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Differential geometry/Mathematical problems in mechanics

$W^{2,p}$-estimates for surfaces in terms of their two fundamental forms

Estimations dans $W^{2,p}$ pour des surfaces à partir de leurs deux formes fondamentales

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ABSTRACT

Let $p > 2$. We show how the fundamental theorem of surface theory for surfaces of class $W^{2,p}_{\text{loc}}(\omega)$ over a simply-connected open subset of $\mathbb{R}^2$ established in 2005 by S. Mardare can be extended to surfaces of class $W^{2,p}(\omega)$ when $\omega$ is in addition bounded and has a Lipschitz-continuous boundary. Then we establish a nonlinear Korn inequality for surfaces of class $W^{2,p}(\omega)$. Finally, we show that the mapping that defines in this fashion a surface of class $W^{2,p}(\omega)$, unique up to proper isometries of $\mathbb{E}^3$, in terms of its two fundamental forms is locally Lipschitz-continuous.

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RÉSUMÉ

Soit $p > 2$. Nous montrons comment le théorème fondamental de la théorie des surfaces de classe $W^{2,p}_{\text{loc}}(\omega)$ sur un ouvert simplement connexe $\omega$ de $\mathbb{R}^2$ établi par S. Mardare en 2005 peut être étendu à des surfaces de classe $W^{2,p}(\omega)$ lorsque $\omega$ est de plus borné et de frontière lipschitzienne. Ensuite, nous établissons une inégalité de Korn non linéaire pour des surfaces de classe $W^{2,p}(\omega)$. Nous établissons enfin que l’application qui définit une surface de classe $W^{2,p}(\omega)$ à une isométrie propre de $\mathbb{E}^3$ près en fonction de ses deux formes fondamentales est localement lipschitzienne.

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1. Preliminaries

In what follows, Greek indices and exponents, except $\varepsilon$ and $\delta$, vary in the set $\{1, 2\}$. Latin indices vary in the set $\{1, 2, 3\}$, and the summation convention for repeated indices and exponents is used. Boldface letters denote vector and matrix fields.

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The three-dimensional Euclidean space is denoted $\mathbb{E}^3$. The inner product, exterior product, and norm, in $\mathbb{E}^3$ are respectively denoted $\cdot$, $\wedge$, and $|\cdot|$. The set of all proper isometries of $\mathbb{E}^3$ is denoted and defined by

$$\text{Isom}_+ (\mathbb{E}^3) := \{ r : \mathbb{E}^3 \to \mathbb{E}^3, \ r(x) = Rx + a, \ x \in \mathbb{E}^3; \ R \in O^+_3, \ a \in \mathbb{E}^3 \}.$$  

where $O^+_3$ denotes the set of all real $3 \times 3$ proper orthogonal matrices.

**Remark 1.** The set $\text{Isom}_+ (\mathbb{E}^3)$ is in effect a smooth submanifold of dimension six of the space of all $3 \times 3$ real matrices and its tangent space at the identity mapping $\text{id} \in \text{Isom}_+ (\mathbb{E}^3)$ is the space of all "infinitesimal rigid displacements of $\mathbb{E}^3$", which is denoted and defined by

$$\text{Rig}(\mathbb{E}^3) = T_{\text{id}} \text{Isom}_+ (\mathbb{E}^3) := \{ \zeta : \mathbb{E}^3 \to \mathbb{E}^3, \ \zeta(x) = Ax + b, \ x \in \mathbb{E}^3; \ A \in \mathbb{A}^3, \ b \in \mathbb{E}^3 \},$$  

where $\mathbb{A}^3$ denotes the set of all real $3 \times 3$ antisymmetric matrices.  

Given an open subset $\omega$ of $\mathbb{R}^2$, we let $y = (y_\alpha)$ denote a generic point in $\omega$, and we let $\partial_\alpha := \partial / \partial y_\alpha$ and $\partial_{\alpha \beta} := \partial^2 / \partial y_\alpha \partial y_\beta$.

The space of distributions over an open subset $\omega$ of $\mathbb{R}^2$ is denoted $\mathcal{D}' (\omega)$. For each integer $m \geq 1$ and each real number $p \geq 1$, $\mathcal{C}^m (\omega)$ denotes the subspace of $\mathcal{C}^0 (\omega)$ of functions that possess continuous partial derivatives up to order $m$, and $W^{m,p} (\omega)$ denotes the usual Sobolev space.

The notation $L^p_{\text{loc}} (\omega)$, resp. $W^{m,p}_{\text{loc}} (\omega)$, denotes the space of functions $f : \omega \to \mathbb{R}$ such that $f|_U \in L^p (U)$, resp. $f|_U \in W^{m,p} (U)$, for all open sets $U \subset \omega$, where $f|_U$ denotes the restriction of $f$ to $U$ and the notation $U \in \omega$ means that the closure of the set $U$ is a compact subset of $\omega$. Given any finite dimensional real space $\mathcal{Y}$, the notation $L^p_{\text{loc}} (\omega; \mathcal{Y})$, resp. $W^{m,p}_{\text{loc}} (\omega; \mathcal{Y})$, denotes the space of $\mathcal{Y}$-valued fields with components in $L^p_{\text{loc}} (\omega)$, resp. $W^{m,p}_{\text{loc}} (\omega)$. Other similar notations with self-explanatory definitions will be used.

An immersion from $\omega$ into $\mathbb{E}^3$ is a smooth enough mapping $\theta : \omega \to \mathbb{E}^3$ such that the two vector fields $\partial_\alpha \theta : \omega \to \mathbb{E}^3$ are linearly independent at each point of $\omega$. Given an immersion $\theta : \omega \to \mathbb{E}^3$, define the functions

$$\hat{\alpha}_{\alpha \beta} (\theta) := \hat{\alpha}_\alpha (\theta) \cdot \hat{\alpha}_\beta (\theta) \quad \text{and} \quad \hat{\beta}_{\alpha \beta} (\theta) := \partial_\alpha \hat{\alpha}_\beta (\theta) \cdot \hat{\alpha}_3 (\theta),$$

where

$$\hat{\alpha}_\alpha (\theta) := \partial_\alpha \theta \quad \text{and} \quad \hat{\alpha}_3 (\theta) := \frac{\partial_1 \theta \wedge \partial_2 \theta}{|\partial_1 \theta \wedge \partial_2 \theta|}.$$  

The image $S = \theta (\omega)$ is thus a surface in $\mathbb{E}^3$ and the functions $\hat{\alpha}_{\alpha \beta} (\theta)$ and $\hat{\beta}_{\alpha \beta} (\theta)$ are the covariant components of the first and second fundamental forms of $S$.

The space of real $2 \times 2$ symmetric matrices is denoted $\mathbb{S}^2$; its subset formed by all positive-definite matrices is denoted $\mathbb{S}^2_+$. An open subset $\omega$ of $\mathbb{R}^2$ satisfies the uniform interior cone property if there exists a bounded open cone $V \subset \mathbb{R}^2$ such that any point $y \in \omega$ is the vertex of a cone $V_y$ congruent with $V$ and contained in $\omega$. An open subset $\omega$ of $\mathbb{R}^2$ is a domain if it is bounded and has a Lipschitz-continuous boundary.

Detailed proofs of the results announced here will be found in [4].

2. The fundamental theorem of surface theory in the spaces $W^{2,p}_{\text{loc}} (\omega)$ and $W^{2,p} (\omega)$

The fundamental theorem of surface theory, which is classically established in the spaces of continuously differentiable functions (cf., e.g., [5, Theorem 3.8.8], [1, Appendix to Chapter 4], [2, Theorems 8.16-1 and 8.17-1]), has been shown to hold in function spaces with little regularity, according to the following remarkable result, due to S. Mardare [6, Theorem 9]:

**Theorem 1.** Let $\omega$ be a simply-connected open subset of $\mathbb{R}^2$, let $p > 2$, and let a matrix field $(a_{\alpha \beta}) \in W^{1,p}_{\text{loc}} (\omega; \mathbb{S}^2)$ and a matrix field $(b_{\alpha \beta}) \in L^p_{\text{loc}} (\omega; \mathbb{S}^2)$ be given that satisfy the Gauss and Codazzi-Mainardi equations, viz.

$$R^\sigma_{\alpha \beta \tau} := \partial_\tau \Gamma^\sigma_{\alpha \beta} - \partial_\beta \Gamma^\sigma_{\alpha \tau} + \Gamma^\tau_{\alpha \beta} \Gamma^\sigma_{\tau \gamma} - \Gamma^\gamma_{\alpha \tau} \Gamma^\sigma_{\beta \gamma} - b_{\alpha \gamma} b^\gamma_{\tau} + b_{\alpha \tau} b^\tau_{\beta} = 0 \quad \text{in} \ \mathcal{D}' (\omega)$$

and

$$R^3_{\alpha \beta \tau} := \partial_\tau b_{\alpha \beta} - \partial_\beta b_{\alpha \tau} + \Gamma^\gamma_{\alpha \beta} b_{\tau \gamma} - \Gamma^\gamma_{\alpha \tau} b_{\beta \gamma} = 0 \quad \text{in} \ \mathcal{D}' (\omega),$$

where the functions $\Gamma^\sigma_{\alpha \beta} \in L^p_{\text{loc}} (\omega)$ and $b_{\alpha \beta} \in L^p_{\text{loc}} (\omega)$ are defined by

$$\Gamma^\sigma_{\alpha \beta} := \frac{1}{2} a^\sigma_{\tau \gamma} (\partial_\tau a_{\beta \gamma} + \partial_\gamma a_{\alpha \tau} - \partial_\alpha a_{\beta \gamma}) \quad \text{and} \quad b_{\alpha \beta} := a_{\alpha \tau} b_{\tau \beta}, \quad \text{where} \quad (a_{\alpha \tau}) := (a_{\alpha \beta})^{-1}.$$
Then there exists an immersion \( \theta \in W^{2,p}_{\text{loc}}(\omega; \mathbb{E}^3) \) such that
\[
\hat{a}_{\alpha\beta}(\theta) = a_{\alpha\beta} \quad \text{and} \quad \hat{b}_{\alpha\beta}(\theta) = b_{\alpha\beta} \quad \text{a.e. in } \omega.
\]

Besides, an immersion \( \psi \in W^{2,p}_{\text{loc}}(\omega; \mathbb{E}^3) \) satisfies
\[
\hat{a}_{\alpha\beta}(\psi) = \hat{a}_{\alpha\beta}(\theta) \quad \text{and} \quad \hat{b}_{\alpha\beta}(\psi) = \hat{b}_{\alpha\beta}(\theta) \quad \text{a.e. in } \omega
\]
if and only if there exists an isometry \( r \in \text{Isom}^+(\mathbb{E}^3) \) such that
\[
\psi = r \circ \theta \quad \text{in } \omega.
\]

Our first objective (Theorem 2) consists in showing that an existence and uniqueness theorem similar to Theorem 1 holds in the spaces \( W^{m,p}(\omega) \) instead of the spaces \( W^{m,p}_{\text{loc}}(\omega) \) if the open set \( \omega \) is in addition a domain.

**Theorem 2.** Let \( \omega \) be a simply-connected domain in \( \mathbb{R}^2 \), let \( p > 2 \), and let a matrix field \( (a_{\alpha\beta}) \in W^{1,p}(\omega; \mathbb{S}^2) \) and a matrix field \( (b_{\alpha\beta}) \in L^p(\omega; \mathbb{S}^2) \) be given that satisfy the equations
\[
R^\sigma_{\alpha\beta\tau} = 0 \quad \text{and} \quad R^3_{\alpha\beta\tau} = 0 \quad \text{in } D'(\omega).
\]
Then there exists an immersion \( \theta \in W^{2,p}(\omega; \mathbb{E}^3) \) such that
\[
\hat{a}_{\alpha\beta}(\theta) = a_{\alpha\beta} \quad \text{and} \quad \hat{b}_{\alpha\beta}(\theta) = b_{\alpha\beta} \quad \text{a.e. in } \omega.
\]

Besides, an immersion \( \psi \in W^{2,p}(\omega; \mathbb{E}^3) \) satisfies
\[
\hat{a}_{\alpha\beta}(\psi) = \hat{a}_{\alpha\beta}(\theta) \quad \text{and} \quad \hat{b}_{\alpha\beta}(\psi) = \hat{b}_{\alpha\beta}(\theta) \quad \text{a.e. in } \omega
\]
if and only if there exists an isometry \( r \in \text{Isom}^+(\mathbb{E}^3) \) such that
\[
\psi = r \circ \theta \quad \text{in } \omega.
\]

**Sketch of proof.** Since \( p > 2 \) and \( \omega \) is a domain, \( W^{1,p}(\omega) \) is a Banach algebra and the canonical injection from \( W^{1,p}(\omega) \) into \( C^0(\bar{\omega}) \) is continuous. Combining these two observations with the **Gauss equations**
\[
\hat{a}_{\alpha}(\theta) = 1^\sigma_{\alpha\beta} \hat{a}_{\beta}(\theta) + b_{\alpha\beta} \hat{a}_3(\theta) \quad \text{a.e. in } \omega
\]
and the relations
\[
|\hat{a}_{\alpha}(\theta)| = \sqrt{a_{\alpha\alpha}} \quad \text{(no summation on } \alpha \text{ here)} \quad \text{and} \quad |\hat{a}_3(\theta)| = 1 \quad \text{a.e. in } \omega,
\]
where \( \theta \in W^{2,p}_{\text{loc}}(\omega; \mathbb{E}^3) \) denotes the immersion found in Theorem 1 and the functions \( \Gamma^\sigma_{\alpha\beta} \) are defined as in Theorem 1 (in effect the Christoffel symbols associated with \( \theta \)), shows that the three vector fields \( \hat{a}_i(\theta) \) belong to \( L^\infty(\omega; \mathbb{E}^3) \), which in turn implies that \( \partial_\theta \theta \in L^\infty(\omega; \mathbb{E}^3) \) and \( \partial_\beta \theta \in L^p(\omega; \mathbb{E}^3) \). It is then an easy matter to conclude that \( \theta \in L^p(\omega; \mathbb{E}^3) \), hence that \( \theta \in W^{2,p}(\omega; \mathbb{E}^3) \). The uniqueness up to isometries follows immediately from Theorem 1.

**3. A nonlinear Korn inequality for surfaces of class \( W^{2,p} \)**

The second objective of this Note is to complement the existence and uniqueness result of Theorem 2 by a stability result (Theorem 3 below), showing that the distance modulo a proper isometry between two surfaces in \( W^{2,p} \)-norm is bounded by the distance between their first fundamental forms in the \( W^{1,p} \)-norm and the distance between their second fundamental forms in the \( L^p \)-norm. A notation such as \( c = c(\omega, p, \varepsilon) \) means that \( c \) is a real constant that depends on \( \omega, p \) and \( \varepsilon \).

**Theorem 3.** Let \( \omega \) be a bounded and connected open subset of \( \mathbb{R}^2 \) that satisfies the uniform interior cone property. Given any \( p > 2 \) and \( \varepsilon > 0 \), let
\[
V_\varepsilon(\omega; \mathbb{E}^3) := \{ \theta \in W^{2,p}(\omega; \mathbb{E}^3); \|\theta\|_{W^{2,p}(\omega; \mathbb{E}^3)} \leq 1/\varepsilon \quad \text{and} \quad |\partial_1 \theta \wedge \partial_2 \theta| \geq \varepsilon \quad \text{in } \omega \}.
\]

Then there exists a constant \( c = c(\omega, p, \varepsilon) \) such that
\[
\inf_{r \in \text{Isom}^+(\mathbb{E}^3)} \|\varphi - r \circ \psi\|_{W^{2,p}(\omega; \mathbb{E}^3)} \leq c \left\{ \|\hat{a}_{\alpha\beta}(\varphi) - \hat{a}_{\alpha\beta}(\psi)\|_{W^{1,p}(\omega; \mathbb{S}^2)} + \|\hat{b}_{\alpha\beta}(\varphi) - \hat{b}_{\alpha\beta}(\psi)\|_{L^p(\omega; \mathbb{S}^2)} \right\}
\]
for all \( \varphi \in V_\varepsilon(\omega; \mathbb{E}^3) \) and \( \psi \in V_\varepsilon(\omega; \mathbb{E}^3) \).
Remark 2. The above inequality can indeed be seen as a nonlinear Korn inequality for surfaces of class $W^{2,p}$, since a formal linearization (such a linearization consists first in letting in the above nonlinear inequality $\phi := \theta + \eta$ and $\psi := \theta$, where $\theta \in W^{2,p}(\omega; \mathbb{E}^3)$ is a given immersion considered as “fixed”, and $\eta \in W^{2,p}(\omega; \mathbb{E}^3)$ is an arbitrary vector field, then in canceling all the terms that depend nonlinearly on $\eta$) yields the following linear Korn inequality on the surface $S = \theta(\omega)$: There exists a constant $c_0 = c_0(\theta, \omega)$ such that (the space $\text{Rig}(\mathbb{E}^3)$ is defined in Remark 1)

$$
\inf_{\xi \in \text{Rig}(\mathbb{E}^3)} \| \eta - \xi \|_{W^{2,p}(\omega; \mathbb{E}^3)} \leq c_0 \left\{ \| (\gamma_{\alpha\beta}(\eta)) \|_{W^{1,p}(\omega; \mathbb{S}^2)} + \| (\rho_{\alpha\beta}(\eta)) \|_{L^p(\omega; \mathbb{S}^2)} \right\}
$$

for all $\eta \in W^{2,p}(\omega; \mathbb{E}^3)$, where

$$
\gamma_{\alpha\beta}(\eta) := \frac{1}{2} \left[ \hat{a}_{\alpha\beta}(\theta + \eta) - \hat{a}_{\alpha\beta}(\theta) \right] \text{lin} \quad \text{and} \quad \rho_{\alpha\beta}(\eta) := \left[ \hat{b}_{\alpha\beta}(\theta + \eta) - \hat{b}_{\alpha\beta}(\theta) \right] \text{lin}
$$

designate the linear parts with respect to $\eta$ of the tensors appearing in the right-hand side of the inequality of Theorem 3.

The proof of Theorem 3 relies on a comparison theorem between solutions to general Pfaff systems due to the first author and S. Mardare (see Theorem 3.1 and Remark 3.1 in [3] and Theorem 4.1 in [7]), which we state below only in the particular case needed here. The notations $M^3$ and $\cdot$ used in the next theorem respectively denote the space of $3 \times 3$ real matrices and the Frobenius norm in this space. The notation $(a \mid b \mid c) \in \mathbb{E}^3$ denotes the matrix in $M^3$ with column vectors $a, b, c \in \mathbb{E}^3$.

Theorem 4. Let $\omega$ be a bounded and connected open subset of $\mathbb{R}^2$ that satisfies the uniform interior cone property. Given any $p > 2$, $\varepsilon > 0$, and $y_0 \in \omega$, there exists a constant $c_1 = c_1(\omega, p, \varepsilon, y_0)$ such that

$$
\| F - \tilde{F} \|_{W^{1,p}(\omega; \mathbb{M}^3)} \leq c_1 \left( |F(y_0) - \tilde{F}(y_0)| + \sum_{\alpha} \| \Gamma_{\alpha} - \tilde{\Gamma}_{\alpha} \|_{L^p(\omega; \mathbb{M}^3)} \right)
$$

for all matrix fields $F, \tilde{F} \in W^{1,p}(\omega; \mathbb{M}^3)$ and $\Gamma_{\alpha}, \tilde{\Gamma}_{\alpha} \in L^p(\omega; \mathbb{M}^3)$ that satisfy

$$
|F(y_0)| + \sum_{\alpha} \| \Gamma_{\alpha} \|_{L^p(\omega; \mathbb{M}^3)} \leq \frac{1}{\varepsilon} \quad \text{and} \quad |\tilde{F}(y_0)| + \sum_{\alpha} \| \tilde{\Gamma}_{\alpha} \|_{L^p(\omega; \mathbb{M}^3)} \leq \frac{1}{\varepsilon},
$$

and

$$
\partial_{\alpha} F = F \Gamma_{\alpha} \quad \text{and} \quad \partial_{\alpha} \tilde{F} = \tilde{F} \tilde{\Gamma}_{\alpha} \quad \text{a.e. in } \omega.
$$

Sketch of the proof of Theorem 3. With any immersion $\phi \in W^{2,p}(\omega; \mathbb{E}^3)$, we associate: the proper isometry $r(\phi, y_0)$ of $\mathbb{E}^3$ defined by

$$
\mathbf{r}(\phi, y_0)(x) := (B^T B)^{1/2} B^{-1} (x - \phi(y_0)) \text{ for all } x \in \mathbb{E}^3,
$$

where

$$
B := \langle \hat{a}_1(\phi)(y_0) \mid \hat{a}_2(\phi)(y_0) \mid \hat{a}_3(\phi)(y_0) \rangle
$$

the immersion

$$
\theta(\phi, y_0) := r(\phi, y_0) \circ \phi \in W^{2,p}(\omega; \mathbb{E}^3);
$$

and the matrix fields

$$
F(\phi, y_0) := \langle \hat{a}_1(\theta(\phi, y_0)) \mid \hat{a}_2(\theta(\phi, y_0)) \mid \hat{a}_3(\theta(\phi, y_0)) \rangle
$$

and

$$
A(\phi) := \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Gamma_{\alpha}(\phi) := \begin{pmatrix} \Gamma_{\alpha 1}^1 & \Gamma_{\alpha 2}^1 & -b_{\alpha 1}^1 \\ \Gamma_{\alpha 1}^2 & \Gamma_{\alpha 2}^2 & -b_{\alpha 2}^1 \\ b_{\alpha 1} & b_{\alpha 2} & 0 \end{pmatrix},
$$

where

$$
a_{\alpha\beta} := \hat{a}_{\alpha\beta}(\phi), \quad b_{\alpha\beta} := \hat{b}_{\alpha\beta}(\phi), \quad b_{\alpha} := a^{\alpha\sigma} b_{\sigma\beta}, \quad (a^{\alpha\tau}) := (a_{\alpha\beta})^{-1},
$$

and

$$
\Gamma_{\alpha\beta} := \frac{1}{2} a^{\alpha\tau} (\partial_{\alpha} a_{\tau\beta} + \partial_{\beta} a_{\tau\alpha} - \partial_{\tau} a_{\alpha\beta}.
$$
These matrix fields satisfy the Pfaff system
\[ \partial_\alpha F(\varphi, y_0) = F(\varphi, y_0) \Gamma_\alpha(\varphi) \text{ a.e. in } \omega, \]
and the “initial condition”
\[ (F(\varphi, y_0))(y_0) = (A(\varphi)(y_0))^{1/2} \in \mathbb{S}^3. \]
Note in passing that the above Pfaff system is equivalent to the equations of Gauss and Weingarten associated with the immersion \( \theta(\varphi, y_0). \)

In addition, if \( \varphi \in V_\varepsilon(\omega; \mathbb{E}^3) \) for some \( \varepsilon > 0 \) (the set \( V_\varepsilon(\omega; \mathbb{E}^3) \) is defined in the statement of Theorem 3), then
\[ F(\varphi, y_0) \in W^{1, p}(\omega; \mathbb{S}^3) \text{ and } \Gamma_\alpha(\varphi) \in L^p(\omega; \mathbb{M}^3), \]
and there exists a constant \( c_1 = c_1(\omega, p, \varepsilon) \) such that
\[ \| (F(\varphi, y_0))(y_0) \| + \| \Gamma_\alpha(\varphi) \|_{L^p(\omega; \mathbb{M}^3)} \leq c_1. \]
This allows us to apply Theorem 4 and to deduce that there exists a constant \( c_2 = c_2(\omega, y_0, p, \varepsilon) \) such that
\[ \| F(\varphi, y_0) - F(\psi, y_0) \|_{W^{1, p}(\omega; \mathbb{S}^3)} \leq c_2 \left( \| (A(\varphi))(y_0) - (A(\psi))(y_0) \| + \sum_\alpha \| \Gamma_\alpha(\varphi) - \Gamma_\alpha(\psi) \|_{L^p(\omega; \mathbb{M}^3)} \right) \]
for all immersions \( \varphi \) and \( \psi \) that belong to the set \( V_\varepsilon(\omega; \mathbb{E}^3) \).

Next, using the expressions of the matrix fields appearing in the right-hand side of the above inequality in terms of the fundamental forms associated with the immersions \( \varphi \) and \( \psi \), we deduce after a series of straightforward, but somewhat technical, computations that there exist two constants \( c_3 = c_3(\omega, p, \varepsilon) \) and \( c_4 = c_4(\omega, p, \varepsilon) \) such that
\[ \| (A(\varphi))(y_0) - (A(\psi))(y_0) \| \leq c_3 \| (\hat{\alpha}_\alpha(\varphi) - \hat{\alpha}_\alpha(\psi)) \|_{W^{1, p}(\omega; \mathbb{S}^2)}, \]
and
\[ \| \Gamma_\alpha(\varphi) - \Gamma_\alpha(\psi) \|_{L^p(\omega; \mathbb{M}^3)} \leq c_4 \left( \| (\hat{\alpha}_\alpha(\varphi) - \hat{\alpha}_\alpha(\psi)) \|_{W^{1, p}(\omega; \mathbb{S}^2)} + \| (\hat{\beta}_\alpha(\varphi) - \hat{\beta}_\alpha(\psi)) \|_{L^p(\omega; \mathbb{S}^2)} \right). \]

Finally, the definition of the immersions \( \theta(\varphi, y_0) \) and \( \theta(\psi, y_0) \) implies that the vector field
\[ \eta := \theta(\varphi, y_0) - \theta(\psi, y_0) \in W^{2, p}(\omega; \mathbb{E}^3) \]
satisfies the Poincaré system (the notation \( [\cdot \cdot ;]_\alpha \) denotes the \( \alpha \)-th column vector of the matrix appearing between the brackets)
\[ \partial_\alpha \eta = [F(\varphi, y_0) - F(\psi, y_0)]_\alpha \text{ in } \omega \]
and the “initial condition”
\[ \eta(y_0) = 0. \]

Using an inequality of Poincaré’s type, we infer from the above system and initial condition that there exists a constant \( c_5 = c_5(\omega, p) \) such that
\[ \| \eta \|_{W^{2, p}(\omega; \mathbb{E}^3)} \leq c_5 \| F(\varphi, y_0) - F(\psi, y_0) \|_{W^{1, p}(\omega; \mathbb{M}^3)}. \]

The conclusion follows by combining the above inequalities and by noting that, thanks to the invariance under rotations of the Euclidean and Frobenius norms,
\[ \| \eta \|_{W^{2, p}(\omega; \mathbb{E}^3)} = \| \theta(\varphi, y_0) - \theta(\psi, y_0) \|_{W^{2, p}(\omega; \mathbb{E}^3)} \geq \inf_{r \in \text{Isom}_+^+(\mathbb{E}^3)} \| \varphi - r \circ \psi \|_{W^{2, p}(\omega; \mathbb{E}^3)}. \]
4. Local Lipschitz-continuity of the mapping defining a surface of class $W^{2,p}$, $p > 2$, in terms of its fundamental forms

Let $\omega$ be an open subset of $\mathbb{R}^2$. Given two symmetric matrix fields $A = (a_{\alpha\beta}) \in W^{1,p}_{\text{loc}}(\omega; \mathbb{S}^2)$ and $B = (b_{\alpha\beta}) \in L^p_{\text{loc}}(\omega; \mathbb{S}^2)$, $p > 2$, such that $A(y) \in \mathbb{S}^2_-$ for all $y \in \partial \omega$, define the distributions

$$R^\sigma_{\alpha\beta\tau}(A, B) := \partial_\tau^\sigma \Gamma^\sigma_{\alpha\beta} - \partial_\beta \Gamma^\sigma_{\alpha\tau} + \Gamma^{\sigma\gamma}_{\alpha\beta} \Gamma^\gamma_{\tau\tau} - \Gamma^{\sigma\gamma}_{\alpha\tau} \Gamma^\gamma_{\beta\tau} - b_{\alpha\beta} b^\sigma_{\tau} + b_{\alpha\tau} b^\sigma_{\beta} \in D'(\omega),$$

$$R^2_{\alpha\beta\tau}(A, B) := \partial_\tau \partial^\sigma b_{\alpha\beta} - \partial_\beta b a_{\alpha\tau} + \Gamma^{\gamma\sigma}_{\alpha\beta} b r_{\tau r} - \Gamma^{\gamma\sigma}_{\alpha\tau} b r_{\beta r} \in D'(\omega),$$

where

$$\Gamma^\sigma_{\alpha\beta} = \Gamma^\sigma_{\alpha\beta}(A) := \frac{1}{2} a^{\alpha\tau} (\partial_\tau a_{\beta r} + \partial_\beta a_{\alpha r} - \partial_\tau a_{\alpha\beta}) \in L^p_{\text{loc}}(\omega),$$

$$b^\sigma_{\beta} := \partial^\sigma b r_{\beta} \in L^p_{\text{loc}}(\omega),$$

$$\left(\Gamma^\sigma_{\alpha\beta}ight) := (a_{\alpha\beta})^{-1} \in W^{1,p}_{\text{loc}}(\omega).$$

Remark 3. The above regularity assumptions on the fields $A$ and $B$ are the minimal possible in order that the definitions of the distributions $R^1_{\alpha\beta\tau}(A, B)$ make sense: combined with the Sobolev embedding $W^{1,p}_{\text{loc}}(\omega) \subset C^0(\omega)$, they ensure that $\det A$ is a continuous positive function over $\omega$, which in turn implies that $a^{\alpha\tau} \in C^0(\omega)$ and so the products appearing in the definitions of $\Gamma^\sigma_{\alpha\beta}$ and $b^\sigma_{\beta}$ belong to $L^p_{\text{loc}}(\omega)$; this allows to define the partial derivatives of $\Gamma^\sigma_{\alpha\beta}$ and $b^\sigma_{\beta}$ appearing in the above definition of $R^1_{\alpha\beta\tau}(A, B)$ as distributions in $D'(\omega)$. \(\square\)

The third objective of this Note is to establish, as a consequence of the nonlinear Korn inequality of Theorem 3, the following "existence, uniqueness, and stability theorem" for the reconstruction of a surface from its fundamental forms in the spaces $W^{1,p}(\omega; \mathbb{S}^2)$ and $L^p(\omega; \mathbb{S}^2)$.

In Theorem 5 below, the set $W^{2,p}(\omega; \mathbb{E}^3)$ is the quotient set of the space $W^{2,p}(\omega; \mathbb{E}^3)$ by the equivalence relation between isometrically equivalent immersions, and the set $T(\omega)$ is the subset of the space $W^{1,p}(\omega; \mathbb{S}^2) \times L^p(\omega; \mathbb{S}^2)$ formed by all pairs of a positive-definite symmetric matrix field and a symmetric matrix field that satisfy together the equations of Gauss and Codazzi–Mainardi in the distributional sense. As such, the sets $W^{2,p}(\omega; \mathbb{E}^3)$ and $T(\omega)$ are metric spaces equipped respectively with the distances defined by

$$\text{dist}_{W^{2,p}(\omega; \mathbb{E}^3)}(\hat{\theta}, \hat{\psi}) := \inf_{\tilde{\theta} \in \hat{\theta}, \tilde{\psi} \in \hat{\psi}} \|\tilde{\theta} - \tilde{\psi}\|_{W^{2,p}(\omega; \mathbb{E}^3)} = \inf_{\mathbf{r} \in \text{Isom}_+(\mathbb{E}^3)} \|\mathbf{r} \circ \theta - \theta\|_{W^{2,p}(\omega; \mathbb{E}^3)}$$

for all $\hat{\theta}$ and $\hat{\psi}$ in $W^{2,p}(\omega; \mathbb{E}^3)$, and by

$$\text{dist}_{T(\omega)}((A, B), (\hat{A}, \hat{B})) := \|A - \hat{A}\|_{W^{1,p}(\omega; \mathbb{S}^2)} + \|B - \hat{B}\|_{L^p(\omega; \mathbb{S}^2)}$$

for all $(A, B)$ and $(\hat{A}, \hat{B})$ in $T(\omega)$.

Theorem 5. Let $\omega$ be a domain in $\mathbb{R}^2$. Given any $p > 2$, define the sets

$$\hat{W}^{2,p}(\omega; \mathbb{E}^3) := \{\hat{\theta} = (\mathbf{r} \circ \theta; \mathbf{r} \in \text{Isom}_+(\mathbb{E}^3)); \theta \in W^{2,p}(\omega; \mathbb{E}^3)\}$$

and

$$\hat{T}(\omega) := \{(A, B) \in W^{1,p}(\omega; \mathbb{S}^2) \times L^p(\omega; \mathbb{S}^2) ; A(y) \in \mathbb{S}^2_- \text{ at each } y \in \partial \omega, R^1_{\alpha\beta\tau}(A, B) = 0 \text{ in } D'(\omega)\}.$$

Then the following assertions are true:

(a) Two matrix fields $A = (a_{\alpha\beta})$ and $B = (b_{\alpha\beta})$ satisfy

$(A, B) \in \hat{T}(\omega)$

if and only if there exists an immersion $\theta \in W^{2,p}(\omega; \mathbb{E}^3)$ such that

$\hat{a}_{\alpha\beta}(\theta) = a_{\alpha\beta}$ in $\omega$ and $\hat{b}_{\alpha\beta}(\theta) = b_{\alpha\beta}$ a.e. in $\omega$.

(b) Two immersions $\theta \in W^{2,p}(\omega; \mathbb{E}^3)$ and $\psi \in W^{2,p}(\omega; \mathbb{E}^3)$ satisfy the relations

$\hat{a}_{\alpha\beta}(\theta) = \hat{a}_{\alpha\beta}(\psi)$ in $\omega$ and $\hat{b}_{\alpha\beta}(\theta) = \hat{b}_{\alpha\beta}(\psi)$ a.e. in $\omega.$
if and only if there exists a proper isometry $r$ of $\mathbb{E}^3$ such that

$$\psi = r \circ \theta$$

in $\omega$.

(c) The mapping defined by (a) and (b), namely

$$G : (A, B) \in T(\omega) \to G((A, B)) : = \hat{\theta} \in W^{2,p}(\omega; \mathbb{E}^3),$$

where $\theta \in W^{2,p}(\omega; \mathbb{E}^3)$ is any immersion that satisfies

$$(\hat{a}_{\alpha\beta}(\theta)) = A \text{ and } (\hat{b}_{\alpha\beta}(\theta)) = B \text{ a.e. in } \omega,$$

is locally Lipschitz-continuous. □

Sketch of proof. Parts (a) and (b) are just a re-statement of Theorem 2. Otherwise, the rest of the proof follows a strategy introduced by the first author and S. Mardare in [3]. More precisely, part (c) of Theorem 5 is deduced from Theorem 3 as follows.

On the one hand, the Sobolev embedding $W^{1,p}(\omega_\delta) \subset C^0(\overline{\omega})$ implies that, given any $(A, B) \in T(\omega)$, there exists $\delta = \delta(A, B) > 0$ such that the set

$$T_\delta(\omega) := \left\{ (\tilde{A}, \tilde{B}) \in T(\omega) ; \det \tilde{A} \geq \delta \text{ in } \omega, \|\tilde{A}\|_{W^{1,p}(\omega; \mathbb{S}^2)} \leq 1/\delta, \text{ and } \|\tilde{B}\|_{L^p(\omega; \mathbb{S}^2)} \leq 1/\delta \right\}$$

is a neighborhood of $(A, B)$ in the metric space $T(\omega)$. It also implies that

$$T(\omega) = \bigcup_{\delta > 0} T_\delta(\omega).$$

Besides, for each $\delta > 0$, there exists $\varepsilon(\delta) > 0$ such that

$$G(T_\delta(\omega)) \subset \{ \hat{\theta} \in W^{2,p}(\omega; \mathbb{E}^3) ; \theta \in V_{\varepsilon(\delta)}(\omega; \mathbb{E}^3) \},$$

where $G$ denotes the mapping defined in part (c) of the statement of the theorem and $V_{\varepsilon(\delta)}(\omega; \mathbb{E}^3)$ is defined as in Theorem 3.

On the other hand, Theorem 3 implies that there exists a constant $c = c(\omega, p, \varepsilon(\delta))$ such that

$$\inf_{\mathsf{resom}_1(\mathbb{E}^3)} \|\varphi - r \circ \psi\|_{W^{2,p}(\omega; \mathbb{E}^3)} \leq c \left\| (\hat{a}_{\alpha\beta}(\varphi) - \hat{a}_{\alpha\beta}(\psi)) \right\|_{W^{1,p}(\omega; \mathbb{S}^2)} + \left\| (\hat{b}_{\alpha\beta}(\varphi) - \hat{b}_{\alpha\beta}(\psi)) \right\|_{L^p(\omega; \mathbb{S}^2)}$$

for all mappings $\varphi \in V_{\varepsilon(\delta)}(\omega; \mathbb{E}^3)$ and $\psi \in V_{\varepsilon(\delta)}(\omega; \mathbb{E}^3)$ (note that Theorem 3 can be applied under the assumptions of Theorem 5 since a domain satisfies the uniform interior cone property).

We then infer from the observations above that, given any mappings $\varphi \in V_{\varepsilon(\delta)}(\omega; \mathbb{E}^3)$ and $\tilde{\varphi} \in V_{\varepsilon(\delta)}(\omega; \mathbb{E}^3)$ such that $\tilde{\varphi} = G(A, B)$ and $\tilde{\varphi} = G(\tilde{A}, \tilde{B})$ for some $(A, B) \in T_\delta(\omega)$ and $(\tilde{A}, \tilde{B}) \in T_\delta(\omega)$,

$$\text{dist}_{W^{2,p}(\omega; \mathbb{E}^3)}(\tilde{\varphi}, \tilde{\varphi}) \leq c \text{ dist}_{T(\omega)}((A, B), (\tilde{A}, \tilde{B})).$$

This shows that the restriction of the mapping $G$ to the set $T_\delta(\omega)$ is Lipschitz-continuous. □

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References