Pi_r(\widetilde{u_{\alpha,\delta}})\rightharpoonup\mathcal{U}_\alpha-X_3\frac{\partial\mathcal{U}_3}{\partial x_\alpha},\quad \Pi_r(\widetilde{u_{3,\delta}})\rightharpoonup\mathcal{U}_3 \quad \text{weakly in } L^2(\omega;H^1(B^{\mathrm{be}})),$$

$$\frac{1}{\delta}\Pi_\delta(u_{\alpha,\delta})\rightharpoonup\mathcal{U}_\alpha-X_3\frac{\partial\mathcal{U}_3}{\partial x_\alpha},\quad \Pi_\delta(u_{3,\delta})\rightharpoonup\mathcal{U}_3 \qquad \text{weakly in } H^1(\Omega^d)$$

The limit displacement is of Kirchhoff-Love type.

8 | ASYMPTOTIC BEHAVIOR OF THE STRUCTURE

Now, let χ be in $D(\mathbb{R}^2)$ satisfying $\chi = 1$ in D_1, $\chi(x') \in [0,1]$ for all $x' \in \mathbb{R}^2$. Below, one recalls two classical approximation results.

Lemma 5. *Let ϕ be in $W^{1,\infty}(\omega)$. Define $\phi_{\varepsilon,r}$ by*

$$\forall x' \in \omega, \qquad \phi_{\varepsilon,r}(x') = \chi\Big(\frac{\varepsilon}{r}\Big\{\frac{x'}{\varepsilon}\Big\}\Big)\phi\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]\Big) + \Big[1 - \chi\Big(\frac{\varepsilon}{r}\Big\{\frac{x'}{\varepsilon}\Big\}\Big)\Big]\phi(x').$$

If $\frac{r}{\varepsilon}$ goes to 0 then for every $p \in [1,+\infty)$

$$\phi_{\varepsilon,r} \to \phi \quad \textit{strongly in } W^{1,p}(\omega). \tag{8.1}$$

Let Φ be in $W^{2,\infty}(\omega)$. Define $\widehat{\Phi}_{\varepsilon,r}$ by

$$\forall x' \in \omega, \quad \widehat{\Phi}_{\varepsilon,r}(x') = \chi\Big(\frac{\varepsilon}{r}\Big\{\frac{x'}{\varepsilon}\Big\}\Big)\Big[\Phi\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]\Big) + \varepsilon\Big\{\frac{x'}{\varepsilon}\Big\}\cdot\nabla\Phi\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]\Big)\Big] + \Big[1 - \chi\Big(\frac{\varepsilon}{r}\Big\{\frac{x'}{\varepsilon}\Big\}\Big)\Big]\Phi(x').$$

If $\frac{r}{\varepsilon}$ goes to 0 then for every $p \in [1,+\infty)$

$$\widehat{\Phi}_{\varepsilon,r} \to \Phi \quad \textit{strongly in } W^{2,p}(\omega). \tag{8.2}$$

8.1 | Weak convergences of the strain tensor fields

As immediate consequence of the convergences in Theorem 1 and the expressions (3.5) to (3.16) of the symmetric gradient in the beams and in the plates, we obtain the following proposition.

Proposition 1. *Under the hypotheses of Theorem 1, in the set of beams one has the weak convergence*

$$\frac{r}{\varepsilon\delta}\Pi_r\big(\big(\widetilde{\nabla u_\delta}\big)_S\big) \rightharpoonup \Gamma \qquad \textit{weakly in } [L^2(\omega\times B^{be})]^{3\times 3} \tag{8.3}$$

where the components of the symmetric matrix Γ are given by

$$\Gamma_{\alpha\beta} = \frac{1}{2}\Big(\frac{\partial\widetilde{\ddot{u}}_\alpha}{\partial X_\beta} + \frac{\partial\widetilde{\ddot{u}}_\beta}{\partial X_\alpha}\Big), \qquad \Gamma_{33} = \frac{\partial\widehat{U}_3}{\partial X_3} - X_1\frac{\partial\widehat{R}_2}{\partial X_3} + X_2\frac{\partial\widehat{R}_1}{\partial X_3},$$
$$\Gamma_{13} = \frac{1}{2}\Big(\widetilde{Z}_1 - X_2\frac{\partial\widetilde{R}_3}{\partial X_3} + \frac{\partial\widetilde{\ddot{u}}_3}{\partial X_1}\Big), \qquad \Gamma_{23} = \frac{1}{2}\Big(\widetilde{Z}_2 + X_1\frac{\partial\widetilde{R}_3}{\partial X_3} + \frac{\partial\widetilde{\ddot{u}}_3}{\partial X_2}\Big).$$

In the plates one obtains the following weak convergences ($d \in \{a,b\}$):

$$\frac{1}{\delta}\Pi_\delta\big(\big(\nabla u_\delta\big)_S\big) \rightharpoonup \Gamma^d \qquad \textit{weakly in } [L^2(\Omega^d)]^{3\times 3} \tag{8.4}$$

where the components of the symmetric matrix Γ^d are given by ($\mathcal{U}_m = \mathcal{U}_1\mathbf{e}_1 + \mathcal{U}_2\mathbf{e}_2$):

$$\Gamma^d_{\alpha\beta} = \gamma_{\alpha\beta}(\mathcal{U}_m) - X_3\frac{\partial^2\mathcal{U}_3}{\partial x_\alpha\partial x_\beta}, \quad \Gamma^d_{\alpha 3} = \frac{1}{2}\Big(Z^d_\alpha + \frac{\partial\overline{u}^d_\alpha}{\partial X_3}\Big), \quad \Gamma^d_{33} = \frac{\partial\overline{u}^d_3}{\partial X_3}. \tag{8.5}$$

8.2 | Determination of the strain tensor in the set of beams

To determine the $\widetilde{Z}_\alpha$'s and the warping $\widetilde{\ddot{u}}$ one proceeds as in section 6.1 of Blanchard et al[10] and section 8.1 of Blanchard et al,[11] one first derives $\widetilde{\ddot{u}}_\alpha$ and $\widetilde{Z}_\alpha$. That gives (a.e. in $\omega\times B^{\mathrm{be}}$),

$$\begin{aligned}
\widetilde{\ddot{u}}_1(\cdot,X) &= \nu^{be}\Big\{-X_1\frac{\partial\widehat{U}_3}{\partial X_3}(\cdot,X_3) + \frac{X_1^2 - X_2^2}{2}\frac{\partial\widehat{R}_2}{\partial X_3}(\cdot,X_3) - X_1X_2\frac{\partial\widehat{R}_1}{\partial X_3}(\cdot,X_3)\Big\},\\
\widetilde{\ddot{u}}_2(\cdot,X) &= \nu^{be}\Big\{-X_2\frac{\partial\widehat{U}_3}{\partial X_3}(\cdot,X_3) + X_1X_2\frac{\partial\widehat{R}_2}{\partial X_3}(\cdot,X_3) - \frac{X_2^2 - X_1^2}{2}\frac{\partial\widehat{R}_1}{\partial X_3}(\cdot,X_3)\Big\},\\
\widetilde{\ddot{u}}_3(\cdot,X) &= 0, \qquad \widetilde{Z}_\alpha = 0,
\end{aligned} \tag{8.6}$$

where $\nu^{be} = \frac{\lambda^{be}}{2(\lambda^{be}+\mu^{be})}$ is the Poisson coefficient of the material of the beams.

From the expression (2.2) of the stress tensor field, the convergence (8.3) and the expressions (8.6) of $\widetilde{\ddot{u}}$ and $\widetilde{Z}_\alpha$ one obtains

$$\frac{r}{\varepsilon\delta}\Pi_r(\widetilde{\sigma}_\delta) \rightharpoonup \Sigma \qquad \text{weakly in } [L^2(\omega\times B^{\mathrm{be}})]^{3\times 3} \tag{8.7}$$

where

$$\begin{aligned}
&\Sigma_{11}(\cdot,X)=\Sigma_{22}(\cdot,X)=\Sigma_{12}(\cdot,X)=0,\\
&\Sigma_{13}(\cdot,X)=-\mu^{be}X_2\frac{\partial\widetilde{R}_3}{\partial X_3}(\cdot,X_3),\quad \Sigma_{23}(\cdot,X)=\mu^{\Sigma_{23}(\cdot,X)=\mu^{be}X_1\frac{\partial\widetilde{R}_3}{\partial X_3}(\cdot,X_3)},\quad \text{a.e. in } \omega\times B^{\mathrm{be}}.\\
&\Sigma_{33}(\cdot,X)=E^{be}\Big(\frac{\partial\widehat{U}_3}{\partial X_3}(\cdot,X_3)-X_1\frac{\partial\widehat{R}_2}{\partial X_3}(\cdot,X_3)+X_2\frac{\partial\widehat{R}_1}{\partial X_3}(\cdot,X_3)\Big),
\end{aligned}\tag{8.8}$$

$E^{be}=\frac{\mu^{be}(3\lambda^{be}+2\mu^{be})}{\lambda^{be}+\mu^{be}}$ is the Young modulus of the elastic material of the beams. In Section 9, we will prove that

$$\Sigma=0\quad\text{in}\quad\omega\times B^{\mathrm{be}}.$$

8.3 | Determination of the Z_α^d's and the warpings $\overline{u}^d$, $d\in\{a,b\}$

Proceeding as in section 5.2 in Griso[5] one first deduces that the functions $\overline{u}_\alpha^d$ and Z_α^d are equal to zero and

$$\begin{aligned}
\frac{\partial\overline{u}_3^d}{\partial X_3}(\cdot,X_3)&=-\frac{\lambda^{pl}}{\lambda^{pl}+2\mu^{pl}}\Big(\gamma_{\alpha\alpha}(\mathcal{U}_m^d)-(X_3-\mathbf{s})\frac{\partial^2\mathcal{U}_3}{\partial x_\alpha\partial x_\alpha}\Big),\\
&=-\frac{\lambda^{pl}}{\lambda^{pl}+2\mu^{pl}}\Big(\gamma_{\alpha\alpha}(\mathcal{U}_m)-X_3\frac{\partial^2\mathcal{U}_3}{\partial x_\alpha\partial x_\alpha}\Big),
\end{aligned}\qquad \text{a.e. in } \Omega^d,\quad d\in\{a,b\}.$$

Recall that from (3.14), one has $\int_{I^d}\overline{u}_3^d(x',X_3)dX_3=0$, a.e. in ω. This equality allows to derive the function $\overline{u}_3^d$ in terms of the fields $\mathcal{U}_m$ and $\mathcal{U}_3$. But these expressions are useless; to give the limit of the stress tensors, we only need the knowledge of the partial derivative of $\overline{u}_3^d$ with respect to X_3. Again, from the expression (2.2) of the stress tensor field and now using the convergence (8.4) one gets

$$\frac{1}{\delta}\Pi_\delta(\sigma_\delta^d)\rightharpoonup\Sigma^d\quad\text{weakly in } [L^2(\Omega^d)]^{3\times3}.\tag{8.9}$$

Inserting the above expression of $\frac{\partial\overline{u}^d}{\partial X_3}$ and taking into account that $Z_\alpha^d=0$ lead to

$$\begin{aligned}
\Sigma_{11}^d&=\frac{E^{pl}}{1-(\nu^{pl})^2}\Big[\gamma_{11}(\mathcal{U}_m)-X_3\frac{\partial^2\mathcal{U}_3}{\partial x_1^2}+\nu^{pl}\Big(\gamma_{22}(\mathcal{U}_m)-X_3\frac{\partial^2\mathcal{U}_3}{\partial x_2^2}\Big)\Big],\\
\Sigma_{22}^d&=\frac{E^{pl}}{1-(\nu^{pl})^2}\Big[\gamma_{22}(\mathcal{U}_m)-X_3\frac{\partial^2\mathcal{U}_3}{\partial x_2^2}+\nu^{pl}\Big(\gamma_{11}(\mathcal{U}_m)-X_3\frac{\partial^2\mathcal{U}_3}{\partial x_1^2}\Big)\Big],\\
\Sigma_{12}^d&=\frac{E^{pl}}{1+\nu^{pl}}\Big[\gamma_{12}(\mathcal{U}_m)-X_3\frac{\partial^2\mathcal{U}_3}{\partial x_1\partial x_2}\Big],\qquad \Sigma_{i3}^d=0\quad\text{a.e. in } \Omega^d
\end{aligned}\tag{8.10}$$

where $\nu^{pl}=\frac{\lambda^{pl}}{2(\lambda^{pl}+\mu^{pl})}$ is the Poisson coefficient and $E^{pl}=\frac{\mu^{pl}(3\lambda^{pl}+2\mu^{pl})}{\lambda^{pl}+\mu^{pl}}$ is the Young modulus of the elastic material of the plates.

8.4 | The limit problem

Set

$$\begin{aligned}
f_\alpha&=\int_{I^a}f_\alpha^a(\cdot,X_3)dX_3+\int_{I^b}f_\alpha^b(\cdot,X_3)dX_3+\int_{B^{\mathrm{be}}}f_\alpha^{\mathrm{be}}(\cdot,X)dX,\\
f_3&=\Big(\int_{I^a}X_3\frac{\partial f_\alpha^a}{\partial x_\alpha}(\cdot,X_3)dX_3+\int_{I^b}X_3\frac{\partial f_\alpha^b}{\partial x_\alpha}(\cdot,X_3)dX_3+\int_{B^{\mathrm{be}}}X_3\frac{\partial f_\alpha^{\mathrm{be}}}{\partial x_\alpha}(\cdot,X)dX\Big)\\
&\quad+\int_{I^a}f_3^a(\cdot,X_3)dX_3+\int_{I^b}f_3^b(\cdot,X_3)dX_3+\int_{B^{\mathrm{be}}}f_3^{\mathrm{be}}(\cdot,X)dX.
\end{aligned}$$

One has $f_i\in L^2(\omega)$.

Theorem 2. *The triplet $(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3)$ is the unique solution of the variational problem:*

$$
\begin{aligned}
&(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3) \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega), \frac{2(\kappa_a + \kappa_b)E^{pl}}{1-(\nu^{pl})^2} \int_\omega \Big[(1-\nu^{pl})\gamma_{\alpha\beta}(\mathcal{U}_m)\gamma_{\alpha\beta}(\phi) + \nu^{pl}\big(\gamma_{kk}(\mathcal{U}_m)\big)\big(\gamma_{kk}(\phi)\big)\Big]dx' \\
&\quad + \frac{\eta E^{pl}}{1-(\nu^{pl})^2} \int_\omega \Big[(1-\nu^{pl})\frac{\partial^2 \mathcal{U}_3}{\partial x_\alpha \partial x_\beta}\frac{\partial^2 \Phi}{\partial x_\alpha \partial x_\beta} + \nu^{pl}\Delta \mathcal{U}_3 \Delta\Phi\Big]dx' \\
&\quad - \frac{2(\kappa_a - \kappa_b)E^{pl}}{1-(\nu^{pl})^2} \int_\omega \Big[(1-\nu^{pl})\frac{\partial^2 \mathcal{U}_3}{\partial x_\alpha \partial x_\beta}\gamma_{\alpha\beta}(\phi) + \nu^{pl}\Delta \mathcal{U}_3\big(\gamma_{kk}(\phi)\big)\Big]dx' \\
&\quad - \frac{2(\kappa_a - \kappa_b)E^{pl}}{1-(\nu^{pl})^2} \int_\omega \Big[(1-\nu^{pl})\gamma_{\alpha\beta}(\mathcal{U}_m)\frac{\partial^2 \Phi}{\partial x_\alpha \partial x_\beta} + \nu^{pl}\big(\gamma_{kk}(\mathcal{U}_m)\big)\Delta\Phi\Big]dx' \\
&= \int_\omega f_\alpha \phi_\alpha dx' + \int_\omega f_3 \Phi dx', \quad \forall (\phi_1, \phi_2, \Phi) \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega),
\end{aligned}
\tag{8.11}
$$

where

$$
\eta = \frac{1}{3}\Big[2(\kappa_a^3 + \kappa_b^3) + 6(\kappa_a + \kappa_b)\Big].
$$

Proof. Let ϕ_1, ϕ_2, and Φ be in $\mathcal{D}(\omega)$. The test displacement v_δ is defined by

$$
\begin{aligned}
v_{\delta,\alpha}(x) &= \tfrac{1}{\delta^2}\Big(\phi_{\alpha,\varepsilon,r}(x') - \tfrac{x_3}{\delta}\tfrac{\partial \widehat{\Phi}_{\varepsilon,r}}{\partial x_\alpha}(x')\Big), \\
v_{\delta,3}(x) &= \tfrac{1}{\delta^3}\widehat{\Phi}_{\varepsilon,r}(x'),
\end{aligned}
\qquad x \in \Omega_{\delta,\varepsilon,r}.
$$

where $\phi_{\alpha,\varepsilon,r}$, $\widehat{\Phi}_{\varepsilon,r}$ are defined in Lemma 5. Because $\chi = 1$ in D_1, in every beam the displacement v_δ coincides with a rigid displacement. Hence $(\nabla v^\delta)_S = 0$ a.e. in $B_{\delta,\varepsilon,r}$.

In Ω_δ^d one has

$$
\gamma_{\alpha\beta}(v_\delta) = \frac{1}{\delta^2}\Big(\gamma_{\alpha\beta}(\phi_{\varepsilon,r}) - \frac{x_3}{\delta}\frac{\partial^2 \widehat{\Phi}_{\varepsilon,r}}{\partial x_\alpha \partial x_\beta}\Big), \qquad \gamma_{i3}(v_\delta) = 0.
$$

Applying the operator Π_δ, then using Lemma 5, and passing to the limit give

$$
\delta^2 \Pi_\delta(\gamma_{\alpha\beta}(v_\delta)) \to \gamma_{\alpha\beta}(\phi) - X_3\frac{\partial^2 \Phi}{\partial x_\alpha \partial x_\beta} \qquad \text{strongly in } L^2(\Omega^d). \tag{8.12}
$$

For all x in $B_{\delta,\varepsilon,r}$ one has

$$
\begin{aligned}
v_{\alpha,\delta}(x) &= \frac{1}{\delta^2}\Big(\phi_{\alpha,\varepsilon,r}\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]\Big) - \frac{x_3}{\delta}\frac{\partial \Phi}{\partial x_\alpha}\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]\Big)\Big), \\
v_{3,\delta}(x) &= \frac{1}{\delta^3}\Big(\Phi_{\varepsilon,r}\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]\Big) + \varepsilon\Big\{\frac{x'}{\varepsilon}\Big\} \cdot \nabla\Phi\Big(\Big[\frac{x'}{\varepsilon}\Big]\Big)\Big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\delta^2 \Pi_r(\widetilde{v_{\alpha,\delta}}) \to \phi_\alpha - X_3\frac{\partial \Phi}{\partial x_\alpha} \qquad \text{strongly in } L^2(\omega \times B^{\text{be}}), \\
&\delta^3 \Pi_r(\widetilde{v_{3,\delta}}) \to \Phi \qquad \text{strongly in } L^2(\omega \times B^{\text{be}}), \\
&\delta^2 \Pi_\delta(v_{\alpha,\delta}) \to \phi_\alpha - X_3\frac{\partial \Phi}{\partial x_\alpha} \qquad \text{strongly in } L^2(\Omega^d), \\
&\delta^3 \Pi_\delta(v_{3,\delta}) \to \Phi \qquad \text{strongly in } L^2(\Omega^d).
\end{aligned}
\tag{8.13}
$$

Choose v_δ as test function in (2.6), then transform with the operators Π_δ, $\widetilde{}$ and Π_r. That yields

$$
\begin{aligned}
\int_{\Omega^a \cup \Omega^b} \frac{1}{\delta}\Pi_\delta(\sigma_{ij,\delta})\delta^2 \Pi_\delta(\gamma_{ij}(v_\delta))dx'dX_3 &= \int_{\Omega^a \cup \Omega^b} f_\alpha^d\ \delta^2 \Pi_\delta(v_{\alpha,\delta})dx'dX_3 + \int_{\Omega^a \cup \Omega^b} f_3^d\ \delta^3 \Pi_\delta(v_{3,\delta})dx'dX_3 \\
&\quad + \int_{\omega \times B^{\text{be}}} \frac{r^2}{\varepsilon^2 \delta}\Pi_r(\widetilde{f_{\alpha,\delta}})\delta^2 \Pi_r(\widetilde{v_{\alpha,\delta}})dx'dX + \int_{\omega \times B^{\text{be}}} \frac{r^2}{\varepsilon^2 \delta^2}\Pi_r(\widetilde{f_{3,\delta}})\delta^3 \Pi_r(\widetilde{v_{3,\delta}})dx'dX.
\end{aligned}
\tag{8.14}
$$

Because of the strong convergences (8.12) and (8.13), passing to the limit leads to

$$
\begin{aligned}
\sum_{d \in \{a,b\}} \int_{\Omega^d} \Sigma_{\alpha\beta}^d\Big(\gamma_{\alpha\beta}(\phi) - X_3\frac{\partial^2 \Phi}{\partial x_\alpha \partial x_\beta}\Big)dx'dX_3 &= \sum_{d \in \{a,b\}} \Big(\int_{\Omega^d} f_\alpha^d\Big(\phi_\alpha - X_3\frac{\partial \Phi}{\partial x_\alpha}\Big)dx'dX_3 + \int_{\Omega^d} f_3^d \Phi dx'dX_3\Big) \\
&\quad + \int_{\omega \times B^{\text{be}}} f_\alpha^{\text{be}}\Big(\phi_\alpha - X_3\frac{\partial \Phi}{\partial x_\alpha}\Big)dx'dX + \int_{\omega \times B^{\text{be}}} f_3^{\text{be}}\Phi_3 dx'dX.
\end{aligned}
\tag{8.15}
$$

Then one substitutes (8.10) in (8.15). Simplifying the obtained equality gives (8.11) for every $(\phi_1, \phi_2, \Phi) \in [\mathcal{D}(\omega)]^3$. Because $\mathcal{D}(\omega)$ is dense in $H_0^1(\omega)$ and $H_0^2(\omega)$, one gets (8.11) for every $(\phi_1, \phi_2, \Phi) \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$.

9 | CONVERGENCE OF THE TOTAL ELASTIC ENERGY

In this section one proves that the total elastic energy $\frac{1}{\delta^3}\mathcal{E}(u_\delta)$ converges. Choose u_δ as test function in (2.6) and use equality (2.7). That gives

$$\frac{1}{\delta^3}\mathcal{E}(u_\delta) = \sum_{d\in\{a,b\}} \int_{\Omega^d} \Big[\lambda^{pl}\Big(\frac{1}{\delta}\Pi_\delta(\gamma_{kk}(u_\delta))\Big)^2 + 2\mu^{pl}\Big(\frac{1}{\delta}\Pi_\delta(\gamma_{ij}(u_\delta))\Big)\Big(\frac{1}{\delta}\Pi_\delta(\gamma_{ij}(u_\delta))\Big)\Big]dx'dX_3$$
$$+ \int_{\omega\times B^{\mathrm{be}}} \Big[\lambda^{be}\Big(\frac{r}{\varepsilon\delta}\Pi_r(\widetilde{\gamma_{kk}(u_\delta)})\Big)^2 + 2\mu^{be}\Big(\frac{r}{\varepsilon\delta}\Pi_r(\widetilde{\gamma_{ij}(u_\delta)})\Big)\Big(\frac{r}{\varepsilon\delta}\Pi_r(\widetilde{\gamma_{ij}(u_\delta)})\Big)\Big]dx'dX$$

and

$$\frac{1}{\delta^3}\mathcal{E}(u_\delta) = \sum_{d\in\{a,b\}} \Big(\int_{\Omega^d} f_\alpha^d \delta^2 \Pi_\delta(u_{\alpha,\delta})dx'dX_3 + \int_{\Omega^d} f_3^d \delta^3 \Pi_\delta(u_{3,\delta})dx'dX_3\Big)$$
$$+ \int_{\omega\times B^{\mathrm{be}}} \frac{r^2}{\varepsilon^2\delta}\Pi_r(\widetilde{f_{\alpha,\delta}})\delta^2\Pi_r(\widetilde{u_{\alpha,\delta}})dx'dX + \int_{\omega\times B^{\mathrm{be}}} \frac{r^2}{\varepsilon^2\delta^2}\Pi_r(\widetilde{f_{3,\delta}})\delta^3\Pi_r(\widetilde{u_{3,\delta}})dx'dX.$$

Convergences (7.5), (7.6), (8.7) to (8.9), equalities (7.8), and the fact that the convex functional $v \mapsto \mathcal{E}(v)$ is lower-semicontinuous on $V_{\delta,\varepsilon,r}$ allow to pass to the limit in the above equalities. One obtains

$$\sum_{d\in\{a,b\}} \int_{\Omega^d} \Big[\lambda^{pl}\Big(\Gamma_{kk}^d\Big)^2 + 2\mu^{pl}\big(\Gamma_{ij}^d\big)\big(\Gamma_{ij}^d\big)\Big]dx'dX_3 + \int_{\omega\times B^{\mathrm{be}}} \Big[\lambda^{be}\Big(\Gamma_{kk}\Big)^2 + 2\mu^{be}\big(\Gamma_{ij}\big)\big(\Gamma_{ij}\big)\Big]dx'dX$$
$$\le \liminf_{\delta\to 0} \frac{1}{\delta^3}\mathcal{E}(u_\delta) \le \limsup_{\delta\to 0} \frac{1}{\delta^3}\mathcal{E}(u_\delta)$$
$$\le \lim_{\delta\to 0}\Big[\sum_{d\in\{a,b\}} \Big(\int_{\Omega^d} f_\alpha^d \delta^2 \Pi_\delta(u_{\alpha,\delta})dx'dX_3 + \int_{\Omega^d} f_3^d \delta^3 \Pi_\delta(u_{3,\delta})dx'dX_3\Big)$$
$$+ \int_{\omega\times B^{\mathrm{be}}} \frac{r^2}{\varepsilon^2\delta}\Pi_r(\widetilde{f_{\alpha,\delta}})\delta^2\Pi_r(\widetilde{u_{\alpha,\delta}})dx'dX + \int_{\omega\times B^{\mathrm{be}}} \frac{r^2}{\varepsilon^2\delta^2}\Pi_r(\widetilde{f_{3,\delta}})\delta^3\Pi_r(\widetilde{u_{3,\delta}})dx'dX\Big]$$
$$= \sum_{d\in\{a,b\}} \Big[\int_{\Omega^d} f_\alpha^d\Big(\mathcal{U}_\alpha - X_3\frac{\partial \mathcal{U}_3}{\partial x_\alpha}\Big)dx'dX_3 + \int_{\Omega^d} f_3^d\ \mathcal{U}_3 dx'dX_3\Big]$$
$$+ \int_{\omega\times B^{\mathrm{be}}} f_\alpha^{\mathrm{be}}\widetilde{U}_\alpha dx'dX + \int_{\omega\times B^{\mathrm{be}}} f_3^{\mathrm{be}}\widetilde{U}_3 dx'dX = \int_\omega f_\alpha\ \mathcal{U}_\alpha dx' + \int_\omega f_3\ \mathcal{U}_3 dx'.$$

Taking $(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3)$ as a test function in (8.11) allow to replace all the above inequalities by equalities. That gives the convergence of the total elastic energy

$$\lim_{\delta\to 0}\frac{1}{\delta^3}\mathcal{E}(u_\delta) = \sum_{d\in\{a,b\}} \int_{\Omega^d} \big[\lambda^{pl}\big(\Gamma_{kk}^d\big)^2 + 2\mu^{pl}\Gamma_{ij}^d\Gamma_{ij}^d\big]dx'dX_3,$$
$$\text{and also} \quad \int_{\omega\times B^{\mathrm{be}}} \big[\lambda^{be}\big(\Gamma_{kk}\big)\big)^2 + 2\mu^{be}\Gamma_{ij}\Gamma_{ij}\big]dx'dX = 0.$$

As immediate consequence of the above equality, one has $\Sigma = 0 \text{in} \omega \times B^{\mathrm{be}}$. Moreover, the weak convergences (8.3) and (8.4) are strong convergences.

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APPENDIX A

Lemma 6. *Let Ω be a bounded domain in $\mathbb{R}^2$ whose diameter is less than R. We assume that Ω is starshaped with respect to the ball $B(O;1)$. For every $\phi \in H^1(\Omega)$ one has*

$$\begin{aligned} \left|\left|\phi - \mathcal{M}(\phi)\right|\right|_{L^2(\Omega)} &\leq CR\sqrt{\ln\left(R\right)}\left|\left|\nabla\phi\right|\right|_{[L^2(\Omega)]^2} \\ \mathcal{M}(\phi) &= \frac{1}{|B(O;1)|}\int_{B(O;1)}\phi(x)dx \end{aligned} \tag{A1}$$

where C does not depend on R.

Proof. Let ϕ be in $C^1(\overline{\Omega})$. Consider the segment joining O to P on the boundary of Ω. Its direction is given by the unit vector $y' \in \mathbb{R}^2$. We have

$$|\phi(ty')| \leq |\phi(y')| + \int_1^t \left|\frac{\partial\phi}{\partial r}(ry')\right|dr, \qquad [y', ty'] \subset [O, P], \qquad t \in [1, |P|], \qquad P \in \partial\Omega.$$

Hence

$$\begin{aligned} |\phi(ty')|^2 &\leq 2\Big(|\phi(y')|^2 + \int_1^t \frac{dr}{r}\int_1^t \left|\frac{\partial\phi}{\partial r}(ry')\right|^2 r\,dr\Big), \\ \Longrightarrow\ |\phi(ty')|^2 &\leq 2\big(|\phi(y')|^2 + \ln\left(R\right)\ \int_1^t \left|\frac{\partial\phi}{\partial r}(ry')\right|^2 r\,dr\big). \end{aligned}$$

Multiply by t, then integrate between 1 and $|P|$ (y' being fixed), and finally integrate over the unit sphere of $\mathbb{R}^2$. That gives ($t \leq R$)

$$\begin{aligned} &\|\phi\|^2_{L^2(\Omega\setminus\overline{B(O;1)})} \leq R^2\big(\|\phi\|^2_{L^2(\partial B(O;1))} + \ln\big(R\big)\ \|\nabla\phi\|^2_{[L^2(\Omega)]^2}\big), \\ &\implies \|\phi\|^2_{L^2(\Omega)} \leq 2\|\phi\|^2_{L^2(B(O;1))} + 2R^2\big(\|\phi\|^2_{L^2(\partial B(O;1))} + \ln\big(R\big)\ \|\nabla\phi\|^2_{[L^2(\Omega)]^2}\big). \end{aligned} \tag{A4}$$

By density of $C^1(\overline{\Omega})$ in $H^1(\Omega)$ the above inequality holds for every ϕ in $H^1(\Omega)$.

Now, choose ϕ in $H^1(\Omega)$. The Poincaré-Wirtinger inequality gives

$$\|\phi - \mathcal{M}(\phi)\|_{L^2(B(O;1))} \leq C\|\nabla\phi\|_{[L^2(B(O;1))]^2}. \tag{A5}$$

The trace theorem and the above estimate yield

$$\begin{aligned} \|\phi - \mathcal{M}(\phi)\|^2_{L^2(\partial B(O;1))} &\leq C\big(\|\phi - \mathcal{M}(\phi)\|^2_{L^2(B(O;1))} + \|\nabla\phi\|^2_{[L^2(B(O;1))]^2}\big) \\ &\leq C\|\nabla\phi\|_{[L^2(B(O;1))]^2}. \end{aligned} \tag{A6}$$

Replace ϕ by $\phi - \mathcal{M}(\phi)$ in $(A4)_2$. Inequalities (A5) to (A6) lead to (A1). □

Choose $\Omega = Y_\varepsilon \subset \mathbb{R}^2$ (the diameter is equal to $\sqrt{2}\varepsilon$), this domain is starshaped with respect to the disc $D_r (r < \varepsilon/2)$. Let ϕ be in $H^1(Y_\varepsilon)$, and denote $\mathcal{M}_r(\phi)$ the mean value of ϕ in D_r. We apply the above lemma with the function $\psi(x) = \phi(x/r)$. That gives

$$\|\phi - \mathcal{M}_r(\phi)\|_{L^2(Y_\varepsilon)} \leq C\varepsilon\sqrt{\ln\left(\frac{\varepsilon}{r}\right)}\|\nabla\phi\|_{[L^2(Y_\varepsilon)]^2}. \tag{A7}$$

The constant does not depend on ε and r.