A Fuss-type family of positive definite sequences
Wojciech Mlotkowski, Karol A. Penson

To cite this version:
Wojciech Mlotkowski, Karol A. Penson. A Fuss-type family of positive definite sequences. Colloquium Mathematicum, 2018, 151 (2), pp.289 - 304. 10.4064/cm6894-2-2017. hal-01744071

HAL Id: hal-01744071
https://hal.sorbonne-universite.fr/hal-01744071
Submitted on 27 Mar 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution 4.0 International License
A FUSS-TYPE FAMILY OF POSITIVE DEFINITE SEQUENCES

BY

WOJCIECH MŁOTKOWSKI (Wrocław) and KAROL A. PENSON (Paris)

Abstract. We study a two-parameter family $a_n(p,t)$ of deformations of the Fuss numbers. We show a sufficient condition for positive definiteness of $a_n(p,t)$ and prove that some of the corresponding probability measures are infinitely divisible with respect to the additive free convolution.

1. Introduction. The aim of the paper is to study a two-parameter family of sequences

$$a_n(p,t) := \binom{np}{n} \binom{n(2p - t - pt) + 2}{(np - n + 1)(np - n + 2)}^t, \quad p, t \in \mathbb{R}$$

which can be regarded as deformation of the Fuss numbers. Assuming that $p \geq 0$ we prove that the sequence $a_n(p,t)$ is positive definite if and only if $p \geq 1$ and $g(p) \leq t \leq 2p/(p + 1)$, where $g(p)$ is defined by (3.9). We conjecture that the assumption $p \geq 0$ is redundant.

The case $t = 2p/(p + 1)$ is particularly interesting due to connections with the work [6] of M. Bousquet-Mélou and G. Schaeffer. They introduced the notion of constellation as a tool for studying factorization problems in symmetric groups. For $p \geq 2$ a $p$-constellation is a 2-cell decomposition of the oriented sphere into vertices, edges and faces, with faces colored black and white in such a way that:

- all faces adjacent to a given white face are black and vice versa,
- the degree of any black face is $p$,
- the degree of any white face is a multiple of $p$.

A constellation is called rooted if one of the edges is distinguished.

The number of rooted $p$-constellations formed of $n$ polygons, counted up to isomorphism, is given by

$$C_p(n) := \binom{np}{n} \frac{(p + 1)p^{n-1}}{(np - n + 1)(np - n + 2)}, \quad p \geq 2, \ n \geq 1$$

2010 Mathematics Subject Classification: Primary 44A60; Secondary 33C20, 46L54.

Key words and phrases: Fuss numbers, Mellin convolution, Meijer $G$-function, free probability.

Received 11 February 2016; revised 9 February 2017.

Published online 12 January 2018.

DOI: 10.4064/cm6894-2-2017
Some of these sequences appear in the On-Line Encyclopedia of Integer Sequences (OEIS) \[27\], namely: \(C_2 = A000257\), \(C_3 = A069726\), \(C_4 = A090374\).

We will prove that the probability distribution \(\eta(p, t)\) corresponding to a positive definite sequence \(a_n(p, t)\) is absolutely continuous, except for \(\eta(1, 1) = \delta_1\), and the support of \(\eta(p, t)\) is \([0, p^p(p - 1)^{1-p}]\). The density function will be denoted \(f_{p,t}(x)\). For \(p = 2\) and \(p = 3\) we compute the \(R\)-transform of \(\eta(p, t)\). We prove that \(\eta(2, t)\) (resp. \(\eta(3, t)\)) is infinitely divisible with respect to the additive free convolution if and only if \(1 \leq t \leq 4/3\) (resp. \(1/2 \leq t \leq 3/2\)). The general problem of determining for which parameters \(p, t\) the distribution \(\eta(p, t)\) is infinitely divisible with respect to the additive free convolution remains open.

Finally, let us record some other sequences from OEIS which are related to this work: A005807: \(2a_n(2, 1/2)\) (sums of adjacent Catalan numbers), A007226: \(2a_n(3, 1/2)\) (studied in [15]), A007054: \(3a_n(2, 4/3)\) (super ballot numbers), A038629: \(3a_n(2, 2/3)\), A000139: \(2a_n(3, 3/2)\), A197271: \(5a_n(4, 8/5)\), A197272: \(3a_n(5, 5/3)\). In Section 4 we also encounter sequences A022558 and A220910.

2. Fuss numbers. The Fuss–Catalan numbers \(\frac{(np+1)_n}{np+1}\) have several combinatorial applications [9, 7, 2, 25, 6, 24]. They count for example:
1. the number of ways of dividing a convex polygon with \(n(p-1)+2\) vertices into \(n\) disjoint \((p+1)\)-gons by means of nonintersecting diagonals,
2. the number of sequences \((a_1, \ldots, a_{np})\), where \(a_i \in \{1, 1-p\}\), with all partial sums \(a_1 + \cdots + a_k\) nonnegative and with \(a_1 + \cdots + a_{np} = 0\),
3. the number of noncrossing partitions \(\pi\) of \(\{1, 2, \ldots, n(p-1)\}\), such that \(p-1\) divides the cardinality of every block of \(\pi\),
4. the number of \(p\)-cacti formed of \(n\) polygons [6].

The generating function

\[
B_p(z) := \sum_{n=0}^{\infty} \binom{np+1}{n} \frac{z^n}{np+1}
\]

satisfies

\[
B_p(z) = 1 + zB_p(z)^p.
\]

Recall also Lambert’s formula for the Taylor expansion of the powers of \(B_p(z)\):

\[
B_p(z)^r = \sum_{n=0}^{\infty} \binom{np+r}{n} \frac{rz^n}{np+r}.
\]

These formulas remain true for \(p, r \in \mathbb{R}\) and the coefficients \(\binom{np+r}{n} \frac{r}{np+r}\).
POSITIVE DEFINITE SEQUENCES 291

(understood to be 1 for \( n = 0 \) and \( \frac{r}{n!} \prod_{i=1}^{n-1} (np + r - i) \) for \( n \geq 1 \) are called two-parameter Fuss numbers or Raney numbers \([9, 13, 22, 12, 8]\).

In some cases the function \( \mathcal{B}_p \) can be written explicitly, for example

\[
\mathcal{B}_2(z) = \frac{2}{1 + \sqrt{1 - 4z}} = \frac{1 - \sqrt{1 - 4z}}{2z},
\]

\[
\mathcal{B}_3(z) = \frac{3}{3 - 4\sin^2 \alpha},
\]

\[
\mathcal{B}_{3/2}(z) = \frac{3}{(\sqrt{3}\cos \beta - \sin \beta)^2},
\]

where \( \alpha = \frac{1}{3} \arcsin(\sqrt{27z/4}) \), \( \beta = \frac{1}{3} \arcsin(3z\sqrt{3}/2) \) \([16]\).

Fuss numbers also have applications in free probability and in the theory of random matrices, as moments of the multiplicative free powers of the Marchenko–Pastur distribution \([1, 3, 13, 14, 18]\). This implies that for \( p \geq 1 \) the sequence \( (np + 1)n \) is positive definite. More generally, the sequence \( (np + r)n \) is positive definite if and only if either \( p \geq 0 \), \( 0 \leq r \leq p \), or \( p \leq 0 \), \( p - 1 \leq r \leq 0 \) or \( r = 0 \) \([11, 13, 16, 12, 8, 21]\). The case \( r = 0 \) is trivial, as it gives the sequence \((1, 0, 0, 0, \ldots)\), the moments of \( \delta_0 \). The distributions corresponding to the second case, \( p \leq 0 \), \( p - 1 \leq r \leq 0 \), are just reflections of those corresponding to \( p \geq 0 \), \( 0 \leq r \leq p \). This is a consequence of the identity

\[
\left(\frac{np + r}{n}\right) \frac{(-1)^n}{np + r} = \left(\frac{n(1-p) - r}{n}\right) \frac{-r}{n(1-p) - r}.
\]

For \( p > 1 \) and \( r > 0 \) we have the integral representation

\[
\left(\frac{np + r}{n}\right) \frac{r}{np + r} = \int_0^{c(p)} x^n W_{p,r}(x) \, dx,
\]

where \( c(p) := p^p(p - 1)^{1-p} \), and \( W_{p,r} \) is given by

\[
W_{p,r}(x) = \frac{(\sin (p - 1) \phi)^{p-r-1} \sin \phi \sin r \phi}{\pi (\sin p \phi)^{p-r}},
\]

where

\[
x = \rho(\phi) = \frac{(\sin p \phi)^p}{\sin \phi (\sin (p - 1) \phi)^{p-1}}, \quad 0 < \phi < \pi/p.
\]

This function is nonnegative if and only if \( r \leq p \) \([10, 18, 8]\).

If \( p = k/l \) is a rational number, \( 1 \leq l < k \), then \( W_{p,r} \) can be expressed in terms of the Meijer G-function \( [22, 17]\)

\[
W_{p,r}(x) = \frac{x^{r/p}}{x(p - 1)^{r+1/2} \sqrt{2k\pi}} G_{k,k}^{k,0} \left( \frac{x^l}{c(p)^l} \left\{ \alpha_1, \ldots, \alpha_k \right\}, \left\{ \beta_1, \ldots, \beta_k \right\} \right).
\]
where \( x \in (0, c(p)) \) and the parameters \( \alpha_j, \beta_j \) are given by

\[
\alpha_j = \begin{cases} 
\frac{j}{l} & \text{if } 1 \leq j \leq l, \\
\frac{r + j - l}{k - l} & \text{if } l + 1 \leq j \leq k,
\end{cases}
\]

(2.8)

\[
\beta_j = \frac{r + j - 1}{k}, \quad 1 \leq j \leq k.
\]

(2.9)

**EXAMPLES.** Let us record formulas for the functions \( W_{p,r} \) for \( p = 2, 3, 3/2 \) and \( r = 1, 2 \). In these cases \( W_{p,r} \) can be expressed as an elementary function \[21, 22, 17\]:

\[
W_{2,1}(x) = \frac{1}{2\pi} \sqrt{\frac{4 - x}{x}},
\]

(2.10)

\[
W_{2,2}(x) = \frac{1}{2\pi} \sqrt{x(4 - x)},
\]

(2.11)

where \( x \in (0, 4) \). \( W_{2,1} \) is the density of the Marchenko–Pastur distribution and \( W_{2,2} \) is Wigner’s semicircle law translated by 2. Moreover,

\[
W_{3,1}(x) = \frac{3(1 + \sqrt{1 - 4x/27})^{2/3} - (4x)^{1/3}}{3^{1/2}2\pi(4x)^{2/3}(1 + \sqrt{1 - 4x/27})^{1/3}},
\]

(2.12)

\[
W_{3,2}(x) = \frac{9(1 + \sqrt{1 - 4x/27})^{4/3} - (4x)^{2/3}}{2\pi3^{3/2}(4x)^{1/3}(1 + \sqrt{1 - 4x/27})^{2/3}},
\]

(2.13)

where \( x \in (0, 27/4) \), and

\[
W_{3/2,1}(x) = \frac{3^{1/2}(1 + \sqrt{1 - 4x^2/27})^{1/3} - (1 - \sqrt{1 - 4x^2/27})^{1/3}}{2(2x)^{1/3}\pi} + \frac{3^{1/2}(2x)^{1/3}(1 + \sqrt{1 - 4x^2/27})^{2/3} - (1 - \sqrt{1 - 4x^2/27})^{2/3}}{4\pi},
\]

(2.14)

\[
W_{3/2,2}(x) = \frac{3^{1/2}(2x)^{5/3}}{8\pi}((1 + \sqrt{1 - 4x^2/27})^{1/3} - (1 - \sqrt{1 - 4x^2/27})^{1/3}) + \frac{3^{1/2}(2x)^{1/3}(x^2 - 1)((1 + \sqrt{1 - 4x^2/27})^{2/3} - (1 - \sqrt{1 - 4x^2/27})^{2/3})}{4\pi},
\]

(2.15)

where \( x \in (0, 3\sqrt{3}/2) \). The function \( W_{3/2,2}(x) \) is not nonnegative on its domain.
3. A family of sequences. For \( p, t \in \mathbb{R} \) define \( a_n(p, t) \) as an affine combination of \( \left( \frac{np + 1}{n} \right) \frac{t}{np + 1} \) and \( \left( \frac{np + 2}{n} \right) \frac{2(1 - t)}{np + 2} \):

\[
(3.1) \quad a_n(p, t) := \left( \frac{np + 1}{n} \right) \frac{t}{np + 1} + \left( \frac{np + 2}{n} \right) \frac{2(1 - t)}{np + 2}
\]

\[
(3.2) \quad = \left( \frac{np}{n} \right) \frac{n(2p - t - pt) + 2}{(np - n + 1)(np - n + 2)},
\]

in particular \( a_0(p, t) = 1 \).

The generating function is

\[
(3.3) \quad tB_p(z) + (1 - t)B_p(z)^2 = \sum_{n=0}^{\infty} a_n(p, t) z^n.
\]

For example,

\[
tB_2(z) + (1 - t)B_2(z)^2 = \frac{1 - t + 3tz - 2z - (1 - t + tz)\sqrt{1 - 4z}}{2z^2},
\]

\[
tB_3(z) + (1 - t)B_3(z)^2 = \frac{9 - 12t\sin^2 \alpha}{(3 - 4\sin^2 \alpha)^2},
\]

\[
tB_{3/2}(z) + (1 - t)B_{3/2}(z)^2 = \frac{9 - 6t\sin^2 \beta + 6t\sqrt{3}\sin \beta \cos \beta}{(\sqrt{3}\cos \beta - \sin \beta)^4}
\]

where \( \alpha = \frac{1}{3} \arcsin(\sqrt{27z/4}) \), \( \beta = \frac{1}{3} \arcsin(3z\sqrt{3}/2) \).

We are going to study positive definiteness of \( a_n(p, t) \). First we observe

**Proposition 3.1.** If the sequence \( a_n(p, t) \) is positive definite then

\[
(3.4) \quad 2p - pt - t^2 + 3t - 3 \geq 0.
\]

In particular \( t \neq 2 \) and either \( p \leq -3 \) or \( p \geq 1 \).

**Proof.** The left hand side is just \( a_2(p, t) - a_1(p, t)^2 \). ■

**Examples.** 1. For \( p = 1 \) we have \( a_n(1, t) = 1 + nt \). Since

\[
a_2(1, t) - a_1(1, t)^2 = -(t - 1)^2,
\]

the sequence \( a_n(1, t) \) is positive definite if and only if \( t = 1 \). Note that \( a_n(1, 1) = 1 \) is the moment sequence of the one-point measure \( \delta_1 \).

2. For \( t = 2/(p + 1) \) we get

\[
a_n(p, 2/(p + 1)) = \left( \frac{np}{n} \right) \frac{2}{np - n + 2}.
\]

If \( p > 1 \) then this is a product of two positive definite sequences: \( \left( \frac{np}{n} \right) \) (see [16, 26]) and \( 2/(np - n + 2) \).
3. Similarly, for \( p > 1 \) and \( t = 2p/(p + 1) \) the sequence
\[
a_n(p, 2p/(p + 1)) = \left( \frac{np}{n} \right) \frac{2}{(np - n + 1)(np - n + 2)}.
\]
is positive definite. Note that from (1.2) we have
\[
C_p(n) = \frac{(p + 1)p^n}{2p} a_n\left( p, \frac{2p}{p + 1} \right),
\]
so for \( p \geq 1 \) the sequence \( C_p(n) \) is positive definite.

As already noted, the sequence \( a_n(p, t) \) is an affine combination of \( \left( \frac{np}{n} + 1 \right) \frac{1}{np+1} \) and \( \left( \frac{np+2}{n} \right) \frac{2}{np+2} \). The former is positive definite for \( p \geq 1 \) and the latter for \( p \geq 2 \). This implies that \( a_n(p, t) \) is positive definite for \( p \geq 2 \) and \( 0 \leq t \leq 1 \). We are going to prove something stronger. Note that if \( t_1 \leq t_2 \leq t_3 \) and the sequences \( a_n(p, t_1), a_n(p, t_3) \) are positive definite then so is \( a_n(p, t_2) \) as their convex combination.

If we assume that \( p > 1 \) then
\[
a_n(p, t) = \int_0^{c(p)} x^n f_{p,t}(x) \, dx,
\]
where
\[
f_{p,t}(x) = tW_{p,1}(x) + (1 - t)W_{p,2}(x).
\]
Then the positive definiteness of \( a_n(p, t) \) is equivalent to \( f_{p,t} \) being nonnegative on \((0, c(p))\). For example, the function
\[
f_{2, t}(x) = \frac{t + x - tx}{2\pi} \sqrt{\frac{4 - x}{x}}
\]
is nonnegative on \((0, 4)\) if and only if \( 0 \leq t \leq 4/3 \).

![Fig. 1. The density function \( f_{3/2,1/5}(x) \)](image_url)
By (2.5) we can write
\[(3.7) \quad f_{p,t}(x) = \frac{\sin^2 \phi (\sin (p-1)\phi)^{p-3}[t \sin (p-1)\phi + 2(1-t) \sin p\phi \cos \phi]}{\pi (\sin p\phi)^{p-1}}\]
for \(x\) as in (2.6). Define
\[(3.8) \quad \Psi_{p,t}(\phi) = t \sin (1 - 1/p)\phi + 2(1-t) \sin \phi \cos \phi/p
\]
\[= (2-t) \sin \phi \cos \phi/p - t \cos \phi \sin \phi/p
\]
\[= (1-t) \sin (1 + 1/p)\phi + \sin (1 - 1/p)\phi.\]

Then the sequence \(a_n(p,t)\) is positive definite if and only if \(\Psi_{p,t}(\phi) \geq 0\) for \(\phi \in [0,\pi]\). For \(p \geq 1\) set
\[(3.9) \quad g(p) := \min\{t \in \mathbb{R} : \Psi_{p,t}(\phi) \geq 0 \text{ for all } 0 < \phi < \pi\}.\]

Since \(\Psi_{p,t}(\pi) = t \sin (\pi/p)\) and \(\Psi_{p,1}(\phi) = \sin (1 - 1/p)\phi\), we have \(0 \leq g(p) \leq 1\) for all \(p \geq 1\).

**Proposition 3.2.** The function \(g\) is continuous on \([1, \infty)\), strictly decreasing on \([1, 2)\), \(g(p) = 0\) for \(p \geq 2\) and \(g(1) = 1\), \(g(3/2) = 1/5\).

**Proof.** For \(p = 1\) we have \(\Psi_{1,t}(\phi) = (1-t) \sin 2\phi\), which implies \(g(1) = 1\). If \(p \geq 2\) then \(\Psi_{p,0}(\phi) = 2 \sin \phi \cos \phi/p\) is nonnegative for \(\phi \in [0,\pi]\), which yields \(g(p) = 0\).

Now observe that for fixed \(t, \phi\) with \(0 \leq t \leq 1\), \(0 < \phi \leq \pi\), the function \(p \mapsto \Psi_{p,t}(\phi)\) is strictly increasing on \([1,2]\). Indeed, we can write
\[\Psi_{p,t}(\phi) = 2(1-t) \sin \phi \cos \phi/p + t \sin (\phi - \phi/p),\]
and if \(0 < \phi \leq \pi\) then both the summands are increasing with \(p \in [1,2]\). This implies that \(g(p)\) is strictly decreasing on \([1,2]\).

To prove continuity of \(g\) assume that \(1 \leq p_1 < p_2 \leq 2\) and set \(t_1 := g(p_1)\) and \(t_2 := g(p_2)\). Then \(t_1 > t_2\), \(\Psi_{p_1,t_1}(\phi) \geq 0\) for all \(\phi \in [0,\pi]\) and there is \(\phi_1\) with \(p_1\pi/(1 + p_1) < \phi_1 < \pi\) such that \(\Psi_{p_1,t_1}(\phi_1) = 0\). Then \(\Psi_{p_2,t_1}(\phi) > 0\) for all \(\phi \in (0,\pi]\). From the third expression in (3.8) we have
\[-c_1 := \sin (1 + 1/p_1)\phi_1 < 0.\]
If we assume that \((p_2 - p_1)\phi_1 < c_1/2\) then
\[|\sin (1 + 1/p_1)\phi_1 - \sin (1 + 1/p_2)\phi_1| \leq (1/p_1 - 1/p_2)\phi_1 < c_1/2,\]
and consequently \(\sin (1 + 1/p_2)\phi_1 < -c_1/2\).

If we take \(t\) with \(0 \leq t < t_1\), then
\[\Psi_{p_2,t}(\phi_1) = \Psi_{p_2,t}(\phi_1) - \Psi_{p_1,t_1}(\phi_1)
\]
\[= (1 - t_1)(\sin (1 + 1/p_2)\phi_1 - \sin (1 + 1/p_1)\phi_1)
\]
\[+ (\sin (1 - 1/p_2)\phi_1 - \sin (1 - 1/p_1)\phi_1) + (t_1 - t) \sin (1 + 1/p_2)\phi_1
\]
\[\leq (2 - t_1)(p_2 - p_1)\phi_1 - (t_1 - t)c_1/2.\]
Hence, if 

\[(2 - t_1)(p_2 - p_1)\phi_1 < (t_1 - t)c_1/2\]

then \(\Psi_{p_2,t}(\phi_1) < 0\). This implies that

\[g(p_1) - g(p_2) = t_1 - t_2 \leq 2(2 - t_1)(p_2 - p_1)\phi_1/c_1,\]

and proves continuity of \(g\).

For \(p = 3/2\) we can write

\[\Psi_{3/2,t}(\phi) = \frac{\sin \phi/3}{4}[(1 - t)(5 - 8\sin^2 \phi/3)^2 + 5t - 1].\]

Note that \(\sqrt{5/8} < \sqrt{3/2} = \sin \pi/3\), so, assuming that \(0 \leq t \leq 1\), \(\Psi_{3/2,t}\) attains its minimum on \([0, \pi]\) at \(\phi = 3\arcsin \sqrt{5/8}\). This yields \(g(3/2) = 1/5\).

Now we are able to describe the domain of positive definiteness of the sequence \(a_n(p,t)\) (see Fig. 2). The density function for the particular case \(p = 3/2, t = 1/5\) is illustrated in Fig. 1.

\[\text{Fig. 2. Domain of positive definiteness of the sequence } a_n(p,t)\]

**Theorem 3.3.** Suppose that \(p \geq 0\). Then the sequence \(a_n(p,t)\) is positive definite if and only if \(p \geq 1\) and

\[(3.10) \quad g(p) \leq t \leq \frac{2p}{1 + p}.\]

**Proof.** Fix \(p \geq 1\). By the definition of \(g(p)\) the sequence \(a_n(p,t)\) is positive definite for \(t = g(p)\) and not positive definite for \(t < g(p)\).

We have already observed that for \(p \geq 1\) the sequence \(a_n(p,2p/(p + 1))\) is positive definite. If \(t > 2p/(p + 1)\) then \(n(2p - t - pt) + 2 < 0\), and consequently \(a_n(p,t) < 0\) for all \(n\) sufficiently large. Alternatively, we have \(\Psi_{p,t}'(0) = 2p - pt - t < 0\) in this case, which implies \(\Psi_{p,t}(x) < 0\) for some \(x \in (0, \pi/p)\).
4. Free transforms. Throughout this section we assume that \( p \geq 1 \) and the sequence \( a_n(p, t) \) is positive definite, i.e. \( g(p) \leq t \leq 2p/(p+1) \). Denote by \( \eta(p, t) \) the corresponding distribution, i.e. \( \eta(1, 1) = \delta_1 \) and \( \eta(p, t) = f_{p,t}(x) \, dx \) on \([0, p^p(p - 1)^{1-p}]\) for \( p > 1 \). We are going to study relations of these measures with free probability.

Recall that for a compactly supported probability measure \( \mu \) on \( \mathbb{R} \) with the moment generating function

\[
M_{\mu}(z) := \sum_{n=0}^{\infty} z^n \int_{\mathbb{R}} x^n \, d\mu(x) = \int_{\mathbb{R}} \frac{1}{1-xz} \, d\mu(x),
\]

the \( S \) - and \( R \)-transforms are defined by

\[
M_{\mu}\left(\frac{z}{1+z} S_{\mu}(z)\right) = 1 + z, \tag{4.2}
\]

\[
1 + R_{\mu}(z M_{\mu}(z)) = M_{\mu}(z). \tag{4.3}
\]

Moreover, we have the relation

\[
R_{\mu}(z S_{\mu}(z)) = z. \tag{4.4}
\]

The coefficients \( r_n(\mu) \) in the Taylor expansion \( R_{\mu}(z) = \sum_{n=1}^{\infty} r_n(\mu)z^n \) are called the free cumulants of \( \mu \). It is known that \( \mu \) is infinitely divisible with respect to the additive free convolution if and only if the sequence \( \{r_{n+2}(\mu)\}_{n=0}^{\infty} \) is positive definite [28, 19].

For the distributions \( \eta(p, t) \) we have

\[
M_{\eta(p,t)}(z) := \sum_{n=0}^{\infty} a_n(p, t)z^n = tB_p(z) + (1-t)B_p(z)^2.
\]

Now we are going to compute the \( S \)-transform of \( \eta(p, t) \).

**PROPOSITION 4.1.** For \( p > 1 \) and \( g(p) \leq t \leq 2p/(p+1) \) we have

\[
S_{\eta(p,t)}(w) = (2 + 2w)^{1-p} \frac{(2-t)^2 + 4(1-t)w + t}{\sqrt{(2-t)^2 + 4(1-t)w + 2 - t}}. \tag{4.5}
\]

**Proof.** From (2.2) we can derive the relation

\[
B_p(z(1+z)^{-p}) = 1 + z
\]

(see [13]). Therefore

\[
M_{\eta(p,t)}(z(1+z)^{-p}) = t(1+z) + (1-t)(1+z)^2.
\]

If we substitute

\[
t(1+z) + (1-t)(1+z)^2 = 1 + w
\]

then

\[
z = \frac{\sqrt{(2-t)^2 + 4(1-t)w + 2 - t}}{2(1-t)} = \frac{2w}{\sqrt{(2-t)^2 + 4(1-t)w + 2 - t}}
\]
and
\[ 1 + z = \frac{\sqrt{(2-t)^2 + 4(1-t)w} - t}{2(1-t)} = \frac{2(1+w)}{\sqrt{(2-t)^2 + 4(1-t)w} + t}, \]

which combined with (4.2) yields (4.5).

Now we are going to compute the \( R \)-transform of \( \eta(p,t) \) for \( p = 2 \) and \( p = 3 \). We will denote \( r_n(p,t) := r_n(\eta(p,t)) \).

**4.1. The case \( p = 2 \).** The density function \( f_{2,t} \) is given by (3.6), \( 0 \leq t \leq 4/3 \). From (4.5) we can compute the \( R \)-transform for \( p = 2 \):

**Proposition 4.2.** \( R_{\eta(2,1)} = z/(1-z) \), and for \( t \neq 1 \),
\[
R_{\eta(2,t)}(z) = \frac{1 - t - 2z + 3tz - z^2 + (t - 1 - z)\sqrt{1 + z(2-4t) + z^2}}{2(t-1)}.
\]

Moreover, \( \eta(2,t) \) is infinitely divisible with respect to the additive free convolution if and only if either \( t = 0 \) or \( 1 \leq t \leq 4/3 \).

**Proof.** First we find \( R_{\eta(2,t)}(z) \) by solving the equation
\[
S_{\eta(2,t)}(R_{\eta(2,t)}(z))R_{\eta(2,t)}(z) = z,
\]
equivalent to (4.4), with the condition \( R_{\eta(2,0)}(0) = 0 \). In particular we have \( R_{\eta(2,0)}(z) = 2z + z^2 \), which implies that \( \eta(2,0) \) is infinitely divisible with respect to the additive free convolution.

Now we can find
\[
\begin{align*}
r_1(2,t) &= 2 - t, \\
r_2(2,t) &= 1 + t - t^2, \\
r_3(2,t) &= 3t^2 - 2t^3, \\
r_4(2,t) &= -4t^2 + 10t^3 - 5t^4.
\end{align*}
\]

Since
\[
r_2(2,t)r_4(2,t) - r_3(2,t)^2 = t^2(t-1)(t-2)(t^2-2),
\]
for \( 0 < t < 1 \) the distribution \( \eta(2,t) \) is not infinitely divisible with respect to the additive free convolution.

For \( t \neq 1 \) we have
\[
1 + R_{\eta(2,t)}(z) = \frac{t - 1 - 2z + 3tz - z^2 + (t - 1 - z)\sqrt{1 + z(2-4t) + z^2}}{2(t-1)},
\]
and \( 1 + R_{\eta(2,1)}(z) = 1/(1-z) \). Then for \( 1 < t \leq 3/2 \) the function
\[
1 + R_{\eta(2,t)}(1/z) = \frac{(t-1)z^2 - 2z + 3tz - 1 + (z(t-1) - 1)\sqrt{1 + z(2-4t) + z^2}}{2(t-1)z^3}
\]

is
\[
\frac{1}{z} = \frac{(t-1)z^2 - 2z + 3tz - 1 + (z(t-1) - 1)\sqrt{1 + z(2-4t) + z^2}}{2(t-1)z^3}.\]
is the Cauchy transform of the probability distribution
\[
\frac{(1 - tx + x)\sqrt{4t(t - 1) - (x - 2t + 1)^2}}{2\pi(t - 1)x^3}
\]
dx
on the interval
\[
x \in [2t - 1 - 2\sqrt{t^2 - t}, 2t - 1 + 2\sqrt{t^2 - t}].
\]
Therefore for \(1 < t \leq 4/3\),
\[
\text{(4.6)} \quad r_n(2, t)
\]
\[
= \int_{2t - 1 - 2\sqrt{t^2 - t}}^{2t - 1 + 2\sqrt{t^2 - t}} x^n \frac{(1 - tx + x)\sqrt{4t(t - 1) - (x - 2t + 1)^2}}{2\pi(t - 1)x^3}
\]
dx,
which proves that the sequence \(\{r_{n+2}(2, t)\}_{n=0}^{\infty}\) is positive definite.

Remark. Note that for \(\eta(2, 0)\) the cumulant sequence is \((2, 1, 0, 0, \ldots)\), so the sequence \(\{r_{n+2}(2, 0)\}_{n=0}^{\infty} = (1, 0, 0, \ldots)\) is positive definite. Actually, \(\eta(2, 0)\), given by (2.11), is a translation of the Wigner semicircle distribution \(\frac{1}{2\pi}\sqrt{4 - x^2} dx, x \in [-2, 2]\). The free additive infinite divisibility of \(\eta(2, 0)\) was overlooked in [16, Corollary 7.1], where \(\eta(2, 0)\) was denoted \(\mu(2, 2)\).

Example 1. Define \(a_0 := 1\) and \(a_n := 3^n \cdot r_n(2, 4/3)\) for \(n \geq 1\), yielding the sequence
\[
1, 2, 5, 16, 64, 304, 1632, 9552, 59520, 388720, 2632864, \ldots.
\]
Applying (4.6) for \(t = 4/3\) we obtain
\[
\text{(4.7)} \quad a_n = \int_1^9 x^n \frac{\sqrt{(x - 1)(9 - x)^3}}{2\pi x^3} dx.
\]

Its generating function is
\[
\sum_{n=0}^{\infty} a_n z^n = 1 + R_{\eta(2, 4/3)}(3z) = \frac{1 + 18z - 27z^2 + \sqrt{(1 - z)(1 - 9z)^3}}{2}.
\]

Example 2. Now consider the binomial transform of \(a_n\):
\[
b_n := \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k,
\]
for which the density function is that of the sequence \(a_n\) translated by \(-1\):
\[
\text{(4.9)} \quad b_n = \int_0^8 x^n \frac{\sqrt{x(8 - x)^3}}{2\pi(x + 1)^3} dx.
\]
For the generating function we have
\[\sum_{n=0}^{\infty} b_n z^n = \sum_{k=0}^{\infty} a_k (-1)^k \sum_{n=k}^{\infty} \binom{n}{k} (-z)^n = \sum_{k=0}^{\infty} a_k \frac{z^k}{(1+z)^{k+1}} = \frac{1}{1+z} \left(1 + R_{\eta(2,4/3)}(3z/(1+z))\right),\]
so from (4.8),
\[\sum_{n=0}^{\infty} b_n z^n = 1 + 20z - 8z^2 + \sqrt{(1-8z)^3} = \frac{2(1+z)^3}{1 + z^2(\eta(2,4/3)(2,4/3)3z/(1+z))},\]
This proves that \(b_n\) coincides with \(A022558\) of OEIS:
\[1, 1, 2, 6, 23, 103, 512, 2740, 15485, 91245, 555662, \ldots,\]
which counts the permutations of length \(n\) which avoid the pattern 1342 [3, Theorem 2].

4.2. The case \(p = 3\)

**Proposition 4.3.** We have
\[R_{\eta(3,t)}(z) = z(4 - 7t + 4t^2 - 2z) - (t - 1)^2 + (1 - 2t + t^2 - tz)\sqrt{1-4tz} = \frac{1}{2(t + z - 1)^2},\]
and the distribution \(\eta(3,t)\) is infinitely divisible with respect to the additive free convolution if and only if \(1/2 \leq t \leq 3/2\).

**Proof.** The proof is similar to that for \(p = 2\). First we find \(R_{\eta(3,t)}\) by solving the equation
\[S_{\eta(3,t)}(R_{\eta(3,t)}(z))R_{\eta(3,t)}(z) = z\]
with the condition \(R_{\eta(3,t)}(0) = 0\). Then we find that
\[1 + R_{\eta(3,t)}(z) = \frac{(t - 1)^2 + tz(4t - 3) + (1 - 2t + t^2 - tz)\sqrt{1-4tz}}{2(t + z - 1)^2}\]
is the moment generating function for the density
\[\frac{(t - x(t - 1)^2)\sqrt{4t - x}}{2\pi(tx - x + 1)^2\sqrt{x}}, \quad x \in [0, 4t],\]
which is positive provided \(1/2 \leq t \leq 3/2\). ■

**Example.** The sequence \(a_n = A220910(n)\):
\[1, 1, 3, 14, 83, 570, 4318, 35068, 299907, 2668994, 24513578, \ldots\]
counts matchings avoiding the pattern 231 (see [4] for details). Its generating function equals

\[ M(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{1 + 36z + \sqrt{(1 - 12z)^3}}{2(1 + 4z)^2} = 1 + R_{\eta(3,3/2)}(2z), \]

so \( a_n = 2^n \cdot r_n(3,3/2) \) for \( n \geq 1 \). Therefore these numbers can be represented as moments:

\[ a_n = \int_0^{12} x^n \frac{\sqrt{(12 - x)^3}}{2\pi(x + 4)^2\sqrt{x}} \, dx. \]

Now we are going to prove a recurrence relation, which was conjectured by R. J. Mathar (see OEIS, entry A220910, Aug. 04 2013).

**Proposition 4.4.** For \( n \geq 2 \) we have

\[ na_n = (8n - 34)a_{n-1} + 24(2n - 3)a_{n-2}. \]

**Proof.** One can check that the generating function satisfies the differential equation

\[ (1 - 8z - 48z^2)M'(z) + (26 - 24z)M(z) = 27. \]

The coefficient of \( z^{n-1} \) on the left hand side is equal to

\[ na_n - 8(n - 1)a_{n-1} - 48(n - 2)a_{n-2} + 26a_{n-1} - 24a_{n-2} \]

for \( n \geq 2 \), which gives (4.15).

Now we will provide two formulas for \( a_n = A220910(n) \).

**Proposition 4.5.**

\[ a_n = \frac{1 - 8n}{2}(-4)^n + \binom{2n}{n} \sum_{k=0}^{n} \frac{8^{-k+1}}{(1 - 3k)^{k+1}} (-3)^k \prod_{i=0}^{k-1} (n - i - 1/2). \]

**Proof.** Setting \( x = 12t \) in (4.14) and applying [20] (15.6.1) we get

\[ a_n = \frac{27(2n)!3^n}{8n!(n + 2)!} F_1(2, n + 1/2; n + 3| -3). \]

From [20] (15.8.2) and the identities

\[ \frac{\Gamma(n - 3/2)}{\Gamma(n + 1/2)} = \frac{4}{(2n - 3)(2n - 1)}, \quad \frac{\Gamma(3/2 - n)}{\Gamma(5/2)} = \frac{(-2)^{n+1}(2n - 1)}{3(2n - 1)!!}, \]

we have
we have
\[
\binom{2}{n+1/2; n+3 | -3} = \frac{4(n+2)!}{9n!(2n-1)(2n-3)} \binom{2}{2, -n; 5/2 - n | -1/3} + \frac{(-2)^{n+1}(n+2)!(2n-1)}{3^{n+3/2}(2n-1)!!} \binom{2}{n+1/2, -3/2; n-1/2 | -1/3}.
\]

Since
\[
\binom{2}{2, -n; 5/2 - n | z} = \sum_{k=0}^{n} (k+1)z^k \prod_{i=0}^{k-1} \frac{n-i}{n-5/2-i}
\]
and
\[
\binom{2}{n+1/2, -3/2; n-1/2 | z} = \frac{(2n-2nz-2z-1)\sqrt{1-z}}{2n-1}
\]
(see [20], (15.4.9)), we obtain
\[
\binom{2}{2, n+1/2; n+3 | -3} = \frac{n!(n+2)!(8n-1)(-4)^{n+1}}{(2n)!3^{n+3}} + \frac{4(n+1)(n+2)}{9(2n-1)(2n-3)} \sum_{k=0}^{n} \frac{k+1}{(-3)^k} \prod_{i=0}^{k-1} \frac{n-i}{n-5/2-i},
\]
which leads to (4.16).

For the second formula we apply the identity
\[
\binom{2}{2, b; c | z}(1-z) = (bz - z - c + 2) \binom{2}{1, b; c | z} + c - 1
\]
(see [20], (15.5.11)) to (4.19) and get
\[
\binom{2}{2, n+1/2; n+3 | -3} = \frac{1-8n}{8} \binom{2}{1, n+1/2; n+3 | -3} + \frac{n+2}{4}.
\]

Applying, [23] (123), p. 462):
\[
\binom{2}{1, b; m+1 | z} = \frac{m!}{z^m(b-1)\ldots(b-m)} \left( (1-z)^{m-b} - \sum_{k=0}^{m-1} \frac{z^k}{k!} \prod_{i=0}^{k-1} (b+i-m) \right)
\]
with \( b = n+1/2, m = n+2, z = -3 \), and using the identity
\[
4^{n+1}n!(n+1/2-1)\ldots(n+1/2-n-2) = 3(2n)!
\]
we get (4.17). ■

Acknowledgments. W. M. is supported by the Polish National Science Center grant No. 2012/05/B/ST1/00626.
REFERENCES


Wojciech Młotkowski
Instytut Matematyczny
Uniwersytet Wrocławski
Plac Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: mlotkow@math.uni.wroc.pl

Karol A. Penson
Sorbonne Universités
Université Pierre et Marie Curie
Laboratoire de Physique Théorique de la Matière Condensée
CNRS UMR 7600, Tour 13, 5ième ét.
Boîte Courrier 121, 4 place Jussieu
F-75252 Paris Cedex 05, France
E-mail: penson@lptl.jussieu.fr