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HIGHER-ORDER OPTIMALITY CONDITIONS AND HIGHER-ORDER TANGENT SETS*

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Abstract. We present a simple approach to an analysis of higher order approximations to sets and functions. The objects we study are not of a specific order; they include objects of order 2 and m with m not necessarily an integer. We deduce from these concepts optimality conditions of higher order and we establish some calculus rules.

Key words. generalized derivatives, tangent cones, mathematical programming, multipliers, optimality conditions

AMS subject classifications. 49K27, 46A20, 46N10, 52A05, 52A40, 90C30

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1. Introduction.

Numerous papers have been devoted to higher-order conditions in optimization. (See [3, 4], [6, 7, 8], [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29], [30, 31, 32, 33, 34, 35, 36, 37, 38, 39], [41, 42, 43, 44, 45, 46, 47, 48], [50, 51, 52, 53, 54, 55, 56, 57, 58] among many other references.) Usually they deal with a specific order of differentiability, most often the second-order. It is the purpose of the present paper to consider the case the higher-order effect is not of a specific order. Such a case has already been dealt with in [43] and some of its references. However the presentation we gave in [43] relied on a projective formalism which made it somewhat aloof from usual methods in optimization. Here we present a direct approach that is certainly more natural for optimization problems.

We also complete the study undertaken in [43] by analyzing higher-order tangent sets of different kinds and by presenting calculus rules (section 2). We also introduce higher-order derivatives for nonsmooth functions and multifunctions (section 3). The application to optimality conditions for unconstrained problems is the object of section 4. In particular we obtain a variant of conditions devised by Dedieu and Janin [20] in the smooth case. We revisit the case of mathematical programming problems (section 5), showing that the conditions presented in [43] and [4] are quite natural. We conclude with a comparison with some classical second-order notions. For more complete comparisons with previous studies about m-order notions, we refer to [21], [24], [25], [32], [38], [43], [41], [53].

The notions we introduce are likely to play a role in other topics such as sensitivity analysis, bifurcation, and dynamical systems, but we limit our study to unconstrained and constrained optimization. An application to semidefinite programming can be found in [45].

For the notation and terminology we keep close to the books [5, 44, 49, 52]; $\overline{\mathbb{R}}$ denotes the set of extended real numbers, $\mathbb{R}_+$ denotes the set of nonnegative real numbers, and $\mathbb{P}$ denotes the set of positive real numbers. We often identify a set-valued map F with its graph gph$F$ when there is no risk of confusion.

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2. Higher-order tangent sets. We start our study with a geometric approach as it seems to be the simplest point of view. We use the classical notion of outer limit (limsup) of a family of sets described in [1, Def. 1.1.1], [44, Def. 1.41], [52, Def. 4.1], for instance.

DEFINITION 1. *Given a subset F of a topological vector space X, $x \in \mathrm{cl}(F)$, the closure of F and $v \in X$ the higher-order tangent cone to F at x in the direction v is the set*

$$T^h(F,x,v) := \limsup_{(t,s)\to(0_+,0_+)} \frac{1}{st}(F - x - tv).$$

Thus, $w \in T^h(F,x,v)$ if and only if there exists a net $((t_i,s_i,w_i)) \to (0,0,w)$ in $\mathbb{P}\times\mathbb{P}\times X$ with $\mathbb{P} :=]0,\infty[$, such that $x + t_i v + t_i s_i w_i \in F$ for all $i \in I$. In what follows we suppose X is a normed space, so that we use sequences instead of nets. Moreover, in such a case we have

$$w \in T^h(F,x,v) \Longleftrightarrow \liminf_{s,\ t\to 0_+} \frac{1}{st} d(x + tv + stw, F) = 0,$$

where $d(y,F) := \inf\{\|y - z\| : z \in F\}$. We observe that this notion captures the classical notion of (outer) *second-order tangent set* defined by

$$T^2(F,x,v) := \limsup_{t\to 0_+} \frac{1}{2t^2}(F - x - tv)$$

in the sense that $T^2(F,x,v) \subset T^h(F,x,v)$. Moreover, for any $m > 1$ we have $T^m(F,x,v) \subset T^h(F,x,v)$, where

$$T^m(F,x,v) := \limsup_{t\to 0_+} \frac{1}{mt^m}(F - x - tv),$$

as we can see by setting $s_n := mt_n^{m-1}$ if $x_n := x + t_n v + mt_n^m w_n \in F$ and $(t_n) \to 0_+$, $(w_n) \to w$. However, for $m \in \mathbb{N}\backslash\{0,1\}$, the tangent set of order m given by

$$T^m(F,x,v_1,\dots,v_{k-1}) := \limsup_{t\to 0_+} \frac{1}{k!t^k}\left(F - x - tv_1 \cdots - \frac{t^{k-1}}{(k-1)!}v_{k-1}\right)$$

carries a more precise information than $T^h(F,x,v_1)$, even if it is smaller.

We observe that $T^h(F,x,v)$ is a cone in the sense that for any $w \in T^h(F,x,v)$ and any $\lambda \in \mathbb{P}$ we have $\lambda w \in T^h(F,x,v)$. We also note that if $w \in T^h(F,x,v)$, then

$$v \in T(F,x) := \limsup_{t\to 0_+}(1/t)(F - x),$$

the *tangent cone* (or contingent cone) to F at x since for $v_n := v + s_n w_n$ we have $(v_n) \to v$ and $x + t_n v_n \in F$ whenever $x + t_n v + t_n s_n w_n \in F$ for all n. For $v = 0$ we have $T^h(F,x,v) = T(F,x)$, so that usually we exclude the case $v = 0$.

A crucial advantage of the notion of higher-order tangent cone is the following nonemptiness property. Such a property could be extended to reflexive Banach spaces, provided weak convergence is substituted to norm convergence. For the sake of simplicity, we discard such an extension.

PROPOSITION 2. *For any nonempty subset F of a finite dimensional normed space, any $x \in \mathrm{cl}(F)$, and any $v \in T(F,x)$ the set $T^h(F,x,v)$ is nonempty.*

Proof. Since $v \in T(F,x)$, there exists a sequence $((t_n, v_n)) \to (0_+, v)$ in $\mathbb{P} \times X$ such that $x_n := x + t_n v_n \in F$ for all $n \in \mathbb{N}$. Let $q_n := \|v_n - v\|$ and let $N := \{n \in \mathbb{N} : q_n = 0\}$. If N is infinite, taking an increasing function $k : \mathbb{N} \to \mathbb{N}$ such that $N = \{k(n) : n \in \mathbb{N}\}$, taking $w_{k(n)} := 0$ and an arbitrary sequence $(s_{k(n)}) \to 0_+$, we can write $x_{k(n)} = x + t_{k(n)} v + t_{k(n)} s_{k(n)} w_{k(n)}$, so that $0 \in T^h(F,x,v)$. If N is finite, setting for $n > \sup N$, $s_n := q_n$, $w_n := s_n^{-1}(v_n - v)$ we see that $\|w_n\| = 1$ and we can find a subsequence of (w_n) that converges to some unit vector w of X. Then we have $x_n = x + t_n v + s_n t_n w_n$, so that $w \in T^h(F,x,v)$. ∎

When F is convex the definition of $T^h(F,x,v)$ can be given in terms of curves.

PROPOSITION 3. *Let F be a convex subset of a normed space X, and let $x \in F$, $v \in X$. Then $w \in T^h(F,x,v)$ if and only if there exist $\tau > 0$ and maps $s :]0,\tau] \to]0,1]$, $z :]0,\tau] \to X$ such that $\lim_{t\to 0_+} s(t) = 0$, $\lim_{t\to 0_+} z(t) = w$ and $x + tv + ts(t)z(t) \in F$ for all $t \in]0,\tau]$.*

Proof. The condition is obviously sufficient. Let us show it is necessary. Given $w \in T^h(F,x,v)$, let (t_n), $(s_n) \to 0_+$, $(w_n) \to w$ be such that $x + t_n v + t_n s_n w_n \in F$ for all $n \in \mathbb{N}$. Without loss of generality we may suppose (t_n) is decreasing. Then we set $\tau := t_0$, and, for $t \in]t_n, t_{n-1}]$, $s(t) := s_n$, $z(t) := w_n$, so that $\lim_{t\to 0_+} s(t) = 0$, $\lim_{t\to 0_+} z(t) = w$, and

$$x + tv + ts(t)z(t) = \left(1 - tt_n^{-1}\right) x + tt_n^{-1}(x + t_n v + t_n s_n w_n) \in F \ \forall t \in]t_{n+1}, t_n].$$

Thus $w \in T^h(F,x,v)$. ∎

The related concept of *incident higher-order tangent cone* or in short the *higher-order incident cone*

$$T^{hi}(F,x,v) := \liminf_{(s,t)\to(0_+,0_+)} \frac{1}{st}(F - x - tv)$$

does not have the same importance because the condition $w \in T^{hi}(F,x,v)$ means that for any sequences (t_n), $(s_n) \to 0_+$ there exists a sequence $(w_n) \to w$ such that $x + t_n v + t_n s_n w_n \in F$ for all $n \in \mathbb{N}$, a rather stringent condition. Thus the set $T^{hi}(F,x,v)$ is often empty; but the next examples show it can play some role.

Example 1. Suppose F is a convex subset of X with a nonempty interior $\mathrm{int}(F)$. Then for $x \in F$, $v \in \mathbb{P}(\mathrm{int}(F) - x)$ one has $T^{hi}(F,x,v) = X$. In particular, if F is some interval $[x, y[$ of $\mathbb{R}$, for $v \in \mathbb{P}$ one has $T^{hi}(F,x,v) = \mathbb{R}$ (and $T^{hi}(F,x,0) = \mathbb{R}_+$).

Example 2. Let $f : W \to \mathbb{R}$ be a function on an open subset W of a normed space X that is quiet at $a \in W$ in the sense that there exists some $c \in \mathbb{R}_+$ such that $f(w) - f(a) \le c\,\|w - a\|$ for all w in a neighborhood of a. Let $F := \{(w,r) \in W \times \mathbb{R} : r \ge f(w)\}$ be the epigraph of f. Then for $v := (0,1)$ or more generally for $v := (w,q)$ with $q > c\,\|w\|$ one has $T^{hi}(F,x,v) = X \times \mathbb{R}$ for $x := (a, f(a))$.

This notion can be useful for some purposes, as the following statements show.

PROPOSITION 4. *If F is a convex subset of a normed vector space, if $x \in \mathrm{cl}(F)$, $v \in T(F,x)$, the cone $T^{hi}(F,x,v)$ is a convex cone. Moreover, one has*

$$T^h(F,x,v) + T^{hi}(F,x,v) \subset T^h(F,x,v).$$

Proof. The proof is immediate, using the fact that $w \in T^{hi}(F,x,v)$ if and only if for any sequences (t_n), $(s_n) \to 0_+$ there exists a sequence $(w_n) \to w$ such that

$x_n := x + t_n v + 2t_n s_n w_n \in F$ for all $n \in \mathbb{N}$. Thus for w, $w' \in T^{hi}(F, x, v)$, taking $(w'_n) \to w$ such that $x'_n := x + t_n v + 2t_n s_n w'_n \in F$ one has $x + t_n v + t_n s_n (w_n + w'_n) = \frac{1}{2}x_n + \frac{1}{2}x'_n \in F$, and hence $w + w' \in T^{hi}(F, x, v)$.

Given $w \in T^h(F, x, v)$, $z \in T^{hi}(F, x, v)$ we can find sequences (t_n), $(s_n) \to 0_+$, $(w_n) \to w$, $(z_n) \to z$ such that $x + t_n v + t_n s_n w_n \in F$, $x + t_n v + t_n s_n z_n \in F$ for all $n \in \mathbb{N}$, so that for $s'_n := s_n/2$ we have $x + t_n v + t_n s'_n (w_n + z_n) \in F$ for all $n \in \mathbb{N}$. ∎

Remark. Another notion can be introduced between $T^{hi}(F, x, v)$ and $T^h(F, x, v)$. It is

$$T^{ih}(F, x, v) := \{w \in X : \forall (t_n) \to 0_+ \ \exists (s_n) \to 0_+, \ (w_n) \to w, \ x + t_n v + t_n s_n w_n \in F \ \forall n\}.$$

When F is convex one can show that $T^{ih}(F, x, v)$ is convex and $T^h(F, x, v) + T^{ih}(F, x, v) \subset T^h(F, x, v)$. However, the role of this cone seems to be more limited than the ones of $T^h(F, x, v)$ and $T^{hi}(F, x, v)$. ∎

PROPOSITION 5. *For any subset F (resp., F') of a normed space X (resp., X'), for any $(x, x') \in \mathrm{cl}(F \times F')$, and any $(v, v') \in T(F \times F', (x, x'))$ one has*

$$T^h(F, x, v) \times T^{hi}(F', x', v') \subset T^h(F \times F', (x, x'), (v, v')) \subset T^h(F, x, v) \times T^h(F', x', v').$$

Proof. The first inclusion is a direct consequence of the definitions. The second one follows from the next result by using the canonical projections. ∎

PROPOSITION 6. *Let $A : X \to Y$ be a continuous linear map between normed vector space and let $B \subset X$, $C \subset Y$ be such that $A(B) \subset C$. Then, for any $x \in \mathrm{cl}(B)$, $v \in X$ one has $A(T^h(B, x, v)) \subset T^h(C, A(x), A(v))$ and $A(T^{hi}(B, x, v)) \subset T^{hi}(C, A(x), A(v))$.*

Proof. Given sequences (t_n), $(s_n) \to 0_+$, $(w_n) \to w$ in X such that $x + t_n v + t_n s_n w_n \in A$ for all $n \in \mathbb{N}$, one has $Ax + t_n A(v) + t_n s_n A(w_n) \in A(B) \subset C$ for all $n \in \mathbb{N}$, so that $A(w) \in T^h(C, A(x), A(v))$. A similar argument yields the second inclusion. ∎

PROPOSITION 7. *Let E and F be two subsets of a normed space X and let $x \in E \cap F$, $v \in X$. Suppose that one of the following directional metric regularity conditions is satisfied: there exist some $\rho > 0$ such that for $t \in]0, \rho[$, $u \in B(v, \rho)$ one has*

$$d(x + tu, E \cap F) \le c d(x + tu, F) \text{ whenever } x + tu \in E, \tag{1}$$

$$d(x + tu, E \cap F) \le c d(x + tu, E) + c d(x + tu, F). \tag{2}$$

Then

$$T^h(E, x, v) \cap T^{hi}(F, x, v) \subset T^h(E \cap F, x, v) \subset T^h(E, x, v) \cap T^h(F, x, v),$$
$$T^{hi}(E \cap F, x, v) = T^{hi}(E, x, v) \cap T^{hi}(F, x, v).$$

Proof. Clearly (1) is a consequence of (2), so that we assume it. The second inclusion is obvious. Given $w \in T^h(E, x, v) \cap T^{hi}(F, x, v)$ we can find sequences (t_n), $(s_n) \to 0_+$, $(w_n) \to w$ such that $x_n := x + t_n v + t_n s_n w_n \in E$ for all $n \in \mathbb{N}$. For n large enough we have $t_n < \rho$, $v_n := v + s_n w_n \in B(v, \rho)$. Thus

$$d(x + t_n v_n, E \cap F) \le c d(x + t_n v_n, F).$$

Since $w \in T^{hi}(F, x, v)$ we have $\lim_n t_n^{-1} s_n^{-1} d(x + t_n v + t_n s_n w_n, F) = 0$. Thus $\lim_n t_n^{-1} s_n^{-1} d(x + t_n v + t_n s_n w_n, E \cap F) = 0$ and $w \in T^h(E \cap F, x, v)$.

The inclusion $T^{hi}(E \cap F, x, v) \subset T^{hi}(E, x, v) \cap T^{hi}(F, x, v)$ is valid without any assumption. Let $w \in T^{hi}(E, x, v) \cap T^{hi}(F, x, v)$. Then for any sequences (t_n), $(s_n) \to 0_+$, there exists a sequence $(w_n) \to w$ such that $x_n := x + t_n v + t_n s_n w_n \in E$ for all $n \in \mathbb{N}$. By the preceding inequality we get

$$\limsup_n \frac{1}{t_n s_n} d(x + t_n v + t_n s_n w_n, E \cap F) \leq \lim_n \frac{1}{t_n s_n} c d(x + t_n v + s_n t_n w_n, F) = 0$$

so that $w \in T^{hi}(E \cap F, x, v)$. ∎

In order to consider the case of the image of a set by a nonlinear map, we split the higher-order tangent cone into a family of subsets indexed by $r \in \overline{\mathbb{R}}_+ := [0, \infty]$ by setting for $F \subset X$, $x \in \mathrm{cl}(F)$, $v \in X$,

$$T_r^h(F, x, v) := \{w \in X : \exists (w_n) \to w, (t_n), (s_n) \to 0_+, \left(\frac{s_n}{t_n}\right) \to \frac{1}{2r}, \\ \forall n \in \mathbb{N}\ x + t_n v + t_n s_n w_n \in F\}.$$

The compactness of $\overline{\mathbb{R}}_+$ ensures that (by extracting subsequences) one has

$$T^h(F, x, v) = \bigcup_{r \in \overline{\mathbb{R}}_+} T_r^h(F, x, v).$$

For $r \in \mathbb{R}_+$ the set $T_r^h(F, x, v)$ is related to the set

$$\widehat{T}^2(F, x, v) := \Big\{(w, r) \in X \times \mathbb{R}_+ : \exists\ (w_n) \to w, (t_n) \to 0_+,\ (r_n) \to r \\ \forall n \in \mathbb{N},\ r_n \in \mathbb{P},\ x + t_n v + \frac{1}{2} t_n^2 r_n^{-1} w_n \in F\Big\}$$

introduced in [43] by the relation

$$T_r^h(F, x, v) = \Big\{w \in X : (w, r) \in \widehat{T}^2(F, x, v)\Big\}$$

as one sees by setting $r_n := \frac{1}{2} s_n^{-1} t_n$ and conversely $s_n = \frac{1}{2} r_n^{-1} t_n$. For $r = 1$ one gets the classical *second-order tangent set* $T^2(F, x, v)$ and for $r \in \mathbb{P}$ one has $T_r^h(F, x, v) = 2r T^2(F, x, v)$. Let us note that if $w \in T_\infty^h(F, x, v)$, then we have $0 \in T^2(F, x, v)$ since when $(w_n) \to w$ and $(r_n) \to \infty$, $(t_n) \to 0_+$ are such that $x_n := x + t_n v + \frac{1}{2} r_n^{-1} t_n^2 w_n \in F$ for all $n \in \mathbb{N}$ we can write $x_n = x + t_n v + \frac{1}{2} t_n^2 z_n$ with $(z_n) := (r_n^{-1} w_n) \to 0$. Still the inclusion $w \in T_\infty^h(F, x, v)$ is more precise than the relation $0 \in T^2(F, x, v)$.

Example 3. For $F_+ := \{(r, r^3) : r \in \mathbb{R}_+\}$, $F_- := \{(r, -r^3) : r \in \mathbb{R}_+\}$ one has $(0, 0) \in T^2(F, x, v)$ for $x := (0, 0)$, $v := (1, 0)$, $F := F_+$ and $F := F_-$; however, for $w := (0, 1)$ one has $w \in T_\infty^h(F_+, x, v) \backslash T_\infty^h(F_-, x, v)$ and $-w \in T_\infty^h(F_-, x, v) \backslash T_\infty^h(F_+, x, v)$.

Since $T_r^h(F, x, v) = 2r T_1^h(F, x, v)$ for $r \in \mathbb{P}$, the information given by $T^h(F, x, v)$ is carried by $T_1^h(F, x, v) = T^2(F, x, v)$, $T_0^h(F, x, v)$, and $T_\infty^h(F, x, v)$. The new sets $T_0^h(F, x, v)$ and $T_\infty^h(F, x, v)$ have a role in the analysis of the infinitesimal behavior of F around x, as shown by the following example.

Example 4. For $c > 1$ let $F := \{(r, r^c) : r \in \mathbb{R}_+\} \subset X := \mathbb{R}$, and let $x := (0, 0)$, $v := (1, 0)$, $w := (0, 1)$. For $c \in]1, 2[$ one has $w \in T_0^h(F, x, v)$. For $c > 2$ one has $w \in T_\infty^h(F, x, v)$ and for $c = 2$ one has $w \in T^2(F, x, v) := T_1^h(F, x, v)$.

Remark. Proposition 7 can be extended by introducing for $r \in \overline{\mathbb{R}}_+$, $x \in \mathrm{cl}(F)$, $v \in X$ the set

$$T_r^{hi}(F,x,v) := \left\{ w \in X : \lim_{t,\ s\to 0_+,\ t/s\to 2r} \frac{1}{ts} d(x+tv+stw,F) = 0 \right\}.$$

By arguments similar to the ones of the preceding proof one can show that assumption (1) implies that

$$T_r^h(E,x,v) \cap T_r^{hi}(F,x,v) \subset T_r^h(E\cap F,x,v) \subset T_r^h(E,x,v) \cap T_r^h(F,x,v),$$
$$T_r^{hi}(E\cap F,x,v) = T_r^{hi}(E,x,v) \cap T_r^{hi}(F,x,v).$$

Examples 3 and 4 show that one may have $T^{hi}(F,x,v) = \varnothing$ but $T_r^{hi}(F,x,v) \neq \varnothing$ for some $r \in \overline{\mathbb{R}}_+$. ∎

The preceding results take a more striking form when assuming that one of the sets, say F, is *higher-order proto-derivable at x in the direction v* in the sense that

$$T^{hi}(F,x,v) = T^h(F,x,v).$$

Then one has $T_r^h(E\cap F,x,v) = T_r^h(E,x,v) \cap T_r^h(F,x,v)$. If more precisely, for some $r \in \overline{\mathbb{R}}_+$ one has

$$T_r^{hi}(F,x,v) = T_r^h(F,x,v),$$

then one has $T_r^h(E\cap F,x,v) = T_r^h(E,x,v) \cap T_r^h(F,x,v)$.

The next properties have been given in [43] in the case $r < \infty$.

PROPOSITION 8. *If F is a convex subset of X, if $x \in F$, $v \in T(F,x)$, then for any $r \in \overline{\mathbb{R}}_+$, $w \in T_r^h(F,x,v)$, $z \in T(T(F,x),v)$ one has $w + z \in T_r^h(F,x,v)$.*

Proof. Let (s_n), $(t_n) \to 0_+$ be such that $(\frac{1}{2}s_n^{-1}t_n) \to r \in \overline{\mathbb{R}}_+$ and let $(w_n) \to w$ be such that $x_n := x + t_n v + t_n s_n w_n \in F$ for all n. For any $y \in F$, p, $q \in \mathbb{R}_+$, by convexity, for n large enough we have

$$x_n' := (1-ps_n)x_n + ps_n(x + qt_n(y-x)) \in F,$$
$$x_n' = x + t_n v + s_n t_n[(1-ps_n)w_n - pv + pq(y-x)].$$

Since $((1-ps_n)w_n) \to w$ it follows that $w + p(q(y-x) - v) \in T_r^h(F,x,v)$. Since $T_r^h(F,x,v)$ is closed and $\mathbb{R}_+(F-x)$ is dense in $T(F,x)$ we get that $w + p(u - v) \in T_r^h(F,x,v)$ for all $p \in \mathbb{R}_+$ and all $u \in T(F,x)$. Since $T_r^h(F,x,v)$ is closed and $\mathbb{R}_+(T(F,x)-v)$ is dense in $T(T(F,x),v)$, we obtain that $w + T(T(F,x),v) \subset T_r^h(F,x,v)$. ∎

PROPOSITION 9. *Let X and Y be normed spaces, let $B \subset X$, $C \subset Y$, $x \in \mathrm{cl}(B)$, $y := g(x)$, and let $g : X \to Y$ be such that $g(B) \subset C$. If g is twice differentiable at x, then for all $r \in \mathbb{R}_+$ and $w \in T_r^h(B,x,v)$ one has $g'(x)w + rg''(x)(v,v) \in T_r^h(C,y,g'(x)v)$. If $r = \infty$ and $w \in T_r^h(B,x,v)$, one has $g''(x)vv \in T^2(C,y,g'(x)v)$.*

Proof. The first assertion follows from a second-order Taylor expansion of g. See also [43, Prop. 2.2]. If $w \in T_\infty^h(B,x,v)$, one has $0 \in T^2(B,x,v) = T_1^h(B,x,v)$, and hence $g''(x)vv \in T_1^h(C,y,g'(x)v) = T^2(C,y,g'(x)v)$. ∎

3. Higher-order derivatives. Since maps and multimaps (or set-valued maps) are determined by their graphs, we interpret a higher-order tangent set to the graph of

a multimap F as higher-order graphical derivatives of F, similarly to the interpretation of the tangent cone to the graph of F being classically interpreted as a graphical derivative of F, namely,

$$\mathrm{gph}(DF(x,x')) = T(\mathrm{gph}F,(x,x')) \qquad \text{for } (x,x') \in \mathrm{gph}F.$$

DEFINITION 10. *Given normed spaces X and X', a multimap $F : X \rightrightarrows X'$ identified with its graph, $(x,x') \in F$, $(v,v') \in X \times X'$ we denote by $D^hF((x,x'),(v,v')) : X \rightrightarrows X'$ the higher-order derivative of F at (x,x') in the direction (v,v') given by*

$$D^hF((x,x'),(v,v'))(w) := \{w' \in X' : (w,w') \in T^h(F,(x,x'),(v,v'))\}.$$

Similarly one sets

$$D^{hi}F((x,x'),(v,v'))(w) := \{w' \in X' : (w,w') \in T^{hi}(F,(x,x'),(v,v'))\},$$
$$D^h_rF((x,x'),(v,v'))(w) := \{w' \in X' : (w,w') \in T^h_r(F,(x,x'),(v,v'))\},$$
$$D^{hi}_rF((x,x'),(v,v'))(w) := \{w' \in X' : (w,w') \in T^{hi}_r(F,(x,x'),(v,v'))\}.$$

These definitions can be applied to graphs and epigraphs of extended real-valued functions, as shown below. We also observe that for $F^{-1} : X' \rightrightarrows X$ given by $F^{-1}(x') := \{x \in X : x' \in F(x)\}$, for any $(x,x') \in \mathrm{gph}F$, $(v,v') \in X \times X'$ one has

$$D^hF^{-1}((x',x),(v',v)) = [D^hF((x,x'),(v,v'))]^{-1}.$$

Different notions are introduced in the next definition, as seen below.

DEFINITION 11. *Given normed spaces X and X', a multimap $F : X \rightrightarrows X'$ identified with its graph, $(x,x') \in F$, (v,v'), $(w,w') \in X \times X'$, $r \in \overline{\mathbb{R}}_+$, we say that F is (higher-order) semiderivable at (x,x') in the direction (v,v') for r, (w,w') if for any sequences $(w_n) \to w$, (t_n), $(s_n) \to 0_+$ such that $(s_n^{-1}t_n) \to 2r$ there exists a sequence $(w'_n) \to w'$ such that*

$$x' + t_nv' + t_ns_nw'_n \in F(x + t_nv + t_ns_nw_n)$$

for infinitely many $n \in \mathbb{N}$. If this property is valid for any $r \in \overline{\mathbb{R}}$, we say that F is (higher-order) semiderivable at (x,x') in the direction (v,v') for (w,w').

We say that F is semiderivable at (x,x') in the direction (v,v') for $w \in X$, $r \in \overline{\mathbb{R}}_+$ if for any $w' \in D^h_rF((x,x'),(v,v'))(w)$ F is (higher-order) semiderivable at (x,x') in the direction (v,v') for r, (w,w').

Example 5. Let F be the graph of a map $g : X \to X'$ that is twice differentiable at x. Then for all $r \in \mathbb{R}_+$, v, $w \in X$, F is semiderivable at (x,x') in the direction $(v,g'(x)v)$ for r, w, and

$$D^h_rF(x,v,v')(w) = g'(x)w + rg''(x)(v,v).$$

If $r = +\infty$, g is quadratic and $g''(x)(v,v) = 0$, F is also semiderivable at (x,x') in the direction $(v,g'(x)v)$ for w, r and $D^h_rF(x,v,v')(w) = g'(x)w$.

Example 6. Let $F : \mathbb{R} \to \mathbb{R}$ be given by $F(x) := |x|^c$ and let $v := 1$, $v' := 0$. For $c \in]1,\infty[$ F is semiderivable at $(0,0)$ in the direction (v,v') for any $r \in \mathbb{R}_+$ and any $(w,0) \in \mathbb{R} \times \{0\}$.

Let us give some calculus rules that can easily be deduced from the definitions.

PROPOSITION 12. *Let X, Y be normed spaces, and let F, G, $H : X \rightrightarrows Y$ be such that $H(x) = F(x) + G(x)$ for all $x \in X$. If F is semiderivable at (x, y) in the direction $(v, v') \in X \times Y$ for $w \in X$, $r \in \overline{\mathbb{R}}_+$, then, for any $z \in G(x)$, $v'' \in Y$ one has*

$$D_r^h F((x,y),(v,v'))(w) + D_r^h G((x,z),(v,v''))(w) \subset D_r^h H((x,x'+x''),(v,v'+v''))(w).$$

PROPOSITION 13. *Let X, X', X'' be normed spaces, and let $F : X \rightrightarrows X'$, $G : X' \rightrightarrows X''$, $H := G \circ F$. Let $r \in \overline{\mathbb{R}}_+$, $(x, x') \in F$, $(x', x'') \in G$, $(v, v', v'') \in X \times X' \times X''$, $w \in X$, $w' \in D_r^h F((x,x'),(v,v'))(w)$, $w'' \in D_r^h G((x',x''),(v',v''))(w')$. If G is semiderivable at (x', x'') in the direction (v', v'') for (w', w''), r, then one has $w'' \in D_r^h H((x,x''),(v,v''))(w)$.*

A variant of the last statement can be given by using Proposition 7 and observing that (the graph of) H is the projection of $(F \times X'') \cap (X \times G)$ on $X \times X''$; then one uses the set $S := \{(x, x', x'') : (x, x') \in F,\ (x', x'') \in G\} = (F \times X'') \cap (X \times G)$.

PROPOSITION 14. *Let F, G, H be as in the preceding proposition, and let $r \in \overline{\mathbb{R}}_+$, $(x, x') \in F$, $(x', x'') \in G$, $(v, v', v'') \in X \times X' \times X''$, $w \in X$, be such that for some $c > 0$, $\rho > 0$ one has*

$$d((x,x',x'') + t(u,u',u''), S) \le cd((x',x'') + t(u',u''), G)$$

for all $t \in]0, \rho[$, $u \in B(v, \rho)$, $u' \in B(v, \rho)$, $u'' \in B(v'', \rho)$ such that $(x, x') + t(u, u') \in F$ or

$$d((x,x',x'') + t(u,u',u''), S) \le cd((x,x') + t(u,u'), F)$$

for all $t \in]0, \rho[$, $u \in B(v, \rho)$, $u' \in B(v, \rho)$, $u'' \in B(v'', \rho)$ such that $(x', x'') + t(u', u'') \in G$. Then one has

$$\begin{aligned}
D_r^h G((x',x''),(v,v'')) \circ D_r^{hi} F((x,x'),(v,v')) &\subset D_r^h H((x,x''),(v,v'')),\\
D_r^{hi} G((x',x''),(v,v'')) \circ D_r^h F((x,x'),(v,v')) &\subset D_r^h H((x,x''),(v,v'')),\\
D_r^{hi} G((x',x''),(v,v'')) \circ D_r^{hi} F((x,x'),(v,v')) &\subset D_r^{hi} H((x,x''),(v,v'')).
\end{aligned}$$

An analysis of the higher-order behavior of the image of a set by a multimap is given in the following proposition. It uses an easy lemma.

LEMMA 15. *Given normed spaces X and X', a multimap $G : X \rightrightarrows X'$ identified with its graph, $(x, x') \in G$, $(v, v') \in X \times X'$, $r \in \overline{\mathbb{R}}_+$ one has*

$$\begin{aligned}
D_r^h G((x,x'),(v,v'))(X) &\subset T_r^h(G(X), x', v'),\\
D^h G((x,x'),(v,v'))(X) &\subset T^h(G(X), x', v'),\\
D^{hi} G((x,x'),(v,v'))(X) &\subset T^{hi}(G(X), x', v').
\end{aligned}$$

Proof. This is a consequence of the definitions of $D_r^h G((x,x'),(v,v'))$, $D^h G((x,x'),(v,v'))$, $D^{hi} G((x,x'),(v,v'))$ and of Proposition 6. ∎

PROPOSITION 16. *Let $F : X \rightrightarrows X'$ be a multimap between two normed spaces, and let $B \subset X$, $x \in B$, $x' \in F(x)$, $(v, v') \in X \times X'$ be such that for some $\rho > 0$, $c > 0$, and all $t \in]0, \rho[$, $u \in B(v, \rho)$, $v' \in B(v', \rho)$ with $x + tu \in B$ one has*

$$d((x,x') + t(u,u'), F \cap (B \times X')) \le cd((x,x') + t(u,u'), F)$$

whenever $x + tu \in B$ or

$$d((x,x') + t(u,u'), F \cap (B \times X')) \le cd(x + tu, B)$$

whenever $(x,x')+t(u,u')\in F$. *Then, for all* $r\in\overline{\mathbb{R}}_+$ *one has*

$$D_r^hF((x,x'),(v,v'))(T_r^{hi}(B,x,v))\subset T_r^h(F(B),x',v'),$$
$$D_r^{hi}F((x,x'),(v,v'))(T_r^h(B,x,v))\subset T_r^h(F(B),x',v'),$$
$$D_r^{hi}F((x,x'),(v,v'))(T_r^{hi}(B,x,v))\subset T_r^{hi}(F(B),x',v').$$

Proof. Setting $E:=B\times X'$, $G:=E\cap F$ and using the preceding lemma, it suffices to observe that these inclusions are consequences of the related inclusions of Proposition 7 and of its following remark. ∎

The result can also be derived from Proposition 14 by composing $J:\{0\}\rightrightarrows X$ with graph $\{0\}\times B$ with F; the following remark can be justified similarly.

Remark. If $B\subset X$, $x\in B$, $x'\in F(x)$, $(v,v')\in T(F,(x,x'))$, and if $F:X\rightrightarrows X'$ is (higher-order) semiderivable at (x,x') in the direction (v,v') for (w,w'), r with $w\in T_r^h(B,x,v)$ (resp., $T_r^{hi}(B,x,v)$), then $w'\in T_r^h(F(B),x',v')$ (resp., $T_r^{hi}(F(B),x',v')$). This assertion is a direct consequence of the definitions. ∎

Replacing F by F^{-1} and B by C, we obtain a result dealing with an inverse image.

COROLLARY 17. *Let* $F:X\rightrightarrows X'$ *be a multimap between two normed spaces, and let* $C\subset X'$, $x\in X$, $x'\in F(x)\cap C$, $(v,v')\in X\times X'$ *be such that for some* $\rho>0$, $c>0$ *and all* $t\in]0,\rho[$, $u\in B(v,\rho)$, $v'\in B(v',\rho)$ *with* $x'+tu'\in C$ *one has*

$$d((x,x')+t(u,u'),F\cap(X\times C))\le cd((x,x')+t(u,u'),F).$$

Then, for all $r\in\overline{\mathbb{R}}_+$ *one has*

$$(D_r^hF((x,x'),(v,v')))^{-1}(T_r^{hi}(C,x',v'))\subset T_r^h\left(F^{-1}(C),x,v\right),$$
$$(D_r^{hi}F((x,x'),(v,v')))^{-1}(T_r^h(C,x',v'))\subset T_r^h\left(F^{-1}(C),x,v\right),$$
$$(D_r^{hi}F((x,x'),(v,v')))^{-1}(T_r^{hi}(C,x',v'))\subset T_r^{hi}\left(F^{-1}(C),x,v\right).$$

Let us turn to functions.

DEFINITION 18. *Given a normed space* X, *a function* $f:X\to\overline{\mathbb{R}}$ *finite at* $x\in X$ *and* v, $w\in X$, $p\in\mathbb{R}$, $r\in\overline{\mathbb{R}}_+$, *one defines the lower higher-order derivatives of* f *at* (x,v,p,w) *by*

$$d_r^hf(x,v,p,w):=\inf\left\{q\in\mathbb{R}:q\in D_r^hF((x,f(x)),(v,p))(w)\right\},$$
$$d^hf(x,v,p,w):=\inf\left\{q\in\mathbb{R}:q\in D^hF((x,f(x)),(v,p))(w)\right\},$$
$$d^{hi}f(x,v,p,w):=\inf\left\{q\in\mathbb{R}:q\in D^{hi}F((x,f(x)),(v,p))(w)\right\},$$

where F *is the epigraph of* f.

Taking into account the fact that $D^hF((x,f(x)),(v,p))(w)$ is closed and stable by addition of nonnegative real numbers, we can write min instead of inf in this definition.

Example 7. If F is the graph of a map $g:X\to X'$ that is twice differentiable at $x\in X$, then for $x':=g(x)$, $v\in X$, $v':=g'(x)v$, and all $w\in X$, $r\in\mathbb{R}_+$ one has

$$g'(x)w+rg''(x)vv\in D^hF((x,x'),(v,v'))(w).$$

In fact, taking a sequence $(t_n) \to 0_+$ and a sequence (r_n) in $\mathbb{P}$ with limit r such that $(s_n) := (\frac{1}{2} t_n r_n^{-1}) \to 0$, setting $v_n := v + s_n w$, $x_n := x + t_n v_n$, the Taylor expansion of g yields a sequence $(z_n) \to 0$ in X' such that

$$g(x_n) = g(x) + t_n g'(x)v + t_n s_n g'(x)w + \frac{1}{2} t_n^2 g''(x)(v_n, v_n) + t_n^2 z_n.$$

Since $t_n = 2r_n s_n$, setting $w'_n := g'(x)w + r_n g''(x)(v_n, v_n) + 2r_n z_n$, we see that $(w'_n) \to w' := g'(x)w + rg''(x)vv$ and $g(x) + t_n g'(x)v + t_n s_n w'_n = g(x + t_n v + t_n s_n w)$, so that $(w, w') \in T_r^h(F, (x, x'), (v, v'))$.

4. Optimality conditions. The preceding analysis enables us to present optimality conditions for various nonsmooth problems. But we start with a smooth, unconstrained case that illustrates the use of higher-order methods. Our statement and our proof are slightly different from the ones in [19], [20] as we distinguish different degrees of differentiability and give separate conditions for each case.

PROPOSITION 19. *Let X be a normed space and let $f : X \to \mathbb{R}$ be a function that attains at $x \in X$ a local minimum and is k times differentiable at x with $k \in \{1, 2, 3, 4\}$.*

If $k = 1$, one has $f'(x) = 0$.

If $k = 2$, one has $f'(x) = 0$, $f''(x)(w, w) \geq 0$ for all $w \in X$, and if $f''(x)(v, v) = 0$ for some $v \in X$, then one has $f''(x)(v, w) = 0$ for all $w \in X$.

If $k = 3$, besides the previous conditions one has $f'''(x)(v, v, v) = 0$ whenever $f''(x)(v, v) = 0$. If for some v, $w \in X$ one has $f''(x)(v, v) = 0$, $f''(x)(w, w) = 0$, then one also has $f'''(x)(w, w, w) = 0$, $f'''(x)(v, v, w) = 0$, and $f'''(x)(v, w, w) = 0$.

If $k = 4$, when for some $v \in X$ one has $f''(x)(v, v) = 0$, then one has $f'''(x)(v, v, v) = 0$, $f^{(iv)}(x)(v, v, v, v) \geq 0$. Moreover, in such a case for every $w \in X$ either one has $f''(x)(w, w) = 0$ and besides the conditions of the case $k = 3$ one has $f^{(iv)}(x)(v, v, v, w) = 0$, $f^{(iv)}(x)(v, w, w, w) = 0$, $f^{(iv)}(x)(v, v, w, w) \geq 0$ whenever $f^{(iv)}(x)(v, v, v, v) = 0$, or $f''(x)(w, w) > 0$ and then one has $f''(x)(w, w) f^{(iv)}(x)(w, w, w, w) \geq 3(f''(x)(v, v, w))^2$.

Proof. The assertions for $k = 1$ and $k = 2$ are classical, the condition $f''(x)(v + sw, v + sw) \geq 0$ for all $s \in \mathbb{R}$, $w \in X$ yielding $f''(x)(v, w) = 0$, $f''(x)(w, w) \geq 0$ whenever $f''(x)(v, v) = 0$. In this occurrence, if $k = 3$, the Taylor expansion of $f(x+tv)$ yields $f'''(x)(v, v, v) = 0$. If moreover for some v, $w \in X$ one has $f''(x)(v, v) = 0$, $f''(x)(w, w) = 0$, then, replacing v by $v+sw$ we obtain $f'''(x)(v+sw, v+sw, v+sw) = 0$ for all $s \in \mathbb{R}$, and hence $f'''(x)(w, w, w) = 0$, $f'''(x)(v, v, w) = 0$, and $f'''(x)(v, w, w) = 0$.

If $k = 4$, if for some $v \in X$ one has $f''(x)(v, v) = 0$, the conditions $f'''(x)(v, v, v) = 0$, $f^{(iv)}(x)(v, v, v, v) \geq 0$ follow from the Taylor expansion of $f(x + tv)$. Moreover, if for some $w \in X$ one has $f''(x)(w, w) = 0$, one can replace v by $v + sw$ to obtain $f^{(iv)}(x)(v + sw, v + sw, v + sw, v + sw) \geq 0$. Since s is arbitrary in $\mathbb{R}$, it follows that when $f^{(iv)}(x)(v, v, v, v) = 0$ one has $f^{(iv)}(x)(v, v, v, w) = 0$, $f^{(iv)}(x)(v, v, w, w) \geq 0$ and similar conclusions with w interchanged with v.

If $k = 4$, if for some $v \in X$, $w \in X$ one has $f''(x)(v, v) = 0$, $f''(x)(w, w) > 0$, the fourth-order Taylor expansion of $f(x+tv+rt^2w)$ with $r \in \mathbb{R}$ yields (after simplification by t^4)

$$\frac{1}{2} f''(x)(w, w)r^2 + \frac{3}{6} f'''(x)(v, v, w)r + \frac{1}{24} f^{(iv)}(x)(v, v, v, v) \geq 0.$$

Since r is arbitrary in $\mathbb{R}$ one must have $3(f'''(x)(v, v, w))^2 - f''(x)(w, w) f^{(iv)}(v, v, v, v) \leq 0$. ∎

Sufficient conditions in which inequalities are replaced by strict inequalities can be derived. We present a proof for the reader's convenience.

PROPOSITION 20 (see [20]). *Let X be a finite dimensional normed space and let $f : X \to \mathbb{R}$ be k times differentiable at $x \in X$ with $f'(x) = 0$. Each of the following conditions ensures that f attains at x a strict local minimum:*

(a) *$k = 2$ and $f''(x)(w, w) > 0$ for all $w \in X\backslash\{0\}$;*

(b) *$k = 4$, $f''(x)(w, w) \geq 0$ for all $w \in X$ and if $f''(x)(v, v) = 0$ for some $v \in X$, then one has $f''(x)(v, w) = 0$ for all $w \in X$, $f'''(x)(v, v, v) = 0$, $f^{(iv)}(x)(v, v, v, v) \geq 0$.*

If for some v, $w \in X$ one has $f''(x)(v, v) = 0$, $f''(x)(w, w) = 0$, then one also has $f'''(x)(w, w, w) = 0$, $f'''(x)(v, v, w) = 0$, $f'''(x)(v, w, w) = 0$, $f^{(iv)}(x)(v, v, v, v) > 0$.

If $f''(x)(v, v) = 0$, $f''(x)(w, w) > 0$, then one has $f''(x)(w, w)f^{(iv)}(x)(w, w, w, w) \geq 3(f''(x)(v, v, w))^2$.

Proof. Suppose on the contrary that there exists a sequence $(x_n) \to x$ in $X\backslash\{0\}$ such that $f(x_n) \leq f(x)$ for all $n \in \mathbb{N}$. Let $t_n := \|x_n - x\|$. In the case $k = 2$, taking a limit point w of the sequence $(t_n^{-1}(x_n - x))$ we obtain $f''(x)(w, w) \leq 0$, a contradiction. Let us consider the case $k = 4$. Since the quadratic form q associated with $f''(x)$ is convex, and since $q \geq 0$, the set $N = \{v \in X : q(v) \leq 0\}$ is a closed linear subspace. Since $N^\perp$ is finite dimensional there exists some $c > 0$ such that $q(u) \geq c\|u\|^2$ for all $u \in N^\perp$. Let $u_n \in N^\perp$, $v_n \in N$ be such that $t_n^{-1}(x_n - x) = u_n + v_n$. A Taylor expansion of $f(x_n) - f(x)$ yields some $(\varepsilon_n) \to 0$ such that, with a symbolic notation,

$$0 \geq \frac{1}{2}t_n^2 f''(x)(u_n + v_n)^{(2)} + \frac{t_n^3}{6}f'''(x)(u_n + v_n)^{(3)} + \frac{t_n^4}{24}f^{(iv)}(x)(u_n + v_n)^{(4)} + t_n^4\varepsilon_n. \tag{3}$$

In view of our assumptions we have $f''(x)(u_n, v_n) = 0$, and hence $f''(x)(u_n + v_n)^{(2)} = f''(x)(u_n)^{(2)}$ and $f'''(x)(v_n)^{(3)} = 0$. Dividing both sides of relation (3) by $t_n^4/24$ we obtain

$$0 \geq 12f''(x)\left(t_n^{-1}u_n\right)^{(2)} + 4f'''(x)\left(t_n^{-1}u_n, u_n, u_n\right) + 12f'''(x)\left(t_n^{-1}u_n, v_n, v_n\right) \tag{4}$$

$$+ 12f'''(x)\left(t_n^{-1}u_n, u_n, v_n\right) + f^{(iv)}(x)(u_n + v_n)^{(4)} + \varepsilon_n, \tag{5}$$

and thus, since $\|u_n + v_n\| = 1$, $\|u_n\| \leq 1, \|v_n\| \leq 1$

$$12c\left\|t_n^{-1}u_n\right\|^2 \leq 28\|f'''(x)\|\left\|t_n^{-1}u_n\right\| + \left\|f^{(iv)}(x)\right\| + \varepsilon_n.$$

It follows that $(t_n^{-1}u_n)$ is bounded and taking a subsequence of $((u_n, v_n, t_n^{-1}u_n))$ if necessary, we may assume this sequence converges to some $(u, v, w) \in N^\perp \times N \times N^\perp$. Passing to the limit in relation (5) and observing that $u = \lim_n t_n(t_n^{-1}u_n) = 0$, we get

$$12f''(x)(w, w) + 12f'''(x)(w, v, v) + f^{(iv)}(x)(v, v, v, v) \leq 0.$$

If $f''(x)(w, w) = 0$, we have $w = 0$ and this relation implies $f^{(iv)}(x)(v, v, v, v) \leq 0$, a contradiction with our assumption. If $a := f''(x)(w, w) > 0$, since for any $b, c, r \in \mathbb{R}$ one has $ar^2 + br + c \geq (4ac - b^2)/4a$, we obtain another contradiction with our assumptions: $0 \geq 48f''(x)(w, w)f^{(iv)}(x)(v, v, v, v) - 144(f''(x)(w, v, v))^2$. ∎

Thus, we see that these proofs are just slightly related to the methods of the present paper, and we do not look for still higher-order conditions.

We can derive optimality conditions from Lemma 15 and Example 1.

PROPOSITION 21. *If $(x,x') \in F$ is a local minimizer of a multimap $F : X \rightrightarrows \mathbb{R}$ in the sense that $x' \le u'$ for all $(u,u') \in F$ close to (x,x'), then for all $v \in X$ one has $DF(x,x')(v) := \{v' \in \mathbb{R} : (v,v') \in T(F,(x,x'))\} \subset \mathbb{R}_+$ and $D^hF((x,x'),(v,0))(X) \subset \mathbb{R}_+$.*

Proof. The first relation is a consequence of the inclusion $T(F,(x,x')) \subset X \times \mathbb{R}_+$ since $F \subset X \times [x',\infty[$. The second one stems from Example 1 and Lemma 15, as announced. ∎

COROLLARY 22. *Let x be a local minimizer of a function $f : X \to \overline{\mathbb{R}}$ that is finite at x. Then, for any $v \in X$, $w \in X$ one has $df(x,v) := \inf\{p \in \mathbb{R} : (v,p) \in T(E_f,(x,f(x))\} \ge 0$, E_f being the epigraph of f, and $d^hf(x,v,0,w) \ge 0$.*

Proof. The condition $df(x,v) \ge 0$ is well known. The condition $d^hf(x,v,0,w) \ge 0$ for all $w \in X$ stems from Proposition 21 by taking $F := E_f$. ∎

Remark. If X is finite dimensional, if x is a local minimizer of the restriction f_B to a subset B of X of a function f that is twice differentiable at x (in the sense that $f_B := f + \iota_B$, where ι_B is the indicator function of B), one recovers the necessary condition of [42, Thm 1.2]: $f'(x)v \ge 0$ for all $v \in T(B,x)$,

$$\frac{1}{2}f''(x)vv + \liminf_{(t,u)\to(0_+,v),\ x+tu\in B} f'(x)t^{-1}(u-v) \ge 0 \qquad \forall v \in \ker f'(x) \cap T(B,x). \tag{6}$$

In fact, taking sequences $(t_n) \to 0_+$, $(v_n) \to v \in \ker f'(x) \cap T(B,x)$ such that $x + t_nv_n \in B$ for all n, $(q_n) := (f'(x)t_n^{-1}(v_n - v)) \to q := \liminf_{(t,u)\to(0_+,v),x+tu\in B} f'(x)t^{-1}(u-v)$ and setting $s_n := \|v_n - v\|$ we can consider two cases. Either $s_n = 0$ for n large enough, in which case one has $v_n = v$ for all those n's, and hence $q = 0$ and $\frac{1}{2}f''(x)vv = \lim_n t_n^{-2}[f(x+t_nv) - f(x)] \ge 0$ since $x + t_nv \in B$, or $s_n > 0$ for infinitely many n in which case we may assume that $(w_n) := (s_n^{-1}(v_n - v))$ converges to some $w \in X$ and then for some sequence $(z_n) \to 0$ in $\mathbb{R}$ we have

$$0 \le f(x+t_nv_n) - f(x) = t_nf'(x)v_n + \frac{1}{2}t_n^2f''(x)v_nv_n + t_n^2z_n.$$

Dividing by t_n^2 and noting that $(t_n^{-1}f'(x)v_n) = (q_n) \to q$ and $(f''(x)v_nv_n) \to f''(x)vv$, we get (6). ∎

PROPOSITION 23. *If $(x,x') \in F$ is a local minimizer on $B \subset X$ of a multimap $F : X \rightrightarrows \mathbb{R}$ in the sense that $x' \le u'$ for all $(u,u') \in (B \times X') \cap F$ close to (x,x'), then for all $v \in X$, $r \in \overline{\mathbb{R}}_+$, $w \in T_r^h(B,x,v)$, $w' \in X'$ such that F is semiderivable at (x,x') in the direction $(v,0)$, for r, (w,w'), one has $w' \ge 0$.*

In particular, if F is semiderivable at (x,x') in the direction $(v,0) \in X \times \mathbb{R}$ for $r \in \overline{\mathbb{R}}_+$, $w \in T_r^h(B,x,v)$, then one has $D^hF((x,x'),(v,0))(w) \subset \mathbb{R}_+$.

Proof. Given $w \in T^h(B,x,v)$ there exist sequences $(w_n) \to w$, (t_n), $(s_n) \to 0_+$ such that $(s_n^{-1}t_n) \to 2r$ such that $x + t_nv + t_ns_nw_n \in B$ for all $n \in \mathbb{N}$. Since F is semiderivable at (x,x') in the direction $(v,0)$ for (w,w'), r there exists a sequence $(w'_n) \to w'$ such that

$$x' + t_ns_nw'_n \in F(x + t_nv + t_ns_nw_n)$$

for infinitely many $n \in \mathbb{N}$. Then we have $x' + t_ns_nw'_n \ge x'$, and hence $w' \ge 0$.

The last assertion is a direct consequence of the first one. ∎

In the next corollary we complete the condition (7) obtained in [43, Prop. 3.1] and extended to the case of a locally Lipschitzian objective function in [18, Cor. 4.2] by condition (8), which is new.

COROLLARY 24. *If x is a local minimizer on $B \subset X$ of a function $f : X \to \mathbb{R}$ that is twice differentiable at x, then for all $v \in T(B,x)$ one has $f'(x)v \geq 0$ and for all $v \in T(B,x) \cap \ker f'(x)$, $r \in \overline{\mathbb{R}}_+$, $w \in T_r^h(B,x,v)$ one has*

$$f'(x)w + rf''(x)(v,v) \geq 0 \quad \text{if } r \in \mathbb{R}_+, \tag{7}$$

$$f''(x)(v,v) \geq 0 \quad \text{if } r = +\infty. \tag{8}$$

Proof. This follows from the fact that f is semiderivable at x in the direction $(v,0)$ for $r \in \mathbb{R}_+$,$w \in X$ by Example 5 with $d_r^h f(x,v,w) = f'(x)w + rf''(x)(v,v)$. If $w \in T_\infty^h(B,x,v)$, one has $0 \in T_1^h(B,x,v)$ and (8) follows from (7). ∎

Example 8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x_1,x_2) := x_2 - x_1^2$ and let $B := \{(t,t^{5/2}) : t \in \mathbb{R}_+\}$. For $x := (0,0)$, $v := (0,1)$, $r \in \mathbb{R}_+$ one has $(0,1) \in T_\infty^h(B,x,v)$ and condition (8) is not satisfied. Thus x is not a local minimizer of f on B.

Corresponding sufficient conditions follow.

PROPOSITION 25. *Let X be a finite dimensional space and let $F : X \rightrightarrows \mathbb{R}$. Suppose that for all $v \in X$, $w \in X\backslash\{0\}$ one has $DF(x,x')(v) \subset \mathbb{R}_+$ and $D^hF(x,x',v,0)(w) \subset \mathbb{P}$. Then x is a local strict minimizer of F in the sense that there exists a neighborhood $U \times U'$ of (x,x') such that for all $(u,u') \in U \times U'$ with $u \neq x$ one has $u' > x'$.*

Proof. Suppose on the contrary that there exist sequences $(x_n) \to x$, $(x'_n) \to x'$ such that $x_n \neq x$, $x'_n \in F(x_n)$, and $x'_n \leq x'$ for all n. Setting $t_n := \|x_n - x\| + |x'_n - x|$ we may assume $((v_n,v'_n)) := ((t_n^{-1}(x_n - x), t_n^{-1}(x'_n - x')))$ converges to some unit vector $(v,v') \in X \times \mathbb{R}$. Since $v'_n \leq 0$ and $v' \in DF(x,x')(v) \subset \mathbb{R}_+$ we have $v' = 0$ and $\|v\| = 1$. Let $s_n := \|(v_n - v, v'_n - v')\|$. If $s_n = 0$ for n in an infinite subset N of $\mathbb{N}$ we have $x'_n = x'$, and hence $t_n = \|x_n - x\|$ for all $n \in N$, and $v \neq 0$, since

$$(x_n, x'_n) = (x,x') + t'_n(v,0) + t'_n s'_n(v,0)$$

for $t'_n := t_n(1-t_n) > 0$ for n large enough and $s'_n := t_n(1-t_n)^{-1}$, so that $(v,0) \in T^h(F,(x,x'),(v,0))$, contradicting $D^hF(x,x',v,0)(v) \subset \mathbb{P}$. Thus, deleting a finite number of terms, we may assume $s_n > 0$ for all $n \in \mathbb{N}$ and that $((w_n,w'_n)) := ((s_n^{-1}(v_n - v), s_n^{-1}(v'_n - v')))$ converges to some $(w,w') \in X \times \mathbb{R}$. Then $(w,w') \in T^h(F,(x,x'),(v,v'))$ or $w' \in D^hF(x,x',v,0)(w) \subset \mathbb{P}$, a contradiction with $w' = \lim_n s_n^{-1}(v'_n - v') = \lim_n s_n^{-1}t_n^{-1}(x'_n - x') \leq 0$. ∎

Taking for F the epigraph of a function f, we get more familiar statements.

COROLLARY 26. *Let X be a finite dimensional space and let $f : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ be finite at x. Suppose that for all v, $w \in X$, $r \in \overline{\mathbb{R}}_+$ one has $df(x,v) \geq 0$ and $d_r^h f(x,v,0,w) > 0$ when $df(x,v) = 0$. Then x is a local strict minimizer of f.*

COROLLARY 27 (see [43]). *Let F be a subset of a finite dimensional space X and let $f : X \to \mathbb{R}$ be twice differentiable at $x \in F$. Suppose that for all $v \in T(F,x)$ one has $f'(x)v \geq 0$ and for all $v \in T(F,x) \cap \ker f'(x)$, $v \neq 0$, $r \in \mathbb{R}_+$, $w \in T_r^h(F,x,v)$ one has $f'(x)w + rf''(x)(v,v) > 0$ provided $(w,r) \neq (0,0)$. Then x is a local strict minimizer of f on F.*

5. Application to mathematical programming. Let us consider the mathematical programming problem

$$(\mathcal{P}) \quad \text{minimize } f(x): \ x \in F := B \cap g^{-1}(C),$$

where the feasible set F is obtained as the intersection of a basic constraint B, a closed convex subset of a Banach space X, for example, an orthant, with an implicit constraint described by a closed convex subset C of a Banach space Z and a map $g : X \to Z$ that is twice differentiable at $x \in F$. Given $v \in T(B, x)$, as in [43] we make a *directional metric regularity* assumption:

(DMR) there exist $\mu > 0$, $\rho > 0$ such that for $t \in]0, \rho[$, $u \in B(v, \rho)$ satisfying $x + tu \in B$ one has

$$d(x + tu, F) \leq \mu d(g(x + tu), C).$$

LEMMA 28. *Under assumption (DMR) one has*

$$T_r^h(F, x, v) = T_r^h(B, x, v) \cap g'(x)^{-1} \left(T_r^h(C, g(x), g'(x)v) - rg''(x)vv\right).$$

The proof of this assertion is similar to the one of [43, Prop. 4.1].

The condition (DMR) (and even a metric regularity property) is a consequence of the relation

$$\mathbb{R}_+(C - g(x)) - \mathbb{R}_+ g'(x)(B - x) = Z, \tag{9}$$

which is a form of the Robinson condition generalizing the Mangasarian–Fromovitz qualification condition. In fact, setting $h := (I_X, g)$, condition (9) implies that

$$\mathbb{R}_+(B \times C - h(x)) - h'(x)(X) = X \times Z$$

since for any $(w, z) \in X \times Z$, using an equivalent form of (9) we can find $r \in \mathbb{R}_+$, $b \in B$, $c \in C$ such that $z - g'(x)w = r(c - g(x)) - rg'(x)(b - x)$ and setting $v := r(b - x) - w$ we obtain $z = r(c - g(x)) - g'(x)(v)$ and $w = r(b - x) - v$. Thus, for some $\mu > 0$, $\sigma > 0$ one has the metric regularity property

$$d(x', F) \leq \mu d(h(x'), B \times C) = \mu d(x', B) + \mu d(g(x'), C) \qquad \forall\, x' \in B(x, \sigma)$$

for the metric d on $X \times Z$ given by $d((x', z'), (x'', z'')) := d(x', x'') + d(z', z'')$.

Let us note the following conditions that weaken relation (9); here v is an element of $T(B, x)$:

$$T(C, g(x)) - g'(x)(T(B, x)) = Z, \tag{10}$$

$$T(T(C, g(x)), g'(x)v) - g'(x)(T(T(B, x), v)) = Z. \tag{11}$$

They can be seen as transversality conditions (see [40]). The second one is the Ben Tal condition [3].

As in [43], we use a duality theorem.

LEMMA 29. *Let P (resp., Q) be a closed convex cone of X (resp., Z), and let $A : X \to Z$ be linear and continuous such that $Z = A(P) - Q$. If for some $m \in \mathbb{R}$, $x^* \in X^*$, $\overline{z} \in Z$ one has*

$$x^*(x) \geq m \ \forall x \in P \cap A^{-1}(Q + \overline{z}),$$

then there exists some $y \in Q^0$, the polar cone of Q such that $x^* + y \circ A \in -P^0$, $-y(\overline{z}) \geq m$.

Enlarging a classical notion, we say that $v \in X$ is a *critical vector* at x, and we write $v \in K(x)$ if $f'(x)v = 0$, $v \in T(B,x)$, $g'(x)v \in T(C, g(x))$.

THEOREM 30. *Let x be a local solution to problem ($\mathcal{P}$). Suppose conditions (DMR) and* (11) *are satisfied. Then for any $v \in K(x)\backslash\{0\}$ and any $r \in \mathbb{R}_+$, $z \in T_r^h(C, g(x), g'(x)v)$ there exists some $y \in N(C, g(x)) = (T(C, g(x)))^0$ such that $\langle y, g'(x)v\rangle = 0$,*

$$f'(x) + y \circ g'(x) \in -N(B,x), \tag{12}$$

$$r(f''(x)vv + \langle y, g''(x)vv\rangle) \geq \langle y, z\rangle. \tag{13}$$

Proof. Let $v \in K(x)\backslash\{0\}$, let $r \in \mathbb{R}_+$, and let $z \in T_r^h(C, g(x), g'(x)v)$. By Proposition 8 for any $w \in T(T(B,x), v)$ such that

$$g'(x)w + rg''(x)vv - z \in T(T(C, g(x)), g'(x)v)$$

we have $g'(x)w + rg''(x)vv \in T_r^h(C, g(x), g'(x)v)$.

It follows from Lemma 28 that $w \in T_r^h(F, x, v)$. Then condition (7) ensures that

$$f'(x)w + rf''(x)vv \geq 0.$$

Using Lemma 29 with $A := g'(x)$, $x^* := f'(x)$, $m := -rf''(x)vv$, $P := T(T(B,x), v)$, $Q := T(T(C, g(x)), g'(x)v)$, $\overline{z} := z - rg''(x)vv$ and noting that $A(P) - Q = Z$ by condition (11), we get some $y \in N(T(C, g(x)), g'(x)v)$ such that

$$\begin{aligned} f'(x) + y \circ g'(x) &\in -N(T(B,x), v) \\ -\langle y, z - rg''(x)vv\rangle &\geq -rf''(x)vv. \end{aligned}$$

Since $T(B,x)$ is a convex cone, $N(T(B,x), v) = \{x^* \in T(B,x)^0 : \langle x^*, v\rangle = 0\}$, so that $x^* := f'(x) + y \circ g'(x) \in -N(B,x)$; in fact this relation is equivalent to $x^* \in -N(T(B,x), v)$ since $f'(x)v = 0$, $\langle y, g'(x)v\rangle \leq 0$ as $y \in N(C, g(x))$, $g'(x)v \in T(C, g(x))$. Similarly since $\langle y, g'(x)v\rangle = \langle x^*, v\rangle \geq 0$ and $\langle y, g'(x)v\rangle \leq 0$ since $g'(x)v \in T(C, g(x))$ and $y \in N(C, g(x))$, we have that $y \in N(T(C, g(x)), g'(x)v)$ is equivalent to $y \in N(C, g(x))$ and $\langle y, g'(x)v\rangle = 0$. The second relation is (13). ∎

The set $M(x)$ of $y \in N(C, g(x))$ satisfying condition (12) is called the set of *multipliers* at x. It also plays a role in the following sufficient condition.

THEOREM 31. *If for problem ($\mathcal{P}$) the following conditions are satisfied, then x is a strict local minimizer of f on F.*

(a) *X is finite dimensional;*
(b) *the set $M(x) := \{y \in N(C, g(x)) : f'(x) + y \circ g'(x) \in -N(B,x)\}$ of multipliers at x is nonempty;*
(c) *for every $v \in T(F,x) \cap \ker f'(x)$, $v \neq 0$, $r \in \mathbb{R}_+$, $w \in T_r^h(F, x, v)$ with $(w, r) \neq (0,0)$ there exists $y \in M(x)$ such that for $z := g'(x)w + rg''(x)vv$ one has*

$$r(f''(x)vv + \langle y, g''(x)vv\rangle) > \langle y, z\rangle.$$

Proof. Given $v \in T(F,x)\backslash\{0\}$, $y \in M(x)$, since $N(B,x) \subset N(F,x)$ and $g'(x)v \in T(C, g(x))$ we have

$$f'(x)v = \langle f'(x) + y \circ g'(x), v\rangle - \langle y, g'(x)v\rangle \geq -\langle y, g'(x)v\rangle \geq 0$$

since $f'(x) + y \circ g'(x) \in -N(B,x)$, $y \in N(C, g(x))$ and $g'(x)v \in T(C, g(x))$. Given $r \in \mathbb{R}_+$, $w \in T_r^h(F,x,v)$ with $(w,r) \neq (0,0)$ we have $z := g'(x)w + rg''(x)vv \in T_r^h(C, g(x), g'(x)v)$ by Proposition 9. Taking $y \in M(x)$ as in assumption (c) we have

$$f'(x)w + rf''(x)vv > f'(x)w + \langle y, z\rangle - \langle y, rg''(x)vv\rangle = \langle f'(x) + y \circ g'(x), w\rangle \geq 0$$

since $x^* := f'(x) + y \circ g'(x) \in -N(T(B,x),v)$, $f'(x)v = 0, \langle y, g'(x)v\rangle = 0$, $w \in T_r^h(B,x,v)$, so that $\langle x^*, v\rangle = 0$ and, taking sequences (s_n), $(t_n) \to 0_+$, $(w_n) \to w$ such that $(s_n^{-1}t_n) \to 2r$, $x_n := x + t_nv + s_nt_nw_n \in F$ for all n, we get $\langle x^*, w\rangle = \lim_n \langle x^*, s_n^{-1}t_n^{-1}(x_n - x - t_nv)\rangle = \lim_n \langle x^*, s_n^{-1}t_n^{-1}(x_n - x)\rangle \geq 0$. Thus the conclusion follows from Corollary 27. ∎

6. Conclusion. It is worth noticing that the presentation of higher-order derivatives of functions through a geometric approach via higher tangent sets can be reversed. In fact it is easy to show that the higher-order tangent set $T^h(S,x,v)$ to a subset S of a normed space X at $x \in S$ in the direction $v \in X$ can be obtained through the higher-order derivative $d^h\iota_S$ of the indicator function ι_S of S. In fact, for all $r \in \overline{\mathbb{R}}_+$ one can show that $d_r^h\iota_S((x,0),(v,0))(\cdot)$ (resp., $d^h\iota_S((x,0),(v,0))(\cdot)$) is the indicator function of $T_r^h(S,x,v)$ (resp., $T^h(S,x,v)$).

In view of the abundance of notions of higher-order generalized derivatives, it is out of the scope of the present article to present comparisons with all existing notions. In particular, we leave aside dual notions such as generalized Hessians (see [33], for instance). However, it is desirable to point out links with concepts that seem to be not too far from the ones we dealt with. In particular it is advisable to compare our notion of higher-order derivative at (x,v,p,w) of a function $f : X \to \overline{\mathbb{R}}$ finite at $x \in X$ with the definition of the *second subderivative of f at x for $x^* \in X^*$* given in [50] and [52, Def. 13.3] by

$$d^2f(x \mid x^*)(v) := \liminf_{(t,u)\to(0_+,v)} \frac{2}{t^2}[f(x+tu) - f(x) - \langle x^*, tu\rangle].$$

Setting $f^D(x,v) := \liminf_{(t,u)\to(0_+,v)} t^{-1}(f(x+tu) - f(x))$ and observing that $d^2f(x,x^*)(v) = +\infty$ if $\langle x, x^*\rangle\langle f^D(x,v)$, $d^2f(x,x^*)(v) = -\infty$ if $\langle x, x^*\rangle\rangle f^D(x,v)$, as shown in [52, Prop. 13.5], we may assume that $\langle x^*, v\rangle = p := f^D(x,v)$. In such a case, for all $w \in X$, $r \in \mathbb{R}_+$ we have

$$rd^2f(x \mid x^*)(v) + \langle x^*, w\rangle \leq d_r^hf(x,v,p,w). \tag{14}$$

In fact, for all $q \in D_r^hF((x,f(x)),(v,p))(w)$, where F is the epigraph of f, we can find sequences (s_n), $(t_n) \to 0_+$, $(w_n) \to w$, $(q_n) \to q$ such that $(s_n^{-1}t_n) \to 2r$ and

$$f(x) + t_np + t_ns_nq_n \geq f(x + t_nv + t_ns_nw_n)$$

and since $(v_n) := (v + s_nw_n) \to v$ and $\langle x^*, t_nv_n\rangle = t_np + t_ns_n\langle x^*, w_n\rangle$ we have

$$q_n \geq \frac{1}{s_nt_n}[f(x + t_nv + t_ns_nw_n) - f(x) - t_np],$$
$$q \geq \liminf_n \frac{2r_n}{t_n^2}[f(x + t_nv + t_ns_nw_n) - f(x) - \langle x^*, t_nv_n\rangle] + \lim_n \langle x^*, w_n\rangle,$$

and hence $q \geq rd^2f(x \mid x^*)(v) + \langle x^*, w\rangle$. Passing to the infimum over

$$q \in D_r^hF((x,f(x)),(v,p))(w)$$

we obtain the announced inequality (14). If $r = \infty$ and $d_\infty^hf(x,v,p,w) \in \mathbb{R}$ (so that

$D_r^h F((x, f(x)), (v, p))(w)$ is nonempty), the preceding calculation shows that

$$d^2 f(x \mid x^*)(v) \leq \inf_{w \in X} \frac{1}{r} \left[d_r^h f(x, v, p, w) - \langle x^*, w \rangle \right].$$

The preceding comparison can be completed. Here we set

$$d_0^2 f(x \mid x^*)(v) := \liminf_{t \to 0_+} \frac{2}{t^2} [f(x + tv) - f(x) - \langle x^*, tv \rangle].$$

PROPOSITION 32. *If X is finite dimensional and if f is finite at x and for $v \in X$, $x^* \in X^*$ with $d^2 f(x \mid x^*)(v)$ finite and different from $d_0^2 f(x \mid x^*)(v)$, there exist $r \in \mathbb{R}_+$ and $w \in X$ such that, for $p := \langle x^*, v \rangle$,*

$$rd^2 f(x \mid x^*)(v) = d_r^h f(x, v, p, w) - \langle x^*, w \rangle = \min_{w' \in X} \left[d_r^h f(x, v, p, w') - \langle x^*, w' \rangle \right]. \tag{15}$$

Proof. Let $a := d^2 f(x \mid x^*)(v)$ and let $(t_n) \to 0_+$, $(v_n) \to v$ be such that $(a_n) \to a$ with

$$a_n := \frac{2}{t_n^2} [f(x + t_n v_n) - f(x) - \langle x^*, t_n v_n \rangle].$$

Let $s_n := \|v_n - v\|$. Since $d^2 f(x \mid x^*)(v) \neq d_0^2 f(x \mid x^*)(v)$ we have $s_n \neq 0$ for n large enough. We may suppose $s_n > 0$ for all $n \in \mathbb{N}$ and assume that for $w_n := s_n^{-1}(v_n - v)$ one has $(w_n) \to w$ for some $w \in X$. We also may assume that for some $r \in \mathbb{R}_+$ we have $(\frac{1}{2} s_n^{-1} t_n) \to r$ since if $(\frac{1}{2} s_n^{-1} t_n) \to \infty$ we have $\lim_n 2t_n^{-2}(\langle x^*, t_n v_n \rangle - \langle x^*, t_n v \rangle) = 0$ and $d^2 f(x \mid x^*)(v) = d_0^2 f(x \mid x^*)(v)$. Then, denoting by F the epigraph of f we have

$$\left(x + t_n v + s_n t_n w_n, f(x) + t_n \langle x^*, v \rangle + s_n t_n \langle x^*, w_n \rangle + \frac{1}{2} t_n^2 a_n \right) \in F,$$

and hence $\langle x^*, w \rangle + ra \in D_r^h F((x, f(x)), (v, \langle x^*, v \rangle))(w)$ and $\langle x^*, w \rangle + ra \geq d_r^h f(x, v, p, w)$ for $p := \langle x^*, v \rangle$. In view of relation (14) we obtain (15). ∎

Another comparison with a classical second-order derivative can be done. When $x^* \in \partial_D f(x)$, the *directional (or contingent) subdifferential* of f at x, i.e., when $x^* \leq f^D(x, \cdot)$, for all v, $w \in X$, $p := \langle x^*, v \rangle$, we deduce from (14) the relation

$$rd^2 f(x)(v) + \langle x^*, w \rangle \leq d_r^h f(x, v, p, w), \tag{16}$$

where $d^2 f(x)(v)$ is the *second derivative of f at x for v* given in [52, Def. 13.3] by

$$d^2 f(x)(v) := \liminf_{(t,u) \to (0_+, v)} \frac{2}{t^2} \left[f(x + tu) - f(x) - tf^D(x, u) \right].$$

Defining a *directional subhessian* of f at x for v as an element A of the set $L_s^2(X, X; \mathbb{R})$ of symmetric continuous bilinear maps from X to $\mathbb{R}$ such that

$$\liminf_{(t,u) \to (0_+, v)} \frac{1}{t^2} \left[f(x + tu) - f(x) - tf^D(x, u) - \frac{1}{2} t^2 A(u, u) \right] \geq 0,$$

so that $A \in L_s^2(X, X; \mathbb{R})$ is a directional subhessian of f at x for v if and only if

$A(v,v) \leq d^2 f(x)(v)$, we deduce from relation (16) the inequality

$$rA(v,v) + \langle x^*, w\rangle \leq d_r^h f(x,v,p,w)$$

for any directional subhessian A of f at x for v.

For other comparisons we refer to [26, 27, 28, 29, 30, 31], [41], [43], [46, 47, 48].

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