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Bruno Machet. 1-Loop mass generation by a constant external magnetic field for an electron propagating in a thin medium. International Journal of Modern Physics B, 2018, 10.1142/S021797921850114X . hal-01723541

**HAL Id: hal-01723541**

**<https://hal.sorbonne-universite.fr/hal-01723541>**

Submitted on 5 Mar 2018

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## 1-LOOP MASS GENERATION BY A CONSTANT EXTERNAL MAGNETIC FIELD FOR AN ELECTRON PROPAGATING IN A THIN MEDIUM

B. Machet<sup>1 2 3 4</sup>

**Abstract:** The 1-loop self-energy of a Dirac electron of mass  $m$  propagating in a thin medium simulating graphene in an external magnetic field  $B$  is investigated in Quantum Field Theory. Equivalence is shown with the so-called reduced  $\text{QED}_{3+1}$  on a 2-brane. Schwinger-like methods are used to calculate the self-mass  $\delta m_{LLL}$  of the electron when it lies in the lowest Landau level. Unlike in standard  $\text{QED}_{3+1}$ , it does not vanish at the limit  $m \rightarrow 0$  :  $\delta m_{LLL} \xrightarrow{m \rightarrow 0} \frac{\alpha}{2} \sqrt{\frac{\pi}{2}} \sqrt{\frac{\hbar |e| B}{c^2}}$  (with  $\alpha = \frac{e^2}{4\pi\hbar c}$ ); all Landau levels of the virtual electron are taken into account and on mass-shell renormalization conditions are implemented. Restricting to the sole lowest Landau level of the virtual electron is explicitly shown to be inadequate. Resummations at higher orders lie beyond the scope of this work.

PACS: 12.15.Lk, 12.20.Ds, 75.70.Ak

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<sup>1</sup>Sorbonne Universités, UPMC Univ Paris 06, UMR 7589, LP THE, F-75005, Paris, France

<sup>2</sup>CNRS, UMR 7589, LP THE, F-75005, Paris, France.

<sup>3</sup>Postal address: LP THE tour 13-14, 4<sup>ème</sup> étage, UPMC Univ Paris 06, BP 126, 4 place Jussieu, F-75252 Paris Cedex 05 (France)

<sup>4</sup>machet@lpthe.jussieu.fr

# 1 Introduction

Graphene is known as a quasi 2+1 dimensional medium with Dirac-like massless electrons (a gapless medium) – see for example [1]. Whether or not and in which circumstances a gap can open has important consequences, for example on electrical and optical properties [2]. In addition to purely theoretical motivations, this is one of the reasons why we study in this work the spontaneous emergence of a gap for a model which can naively mimic graphene in the presence of a constant external magnetic field  $B$ .

While it is generally admitted that the presence of  $B$  is likely to trigger chiral symmetry breaking (see for example [3]), the demonstrations usually rely on various approximations. In standard  $\text{QED}_{3+1}$ , they are often based on the dimensional reduction that operates in the presence of a strong  $B$  [4] [5] and on resummations of a certain class of diagrams [6] [7] (which become suspicious after realizing that only double logs have been taken into account, leaving aside large single logs [8]). Also, various approximations to coupled Dyson-Schwinger equations are invoked, associated to the use of very special gauges to simplify the vertex (see [9]); this makes the demonstrations tedious, not very transparent and possibly controversial. In reduced  $\text{QED}_{3+1}$  on a 2-brane, which is often considered to provide a fair description of graphene, other approximations are invoked, like the dominance of the lowest Landau level [10] while it was shown, for example in [7], that higher levels are important and trigger charge renormalization; moreover the language that is used is often confusing for people working in Quantum Field Theory.

The calculation of the 1-loop self-energy of an electron propagating in an external  $B$  that I present here uses the sole techniques of Quantum Field Theory. The external electron is chosen, for the sake of simplicity, to lie in the lowest Landau level (LLL), and, in this case, analytical (quasi-)exact formulæ can be obtained by using the formalism of Schwinger [11] as it is carefully explained in [12].

I previously tackled the case of standard  $\text{QED}_{3+1}$  in [8] by calculating the integral of Demeur [13] and Jancovici [14] beyond the leading  $\left(\ln \frac{|e|B}{m^2}\right)^2$  approximation. I demonstrated that large logarithms had been overlooked and, then, neglected; they are tightly connected with the counterterms needed to implement suitable renormalization conditions. In this case,  $\delta m \rightarrow 0$  when  $m \rightarrow 0$ .

These calculations are adapted here to a thin graphene-like medium. They are explained step by step such that they should appear fairly easy to reproduce, with no obscure gap to fill. They mostly go along the lines of [12], and differences are outlined. A massive Dirac electron is considered to propagate inside a thin film of thickness  $2a$ , the Hamiltonian of which being deprived of its “ $p_3\gamma_3$ ” term (see for example [2]).  $B$ , supposed to be static and uniform is considered to be directed along the  $z$  axis orthogonal to the medium strip. To make the calculation simpler and more transparent, no Fermi velocity different from the speed of light is introduced, such that I will be dealing with a special avatar of Quantum Electrodynamics, and extra degeneracies present in graphene [2] are eluded. The topic of symmetries will not be dwelt on either (see the review [3] on this subject).

As I will demonstrate by working in position space, this model yields for the electron self-energy the same expression as reduced  $\text{QED}_{3+1}$  on a 2-brane [15] [16]: the effective photon propagator turns out, indeed, to be the one of standard  $\text{QED}_{3+1}$  integrated over its  $k_3$  momentum. For the internal electron propagator in presence of an external  $B$  I use Schwinger’s [17] and Tsai’s [18] expression, which accounts for all Landau levels, adapted to the particular situation and Hamiltonian under consideration. The calculations are (and should) be performed with a non-vanishing electron mass  $m$  before the limit  $m \rightarrow 0$  is taken. In the last part I only take into account the LLL of the internal electron, and show that neglecting higher levels is a bad approximation.

To avoid confusion, let me stress that all spinors and  $\gamma$  matrices that are considered in this work are 4-dimensional. Any eventual connection with  $\text{QED}_{2+1}$ , if any, can accordingly only be quite remote, and we shall not dwell on this any more.

So, though the result that I exhibit will certainly not be a surprise for many, I hope that the rigorous demonstration of a simple and exact formula that anyone can check with standard techniques will bring  $B$ -triggered mass generation from radiative corrections on a more solid ground. Like for  $\text{QED}_{3+1}$ , renormalization conditions and the counterterms that must be introduced to fulfill them play important roles <sup>5</sup>.

A major challenge is also, there, to deal with a strongly coupled theory since a 1-loop result is certainly meaningless when the coupling constant gets of order 1. The necessary resummations look highly non-trivial since they do not only concern double and / or simple logs, but more complicated functions, and they have furthermore, of course, to be performed while satisfying at each order appropriate renormalization conditions. To my knowledge this last requirement has never been satisfied and tackling such formidable tasks lies largely beyond the scope of this work.

## 2 Propagation inside a thin, graphene-like medium; equivalence with reduced $\text{QED}_{3+1}$ on a 2-brane

A general argumentation concerning reduced QED can be found, for example, in [15]. A more down-to-earth determination of the effective photon propagator is nevertheless instructive because it provides a simpler understanding of the mechanisms at work, and also because this approach can be applied to vacuum polarization [19], yielding less-trivial results.

Let us calculate in position space the electron propagator  $G(y, x)$  at 1-loop depicted in Fig. 1 (including external legs). We call  $G_0$  the tree-level electron propagator in the presence of  $B$  <sup>6</sup> (described by the double lines in Fig. 1) and  $\Delta_{\mu\nu}$  the bare photon propagator.

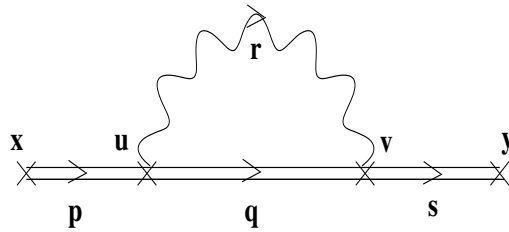


Fig. 1: the 1-loop electron propagator in external  $B$

One has

$$G(x', x'') \equiv i < 0 | T \psi(x') \bar{\psi}(x'') | 0 > = \Phi(x', x'') \int \frac{d^4 p}{(2\pi)^4} e^{ip(x''-x')} G(p) \quad (1)$$

in which the phase [18]

$$\Phi(x', x'') = e^{i|e|q \int_{x''}^{x'} dx_\mu A^\mu(x)} \quad (2)$$

ensures the gauge invariance of the Green function ( $A^\mu$  is the vector potential).

To avoid confusion, the unit of electric charge we note  $|e|$  such that the electron charge is  $-|e|$ . In [11] and in [18], this unit of electric charge is instead noted  $e$ . In [12],  $e$  denotes instead the (negative) electron charge. We shall see that these precisions are important, in particular to get the appropriate propagator for the LLL of an electron.

Like in [11] and [18] we introduce  $q$  such that  $q|e|$  is the electron charge. Therefore  $q = -1$  <sup>7</sup>. This makes the covariant derivative  $D_\mu = \partial_\mu - i|e|qA_\mu$  such that  $\pi_\mu = \frac{1}{i}\partial_\mu = p_\mu + |e|A_\mu$ .

<sup>5</sup>In the work [19] I emphasize their role in the calculation of the photon vacuum polarization for the same graphene-like medium as the one considered here.

<sup>6</sup>The results of this paragraph do not depend whether the external  $B$  is present or not.

<sup>7</sup>This  $q$  should not be confused with the 4-momentum that appears in Fig. 1. The reader will easily make the difference.

For any 4-vector  $v = (v_0, v_1, v_2, v_3)$ , it is useful to introduce the notations  $\hat{v} = (v_0, v_1, v_2, 0)$ ,  $v_{||} = (v_0, 0, 0, v_3)$  and  $v_{\perp} = (0, v_1, v_2, 0)$ .

The 1-loop electron propagator depicted in Fig. 1 writes

$$iG(y, x) = -e^2 e^{i|e|q \int_y^x dt_{\mu} A^{\mu}(t)} \int d^4 u \int d^4 v \int \frac{d^4 p}{(2\pi)^4} e^{ip(u-x)} G_0(p) \int \frac{d^4 q}{(2\pi)^4} e^{iq(v-u)} G_0(q) \gamma^{\mu} \int \frac{d^4 r}{(2\pi)^4} e^{ir(v-u)} \Delta_{\mu\nu}(r) \gamma^{\nu} \int \frac{d^4 s}{(2\pi)^4} e^{is(y-v)} G_0(s). \quad (3)$$

We now specialize to the medium under concern and consider “graphene-like” electrons propagating inside a thin film of thickness  $2a$ . This situation has two consequences:

- \*  $G_0(q) = G_0(\hat{q})$ ,  $G_0(p) = G_0(\hat{p})$ ,  $G_0(s) = G_0(\hat{s})$  get deprived of their  $\gamma_3$  components;
- \* the vertices at which the electron and photon interact being located inside the strip, the integrals on their positions  $u_3$  and  $v_3$  along the  $z$  axis should be truncated to  $\int_{-a}^{+a} du_3 \int_{-a}^{+a} dv_3$ . This gives

$$iG(y, x) = -e^2 e^{i|e|q \int_y^x dt_{\mu} A^{\mu}(t)} \int \frac{dq_3}{2\pi} \int \frac{dp_3}{2\pi} \int \frac{dr_3}{2\pi} \int \frac{ds_3}{2\pi} \int_{-a}^{+a} du_3 e^{iu_3(p_3-q_3-r_3)} \int_{-a}^{+a} dv_3 e^{iv_3(q_3+r_3-s_3)} \int \frac{d^3 \hat{p}}{(2\pi)^3} e^{i\hat{p}(\hat{y}-\hat{x})-ip_3x_3+is_3y_3} G_0(\hat{p}) \int \frac{d^3 \hat{q}}{(2\pi)^3} G_0(\hat{q}) \gamma^{\mu} \Delta_{\mu\nu}(\hat{p}-\hat{q}, r_3) \gamma^{\nu} G_0(\hat{p}). \quad (4)$$

The two integrations  $\int du_3$  and  $\int dv_3$  can be performed since

$$\int_{-a}^{+a} dx e^{itx} = \frac{2 \sin at}{t}, \quad (5)$$

which leads to

$$iG(y, x) = -4e^2 e^{i|e|q \int_y^x dt_{\mu} A^{\mu}(t)} \int \frac{dp_3}{2\pi} \int \frac{dr_3}{2\pi} \int \frac{ds_3}{2\pi} \left[ \int \frac{dq_3}{2\pi} \frac{\sin a(q_3 + r_3 - s_3)}{q_3 + r_3 - s_3} \frac{\sin a(p_3 - q_3 - r_3)}{p_3 - q_3 - r_3} \right] \int \frac{d^3 \hat{p}}{(2\pi)^3} e^{i\hat{p}(\hat{y}-\hat{x})-ip_3x_3+is_3y_3} G_0(\hat{p}) \int \frac{d^3 \hat{q}}{(2\pi)^3} G_0(\hat{q}) \gamma^{\mu} \Delta_{\mu\nu}(\hat{p}-\hat{q}, r_3) \gamma^{\nu} G_0(\hat{p}). \quad (6)$$

In there the integration  $\int dq_3$  can also be done explicitly since

$$\int dq_3 \frac{\sin a(q_3 - \sigma)}{q_3 - \sigma} \frac{\sin a(q_3 - \tau)}{q_3 - \tau} = \pi \frac{\sin a(\sigma - \tau)}{\sigma - \tau}, \quad (7)$$

with  $\sigma = s_3 - r_3, \tau = p_3 - r_3$ , which has the property to be independent of  $r_3$ . We get now

$$iG(y, x) = -2e^2 e^{i|e|q \int_y^x dt_{\mu} A^{\mu}(t)} \int \frac{dp_3}{2\pi} \int \frac{ds_3}{2\pi} \frac{\sin a(s_3 - p_3)}{s_3 - p_3} \int \frac{d^3 \hat{p}}{(2\pi)^3} e^{i\hat{p}(\hat{y}-\hat{x})-ip_3x_3+is_3y_3} G_0(\hat{p}) \int \frac{d^3 \hat{q}}{(2\pi)^3} \int \frac{dr_3}{2\pi} G_0(\hat{q}) \gamma^{\mu} \Delta_{\mu\nu}(\hat{p}-\hat{q}, r_3) \gamma^{\nu} G_0(\hat{p}). \quad (8)$$

Going to the new variables  $h_3 = s_3 + p_3, l_3 = s_3 - p_3 \Rightarrow dp_3 ds_3 = \frac{1}{2} dh_3 dl_3$  yields

$$iG(y, x) = -e^2 e^{i|e|q \int_y^x dt_{\mu} A^{\mu}(t)} \int \frac{dl_3}{2\pi} \frac{\sin al_3}{l_3} e^{il_3 \frac{x_3+y_3}{2}} \int \frac{dh_3}{2\pi} e^{ih_3 \frac{y_3-x_3}{2}} \int \frac{d^3 \hat{p}}{(2\pi)^3} e^{i\hat{p}(\hat{y}-\hat{x})} G_0(\hat{p}) \int \frac{d^3 \hat{q}}{(2\pi)^3} \int \frac{dr_3}{2\pi} G_0(\hat{q}) \gamma^{\mu} \Delta_{\mu\nu}(\hat{p}-\hat{q}, r_3) \gamma^{\nu} G_0(\hat{p}). \quad (9)$$

The condition  $x_3 + y_3 \leq 2a$  is verified because the electrons are constrained to propagate inside the strip, such that

$$\int \frac{dl_3}{2\pi} \frac{\sin al_3}{l_3} e^{il_3 \frac{x_3+y_3}{2}} = \frac{1}{2}. \quad (10)$$

This yields

$$iG(y, x) = -e^2 \int \frac{d^4 p}{(2\pi)^4} e^{ip(y-x)} G_0(\hat{p}) e^{i|e|q \int_y^x dt_\mu A^\mu(t)} \int \frac{d^3 \hat{q}}{(2\pi)^3} \int \frac{dr_3}{2\pi} G_0(\hat{q}) \gamma^\mu \Delta_{\mu\nu}(\hat{p} - \hat{q}, r_3) \gamma^\nu G_0(\hat{p}). \quad (11)$$

The self-energy  $\Sigma$  is obtained from the 1-loop propagator above by chopping off the two external fermion  $iG_0$  propagators, which leads to

$$\Sigma(x, y) = \Phi(x, y) \int \frac{d^4 p}{(2\pi)^4} e^{ip(y-x)} \Sigma(\hat{p}), \quad (12)$$

with the phase  $\Phi$  given in (2) and to

$$i\Sigma(\hat{p}) = e^2 \int \frac{d^3 \hat{k}}{(2\pi)^3} \int \frac{dr_3}{2\pi} G_0(\hat{p} - \hat{k}) \gamma^\mu \Delta_{\mu\nu}(\hat{k}, r_3) \gamma^\nu, \quad (13)$$

in which, to avoid conflicts between notations, we have made the change of variables  $p - q \rightarrow k$  in the momenta, which amounts to labeling them like in [12].

This shows the equivalence with reduced QED<sub>3+1</sub> on a 2-brane, in which the “effective” internal photon propagator is (see [15])

$$\tilde{\Delta}_{\mu\nu}(\hat{k}) = \int \frac{dr_3}{2\pi} \Delta_{\mu\nu}(\hat{k}, r_3). \quad (14)$$

In the Feynman gauge<sup>8</sup> one gets  $\tilde{\Delta}_{\mu\nu}(\hat{k}) = \int \frac{dr_3}{2\pi} \frac{g_{\mu\nu}}{\hat{k}^2 + r_3^2} = \frac{1}{2} \frac{g_{\mu\nu}}{\sqrt{\hat{k}^2}}$  such that

$$i\Sigma(\hat{p}) = -\frac{e^2}{2} \int \frac{d^3 \hat{k}}{(2\pi)^3} \gamma^\mu G_0(\hat{p} - \hat{k}) \frac{g_{\mu\nu}}{\sqrt{\hat{k}^2}} \gamma^\nu. \quad (15)$$

which should be compared with eq. (3.9) of [12].

No dependence on the thickness  $a$  of the medium occurs anymore (unlike for the vacuum polarization [19]). This is easily understood since we constrained the fermion to propagate inside the medium (while, for the vacuum polarization, the photon is allowed to also propagate in the “bulk”).

### 3 The self-energy and self-mass of the electron

In the whole paper, we use the metric  $(- + + +)$  like in [11], [18] and [12].

The conventions for  $\gamma$  matrices and Pauli  $\vec{\sigma}$  matrices are the same as in [18], [12] and [11]. In particular,  $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$ . We shall denote (abusively)  $\sigma^3 \equiv \sigma^{12} = \frac{1}{2}[\gamma^1, \gamma^2] = \text{diag}(1, -1, 1, -1)$ ; it should not be mistaken for the corresponding  $2 \times 2$  Pauli matrix.

With these conventions, for an external magnetic field  $B$  along the  $z$  axis, the wave function of the lowest Landau

level  $|LLL\rangle$  is proportional to  $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$  ([20] [21]) such that  $\sigma^3|LLL\rangle = (-1)|LLL\rangle$  and  $(1 - i\gamma^1\gamma^2)|LLL\rangle \equiv (1 - \sigma^3)|LLL\rangle = 2|LLL\rangle$ .

<sup>8</sup>The choice of a special gauge is of course not optimal but is justified by the property that the formalism of Schwinger is gauge invariant [17].

### 3.1 The self-energy in momentum space

We now proceed to calculating the self-energy expressed in (15), following the procedure given in [12]. To that purpose, we introduce 2 Schwinger parameters:  $s_2$  for the photon and  $s_1$  for the electron.

As far as the photon is concerned, instead of  $\frac{1}{\hat{k}^2 - i\epsilon} = i \int_0^\infty ds_2 e^{-is_2(\hat{k}^2 - i\epsilon)}$  (eq. (3.10) of [12]), that is used to represent the 4-dimensional photon propagator in the Feynman gauge, we shall use now, according to (14) and (15)

$$\frac{1}{\sqrt{\hat{k}^2 - i\epsilon}} = \sqrt{\frac{i}{\pi}} \int_0^\infty ds_2 \frac{e^{-is_2(\hat{k}^2 - i\epsilon)}}{\sqrt{s_2}}. \quad (16)$$

However, it is important (see just above (18)) to use Tsai's [18] formulæ and not the ones used in [12].

As for the electron, in general QED<sub>3+1</sub>, its propagator is given (see eq. (6) of [18]) by

$$G_0(k, B) = i \int_0^\infty ds_1 e^{-is_1(m^2 - i\epsilon + k_\parallel^2 + \frac{\tan z}{z} k_\perp^2)} \frac{e^{iqz\sigma^3}}{\cos z} \left( m - \not{k}_\parallel - \frac{e^{-iqz\sigma^3}}{\cos z} \not{k}_\perp \right), \quad z = |e|Bs_1, \quad (17)$$

and, in position space by equations similar to (1) and (2). These expressions only need to be trivially adapted to the “truncated” momenta  $\hat{p}$  and  $\hat{k}$  (see section 2).

As shown in appendix A, (17) leads to the adequate propagator for the *LLL* at the limit  $B \rightarrow \infty$ . It is in particular proportional to the customary projector  $1 - i\gamma^1\gamma^2$ . This is not the case of eq. (2.47b) of [12] (in there  $e < 0$ ), which involves  $e^{i\sigma^3 z}$  instead of  $e^{iq\sigma^3 z}$  and leads to the wrong projector  $1 + i\gamma^1\gamma^2$  and, later, to confusions and problems.

From (15) and using (16) and (17), one gets instead of (3.11) of [12] (“c.t.” means “counter terms”)

$$\begin{aligned} \Sigma(\hat{p}) = & -\sqrt{\frac{i}{\pi}} \frac{e^2}{2} \int_0^\infty ds_1 \int_0^\infty \frac{ds_2}{\sqrt{s_2}} \int \frac{d^3 \hat{k}}{(2\pi)^3} e^{-is_2(\hat{k}^2 - i\epsilon)} e^{-is_1(m^2 + (\hat{p} - \hat{k})_\parallel^2 + \frac{\tan z}{z} (\hat{p} - \hat{k})_\perp^2)} \\ & \gamma^\mu \frac{e^{iqz\sigma^3}}{\cos z} \left[ m - (\not{p} - \not{\hat{k}})_\parallel - \frac{e^{-iqz\sigma^3}}{\cos z} (\not{p} - \not{\hat{k}})_\perp \right] \gamma_\mu + c.t., \quad \text{with } z = |e|Bs_1, \end{aligned} \quad (18)$$

Since the Hamiltonian of the Dirac electron is presently considered to be deprived of its  $\gamma_3(p - k)_3$  part,  $(\hat{p} - \hat{k})_\parallel^2 = -(p_0 - k_0)^2$ ,  $(\not{p} - \not{\hat{k}})_\parallel = -\gamma_0(p_0 - k_0)$ , while preserving  $(\hat{p} - \hat{k})_\perp^2 = (p_1 - k_1)^2 + (p_2 - k_2)^2$  and  $(\not{p} - \not{\hat{k}})_\perp = \gamma_1(p_1 - k_1) + \gamma_2(p_2 - k_2)$ .

One performs the same change of variable as (3.12) of [12]

$$s_1 = su, \quad s_2 = s(1 - u) \Rightarrow ds_1 \frac{ds_2}{\sqrt{s_2}} = ds \sqrt{s} \frac{du}{\sqrt{1 - u}}, \quad (19)$$

and one still introduces  $y = |e|Bs u$ .

The exponentials are then re-expressed in view of performing the  $\int d^3 \hat{k}$  integration. Following a procedure identical to that in [12] yields, instead of their (3.17)

$$\begin{aligned} \Sigma(\hat{p}) = & -i \frac{e^2}{2} \sqrt{\frac{i}{\pi}} \int_0^\infty ds \sqrt{s} \int_0^1 \frac{du}{\sqrt{1 - u}} \frac{1}{\cos y} \left\{ \int \frac{d^3 \hat{k}}{(2\pi)^3} e^{-is\chi} \right\} \\ & \gamma^\mu e^{iqy\sigma^3} \left[ m - (1 - u)\not{p}_\parallel + \frac{e^{-iqy\sigma^3}}{\cos y} \frac{1 - u}{1 - u + u \tan y/y} \not{p}_\perp \right] \gamma_\mu + c.t., \end{aligned} \quad (20)$$

in which  $\chi$  and  $\varphi$  are still given by (3.14), (3.15) of [12]

$$\begin{aligned} \chi = & um^2 + \varphi + (k_\parallel - up_\parallel)^2 + (1 - u + u \frac{\tan y}{y}) \left[ k_\perp - \frac{u \tan y/y}{1 - u + u \tan y/y} p_\perp \right]^2, \\ \varphi = & u(1 - u) p_\parallel^2 + \frac{u}{y} \frac{(1 - u) \sin y}{(1 - u) \cos y + u \sin y/y} p_\perp^2. \end{aligned} \quad (21)$$

The shifts in the integration variables are naturally  $k_\parallel \rightarrow k_\parallel - up_\parallel$  and  $k_\perp \rightarrow k_\perp - \frac{u \tan y/y}{1 - u + u \tan y/y} p_\perp$ .

One has to redo the  $k$  integrations (which only concerns the integral inside curly brackets in (20)) since it is now  $\int \frac{d^3 k}{(2\pi)^3}$  instead of  $\int \frac{d^4 k}{(2\pi)^4}$  for standard QED<sub>3+1</sub>. This is simple with the aid of the standard integral

$$\int_{-\infty}^{+\infty} dx e^{\pm iAx^2} = e^{\pm i\pi/4} \left(\frac{\pi}{A}\right)^{1/2}, \quad A > 0, \quad (22)$$

and leads to

$$\begin{aligned} \Sigma(\hat{p}) = & -\frac{me^2}{16\pi^2} \int_0^\infty \frac{ds}{s} \int_0^1 \frac{du}{\sqrt{1-u}} \frac{e^{-is(um^2+\varphi)}}{(1-u)\cos y + u\sin y/y} \\ & \gamma^\mu e^{iqy\sigma^3} \left[ 1 - (1-u) \frac{\not{p}_\parallel}{m} + \frac{e^{-iqy\sigma^3}}{\cos y} \frac{1-u}{1-u+u\tan y/y} \frac{\not{p}_\perp}{m} \right] \gamma_\mu + c.t. \end{aligned} \quad (23)$$

It is then simple matter to perform the Dirac algebra, which leads, instead of eq. (3.27) of [12], to

$$\begin{aligned} \Sigma(\hat{p}) = & \frac{\alpha m}{4\pi} \int_0^\infty \frac{ds}{s} \int_0^1 \frac{du}{\sqrt{1-u}} \frac{e^{-is(um^2+\varphi)}}{(1-u)\cos y + u\sin y/y} e^{iqy\sigma^3} \\ & \left[ 1 + e^{-2iqy\sigma^3} + (1-u)e^{-2iqy\sigma^3} \frac{\not{p}_\parallel}{m} + (1-u) \frac{e^{-iqy\sigma^3}}{(1-u)\cos y + u\sin y/y} \frac{\not{p}_\perp}{m} \right] + c.t. \end{aligned} \quad (24)$$

Quite remarkably, in addition to the replacement  $\sigma^3 y \rightarrow q\sigma^3 y$  in the exponentials, which originates from our taking the original Tsai's formula for  $G_0$  instead of that of [12], and to a global factor 1/2, it only differs from (3.27) of [12] by  $\int \frac{du}{\sqrt{1-u}}$  instead of  $\int du$  and by the fact that, in the present situation,  $p_\parallel^2 = -p_0^2$ ,  $\not{p}_\parallel = -\gamma_0 p_0$ . We thus see that, after these lengthy but straightforward transformations have been done, the electron self-energy for QED<sub>3+1</sub> reduced on a 2-brane is formally very close to the one for QED<sub>3+1</sub>. The difference between the two integration measures for  $u$  is however at the origin of the completely different behaviors of the corresponding  $\delta m_{LLL}$  at the limit  $m \rightarrow 0$ , as we shall see in subsection 4.2.

### 3.2 Transforming the space representation of $\Sigma$

Unlike for the vacuum polarization in which the two opposite phases cancel, the phase  $\Phi$ , given in (2), which occurs in the space representation (12) of the self-energy plays an important role. This makes the calculations all the more tedious as, like for QED<sub>3+1</sub>, the integrations on  $s$  and  $u$  for  $\Sigma(\hat{p})$  obtained in (24) cannot be done explicitly.

It is however possible, along the lines of p. 47-52 of [12] to get from the space representation  $\Sigma(x', x'')$  as written in (12) a useful expression for  $\Sigma(\hat{\pi})$  defined by

$$\Sigma(x', x'') = \langle x' | \Sigma(\hat{\pi}) | x'' \rangle. \quad (25)$$

$\Sigma(\hat{\pi})$ , which now depends on the covariant derivative  $\hat{\pi}$ , has somewhat “swallowed” the phase  $\Phi$ , and is the essential ingredient to get the self-mass  $\delta m$  of an electron on mass-shell ( $\not{\hat{\pi}} + m = 0$ ).

The  $\int d^4 p$  in (12), which is at the root of the corresponding formal manipulations stays unchanged. One has to go through the steps of p. 34-36 and p. 47-50 of [12], which use in particular eq. (2.41) of [12]

$$\langle X' | e^{-is\pi^2} | X'' \rangle = \Phi(X', X'') \int \frac{d^4 k}{(2\pi)^4} e^{ik(X'-X'')} \frac{1}{\cos q|e|Bs} e^{-is(k_\parallel^2 + k_\perp^2 \frac{\tan q|e|Bs}{q|e|Bs})} \quad (26)$$

and its avatars, (2.45) and more specially (2.46)

$$\begin{aligned} & \langle X' | e^{-is(a_0\pi_0\pi^0 + a_3\pi_3\pi^3 + a_\perp\pi_\perp^2)} (1, \gamma_0\pi^0, \gamma_3\pi^3, \gamma_\perp\pi_\perp) | X'' \rangle \\ & = \Phi(X', X'') \int \frac{d^4 k}{(2\pi)^4} e^{ik(X'-X'')} \frac{1}{\cos q|e|Bs a_\perp} e^{-is(a_0k_0k^0 + a_3k_3k^3 + a_\perp \frac{\tan q|e|Bs a_\perp}{q|e|Bs a_\perp})} \\ & \quad \left( 1, \gamma_0k^0, \gamma_3k^3, \frac{1}{\cos q|e|a_\perp Bs} e^{-iq|e|Bs a_\perp \sigma^3} \gamma_\perp p_\perp \right). \end{aligned} \quad (27)$$



They entail, by simple changes of variables ( $\varphi$  is given in (21))

$$\begin{aligned} \Phi(X', X'') & \int \frac{d^4 p}{(2\pi)^4} e^{ip(X'-X'')} e^{-is\varphi} = \cos \beta < X' | e^{-isu(1-u)p_{\parallel}^2} e^{-i\frac{\beta}{q|e|B}\pi_{\perp}^2} | X'' >, \\ \Phi(X', X'') & \int \frac{d^4 p}{(2\pi)^4} e^{ip(X'-X'')} e^{-is\varphi} (a \not{p}_{\parallel} + b \not{p}_{\perp}) \\ & = \cos \beta < X' | e^{-isu(1-u)p_{\parallel}^2} e^{-i\frac{\beta}{q|e|B}\pi_{\perp}^2} (a \not{p}_{\parallel} + b \cos \beta e^{iq\sigma^3 \beta} \not{p}_{\perp}) | X'' > \end{aligned} \quad (28)$$

in which  $\Delta(u, y)$  and the angle  $\beta$  have been introduced, which satisfy [12]

$$\begin{aligned} \sin \beta & = \frac{(1-u) \sin y}{\Delta(u, y)^{1/2}}, \quad \cos \beta = \frac{(1-u) \cos y + u \sin y/y}{\Delta(u, y)^{1/2}}, \\ \Delta(u, y) & = (1-u)^2 + 2u(1-u) \frac{\sin y \cos y}{y} + u^2 \left( \frac{\sin y}{y} \right)^2. \end{aligned} \quad (29)$$

After all terms inside (24) have been transformed via (28), one gets the result

$$\begin{aligned} \Sigma(\hat{\pi}) & = \frac{\alpha m}{4\pi} \int_0^\infty \frac{ds}{s} \int_0^1 \frac{du}{\sqrt{1-u}} e^{-isu^2 m^2} \\ & \left[ \frac{e^{-is\Theta}}{\sqrt{\Delta(u, y)}} \left[ 1 + e^{-2iqy\sigma^3} + (1-u)e^{-2iqy\sigma^3} \frac{\not{\pi}}{m} + (1-u) \left( \frac{1-u}{\Delta(u, y)} + \frac{u}{\Delta(u, y)} \frac{\sin y}{y} e^{-iqy\sigma^3} - e^{-2iqy\sigma^3} \right) \frac{\not{\pi}_{\perp}}{m} \right] + c.t. \right], \\ \Theta & = u(1-u)(m^2 - \not{\pi}^2) + \frac{u}{y} (\beta - (1-u)y) \pi_{\perp}^2 - u^2 \frac{|e|q}{2} \sigma_{\mu\nu} F^{\mu\nu}, \end{aligned} \quad (30)$$

which differs from (3.38a) of [12] by the absence of  $\gamma_3 \pi^3$  from  $\not{\pi}$ , the same factor  $\frac{1}{2}$  that we already mentioned concerning (24), and the presence of  $q \equiv -1$  in the exponentials (that was omitted in [12]).

### 3.2.1 Renormalization conditions and counterterms

The electron mass we define as the pole of its propagator, which is the only gauge invariant definition.

We briefly recall here the general procedure to fix the counterterms. It is then straightforwardly adapted to our concern by replacing everywhere  $p$  with  $\hat{p}$  and  $\pi$  with  $\hat{\pi}$  ( $\pi_{\mu} = p_{\mu} + |e|A_{\mu}$ ).

At  $B = 0$ , the renormalized electron mass is defined by

$$m = m_0 + \delta m, \quad \delta m = \Sigma(p)|_{\not{p}+m=0}, \quad (31)$$

in which  $m_0$  is the bare mass and  $\Sigma(p)$  the bare self-energy.

In the presence of and external field  $A^{\mu}$ , the propagator of a Dirac electron is

$$iG = \frac{i}{\not{\pi} + m_0 + \Sigma(\pi)}, \quad (32)$$

and we define, in analogy with (31) the mass of the electron as the pole of its propagator by

$$m = m_0 + \Sigma(\pi)|_{\not{\pi}+m=0}, \quad \delta m = \Sigma(\pi)|_{\not{\pi}+m=0}. \quad (33)$$

$\delta m$  depends on the external  $B$ .

The on mass-shell renormalization conditions write <sup>9</sup>

$$\lim_{\not{\pi}+m=0} \lim_{B \rightarrow 0} \Sigma^{ren}(\pi) = 0, \quad \lim_{\not{\pi}+m=0} \lim_{B \rightarrow 0} \frac{\partial \Sigma^{ren}(\pi)}{\partial \not{\pi}} = 0, \quad (34)$$

---

<sup>9</sup>They are carefully explained p.38-41 of [12].

in which the superscript “*ren*” denotes the renormalized quantities.

They lead to the same counterterms as in [12] but for the simple modifications  $p \rightarrow \hat{p}$ ,  $\pi \rightarrow \hat{\pi}$ , and one gets

$$\Sigma(\hat{\pi}) = \frac{\alpha m}{4\pi} \int_0^\infty \frac{ds}{s} \int_0^1 \frac{du}{\sqrt{1-u}} e^{-isu^2 m^2} \left[ \frac{e^{-is\Theta}}{\sqrt{\Delta}} \left[ 1 + e^{-2iqy\sigma^3} + (1-u)e^{-2iqy\sigma^3} \frac{\not{q}}{m} + (1-u) \left( \frac{1-u}{\Delta} + \frac{u}{\Delta} \frac{\sin y}{y} e^{-iqy\sigma^3} - e^{-2iqy\sigma^3} \right) \frac{\not{q}_\perp}{m} \right] \right. \\ \left. \underbrace{-(1+u) - (m + \not{q}) \left( \frac{1-u}{m} - 2imu(1-u^2)s \right)}_{c.t.} \right], \quad (35)$$

in which  $y, \Theta, \Delta$  are given in (30) and (29).

The 2nd counterterm vanishes on mass-shell (since it must satisfy the 1st renormalization condition), and can therefore be forgotten in the calculation of  $\delta m$ .

### 3.3 The self-mass $\delta m_{LLL}$ for an electron in the lowest Landau level

The spectrum of a Dirac electron in a pure magnetic field directed along  $z$  is [22]

$$\epsilon_n^2 = m^2 + p_z^2 + (2n + 1 + \sigma_z) |e|B, \quad (36)$$

in which  $\sigma_z = \pm 1$  is  $2 \times$  the spin projection of the electron on the  $z$  axis. So, at  $n = 0, \sigma_z = -1, p_z = 0, \epsilon_n = m$ : this is the lowest Landau level.

We can consider  $A_\mu = \begin{pmatrix} A_0 = 0 \\ A_x = 0 \\ A_y = xB \\ A_z = 0 \end{pmatrix}$  such that  $F_{12} = B$  is the only non-vanishing component of the classical external  $F_{\mu\nu}$ . Then, the wave function of the LLL writes [20] [21]

$$\psi_{n=0, s=-1, p_y=p_z=0} = \frac{1}{\sqrt{N}} \left( \frac{|e|B}{\pi} \right)^{1/4} e^{-\frac{|e|B}{2} x^2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad N \stackrel{[21]}{=} \underbrace{L_y L_z}_{\text{dimensions along } y \text{ and } z}. \quad (37)$$

Following (33), in order to determine  $\delta m$  for the (on mass-shell) LLL, we shall sandwich the general self-energy operator (35) between two states  $|\psi\rangle$  defined in (37) and satisfying  $(\not{q} + m)|\psi\rangle = 0$ .

The expression (35) involves  $\not{q}$  that we shall replace by  $-m$ ,  $\Delta$  that needs not be transformed, and  $\Theta$  which involves  $m^2 - \not{q}^2$ ,  $\pi_\perp^2$  and  $\sigma_{\mu\nu} F^{\mu\nu}$ . The only non-vanishing component of  $F^{\mu\nu}$  being  $F^{12} = B$ ,  $\sigma_{\mu\nu} F^{\mu\nu} = \sigma_{12} F^{12} + \sigma_{21} F^{21} = 2\sigma_{12} F^{12} \equiv 2\sigma_3 B$ .

Since the electron is an eigenstate of the Dirac equation in the presence of  $B$ ,  $m^2 - \not{q}^2$  can be taken to vanish.  $\pi_\perp^2 \equiv \pi_1^2 + \pi_2^2$  is also identical, since the LLL has  $p_z = 0$  and we work in a gauge with  $A_z = 0$ , to  $\vec{\pi}^2 \equiv \pi^2 + \pi_0^2$ . One has  $\not{q}^2 = -\pi^2 + \frac{q|e|}{2} \sigma_{\mu\nu} F^{\mu\nu}$  such that  $\pi_\perp^2 = -\not{q}^2 + \pi_0^2 + \sigma_3 q|e|B$ . Since our gauge for the external  $B$  has  $A_0 = 0, \pi_0^2 = p_0^2$ , which is the energy squared of the electron, identical to  $m^2$  for the LLL. Therefore, on mass-shell,  $\pi_\perp^2 = \sigma_3 q|e|B$ . When sandwiched between LLL,

$$\langle \psi | \sigma^3 | \psi \rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \text{diag}(1, -1, 1, -1) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = -1 \text{ such that } \sigma^3 \text{ can be replaced by } (-1). \quad \Theta \text{ shrinks to}$$

$u(\beta/y - 1)q|e|B\sigma^3$ , which gives, replacing  $\sigma^3$  with  $(-1)$ ,  $\Theta \rightarrow u(1 - \beta/y)q|e|B$ .  $\sigma^3$  can also be replaced by  $(-1)$  in the exponentials of (35).

$\Sigma(\hat{\pi})$  in (35) also involves a term proportional to  $\not{\epsilon}_\perp$ . Since the LLL has  $p_z = 0$  and we work at  $A_z = 0$ , this is also equal to  $\vec{\gamma} \cdot \vec{\pi} = \gamma^\mu \hat{\pi}_\mu - \gamma^0 \pi_0 = \not{\epsilon} + \gamma^0 p^0$ .  $\langle \psi | \not{\epsilon} | \psi \rangle = -m$  such that  $\langle \psi | \not{\epsilon}_\perp | \psi \rangle = \langle \psi | -m + \gamma^0 p^0 | \psi \rangle$ . Since  $\gamma^0 = \text{diag}(1, 1, -1, -1)$ , eq. (37) yields  $\langle \psi | \not{\epsilon}_\perp | \psi \rangle = -m + p^0$ . The energy  $p^0$  of the LLL  $|\psi\rangle$  being equal to  $m$ , this term vanishes.

Gathering all information and simplifications leads finally to

$$\delta m_{LLL} \equiv \Sigma(\hat{\pi})_{\not{\epsilon}+m=0} = \frac{\alpha m}{4\pi} \int_0^\infty \frac{ds}{s} \int_0^1 \frac{du}{\sqrt{1-u}} e^{-isu^2 m^2} \left[ \frac{e^{-is\Theta(u,y)}}{\sqrt{\Delta(u,y)}} (1 + u e^{2iqy}) - \underbrace{(1+u)}_{\text{from c.t.}} \right], \quad (38)$$

in which  $y = |e|Bsu$  as before,  $\Delta(u, y)$  is the same as in (30),  $\beta$  the same as in (29), and  $\Theta$  has shrunk down to

$$\Theta(u, y) = uq|e|B \left( 1 - \frac{\beta(u, y)}{y} \right) = uq|e|B - \frac{q\beta(u, y)}{s}. \quad (39)$$

## 4 The “reduced” Demeur-Jancovici integral $\hat{I}(L)$

### 4.1 General expression

We define  $\hat{I}(L)$  by

$$\delta m_{LLL} = \frac{\alpha m}{4\pi} \hat{I}(L) \quad \text{with } L = \frac{|e|B}{m^2}. \quad (40)$$

such that

$$\hat{I}(L) = \int_0^\infty \frac{ds}{s} \int_0^1 \frac{du}{\sqrt{1-u}} e^{-isu^2 m^2} \left[ \frac{e^{-is\Theta(u,y)}}{\sqrt{\Delta(u,y)}} (1 + u e^{2iqy}) - \underbrace{(1+u)}_{\text{from c.t.}} \right] \quad (41)$$

By a successive change of variables, we cast it in a form similar to  $I(L)$  deduced by Jancovici in [14] from the formula obtained by Demeur in [13], and that was revisited in [8]. The calculations, which are detailed in appendix B, lead to

$$\hat{I}(L) = \int_0^\infty dz \int_0^1 \frac{dv}{\sqrt{1-v}} e^{-z \frac{m^2}{|e|B}} \left[ \frac{2(1+v e^{-2z/v})}{2z(1-v) + v^2(1 - e^{-2z/v})} - \underbrace{\frac{1+v}{z}}_{\text{from c.t.}} \right], \quad (42)$$

which is the expression which we shall focus on hereafter.

Calling

$$f(v, z) = \frac{2(1+v e^{-2z/v})}{2z(1-v) + v^2(1 - e^{-2z/v})} - \frac{1+v}{z}, \quad (43)$$

$\hat{I}(L)$  in (42) can be cast in the form

$$\hat{I}(L) = \int_0^\infty dz e^{-z/L} \int_0^1 \frac{dv}{\sqrt{1-v}} f(v, z). \quad (44)$$

That  $\hat{I}(L)$  would be divergent at  $z = 0$  without the counterterm can be easily seen by expanding  $\frac{2(1+v e^{-2z/v})}{2z(1-v) + v^2(1 - e^{-2z/v})} \underset{z \rightarrow 0}{\sim} \frac{1+v}{z} + v - 1 + \mathcal{O}(z)$

## 4.2 Analytical evaluation of $\hat{I}(L)$

We split  $\int_0^\infty dz (\dots)$  in  $\hat{I}(L)$  given by (42) into  $\int_0^{z_0} dz (\dots) + \int_{z_0}^\infty dz (\dots)$ , with:

\*  $z_0$  large enough such that, in the 2nd integral, in which  $z > z_0$ ,  $f(v, z) \simeq \frac{2}{2z(1-v) + v^2} - \frac{1+v}{z}$ , that is, the exponentials can be neglected;

\*  $z_0$  small enough for  $\int_0^{z_0} dz (\dots) \ll \int_0^\infty dz (\dots) \simeq \int_{z_0}^\infty dz (\dots)$  and can be neglected.

In practice,  $z_0 = 1$  fits perfectly and, even down to  $L = 20$ , the ratio of the 2 integrals is  $\leq 1/100$ .

$\int_0^{z_0} dz (\dots)$  involves two canceling divergent integrals, and, for proper numerical evaluation, one has to set the lower bound of integration to  $\epsilon \neq 0$ , checking stability when  $\epsilon$  decreases from  $10^{-3}$  down to  $10^{-12} \dots$

Likewise, to numerically evaluate  $\int_{z_0 \simeq 1}^\infty (\dots)$ , avoiding to deal with too small numbers requires to set the upper bound of integration at a large but finite number (which depends on the value of  $L$ ) instead of infinity and to check stability by varying this bound inside a large interval.

The result is that, for  $L \geq 20$  and  $z_0 \simeq 1$  one can approximate at a precision better than  $1/100$

$$\hat{I}(L) \approx \int_{z_0 \simeq 1}^\infty dz e^{-z/L} \int_0^1 \frac{dv}{\sqrt{1-v}} \left[ \frac{2}{2z(1-v) + v^2} - \frac{1+v}{z} \right]. \quad (45)$$

One has

$$g(z) \equiv \int_0^1 \frac{dv}{\sqrt{1-v}} \frac{2}{2z(1-v) + v^2} = \frac{2}{\sqrt{z(z-2)}} \left[ \frac{\tan^{-1} \frac{1}{\sqrt{-1+z-\sqrt{z(z-2)}}}}{\sqrt{-1+z-\sqrt{z(z-2)}}} - \frac{\tan^{-1} \frac{1}{\sqrt{-1+z+\sqrt{z(z-2)}}}}{\sqrt{-1+z+\sqrt{z(z-2)}}} \right],$$

$$\int_0^1 dv \frac{1+v}{\sqrt{1-v}} = \frac{10}{3},$$

$$\int_{z_0 \simeq 1}^\infty dz \frac{e^{-z/L}}{z} = \Gamma(0, 1/L),$$
(46)

therefore

$$\delta m_{LLL} = \frac{\alpha m}{4\pi} \left( \int_{z_0 \simeq 1}^\infty dz e^{-z/L} g(z) - \frac{10}{3} \Gamma(0, 1/L) \right). \quad (47)$$

On Fig. 2 we compare  $g(z)$  given in (46) (blue) with the one obtained in [8] for standard  $\text{QED}_{3+1}$  ( $g(z) = \ln(z-1 + \sqrt{z(z-2)})/\sqrt{z(z-2)} \simeq \ln z/z + \pi/2z^{1.175}$ ) (yellow).

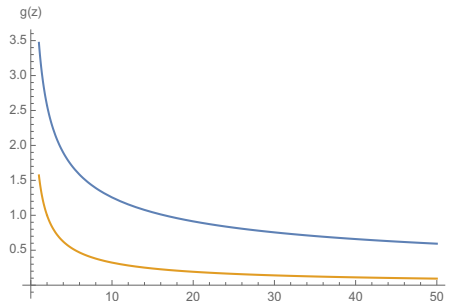


Fig. 2: A comparison between the integrand  $g(z)$  in  $\text{QED}_{3+1}$  (yellow) and in  $\text{QED}_{3+1}$  reduced on a 2-brane (blue)

We now proceed like M.I. Vysotsky in [23] and look for an interpolating function for  $g(z)$ . One has

$$g(1) \approx 3.468,$$

$$g(z) \stackrel{z \rightarrow \infty}{\simeq} \pi \sqrt{\frac{2}{z}} - \frac{2}{z} + \mathcal{O}\left(\frac{1}{z^{3/2}}\right) \simeq \frac{4.443}{\sqrt{z}} + \dots \quad (48)$$

and an excellent fit for  $z \in [z_0 \simeq 1, \infty]$  is

$$g(z) \approx \pi \sqrt{\frac{2}{z}} + \frac{g(1) - \pi\sqrt{2}}{z}. \quad (49)$$

It is plotted in yellow on Fig. 3, while the exact  $g$  is in blue.

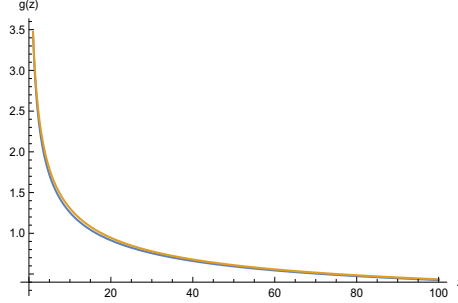


Fig. 3: exact (blue) and approximated (yellow)  $g(z)$  for  $z \geq 1$

This approximation gives (using (40))<sup>10</sup>

$$\delta m_{LLL} \equiv \frac{\alpha m}{4\pi} \hat{I}(L) = \frac{\alpha}{4\pi} \sqrt{|e|B} \left[ \sqrt{2} \pi^{3/2} \text{Erfc}\left(\frac{1}{\sqrt{L}}\right) + \frac{\Gamma(0, \frac{1}{L})}{\sqrt{L}} \left(g(1) - \pi\sqrt{2} - \frac{10}{3}\right) \right]. \quad (50)$$

When  $L \rightarrow \infty$ ,  $\text{Erfc}(\frac{1}{\sqrt{L}}) \simeq 1 - \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{L}} + \dots$  and  $\Gamma(0, \frac{1}{L}) \simeq \ln L - \gamma_E + \dots$  such that

$$\delta m_{LLL} \xrightarrow{L \rightarrow \infty} \frac{\alpha}{2} \sqrt{|e|B} \sqrt{\frac{\pi}{2}} \left[ 1 - \frac{2}{\sqrt{\pi L}} + \frac{1}{\sqrt{2} \pi^{3/2}} \frac{\ln L - \gamma_E}{\sqrt{L}} \left(g(1) - \pi\sqrt{2} - \frac{10}{3}\right) + \dots \right]. \quad (51)$$

The constant term comes from the contribution to  $\hat{I}(L)$  of  $\int_{z_0 \simeq 1}^{\infty} dz e^{-z/L} / \sqrt{z} = \sqrt{\pi L} \text{Erf}(\sqrt{z/L}) \Big|_{z_0 \simeq 1}^{\infty}$  at  $\infty$ . So, it is not sensitive to the precise value of  $z_0 = 1$ , but it is controlled by the leading behavior of  $g(z) \sim 1/\sqrt{z}$  at  $z \rightarrow \infty$ <sup>11</sup>.

It is important to check that, at the limit of large  $L$ , the first integral  $\int_0^{z_0=1} dz(\dots)$  is stable and can still be neglected with respect to the second integral. This is shown on Fig. 4-left, in which we plot the 1st integral as a function of  $L$ . As already mentioned, the numerical cancellation of infinities requires that the lower bound of integration be set not to 0 but to smaller and smaller  $\epsilon$ . The curve in blue corresponds to  $\epsilon = 10^{-3}$ , and the 3 other curves, green, yellow and red, corresponding to  $\epsilon = 10^{-6}, 10^{-9}, 10^{-12}$  are superposed;  $\hat{I}(L)$  as given by (50) is plotted on Fig. 4-right. We see that, even at very large values of  $L$ , the 1st integral can always be safely neglected inside  $\hat{I}(L)$ .

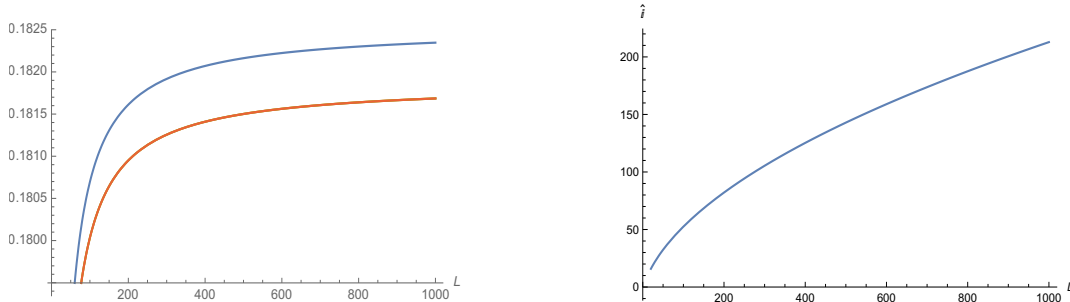


Fig. 4: on the left: value of the 1st (neglected inside  $\hat{I}(L)$ ) integral  $\int_0^{z_0=1} dz(\dots)$  for lower bounds of integrations going from  $10^{-3}$  (blue) to  $10^{-12}$  (yellow, green, red); on the right :  $\hat{I}(L)$

<sup>10</sup> $\text{Erfc}(x) = 1 - \text{Erf}(x)$ .

<sup>11</sup>By comparison, in the case of standard QED<sub>3+1</sub>, the leading behavior of  $g(z)$  when  $z \rightarrow \infty$  being  $g(z) \xrightarrow{z \rightarrow \infty} \ln z/z$ , one gets  $I(L) \sim \int_{z_0 \simeq 1}^{\infty} dz e^{-z/L} \ln z/z \sim \text{constant}$ , which yields  $\delta m_{LLL} \sim \frac{\alpha m}{2\pi} \cdot \text{constant} \xrightarrow{m \rightarrow 0} 0$ .

## 5 A non-vanishing 1-loop $\delta m_{LLL}$ at $m \rightarrow 0$

From (51) one gets immediately (restoring  $\hbar$  and  $c$ )

$$\delta m_{LLL} \xrightarrow{m \rightarrow 0} \frac{\alpha}{2} \sqrt{\frac{\pi}{2}} \sqrt{\frac{\hbar |e| B}{c^2}}, \quad (52)$$

which shows that, in an external magnetic field, this model, equivalent to reduced QED<sub>3+1</sub> on a 2-brane, cannot stay massless at 1-loop. Notice that (52) fulfills the renormalization conditions (34), which are expressed at  $B = 0$ .

Since the role of the counterterms is slightly more subtle than for QED<sub>3+1</sub> (in which they yield the large logs (see [8])), it is useful to make some comments about them.

In (38), the (infinite) counterterm only depends on  $m$  through the exponential  $e^{-isu^2 m^2}$  inside the integrand.

Noting respectively *b.term* and *c.term* the bare term and the counterterm inside the expression (38) of  $\delta m_{LLL}$ , one can write symbolically  $b.term = +\infty + f_1(m, eB)$ ,  $c.term = -\infty + f_2(m)$ , in which  $f_1, f_2$  are finite.

The change of variables (101) introduces a dependence of both on  $L$ , that we write symbolically  $b.term = +\infty + \hat{f}_1(m, eB, L) + \zeta(L) = +\infty + h_1(m, eB, L)$ ,  $c.term = -\infty + \hat{f}_2(m, L) - \zeta(L) = -\infty + h_2(m, L)$ . Therefore, via the change of variable (101), the counterterm has reacted on the bare contribution and the two become entangled (we introduced  $\pm\zeta$  to picture the fact that this dependence globally cancels but, in practice, one cannot “isolate”  $\zeta$ ; also, strictly speaking, these terms are not defined before the infinities are regularized and canceled).

The “educated” splitting [14] of the  $z$  interval of integration  $[0, \infty[ = [0, z_0] + [z_0, \infty[$  brings then  $\hat{I}(L)$  down to the approximation (45). Let us call the integrands in there  $h_1^{z_0}(m, eB, L)$  and  $h_2^{z_0}(m, L)$ . That the limit  $m \rightarrow 0$  yields a constant  $\delta m$ , or, equivalently,  $\hat{I}(L) \propto \sqrt{L}$ , is due to  $h_1^{z_0}$  (and the corresponding  $g(z)$  defined in (46) (47) (49)) which has an asymptotic expansion  $\simeq 1/\sqrt{z}$  at  $z \rightarrow \infty$ . This makes the result insensitive to the precise value of  $z_0$ . By contrast, as we have mentioned, in standard QED<sub>3+1</sub>, the asymptotic behavior of  $g(z)$  is  $\ln z/z$  [8].

$h_1^{z_0}$  no longer represents the bare contribution for the reasons that we just evoked: \* a change of variables introduced an extra dependence on  $L$  that mixes with the counterterm; \* the splitting of the  $z$  interval of integration collects in the neglected (small)  $\int_0^{z_0} dz(\dots)$ , in particular, the two canceling infinite parts of the bare term and of the counterterm, establishing a second connection between the two. In this respect, both play crucial roles in the massless limit of  $\delta_{LLL}$ , that can hardly be disentangled.

Last, let us remark that it is necessary to make the  $z$  integration at  $m \neq 0$  *before taking the limit*  $m \rightarrow 0$ , otherwise, since  $L = |e|B/m^2$ , one gets the undetermined expression  $\frac{1}{0} \times 0$ . Had we started from a massless theory, we would have obtained such an undetermined result. This is why one can only state that *the massless limit* of the 1-loop  $\delta m_{LLL}$  goes to a constant, or, equivalently, that the model under consideration cannot stay massless at 1-loop.

## 6 Restricting to the lowest Landau level of the virtual electron

### 6.1 Basics

The contribution of different Landau levels to the propagator of an electron in a constant uniform external  $B$  has been investigated in [24] and [25]. From eqs.(22,23,24) of [24] one gets

$$\begin{aligned} G(x, x') &= \sum_{n=0}^{\infty} G^n(x, x') = \sum_{n=0}^{\infty} e^{i\omega(x, x')} \hat{G}^n(x - x') \\ &= e^{i\omega(x, x')} \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-x')} \sum_{n=0}^{\infty} \hat{G}^n(p, B), \\ \omega(x, x') &= -\frac{|e|B}{2} (x_1 + x'_1)(x_2 - x'_2), \end{aligned} \quad (53)$$

in which  $x = (x_0, x_1, x_2, x_3)$ ,  $x' = (x'_0, x'_1, x'_2, x'_3)$ . The factor  $e^{i\omega(x, x')}$  is identical to Schwinger's  $\Phi(x, x')$  as written in (2) (see for example [21], chapter 3).

Using the conventions and metric  $(-+++)$  of Schwinger, the contribution of the LLL is

$$-i\hat{G}^{n=0}(p, B) = e^{-p_\perp^2/|e|B} \int_0^\infty ds_1 e^{-is_1(m^2+p_\parallel^2)} (m - \not{p}_\parallel)(1 - i\gamma_1\gamma_2), \quad (54)$$

in which we have introduced the Schwinger's parameter  $s_1$  (see also appendix A).

To determine the contribution of the LLL of the virtual electron to the self-energy, we have to calculate (see (15))

$$i\Sigma^{n=0}(\hat{p}, B) = -\frac{e^2}{2} \int \frac{d^3\hat{p}}{(2\pi)^3} \gamma^\mu \hat{G}^{n=0}(\hat{p} - \hat{k}, B) \frac{g_{\mu\nu}}{\sqrt{\hat{k}^2}} \gamma^\nu \quad (55)$$

One introduces as before (see (16)) the Schwinger parameter  $s_2$  for the photon propagator and, instead of eq. (3.11) of [12], one gets

$$\Sigma^{n=0}(p, B) = -ie^2 \int_0^\infty ds_1 \sqrt{\frac{i}{\pi}} \int_0^\infty \frac{ds_2}{\sqrt{s_2}} \int \frac{d^3\hat{k}}{(2\pi)^3} e^{-\frac{(p-k)_\perp^2}{|e|B}} e^{-is_2(\hat{k}^2 - i\epsilon)} e^{-is_1(m^2 + (\hat{p} - \hat{k})_\parallel^2)} \gamma^\mu (m - (\not{p}_\parallel - \not{k}_\parallel))(1 - i\gamma_1\gamma_2) \gamma_\mu. \quad (56)$$

We use again the change of variables (19) together with

$$z = |e|Bs_1, \quad y = |e|Bs. \quad (57)$$

Like before, aiming at performing the integration  $\int d^3\hat{k}$ , one rewrites the exponentials (watch the “ $i$ ” which now occurs). Since  $s$  cannot be factorized everywhere, we have now included it into the definitions of  $\chi_0$  and  $\varphi_0$ , unlike previously for  $\chi$  and  $\varphi$ .

$$\begin{aligned} & \frac{(p-k)_\perp^2}{i|e|B} + s_2\hat{k}^2 + s_1(m^2 + (\hat{p} - \hat{k})_\parallel^2) \\ &= usm^2 + su(1-u)\hat{p}_\parallel^2 + s(\hat{k} - u\hat{p})_\parallel^2 + \left(s(1-u) + \frac{1}{i|e|B}\right) \left(k_\perp - \frac{p_\perp}{1+i|e|Bs(1-u)}\right)^2 + p_\perp^2 \frac{s(1-u)}{1+i|e|Bs(1-u)} \\ &= \chi_0 + \varphi_0, \\ & \chi_0 = s(\hat{k} - u\hat{p})_\parallel^2 + \left(s(1-u) + \frac{1}{i|e|B}\right) \left(k_\perp - \frac{p_\perp}{1+i|e|Bs(1-u)}\right)^2, \\ & \varphi_0 = usm^2 + su(1-u)\hat{p}_\parallel^2 + p_\perp^2 \frac{s(1-u)}{1+i|e|Bs(1-u)} = usm^2 + b_0\hat{p}_\parallel^2 + b_\perp p_\perp^2, \\ & b_0 = us(1-u), \quad b_\perp = \frac{s(1-u)}{1+i|e|Bs(1-u)}, \end{aligned} \quad (58)$$

such that

$$\Sigma^{n=0}(p, B) = -i\frac{e^2}{2} \int_0^\infty ds \sqrt{s} \sqrt{\frac{i}{\pi}} \int_0^1 \frac{du}{\sqrt{1-u}} \int \frac{d^3\hat{k}}{(2\pi)^3} e^{-i(\chi_0 + \varphi_0)} \gamma^\mu (m - (\not{p}_\parallel - \not{k}_\parallel))(1 - i\gamma_1\gamma_2) \gamma_\mu. \quad (59)$$

One then shifts the variables  $k_\parallel \rightarrow r_\parallel = k_\parallel - up_\parallel$ ,  $k_\perp \rightarrow r_\perp = k_\perp - \frac{p_\perp}{1+i|e|Bs(1-u)}$ . One has  $\chi_0 = s\hat{r}_\parallel^2 + \left(s(1-u) + \frac{1}{i|e|B}\right)r_\perp^2$ .

Then,  $(m - \gamma^0(k^0 - p^0)) = m - \gamma^0(r^0 + (u-1)p^0)$ .  $\chi_0$  being even since it depends on  $r_0^2$ , the odd term  $\propto r_0$  yields a vanishing contribution to the  $\int dk_0$ . One can thus replace  $m - \gamma^0(k^0 - p^0)$  by  $m - (u-1)\gamma^0 p^0$ . One gets

$$\Sigma^{n=0}(p, B) = -i\frac{e^2}{2} \sqrt{\frac{i}{\pi}} \int_0^\infty ds \sqrt{s} \int_0^1 \frac{du}{\sqrt{1-u}} \int \frac{d^3\hat{r}}{(2\pi)^3} e^{-i\varphi_0} e^{-i[s\hat{r}_\parallel^2 + (s(1-u) + 1/i|e|B)r_\perp^2]} \gamma^\mu (m + (u-1)\not{p}_\parallel)(1 - i\gamma_1\gamma_2) \gamma_\mu. \quad (60)$$

With the help of (22) one gets

$$\int d^3\hat{r} e^{-i\chi} = e^{-i\pi/4} \frac{\sqrt{\pi}}{\sqrt{s}} (\sqrt{\pi} e^{-i\pi/4})^2 \frac{1}{s(1-u) + 1/i|e|B}, \quad (61)$$

and, since  $\sqrt{i} = e^{i\pi/4}$ ,

$$\Sigma^{n=0}(p, B) = -\frac{e^2}{16\pi^2} \int_0^\infty ds \int_0^1 \frac{du}{\sqrt{1-u}} e^{-i\varphi_0} \frac{i|e|B}{1+i|e|Bs(1-u)} \gamma^\mu (m + (1-u)\gamma^0 p^0) (1 - i\gamma_1 \gamma_2) \gamma_\mu. \quad (62)$$

Next, one performs the Dirac algebra

$$\begin{aligned} \gamma^\mu (m + (1-u)\gamma^0 p^0) (1 - i\gamma_1 \gamma_2) \gamma_\mu &= -4m + 2im\gamma_1 \gamma_2 + 2(1-u)p^0 \gamma^0 + 2i(1-u)p^0 \gamma^0 \gamma_1 \gamma_2 \\ &= -4m + 2(1-u)p^0 \gamma^0 (1 + i\gamma_1 \gamma_2), \end{aligned} \quad (63)$$

such that

$$\Sigma^{n=0}(p, B) = -\frac{\alpha}{2\pi} \int_0^\infty ds \int_0^1 \frac{du}{\sqrt{1-u}} e^{-isum^2} e^{-i(b_0 \hat{p}_\parallel^2 + b_\perp p_\perp^2)} \frac{i|e|B}{1+i|e|Bs(1-u)} [-2m + (1-u)p^0 \gamma^0 (1 + i\gamma_1 \gamma_2)] + c.t., \quad (64)$$

in which  $b_0$  and  $b_\perp$  are given in (58) and where we have now mentioned the counterterms (c.t.) that need eventually to be introduced to fulfill suitable renormalization conditions.

We are interested in  $\delta m_{LLL}^0$  concerning external electrons in the LLL. To get it we sandwich  $\Sigma(\pi)$  between two LLL eigenstates. Since these are annihilated by  $1 + i\gamma_1 \gamma_2$ , the only term that may play a role is the one proportional to  $m$ . Accordingly, the quantity of interest to us is

$$\Sigma_{LLL}^{n=0}(p, B) = \frac{\alpha m}{\pi} \int_0^\infty ds \int_0^1 \frac{du}{\sqrt{1-u}} e^{-isum^2} e^{-is(1-u)\left(u\hat{p}_\parallel^2 + \frac{p_\perp^2}{1+i|e|Bs(1-u)}\right)} \frac{i|e|B}{1+i|e|Bs(1-u)} + c.t. \quad (65)$$

## 6.2 Transforming the space representation

One needs to determine  $\Sigma(\pi)$  satisfying (25). To that purpose, one must find the suitable change of variables to adapt (2.45) (2.46) of [12] to the present situation, that is to determine  $a_0$  and  $a_\perp$  in (27 (which is the same as (2.46) of [12])). One must have

$$\begin{aligned} \exp[-isa_\perp p_\perp^2 \frac{\tan |e|Bs a_\perp}{|e|Bs a_\perp}] &= \exp[-is(1-u) \frac{p_\perp^2}{1+i|e|Bs(1-u)}] \\ \Leftrightarrow \tan |e|Bs a_\perp &= \frac{|e|Bs(1-u)}{1+i|e|Bs(1-u)} \Leftrightarrow a_\perp = \frac{1}{|e|Bs} \tan^{-1} \frac{|e|Bs(1-u)}{1+i|e|Bs(1-u)}. \end{aligned} \quad (66)$$

Then

$$\cos |e|Bs a_\perp = \cos \tan^{-1} \frac{|e|Bs(1-u)}{1+i|e|Bs(1-u)}. \quad (67)$$

One also has trivially

$$-isa_0(-p_0^2) = -isu(1-u)(-p_0^2) \Leftrightarrow a_0 = u(1-u). \quad (68)$$

This gives

$$\begin{aligned} \Sigma^{n=0}(\pi_0, \pi_\perp) &= \frac{\alpha m}{\pi} \int_0^\infty ds \int_0^1 \frac{du}{\sqrt{1-u}} \left[ \cos \tan^{-1} \frac{|e|Bs(1-u)}{1+i|e|Bs(1-u)} \right] \frac{i|e|B}{1+i|e|Bs(1-u)} \\ &\quad e^{-isum^2} e^{-isu(1-u)(-\pi_0^2)} e^{-is\pi_\perp^2 \left( \frac{1}{|e|Bs} \tan^{-1} \frac{|e|Bs(1-u)}{1+i|e|Bs(1-u)} \right)} + c.t. \end{aligned} \quad (69)$$



### 6.3 Renormalization conditions and counterterms

Let us consider general on mass-shell external electrons. Since renormalization conditions have to be expressed at  $B = 0$ , let us also consider the limit  $B \rightarrow 0$  of  $\Sigma^{n=0}(\pi)$ .

$$\begin{aligned} \Sigma^{n=0}(\pi_0, \pi_\perp) &\stackrel{B \rightarrow 0}{\sim} \frac{\alpha m}{\pi} \int_0^\infty ds \int_0^1 \frac{du}{\sqrt{1-u}} \left[ \cos 0 \right] \frac{i|e|B}{1+0} \\ &\quad e^{-isu m^2} e^{-isu(1-u)(-\pi_0^2)} e^{-is\pi_\perp^2} \left( \frac{1}{|e|Bs} \arctan \frac{|e|Bs(1-u)}{1+0} \right) + \text{terms} \propto (1+i\gamma_1\gamma_2) + c.t \\ &\sim \frac{\alpha m}{\pi} \int_0^\infty ds \int_0^1 \frac{du}{\sqrt{1-u}} i|e|B e^{-isu m^2} e^{isu(1-u)\pi_0^2} e^{-i\pi_\perp^2} \frac{1}{|e|B} |e|Bs(1-u) + \text{terms} \propto (1+i\gamma_1\gamma_2) + c.t \end{aligned} \quad (70)$$

We then go through the successive changes of variables  $(u, s) \rightarrow (u, y = |e|Bs u)$ ,  $t = iy$ , last  $z = ut$ , plus a Wick rotation (see subsection 6.4 below), to get

$$\Sigma^{n=0}(\pi_0, \pi_\perp) \stackrel{B \rightarrow 0}{\sim} \frac{\alpha m}{\pi} \int_0^\infty dz \int_0^1 \frac{du}{u^2 \sqrt{1-u}} e^{-\frac{zm^2}{u|e|B}} e^{\frac{z(1-u)}{u} \frac{\pi_0^2}{|e|B}} \pi_0^2 e^{-z \frac{1-u}{u^2} \frac{\pi_\perp^2}{|e|B}} + \text{terms} \propto (1+i\gamma_1\gamma_2) + c.t \quad (71)$$

If we now go on mass-shell,  $\not{p} + m = 0$ ,  $\not{p}^2 = m^2 = -\pi^2 - \frac{|e|}{2} 2\sigma^3 B \Rightarrow m^2 = \pi_0^2 - \pi_\perp^2 - |e|\sigma^3 B$ , we get

$$\Sigma_{mass-shell}^{n=0}(\pi_0, \pi_\perp) \stackrel{B \rightarrow 0}{\sim} \frac{\alpha m}{\pi} \int_0^\infty dz \int_0^1 \frac{du}{u^2 \sqrt{1-u}} e^{-z \frac{m^2}{|e|B}} e^{\frac{z(1-u)}{u} \sigma^3} e^{-z \frac{(1-u)^2}{u^2} \frac{\pi_\perp^2}{|e|B}} + \text{terms} \propto (1+i\gamma_1\gamma_2) + c.t \quad (72)$$

The 1st renormalization condition in (34) concerns the vanishing, on mass-shell, of  $\Sigma$  at the limit  $B \rightarrow 0$ . We have therefore to introduce a 1st counterterm  $c.t._1$

$$c.t._1 = - \lim_{B \rightarrow 0} \frac{\alpha m}{\pi} \int_0^\infty dz \int_0^1 \frac{du}{u^2 \sqrt{1-u}} e^{-z \frac{m^2}{|e|B}} e^{\frac{z(1-u)}{u} \sigma^3} e^{-z \frac{(1-u)^2}{u^2} \frac{\pi_\perp^2}{|e|B}} + \text{terms} \propto (1+i\gamma_1\gamma_2) \quad (73)$$

(the terms  $\propto (1+i\gamma^1\gamma^2)$  give vanishing contribution only to external LLL).

The second renormalization condition (see (34)) concerns the derivative of  $\Sigma$ . This leads to introducing a second set of counterterms. However, they have to vanish on mass-shell since they must satisfy the 1st renormalization condition. Since, in order to calculate  $\delta m$ , we precisely work on mass-shell, we can forget about the second set of counterterms and proceed now with the calculation of  $\delta m_{LLL}^0$ .

### 6.4 Calculation of the 1-loop self-mass $\delta m_{LLL}^0$ when both external and internal electrons are in the lowest Landau level

When acting on external LLL electrons, and on mass-shell, one has  $\pi_0^2 = m^2$ ,  $\pi_\perp^2 = \sigma^3 eB = -eB = +|e|B$ . From (69) and (73) one then obtains

$$\delta m_{LLL}^0 = \frac{\alpha m}{\pi} \int_0^\infty ds \int_0^1 \frac{du}{\sqrt{1-u}} \left[ \cos \tan^{-1} \frac{|e|Bs(1-u)}{1+i|e|Bs(1-u)} \right] \frac{i|e|B}{1+i|e|Bs(1-u)} e^{-isu^2 m^2} e^{-i \tan^{-1} \frac{|e|Bs(1-u)}{1+i|e|Bs(1-u)}} + c.t._1 \quad (74)$$

After some calculations which are detailed in appendix C, one gets

$$\delta m_{LLL}^0 = \frac{\alpha m}{4\pi} \underbrace{4 \int_0^\infty dz \int_0^1 \frac{du}{\sqrt{1-u}} \left[ \cosh \tanh^{-1} \frac{z(1-u)}{u^2+z(1-u)} \right] \frac{1}{u^2+z(1-u)} e^{-zm^2/|e|B} e^{-\tanh^{-1} \frac{z(1-u)}{u^2+z(1-u)}}}_{\hat{I}^0(L), L=|e|B/m^2} + c.t._1 \quad (75)$$

When  $m = 0$ ,  $\delta m_{LLL}^0 = \frac{\alpha m}{\pi} \int_0^\infty dz \int_0^1 \frac{du}{\sqrt{1-u}} \frac{1}{u^2+z(1-u)} \frac{1}{2} \left( 1 + e^{-2 \tanh^{-1} \frac{z(1-u)}{u^2+z(1-u)}} \right) \sim \frac{\alpha m}{2\pi} \int_0^\infty dz \int_0^1 \frac{du}{\sqrt{1-u}} \frac{1}{u^2+z(1-u)}$  which diverges at  $z \rightarrow \infty$ . Like before, one must eventually take the limit  $m \rightarrow 0$  after the integration has been performed.

The exponential  $e^{-2 \tanh^{-1} \frac{z(1-u)}{u^2+z(1-u)}}$  being bounded by 1 and going to 0 when  $z \rightarrow \infty$ , we have to evaluate

$$\delta m_{LLL}^0 \sim \frac{\alpha m}{4\pi} \underbrace{2 \int_0^\infty dz \int_0^1 \frac{du}{\sqrt{1-u}} \frac{e^{-zm^2/|e|B}}{u^2+z(1-u)}}_{\hat{I}^0(L), L=|e|B/m^2} (1+\epsilon) + c.t._1, \quad 0 \leq \epsilon \leq 1, \quad (76)$$

in which we have, like previously, factorized  $\frac{\alpha m}{4\pi}$ , at the price of introducing an extra factor 2 in front of the integral. One accordingly defines now (compare with (42) (45))

$$\begin{aligned} \hat{I}^0(L) &= 4 \int_0^\infty dz \int_0^1 \frac{du}{\sqrt{1-u}} \left[ \cosh \tanh^{-1} \frac{z(1-u)}{u^2+z(1-u)} \right] \frac{1}{u^2+z(1-u)} e^{-zm^2/|e|B} e^{-\tanh^{-1} \frac{z(1-u)}{u^2+z(1-u)}} \\ &\simeq 2 \int_0^\infty dz \int_0^1 \frac{du}{\sqrt{1-u}} \frac{e^{-zm^2/|e|B}}{u^2+z(1-u)} (1+\epsilon), \quad 0 \leq \epsilon \leq 1. \end{aligned} \quad (77)$$

Note that, unlike when taking all Landau levels of the internal electrons into account, the integral  $\hat{I}_0(L)$  is convergent at  $z = 0$  without introducing any counterterm.

One has

$$g_0(z) \equiv \int_0^1 \frac{du}{\sqrt{1-u}} \frac{2}{u^2+z(1-u)} = - \frac{4\sqrt{2} \left( \frac{\tan^{-1} \left( \frac{\sqrt{2}}{\sqrt{z + \sqrt{(z-4)z-2}}} \right)}{\sqrt{z + \sqrt{(z-4)z-2}}} - \frac{\tan^{-1} \left( \frac{\sqrt{2}}{\sqrt{z - \sqrt{(z-4)z-2}}} \right)}{\sqrt{z - \sqrt{(z-4)z-2}}} \right)}{\sqrt{(z-4)z}}, \quad (78)$$

(to be compared with (46)) such that

$$\delta m_{LLL}^0 \sim \frac{\alpha m}{4\pi} \int_0^\infty dz e^{-zm^2/|e|B} g_0(z) + c.t._1. \quad (79)$$

#### 6.4.1 Contribution of the counterterm to $\delta m_{LLL}^0$

For external LLL,  $\pi_\perp^2 \rightarrow |e|B$ ,  $\sigma^3 \rightarrow -1$ , this counterterm contributes to  $\delta m_{LLL}^0$  by

$$c.t._1 = - \lim_{B \rightarrow 0} \frac{\alpha m}{\pi} \int_0^\infty dz \int_0^1 \frac{du}{u^2 \sqrt{1-u}} e^{-z \frac{m^2}{|e|B}} e^{-z \frac{1-u}{u^2}}, \quad (80)$$

which is convergent. It yields

$$c.t._1 = - \lim_{B \rightarrow 0} \frac{\alpha m}{\pi} \int_0^1 \frac{du}{u^2 \sqrt{1-u}} \frac{1}{\frac{m^2}{|e|B} + \frac{1-u}{u^2}} = - \lim_{B \rightarrow 0} \frac{\alpha}{2\pi} \frac{|e|B}{m} g_0 \left( \frac{|e|B}{m^2} \right), \quad (81)$$

in which  $g_0$  is the same as that defined in (78). At the limit  $z \rightarrow 0$

$$g_0(z) \xrightarrow{z \rightarrow 0} \frac{\pi}{\sqrt{z}} + 2 \ln 2 - \frac{\ln z}{2} + \frac{z}{16} (-\ln z - 1 + 4 \ln 2) + \mathcal{O}(z^{3/2}), \quad (82)$$

such that

$$c.t._1 = - \lim_{B \rightarrow 0} \left( \frac{\alpha}{2} \sqrt{|e|B} + \frac{\alpha}{\pi} \frac{|e|B}{m} \ln 2 \right) + \dots \quad (83)$$

which we shall truncate at the first term since the limit  $m \rightarrow 0$  should be taken afterwards. Accordingly, one finds a vanishing counterterm (which is in particular independent of the external  $B$ )

$$c.t._1 = 0. \quad (84)$$

Collecting (79) and (84) yields

$$\delta m_{LLL}^0 \sim \frac{\alpha m}{4\pi} \int_0^\infty dz e^{-zm^2/|e|B} g_0(z). \quad (85)$$

Notice that the bare  $\delta m_{LLL}^0$  (and, of course, the (vanishing) counterterm) are both finite, unlike when all Landau levels of the internal electron are accounted for.

## 6.5 The limit of $\delta m_{LLL}^0$ when $m \rightarrow 0$

In addition to the limit  $z \rightarrow 0$  given in (82) one has

$$g_0(z) \xrightarrow{z \rightarrow \infty} \frac{2\pi}{\sqrt{z}} - \frac{4}{z} + \frac{2\pi}{z^{3/2}} - \frac{32}{3z^3} + \dots \quad (86)$$

So, splitting the  $z$  interval of integration of (85) into 3 sub-intervals gives

$$\delta m_{LLL}^0 \sim \frac{\alpha m}{4\pi} \left[ \int_0^a dz e^{-zm^2/|e|B} \frac{\pi}{\sqrt{z}} + \underbrace{\int_a^b dz e^{-zm^2/|e|B} g_0(z)}_{\text{constant}} + \int_b^\infty dz e^{-zm^2/|e|B} \frac{2\pi}{\sqrt{z}} \right]. \quad (87)$$

The bounds  $a$  and  $b$  are chosen such that, for  $z \in [0, a]$  the expansion (82) is valid, and for  $z \in [b, \infty]$  the expansion (86) is valid. Since

$$\int dz \frac{e^{-zm^2/|e|B}}{\sqrt{z}} = \frac{\sqrt{\pi} \operatorname{Erf}(\sqrt{m^2/|e|B} \sqrt{z})}{\sqrt{m^2/|e|B}}, \quad (88)$$

one has

$$\begin{aligned} \delta m_{LLL}^0 \sim \frac{\alpha m}{4\pi} & \left[ \pi \frac{\sqrt{\pi}}{\sqrt{m^2/|e|B}} \left( \operatorname{Erf}(\sqrt{m^2/|e|B} \sqrt{a}) - \frac{4}{\sqrt{\pi}} \times 0 \right) \right. \\ & + cst \\ & \left. + 2\pi \frac{\sqrt{\pi}}{\sqrt{m^2/|e|B}} \left( \underbrace{\operatorname{Erf}(\sqrt{m^2/|e|B} \sqrt{z} = \infty)}_1 - \operatorname{Erf}(\sqrt{m^2/|e|B} \sqrt{b}) \right) \right]. \end{aligned} \quad (89)$$

To study the limit  $m \rightarrow 0$  we use

$$\begin{aligned} \operatorname{Erf}(x) & \xrightarrow{x \rightarrow 0} \frac{2x}{\sqrt{\pi}}, \\ \operatorname{Erf}(x) & \xrightarrow{x \rightarrow \infty} 1, \end{aligned} \quad (90)$$

which shows that it is the value at  $z = \infty$  that controls  $\delta m_{LLL}^0$ .

Finally

$$\delta m_{LLL}^0 \xrightarrow{m \rightarrow 0} \frac{\alpha}{2} \sqrt{\pi |e|B} = \sqrt{2} \delta m_{LLL}. \quad (91)$$

## 6.6 An approximate analytical expression for $\delta m_{LLL}^0$ . Comparison with $\delta m_{LLL}$

It is easy to get a fair approximate analytical expression for  $\delta m_{LLL}^0$  given in (85) by using the following simple fit to  $g_0(z)$

$$g_0^{app}(z) \simeq e^{-z/30} \left( \frac{\pi}{\sqrt{z}} + 2 \ln 2 \right) + e^{-30/z} \left( \frac{2\pi}{\sqrt{z}} - \frac{4}{z} \right), \quad (92)$$

which has, in particular, the appropriate limits at  $z \rightarrow 0$  and  $z \rightarrow \infty$ . On fig. 5 the exact  $g_0$  is plotted in blue and the approximate one in yellow.

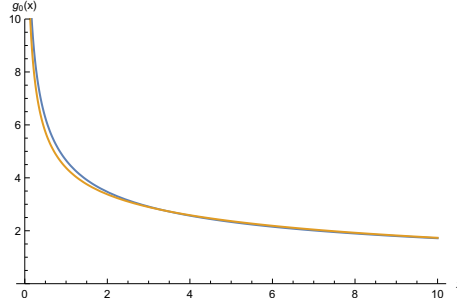


Fig. 5: the exact  $g_0$  given in (78) (blue) and its approximate expression (92) (yellow)

This yields

$$\delta m_{LLL}^0 \approx \frac{\alpha m}{4\pi} \left( \frac{2\pi^{3/2} \exp \left[ -2\sqrt{30} \sqrt{\frac{m^2}{|e|B}} \right]}{\sqrt{\frac{m^2}{|e|B}}} + \frac{\pi^{3/2}}{\sqrt{\frac{m^2}{|e|B} + \frac{1}{30}}} + \frac{60 \ln(2)}{30 \frac{m^2}{|e|B} + 1} - 8 \text{BesselK} \left( 0, 2\sqrt{30} \sqrt{\frac{m^2}{|e|B}} \right) \right), \quad (93)$$

which has the limit (91) when  $m \rightarrow 0$ . Notice also that the second contribution yields a finite  $\delta m_{LLL}^0 \rightarrow \frac{\alpha}{4\pi} \pi^{3/2} \sqrt{|e|B}$  when  $m \rightarrow \infty$ .

On Fig. 6, we plot  $\sqrt{\frac{m^2}{|e|B}} \int_0^\infty dz e^{-zm^2/|e|B} g_0(z)$  in blue together with  $\sqrt{\frac{m^2}{|e|B}} \int_0^\infty dz e^{-zm^2/|e|B} g_0^{app}(z)$  in yellow, which corresponds to  $4\pi\delta m_{LLL}^0/\alpha\sqrt{|e|B}$ . It shows that this rather crude approximation is good at better than 7% for  $\frac{m^2}{|e|B} \geq .4$ , at  $\sim 10\%$  for lower values of  $\frac{m^2}{|e|B}$  and that it has, of course, the appropriate limit  $2\pi^{3/2} \approx 11.14$  at  $\frac{m^2}{|e|B} = 0$ .

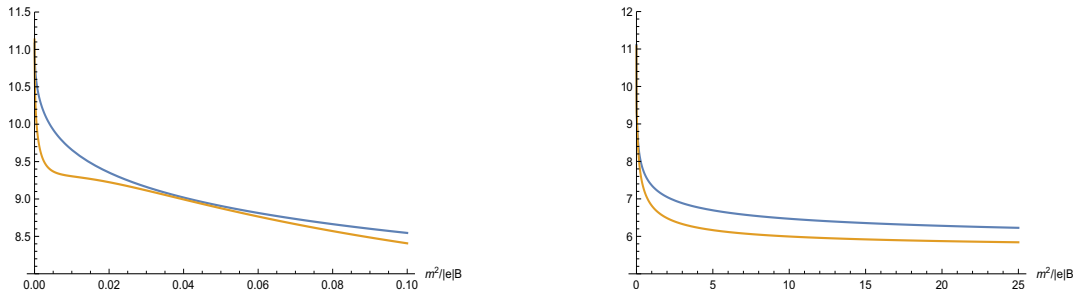


Fig. 6:  $\sqrt{\frac{m^2}{|e|B}} \int_0^\infty dz e^{-zm^2/|e|B} g_0(z)$  in blue and  $\sqrt{\frac{m^2}{|e|B}} \int_0^\infty dz e^{-zm^2/|e|B} g_0^{app}(z)$  in yellow as functions of  $\frac{m^2}{|e|B}$ .

Fig. 6 also shows that this approximation is the worse in the close vicinity of  $\frac{m^2}{|e|B} = 0$ . Including higher orders in the expansions of  $g_0(z)$  at  $z \rightarrow 0$  and  $z \rightarrow \infty$  turns out to improve the situation at large values of  $\frac{m^2}{|e|B}$  but, instead, to worsen it close to 0.

On Fig. 7 we plot  $\frac{4\pi}{\alpha} \frac{\delta m_{LLL}}{\sqrt{|e|B}}$  given in (47) and (46) in blue together with  $\frac{4\pi}{\alpha} \frac{\delta m_{LLL}^0}{\sqrt{|e|B}}$  given in (85) and (78) as functions of  $\frac{m^2}{|e|B}$ . They determine the behavior of the corresponding  $\delta m$ 's at fixed value of  $|e|B$  when  $m$  becomes larger and larger (and not their limits at  $|e|B \rightarrow 0$ , which vanishes for both in virtue of the first renormalization condition). As we see, this behavior is very different for the two cases:  $\frac{4\pi}{\alpha} \frac{\delta m_{LLL}}{\sqrt{|e|B}}$  behaves like  $\frac{e^{-m^2/|e|B}}{\sqrt{m^2/|e|B}} \rightarrow 0$  when  $\frac{m^2}{|e|B} \rightarrow \infty$  while  $\frac{4\pi}{\alpha} \frac{\delta m_{LLL}^0}{\sqrt{|e|B}}$  goes to  $\pi^{3/2}$  at the same limit.

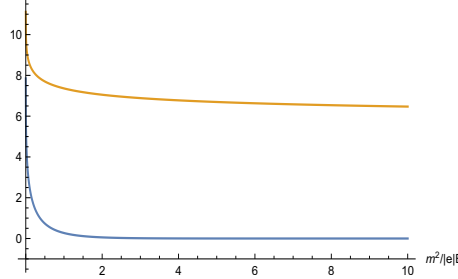


Fig. 7:  $\frac{4\pi}{\alpha} \frac{\delta m_{LLL}}{\sqrt{|e|B}}$  given in (50) (blue) and  $\frac{4\pi}{\alpha} \frac{\delta m_{LLL}^0}{\sqrt{|e|B}}$  given in (93) (yellow) as functions of  $\frac{m^2}{|e|B}$

On Fig. 8<sup>12</sup> we now plot  $\frac{4\pi}{\alpha} \frac{\delta m_{LLL}}{m}$  (in blue) and  $\frac{4\pi}{\alpha} \frac{\delta m_{LLL}^0}{m}$  (in yellow) as functions of  $\frac{|e|B}{m^2}$ . This shows how the  $\delta m$ 's vary with  $B$  at fixed  $m$ . Once more, while we witness as expected their both vanishing at  $B = 0$  according to the 1st renormalization condition, their behavior  $\propto \sqrt{|e|B}$  when  $B$  becomes larger and larger is factorized by different coefficients; as a result  $\delta m_{LLL}^0/m$  is already more than twice  $\delta m_{LLL}/m$  at  $\frac{|e|B}{m^2} = 20$ . Restricting the internal electron to its LLL results accordingly in a very large overestimate of the self-mass.

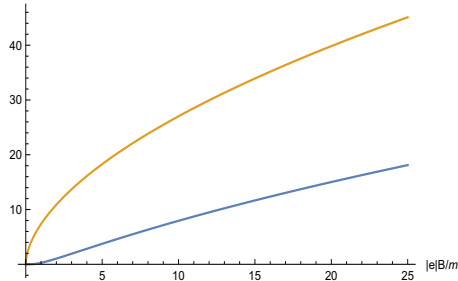


Fig. 8:  $\frac{4\pi}{\alpha} \frac{\delta m_{LLL}}{m}$  (blue) and  $\frac{4\pi}{\alpha} \frac{\delta m_{LLL}^0}{m}$  (yellow) as functions of  $\frac{|e|B}{m^2}$

## 6.7 A few remarks

$\delta m_{LLL}$  and  $\delta m_{LLL}^0$  do not have the same limits at  $m \rightarrow 0$ , nor at  $m \rightarrow \infty$ .

Would  $m \rightarrow 0$  be equivalent to  $eB \rightarrow \infty$ , one could, at first sight, expect that only the LLL plays a role. This would however only be true if the only physical variable was  $|e|B/m^2$ , and if renormalization did not put a grain of salt in such an argumentation.

While it is true that  $G^{n=0}(p, B)$  can indeed be obtained by formally taking the limit  $B \rightarrow \infty$  of  $G_0(p, B)$  (see Appendix A), one should notice that:

\* this limit cannot be applied to the phase  $\Phi$ ;

<sup>12</sup>Figs. 7 and 8 are not plotted with the approximate analytical expressions that we have deduced for the  $\delta m$ 's, but by numerical integration of their exact expressions.

\* the factor  $e^{-k_\perp^2/|e|B}$  is not replaced by 1 inside  $G^{n=0}$  despite  $B \rightarrow \infty$ ; this is because, as the Larmor radius shrinks to 0 at this limit,  $k_\perp$  can extend to  $\infty$ ;

\* the (vanishing) counterterm is determined by taking first the limit  $B \rightarrow 0$ , so as to fulfill renormalization conditions; then, eventually, the non-vanishing limit  $m \rightarrow 0$  is taken; therefore, naively taking the limit  $B \rightarrow \infty$  to “select” the LLL cannot be applied either to the counterterm.

Arguing that the limit  $m \rightarrow 0$  is equivalent to  $B \rightarrow \infty$  can accordingly only be wrong<sup>13</sup>.

The limits at  $m \rightarrow \infty$  (which should not be confused with those at  $B \rightarrow 0$ ) are also very different since  $\delta m_{LLL} \sim \frac{|e|B}{m} e^{-m^2/|e|B} \rightarrow 0$  while  $\delta m_{LLL}^0 \sim cst \times \sqrt{|e|B}$  (see Fig. 7).

Large cancellations therefore occur among multiple Landau levels of the virtual electron. However, they can only be estimated after going through the filter of renormalization, and infinities that need being tamed only arise when one accounts for all levels.

## 7 Conclusion and prospects

Unlike what happens for  $\text{QED}_{3+1}$ , the massless limit of the 1-loop  $\delta m_{LLL}$  in external  $B$  for  $\text{QED}_{3+1}$  reduced on a 2-brane does not vanish. We have shown furthermore that it corresponds to an electron propagating inside a graphene-like medium. The latter cannot therefore stay “gapless” at 1-loop in the presence of a magnetic field. This result has been obtained with special attention paid to the renormalization conditions.

The result is very simple because we have restricted the external electron to lie in the lowest Landau level. For higher levels, the situation is much more intricate and analytical formulæ certainly cannot be obtained.

We have also shown that restricting to the LLL of the internal electron largely overestimates the self-mass; in particular, its value when  $m \rightarrow 0$  triggers a multiplicative factor  $\sqrt{2}$ . Despite the case under concern has the peculiarity that taming infinities and renormalizing is only needed when accounting for all Landau levels, studies based on such an approximation appear rather suspicious. Note that, in the case of standard  $\text{QED}_{3+1}$ , it was shown in [19] that accounting for the sole leading  $(\ln)^2$  terms largely increases the result, too.

I cannot pretend to have dealt with real graphene, in which, in particular, the smallness of the Fermi velocity with respect to the speed of light makes the theory strongly coupled. There, techniques have to be mastered which go beyond perturbative expansions, while respecting appropriate renormalization conditions.

It is also well known that the photon propagator gets modified in the presence of an external  $B$  (see for example [27]). This modification has been included in calculations of the electron self-energy [26] [7] with the result that double logs are turned into single logs. However, the large single logs closely associated with counterterms (see [8]) were not taken into account. Furthermore, this modification of the photon propagator and the eventual screening of the Coulomb potential is obtained by resumming the infinite geometric series of 1-loop vacuum polarizations (see for example [27]); in contrast, Quantum Field Theory stipulates that renormalization conditions and the addition of the corresponding counterterms should be achieved consistently order by order in powers of the coupling constant or in the number of loops. In this framework, screening the Coulomb potential inside the electron self-energy at finite order raises many issues, both technical and conceptual.

*Acknowledgments: It is a great pleasure to thank M.I. Vysotsky for his invaluable assistance and advice.*

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<sup>13</sup>Eventually forcing the identity between the two limits at  $m = 0$  of  $\delta m_{LLL}$  and  $\delta m_{LLL}^0$  as a kind of renormalization condition must be rejected.

## Appendix

### A The propagator $G^{n=0}(p, B)$ of an electron in the lowest Landau level as the limit at $B \rightarrow \infty$ of $G_0(p, B)$ (without the phase (2))

After putting aside the phase  $\Phi$  given in (2), we can get it by taking the limit  $B \rightarrow \infty$  in  $G(p, B)$

Let us consider the general expression (6) of [18] ( $z = |e|Bs_1$ ), which does not include the phase

$$-iG_0(p, B) = \int_0^\infty ds_1 e^{-is_1[m^2 - i\epsilon + p_\parallel^2 + \frac{\tan z}{z}p_\perp^2]} \frac{e^{iq\sigma^3 z}}{\cos z} \left( m - (\gamma p)_\parallel - \frac{e^{-iq\sigma^3 z}}{\cos z} (\gamma p)_\perp \right). \quad (94)$$

Since  $(q\sigma^3)^2 = 1$ ,  $\cos q\sigma^3 z = \cos z$  and  $\sin q\sigma^3 z = q\sigma^3 \sin z$ . As  $\sigma^3 = i\gamma^1\gamma^2$ , if one cancels at the beginning the 2 inverse exponentials one gets

$$-iG_0(p, B) = \int_0^\infty ds_1 e^{-is_1(m^2 - i\epsilon + p_\parallel^2 + \frac{\tan z}{z}p_\perp^2)} \left( (1 - q\gamma^1\gamma^2 \frac{\sin z}{\cos z})(m - (\gamma p)_\parallel) - \frac{(\gamma p)_\perp}{\cos^2 z} \right). \quad (95)$$

To take the limit  $B \rightarrow \infty$  one must first make a Wick rotation  $s_1 = -iy_1$ . Then,  $\sin z = -i \sinh |e|By_1$ ,  $\cos z = \cosh |e|By_1$  and

$$\begin{aligned} -iG_0(p, B) &= -i \int_0^{i\infty} dy_1 e^{-y_1(m^2 - i\epsilon + p_\parallel^2 + \frac{-i \tanh |e|By_1}{-i|e|By_1} p_\perp^2)} \left( (1 - q\gamma^1\gamma^2 \frac{-i \sinh |e|By_1}{\cosh |e|By_1})(m - (\gamma p)_\parallel) - \frac{(\gamma p)_\perp}{\cosh^2 |e|By_1} \right) \\ &= -i \int_0^{i\infty} dy_1 e^{-y_1(m^2 - i\epsilon + p_\parallel^2 + \frac{\tanh |e|By_1}{|e|By_1} p_\perp^2)} \left( (1 + iq\gamma^1\gamma^2 \frac{\sinh |e|By_1}{\cosh |e|By_1})(m - (\gamma p)_\parallel) - \frac{(\gamma p)_\perp}{\cosh^2 |e|By_1} \right). \end{aligned} \quad (96)$$

Then,  $\int_0^{i\infty} + \int_{1/4 \text{ circle}} + \int_\infty^0 dy_1 = \sum \text{residues}$ . If we suppose that  $\int_{1/4 \text{ circle}} = 0$  and that  $\sum \text{residues} = 0$ ,  $\int_0^{i\infty} dy_1 = \int_\infty^0 dy_1$  and

$$-iG_0(p, B) = -i \int_0^\infty dy_1 e^{-y_1(m^2 - i\epsilon + p_\parallel^2 + \frac{\tanh |e|By_1}{|e|By_1} p_\perp^2)} \left( (1 + iq\gamma^1\gamma^2 \frac{\sinh |e|By_1}{\cosh |e|By_1})(m - (\gamma p)_\parallel) - \frac{(\gamma p)_\perp}{\cosh^2 |e|By_1} \right), \quad (97)$$

on which we can now take the limit  $B \rightarrow \infty$ .

$$-iG_0(p, B) \xrightarrow{B \rightarrow \infty} -ie^{-p_\perp^2/|e|B} \int_0^\infty dy_1 e^{-y_1(m^2 + p_\parallel^2)} \left( (1 + iq\gamma^1\gamma^2)(m - (\gamma p)_\parallel) \right). \quad (98)$$

This is the usual result (54) for  $G^{n=0}(p, B)$  since  $q = -1$ .

If we had used instead eq. (2.47b) of [12], in which  $e < 0$ , we would have got the wrong projector  $1 + i\gamma^1\gamma^2$ , while, with their conventions, the wave function of the LLL is the same. The exponentials  $e^{\pm iz\sigma^3}$  of [12], which should in reality be  $e^{\pm iqz\sigma^3}$  with  $q = -1$ . This is one of the rare examples in QED where the sign of the electric charge matters.

### B Demonstration of (42)

In (41) it is interesting to expand  $e^{i\beta}$  into  $\cos \beta + i \sin \beta$  and to use the expressions (29) of  $\cos \beta$  and  $\sin \beta$  to cast  $\delta m$  in the form

$$\begin{aligned} \hat{I} &= \int_0^\infty \frac{ds}{s} \int_0^1 \frac{du}{\sqrt{1-u}} e^{-iuy \frac{m^2}{|e|B}} \left[ \frac{e^{-iqy} ((1-u) \cos y + u \sin y/y + i(1-u) \sin y)}{\Delta(u, y)} (1 + ue^{2iqy}) - (1-u) \right] \\ &= \int_0^\infty \frac{ds}{s} \int_0^1 \frac{du}{\sqrt{1-u}} e^{-iuy \frac{m^2}{|e|B}} \left[ \frac{1-u + u \sin y/y e^{-iqy}}{\Delta(u, y)} (1 + ue^{2iqy}) - (1-u) \right] \end{aligned} \quad (99)$$

then to notice that  $\Delta(u, y) = (1 - u + u \frac{\sin y}{y} e^{+iqy})(1 - u + u \frac{\sin y}{y} e^{-iqy})$  to simplify the previous expression into

$$\hat{I} = \int_0^\infty \frac{ds}{s} \int_0^1 \frac{du}{\sqrt{1-u}} e^{-iuy \frac{m^2}{|e|B}} \left[ \frac{1 + u e^{2iqy}}{1 - u + u \frac{\sin y}{y} e^{+iqy}} - (1 + u) \right] \quad (100)$$

After the change of variables (we shall come back later to this change of variables which introduces in particular a dependence of the counterterm on  $L$ )

$$(u, s) \rightarrow (u, y = |e|Bsu) \Rightarrow \frac{du ds}{s} = \frac{du dy}{y} = du \frac{d(qy)}{qy}, \quad (101)$$

it becomes

$$\hat{I} = \int_0^{q\infty} \frac{d(qy)}{qy} \int_0^1 \frac{du}{\sqrt{1-u}} e^{-iuy \frac{m^2}{|e|B}} \left[ \frac{1 + u e^{2iqy}}{1 - u + u \frac{\sin qy}{qy} e^{+iqy}} - (1 + u) \right] \quad (102)$$

Noticing that, since  $q = -1$ ,  $\sin y/y = \sin qy/qy$  and expressing  $\sin qy$  in the denominator in terms of complex exponentials gives

$$\hat{I}(L) = \int_0^{q\infty} d(qy) \int_0^1 \frac{du}{\sqrt{1-u}} e^{-iuy \frac{m^2}{|e|B}} \left[ \frac{2i(1 + u e^{2iqy})}{2iqy(1 - u) + u(e^{2iqy} - 1)} - \frac{1 + u}{qy} \right]. \quad (103)$$

Going to  $t = -iqy$  yields

$$\hat{I}(L) = \int_0^{-iq\infty} dt \int_0^1 \frac{du}{\sqrt{1-u}} e^{uqt \frac{m^2}{|e|B}} \left[ \frac{2(1 + u e^{-2t})}{2t(1 - u) + u(1 - e^{-2t})} - \frac{1 + u}{t} \right]. \quad (104)$$

Last, we change to  $z = ut \Rightarrow du dt = \frac{du dz}{u}$  and get

$$\begin{aligned} \hat{I}(L) &= \int_0^{-iq\infty} dz \int_0^1 \frac{du}{\sqrt{1-u}} e^{zq \frac{m^2}{|e|B}} \left[ \frac{2(1 + u e^{-2z/u})}{2z(1 - u) + u^2(1 - e^{-2z/u})} - \frac{1 + u}{z} \right] \\ &\stackrel{q=-1}{=} \int_0^{+i\infty} dz \int_0^1 \frac{du}{\sqrt{1-u}} e^{-z \frac{m^2}{|e|B}} \left[ \frac{2(1 + u e^{-2z/u})}{2z(1 - u) + u^2(1 - e^{-2z/u})} - \frac{1 + u}{z} \right]. \end{aligned} \quad (105)$$

The last operation to perform is a Wick rotation.  $\int_0^{+i\infty} + \int_{1/4 \text{ infinite circle}} + \int_\infty^0 = 2i\pi \sum \text{residues}$ . Because of  $e^{-z \frac{m^2}{|e|B}}$  the contribution on the infinite 1/4 circle is vanishing. That the residue at  $z = 0$  vanishes is trivial as long as  $u$  is not strictly vanishing. The expansion of the terms between square brackets in (105) at  $z \rightarrow 0$  writes indeed  $u - 1 + (-\frac{5}{3} + \frac{4}{3u} + u)z + (-\frac{7}{3} - \frac{1}{u^2} + \frac{7}{3u} + u)z^2 + \mathcal{O}(z^3)$ , which seemingly displays poles at  $u = 0$ . However, without expanding, it also writes, then,  $\frac{2}{2z} - \frac{1}{z} = 0$ , which shows that the poles at  $u = 0$  in the expansion at  $z \rightarrow 0$  are fake and that the residue at  $z = 0$  always vanishes. Other poles (we now consider eq. (104)) can only occur when the denominator of the first term inside brackets vanishes. That the corresponding  $u_{\text{pole}} = \frac{2t}{2t + e^{-2t} - 1}$  should be real constrains them to occur at  $t \rightarrow in\pi, n \in \mathbb{N} > 0$  and  $u \rightarrow 1$ . In general, they satisfy  $2t(1 - u) + u(1 - e^{-2t}) = 0$  which, setting  $t = t_1 + it_2, t_1, t_2 \in \mathbb{R}$ , yields the 2 equations  $e^{-2t_1} \cos 2t_2 = 1 + 2\eta t_1, e^{-2t_1} \sin 2t_2 = -2\eta t_2, \eta = \frac{1-u}{u} \geq 0$ . Since  $t_1 \rightarrow 0$ , one may expand the first relation at this limit, which yields  $\cos 2t_2 - 1 = 2t_1(\eta + \cos 2t_2)$ . As  $t_2 \rightarrow n\pi, \cos 2t_2 > 0$  and  $\cos 2t_2 - 1 < 0$ , which, since  $\eta > 0$ , constrains  $t_1$  to stay negative<sup>14</sup>. Therefore, the potentially troublesome poles lie in reality on the left of the imaginary  $t$  axis along which the integration is done and should not be accounted for when doing a Wick rotation. After changing  $u$  into  $v$  to work from now onwards with the same notation as in [14] and ease the comparison, one gets (42).

<sup>14</sup>The 2nd relation then tells us that  $\sin 2t_2 < 0$ , which means that the poles correspond to  $t_2 = n\pi - \epsilon, \epsilon > 0$ .



## C Demonstration of (75)

In (74), we go, like before (see (101)), to the variables  $u, y = |e|Bsu$  such that  $du ds = \frac{du dy}{|e|Bu}$  and get

$$\begin{aligned}\delta m_{LLL}^0 &= \frac{\alpha m}{\pi} \int_0^\infty dy \int_0^1 \frac{du}{|e|B u \sqrt{1-u}} \left[ \cos \tan^{-1} \frac{y(1-u)}{u + iy(1-u)} \right] \frac{i|e|Bu}{u + iy(1-u)} e^{-iyum^2/|e|B} e^{-i \tan^{-1} \frac{y(1-u)}{u + iy(1-u)}} + c.t._1 \\ &= \frac{\alpha m}{\pi} \int_0^\infty dy \int_0^1 \frac{du}{\sqrt{1-u}} \left[ \cos \tan^{-1} \frac{y(1-u)}{u + iy(1-u)} \right] \frac{i}{u + iy(1-u)} e^{-iyum^2/|e|B} e^{-i \tan^{-1} \frac{y(1-u)}{u + iy(1-u)}} + c.t._1\end{aligned}\quad (106)$$

Next, we go to  $t = iy$ . This yields

$$\begin{aligned}\delta m_{LLL}^0 &= \frac{\alpha m}{\pi} \int_0^{+i\infty} dt \int_0^1 \frac{du}{\sqrt{1-u}} \left[ \cos \tan^{-1} \frac{-it(1-u)}{u + t(1-u)} \right] \frac{1}{u + t(1-u)} e^{-tum^2/|e|B} e^{-i \tan^{-1} \frac{-it(1-u)}{u + t(1-u)}} + c.t._1 \\ &= \frac{\alpha m}{\pi} \int_0^{+i\infty} dt \int_0^1 \frac{du}{\sqrt{1-u}} \left[ \cos \tan^{-1} \frac{-it(1-u)}{u + t(1-u)} \right] \frac{1}{u + t(1-u)} e^{-tum^2/|e|B} e^{-i \tan^{-1} \frac{-it(1-u)}{u + t(1-u)}} + c.t._1\end{aligned}\quad (107)$$

Last, as before, we go to  $z = ut \Rightarrow du dt = \frac{du dz}{u}$ .

$$\delta m_{LLL}^0 = \frac{\alpha m}{\pi} \int_0^{i\infty} dz \int_0^1 \frac{du}{\sqrt{1-u}} \left[ \cos \tan^{-1} \frac{-iz(1-u)}{u^2 + z(1-u)} \right] \frac{1}{u^2 + z(1-u)} e^{-zm^2/|e|B} e^{-i \tan^{-1} \frac{-iz(1-u)}{u^2 + z(1-u)}} + c.t._1\quad (108)$$

One has

$$\tan^{-1}(-ix) = (-i) \tanh^{-1} x, \quad \cos(-ix) = \cosh x\quad (109)$$

therefore

$$\delta m_{LLL}^0 = \frac{\alpha m}{\pi} \int_0^{i\infty} dz \int_0^1 \frac{du}{\sqrt{1-u}} \left[ \cosh \tanh^{-1} \frac{z(1-u)}{u^2 + z(1-u)} \right] \frac{1}{u^2 + z(1-u)} e^{-zm^2/|e|B} e^{-\tanh^{-1} \frac{z(1-u)}{u^2 + z(1-u)}} + c.t._1\quad (110)$$

As long as  $m \neq 0$ , the  $e^{-zm^2/|e|B}$  and the  $e^{-\tanh^{-1} \frac{z(1-u)}{u^2 + z(1-u)}}$  ensure the convergence on the infinite 1/4 circle such that, supposing that no pole in the 1/4 quadrant causes problems, one may do a Wick rotation, which yields (75).

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