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**Abstract**

We consider the non-local Fisher-KPP equation on a bounded domain with Neumann boundary conditions. Thanks to a Lyapunov function, we prove that under a general hypothesis on the Kernel involved in the non-local term, the homogenous steady state 1 is globally asymptotically stable. This assumption happens to be linked to some conditions given in the literature, which ensure that travelling waves link 0 to 1.

## 1 Introduction

We consider the so-called non-local Fisher-KPP equation endowed with Neumann boundary conditons

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = \mu\left(1 - \int_\Omega K(x,y)u(t,y)\,dy\right)u(t,x) + \Delta u(t,x), \quad x \in \Omega,\ t > 0, \\ &\frac{\partial u}{\partial n}(t,x) = 0, \quad x \in \partial\Omega,\ t > 0, \\ &\quad u(0,x) = u^0(x) \geqslant 0 \quad x \in \Omega, \end{aligned} \tag{1}$$

where $\Omega$ a regular bounded domain of $\mathbb{R}^d$ and $K > 0$ a Kernel modelling an additional death rate due to non-local interactions.

We will sometimes write in short $K[u] = \int_\Omega K(x,y)u(y)\,dy$ for a generic function $u$.

Assuming

$$\forall y \in \Omega, \ \int_\Omega K(x,y)\,dx = 1, \tag{2}$$

and in the limit $K(x,y) \to \delta_{x-y}$, we recover the classical Fisher KPP-equation

$$\frac{\partial u}{\partial t} = \mu(1-u)u + \Delta u. \tag{3}$$

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The assumption (2) ensures that 1 remains a homogeneous stationary solution of (1).

The classical Fisher-KPP equation (3) is often analysed on the whole space for the investigation of travelling waves, which are known to exist since the pioneering works of Fisher, Kolmogorov, Petrovsky and Piskunov [7] for any speed above $2\sqrt{\mu}$. Furthermore, any-non zero initial condition eventually converges locally uniformly to 1, which is therefore a globally asymptotically stable for non zero initial conditions.

When one adds a non-local term, it does not remain true that travelling waves exist and when they do, whether they link 0 to 1 or another non-homogeneous steady-state of the equation. 1 can indeed become unstable: Türing patterns appear [8, 9].

A natural question is thus to understand under which conditions the status of 1 is changed due to the non-local term. When $K(x, y)$ is given by a convolution $\phi(x-y)$, several results have already been obtained in the full space, in dimension $d = 1$. If the Fourier transform is everywhere positive of if $\mu$ is small enough, it is known that travelling waves necessarily connect 0 to 1 [2]. See also [1, 5]

In this note, we provide a general result on the global asymptotic stability on 1 on a bounded domain, based on a Lyapunov functional. The results holds provided that the following general assumption on the Kernel $K$ is satisfied:

$$\forall f \in L^2(\Omega),\ \int_{\Omega\times\Omega} K(x,y)f(x)f(y)\,dx\,dy \geqslant 0. \tag{4}$$

$K$ is then referred to as being a positive Kernel, and (4) can be thought of as a strong competition assumption. These types of Lyapunov functionals have been used successfully in selection equations in [6, 10, 11] and are inspired by Lyapunov functions for Lotka-Volterra ODEs [4].

It remains an open question to know whether this condition leads to the same conclusion on the whole space. As such, our Lyapunov function requires integrability for $u(t)-1-\ln(u(t))$ which is too much to ask in $\mathbb{R}^d$. We still believe that the condition (4) is highly relevant. Indeed, when $\Omega = \mathbb{R}^d$, and if $K$ is a convolution $K(x, y) = \phi(x - y)$, then condition (4) becomes

$$\forall f \in L^2(\mathbb{R}^d),\ \int_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x-y)f(x)f(y)\,dx\,dy \geqslant 0. \tag{5}$$

It is easy to check that if $\phi$ has a non-negative Fourier transform, then condition is satisfied, see [6]. The converse is almost true, as evidenced by Bochner's Theorem [12]: if $\phi$ is bounded an continuous, then (4) holds if and only if it is the Fourier transform of a finite bounded measure on $\mathbb{R}^d$.

Consequently, condition (4) (or (1)) shows that the condition on the Fourier transform of $\phi$ used in dimension 1 in the literature can be appropriate in any dimension, and may not only be a sufficient but also a necessary condition when it comes to the stability of the state 1.

## 2 The Lyapunov function approach

We make the following regularity assumption on the Kernel $K$:

$$K \in C^{0,1}\big(\overline{\Omega} \times \overline{\Omega}\big), \tag{6}$$

where $C^{0,1}\big(\overline{\Omega} \times \overline{\Omega}\big)$ denotes the set of Lipschitz continuous functions on $\overline{\Omega} \times \overline{\Omega}$.

Under the previous assumption (6), for $u^0 \in L^1(\Omega)$, we know from [3] that there exists a unique non-negative classical solution in $C([0, +\infty), L^1(\Omega)) \cap C^1((0, +\infty), C^{2,\alpha}(\Omega))$, which we denote $t \longmapsto S_t u^0$.

It will also be convenient to introduce the space $Z := \{u \in C^{2,\alpha}(\Omega),\ u \geqslant 0\}$. Finally, we define the non-negative function $H(w) := w - 1 - \ln(w)$ for $w > 0$, and for $u$ in $Z$

$$V(u) := \int_\Omega (u(x) - 1 - \ln(u(x)))\, dx, \tag{7}$$

the last integral possibly being equal to $+\infty$.

Our result is then the following:

**Theorem 1.** *Assume (4), (6), (2). Then for any initial datum $u^0$ in $L^1(\Omega)$, $u^0 \geqslant 0$, $u^0 \neq 0$, the solution to (1) satisfies*

$$u(t, \cdot) \longrightarrow 1$$

*uniformly in $\Omega$.*

*Proof. First step: computation of the Lyapunov functional.*

First, let us remark that by the parabolic strong maximum principle, $u(t, x) > 0$ for all $t > 0$, $x \in \Omega$. Now, let us check that this holds true also for $x \in \partial\Omega$, from which we will infer that $V(u(t))$ is well defined for all $t > 0$. By the parabolic strong maximum principle at the boundary, we have the following alternative for $x \in \partial\Omega$: either $u(t, x) > 0$ or $u(t, x) = 0$ and $\dfrac{\partial u}{\partial n}(t, x) < 0$. Only $u(t, x) > 0$ can hold due to the Neumann boundary conditions.

We now consider $g(t) := V(u(t))$ for $t > 0$, where $\{u(t)\}_{t \geqslant 0}$ is the trajectory given rise to by $u_0$. Let us prove that this is a Lyapunov functional, by computing for $t > 0$

$$\begin{aligned}
g'(t) &= \int_\Omega \frac{\partial u}{\partial t}(t) \left(1 - \frac{1}{u(t)}\right) \\
&= \int_\Omega \Delta u(t) \left(1 - \frac{1}{u(t)}\right) - \mu \int_\Omega (1 - K[u(t)])\,(1 - u(t))) \\
&= -\int_\Omega \frac{|\nabla(u(t, x))|^2}{u^2(t, x)}\, dx - \mu \int_{\Omega^2} K(x, y)\,(1 - u(t, x)))\,(1 - u(t, y))\, dx\, dy.
\end{aligned}$$

after integration by part for the first term. For the second one, we used $1-K[u]=K[1-u]$, owing to (2).

Thanks to (4), this yields $g'(t)\leqslant 0$ *i.e.*, that $g$ is non-increasing over the real line. Since $g\geqslant 0$, we infer the convergence of $g(t)$ as $t$ tends to $+\infty$, and we denote $l$ its limit.

*Second step: compactness of trajectories.*

Since $C^{2,\alpha}(\Omega)$ is compactly embedded into $C\left(\overline{\Omega}\right)$, the trajectory $\{S_t u^0\}_{t\geqslant 0}$ is relatively compact in $C(\overline{\Omega})$, meaning that one can find $\bar{u}\geqslant 0$ in $C(\overline{\Omega})$ and a sequence $(t_k)$ tending to $+\infty$ in $k$, such that $u(t_k)$ converges to $\bar{u}$ as $k$ goes to $+\infty$, in $C\left(\overline{\Omega}\right)$. Note that the limit cannot be identically 0 since otherwise $g(t)$ would go to $+\infty$, in contradiction with its convergence to $l$.

Our aim is to prove that $\bar{u}=1$, which will mean that the whole trajectory converges to $\bar{u}$, hence the expected result.

*Third step: identifying the limit.*

Let us now consider the trajectory starting from the initial datum $\bar{u}$, namely $\{S_t\bar{u}\}_{t\geqslant 0}$, which we also denote by $\{\tilde{u}(t)\}_{t\geqslant 0}$. Because $\bar{u}\geqslant 0$, $\bar{u}\neq 0$, we again have $\tilde{u}(t,x)>0$ for all $t>0$, $x\in\overline{\Omega}$. Let us prove that $V$ is constant along the trajectory $\{S_t\bar{u}\}_{t\geqslant 0}$ for $t>0$.

For this, we write $V(\tilde{u}(t))=V(S_t\bar{u})=V\left(S_t\lim_{k\to+\infty}S_{t_k}u^0\right)=V\left(\lim_{k\to+\infty}S_{t+t_k}u^0\right)$. It is also easy to see that for any $u$ in $C\left(\overline{\Omega}\right)$ which is furthermore positive on $\overline{\Omega}$, $V$ (seen as acting on $C\left(\overline{\Omega}\right)$) is continuous at $u$, and this implies $V(\tilde{u}(t))=\lim_{k\to+\infty}V\left(S_{t+t_k}u^0\right)=l$. As claimed the function $t\longmapsto V(S_t\bar{u})$ is constant (equal to $l$) for $t>0$.

Hence its derivative must be zero for $t>0$: from the computations made in the first step, it must hold that both $\int_\Omega\frac{|\nabla(\tilde{u}(t))|^2}{\tilde{u}^2(t)}$ and $\int_{\Omega^2}K(x,y)\,(\tilde{u}(t,x)-1))\,(\tilde{u}(t,y)-1)\,dx\,dy$ vanish identically for $t>0$. Let us now fix $t>0$, and from the first term, we know that $\tilde{u}(t)$ is a constant. From the second term and owing to $K>0$, this constant must be equal to 1. By continuity of the trajectory, this also holds true at $t=0$, *i.e.*, $\bar{u}=1$, which ends the proof.

□

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