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PHILIPPE G. CIARLET AND CRISTINEL MARDARE

Abstract. In an intrinsic approach to a problem in elasticity, the only unknown is a tensor field representing an appropriate 'measure of strain', instead of the displacement vector field in the classical approach.

The objective of this paper is to study the displacement traction-problem in the special case where the elastic body is a linearly elastic plate of constant thickness, clamped over a portion of its lateral face. In this respect, we first explicitly compute the intrinsic three-dimensional boundary condition of place in terms of the Cartesian components of the linearized strain tensor field, thus avoiding the recourse to covariant components in curvilinear coordinates and providing an interesting example of actual computation of an intrinsic boundary condition of place in three-dimensional elasticity. Second, we perform a rigorous asymptotic analysis of the three-dimensional equations as the thickness of the plate, considered as a parameter, approaches zero. As a result, we identify the intrinsic two-dimensional equations of a linearly elastic plate modeled by the Kirchhoff-Love theory, with the linearized change of metric and change of curvature tensor fields of the middle surface of the plate as the new unknowns, instead of the displacement field of the middle surface in the classical approach.

Keywords: Displacement-traction problem, intrinsic elasticity, intrinsic boundary condition of place

1. The classical and intrinsic three-dimensional equations of a linearly elastic body

In what follows, Latin indices and exponents range in the set $\{1,2,3\}$ save when they are used for indexing sequences; while Greek indices and exponents range in the set $\{1,2\}$ save in the notations ∂_ν and ∂_τ, and the summation convention with respect to repeated indices is systematically used. For brevity, "three-dimensional" and "two-dimensional" will be usually abbreviated as "3d" and "2d", respectively.

All functions, vector fields, etc., considered here are real. As usual, $\delta_i^j = \delta^{ij} := 1$ if $i = j$ and $\delta_i^j = \delta^{ij} := 0$ if $i \neq j$. Spaces of vector fields are denoted by boldface letters while spaces of *symmetric* 3×3 or 2×2 matrix fields are denoted by special Roman capital letters.

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The notation $\mathbb{E}^3$ designates the Euclidean three-dimensional vector space, equipped with an orthonormal basis $(\boldsymbol{e}^i)$. The Euclidean inner product and tensor product of vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}^3$ are respectively denoted $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \otimes \boldsymbol{b}$, and $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ denotes the Euclidean norm of a vector $\boldsymbol{a} \in \mathbb{E}^3$; a unit vector $\boldsymbol{a} \in \mathbb{E}^3$ is one such that $|\boldsymbol{a}| = 1$. The notation $\mathbb{S}^3$, resp. $\mathbb{A}^3$, designates the space of all 3×3 symmetric, resp. antisymmetric, matrices. The notation (a_{ij}) designates a matrix with a_{ij} as its component at the i-th row and j-th column.

Given two vector spaces X, Y and a linear operator $A : X \to Y$, the kernel and image of A are respectively denoted $\operatorname{Ker} A$ and $\operatorname{Im} A$. The notation $(X, \|\cdot\|)$ designates a vector space X equipped with a norm $\|\cdot\|$, then also denoted $\|\cdot\|_X$. Given two normed vector spaces X and Y, the space of continuous linear operators from X into Y is denoted $\mathcal{L}(X; Y)$, and an *isomorphism* $A : X \to Y$ is a continuous linear operator that is one-to-one and onto and such that its inverse operator $A^{-1} : Y \to X$ is also continuous.

A *domain in* $\mathbb{R}^n$, $n \geq 2$, is a connected and bounded open subset of $\mathbb{R}^n$ whose boundary is Lipschitz-continuous in the sense of Nečas [1] or Adams [2], the set Ω being locally on the same side of its boundary.

Let Ω be a domain $\mathbb{E}^3$ with a smooth enough boundary Γ (specific smoothness assumptions on Γ will be made later). The closure $\overline{\Omega}$ of the set Ω is the reference configuration, assumed to be a natural state, of a homogeneous and isotropic linearly elastic body, thus characterized by two Lamé constants $\lambda \geq 0$ and $\mu > 0$. The body is subjected to applied body forces of density $(f^i) : \Omega \to \mathbb{R}^3$ and to a homogeneous boundary condition of place (i.e., of vanishing displacement vector field) on a relatively open subset Γ_0 of the boundary Γ; for simplicity, it is assumed that there are no applied surface forces acting on the remaining portion $\Gamma_1 := \Gamma - \Gamma_0$ of the boundary, but the subsequent analysis can be easily extended to accommodate such applied surface forces.

Let $x = (x_i)$ denote a generic point in the set Ω, let $\partial_i := \partial/\partial x_i$ and $\partial_{ij} := \partial^2/\partial x_i \partial x_j$, and let $(n_i) : \Gamma \to \mathbb{R}^3$ denote the unit outer normal vector along Γ. Then, according to the well-known classical theory of 3d-linearized elasticity, the unknown displacement $\boldsymbol{u} = (u_i) := \overline{\Omega} \to \mathbb{E}^3$ should be the solution, possibly only in a weak sense, of the following boundary value problem, which constitutes the classical 3d-*equations of linearized elasticity*:

$$
\begin{aligned}
-\partial_j(A^{ijk\ell}\varepsilon_{k\ell}(\boldsymbol{u})) &= f^i \text{ in } \Omega, \\
u_i &= 0 \text{ on } \Gamma_0, \\
A^{ijk\ell}\varepsilon_{k\ell}(\boldsymbol{u})n_j &= 0 \text{ on } \Gamma_1,
\end{aligned}
$$

where

$$
\begin{aligned}
A^{ijk\ell} &:= \lambda\delta^{ij}\delta^{k\ell} + \mu(\delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk}), \\
\varepsilon_{ij}(\boldsymbol{u}) &:= \frac{1}{2}(\partial_i u_j + \partial_j u_i)
\end{aligned}
$$

respectively denote the components of the *elasticity tensor* of the material constituting the linearly elastic body under consideration and the components of the *linearized strain tensor* $(\varepsilon_{ij}(\boldsymbol{u}))$ associated with the displacement vector field $\boldsymbol{u} = (u_i)$. The partial differential equations in Ω and the boundary conditions on Γ_1 constitute the 3d-*equations of equilibrium* while the boundary conditions on Γ_0 constitute the (homogeneous) 3d-*boundary condition of place.*

It is classical that, if $\Gamma_0 \neq \phi$ and $f^i \in L^2(\Omega)$, there exists a unique solution

$$\boldsymbol{u} \in \boldsymbol{V}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0\}$$

to the variational formulation of the above boundary value problem, thanks to Korn's inequality and to the Lax-Milgram lemma.

It is less known that the same problem can be modeled through a completely different approach, called the *intrinsic approach.* In such an approach, the idea of which goes back to Chien [3, 4, 5] (who proposed it for modeling linearly elastic plates and shells), appropriate "measures of strain", such as the linearized change of metric and change of curvature tensors in the case of shells for instance, are considered to be the only unknowns, instead of the components of the displacement vector field in the classical approach. When applied to the above 3d-equations of linearized elasticity, the intrinsic approach consists in considering the *components*

$$e_{ij} := \varepsilon_{ij}(\boldsymbol{u})$$

of the linearized strain tensor field as the new, and only, unknowns.

The first mathematical justifications of this intrinsic approach to three-dimensional elasticity was given in 2005 by Ciarlet & Ciarlet, Jr. [6], who applied it to the "pure traction problem", i.e., when $\Gamma_0 = \phi$, and in 2014 by Ciarlet & Mardare [7], who applied it to genuine "displacement-traction problems", i.e., when $\Gamma_0 \neq \phi$. What follows is a brief account of the main results of [6, 7] (more details are provided at the beginning of Sect. 2).

Assume that the set Ω is a simply-connected domain in $\mathbb{R}^3$ and that the open subset Γ_0 of its boundary Γ is connected and of class $\mathcal{C}^4$. Then the matrix field

$$\boldsymbol{e} := (e_{ij}) \in \mathbb{L}^2(\Omega)$$

satisfies the following *intrinsic* 3d-*equations of linearized elasticity*:

$$\begin{aligned}
-\partial_j(A^{ijk\ell} e_{k\ell}) &= f^i \text{ in } \Omega,\\
\partial_{j\ell} e_{ki} + \partial_{ik} e_{\ell j} - \partial_{i\ell} e_{kj} - \partial_{jk} e_{\ell i} &= 0 \text{ in } \Omega,\\
\tilde{\gamma}^\sharp_{\alpha\beta}(\boldsymbol{e}) &= 0 \text{ on } \Gamma_0,\\
\tilde{\rho}^\sharp_{\alpha\beta}(\boldsymbol{e}) &= 0 \text{ on } \Gamma_0,\\
A^{ijk\ell} e_{k\ell} n_j &= 0 \text{ on } \Gamma_1,
\end{aligned}$$

where $(\tilde{\gamma}^\sharp_{\alpha\beta}) : \mathbb{E}(\Omega) \to \mathbb{H}^{-1}(\Gamma_0)$ and $(\tilde{\rho}^\sharp_{\alpha\beta}) : \mathbb{E}(\Omega) \to \mathbb{H}^{-2}(\Gamma_0)$ are specific continuous linear operators (the construction of which is recalled in Sect. 2), the space $\mathbb{E}(\Omega)$ being defined by

$$\mathbb{E}(\Omega) := \{\boldsymbol{t} = (t_{ij}) \in \mathbb{L}^2(\Omega);\ \partial_{j\ell}t_{ki} + \partial_{ik}t_{\ell j} - \partial_{i\ell}t_{kj} \\ -\partial_{jk}t_{\ell i} = 0 \text{ in } H^{-2}(\Omega)\}.$$

The matrix field $\boldsymbol{e}$ thus belongs to the space $\mathbb{V}(\Omega)$ defined by

$$\mathbb{V}(\Omega) := \{\boldsymbol{t} \in \mathbb{E}(\Omega);\ \tilde{\gamma}^\sharp_{\alpha\beta}(\boldsymbol{t}) = 0 \text{ in } H^{-1}(\Gamma_0) \text{ and} \\ \tilde{\rho}^\sharp_{\alpha\beta}(\boldsymbol{t}) = 0 \text{ in } H^{-2}(\Gamma_0)\}.$$

The partial differential equations in Ω and the boundary conditions on Γ_1 constitute the *intrinsic* 3d-*equations of equilibrium* and the boundary conditions on Γ_0 constitute the *intrinsic* 3d-*boundary condition of place.* The relations

$$\partial_{j\ell}e_{ki} + \partial_{ik}e_{\ell j} - \partial_{i\ell}e_{kj} - \partial_{jk}e_{\ell i} = 0 \text{ in } H^{-2}(\Omega)$$

constitute the *Saint-Venant compatibility conditions* that a matrix field $\boldsymbol{e} = (e_{ij}) \in \mathbb{L}^2(\Omega)$ necessarily satisfy if there exists a vector field $\boldsymbol{u} = (u_i) \in \boldsymbol{H}^1(\Omega)$ such that

$$e_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \text{ in } \Omega.$$

While it is classical that the Saint-Venant compatibility conditions become sufficient if Ω is a simply-connected open subset of $\mathbb{R}^3$ and the functions e_{ij} are smooth enough, say in $\mathcal{C}^2(\Omega)$ in which case $\boldsymbol{u} \in \boldsymbol{\mathcal{C}}^3(\Omega)$, the main contribution of [6] was to show that the Saint-Venant compatibility conditions are also *sufficient* for the existence of such a vector field $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ *if Ω is a simply-connected domain in $\mathbb{R}^3$ and the functions e_{ij} are only assumed to be in the space $L^2(\Omega)$.*

The first contribution of [7], whose notations are re-used here for convenience, was to give an *explicit construction of mappings* $(\tilde{\gamma}^\sharp_{\alpha\beta}) \in \mathcal{L}(\mathbb{E}(\Omega);\ \mathbb{H}^{-1}(\Gamma_0))$ *and* $(\tilde{\rho}^\sharp_{\alpha\beta}) \in \mathcal{L}(\mathbb{E}(\Omega);\ \mathbb{H}^{-2}(\Gamma_0))$ such that, *together the intrinsic* 3d-*boundary conditions*

$$(\tilde{\gamma}^\sharp_{\alpha\beta}(\boldsymbol{e})) = 0 \text{ in } \mathbb{H}^{-1}(\Gamma_0) \text{ and } (\tilde{\rho}^\sharp_{\alpha\beta}(\boldsymbol{e})) = 0 \text{ in } H^{-2}(\Gamma_0))$$

are equivalent to the homogeneous boundary condition of place

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_0$$

(up to an infinitesimal rigid displacement) of the classical approach; see Theorem 2.1 for the explicit expressions of the components $\tilde{\gamma}^\sharp_{\alpha\beta}(\boldsymbol{e})$ and $\tilde{\rho}^\sharp_{\alpha\beta}(\boldsymbol{e})$. Otherwise, we refer to [7] for the detailed, and fairly lengthy, derivation of the intrinsic 3d-boundary conditions.

The second contribution of [7] was to establish that, if the domain Ω is simply-connected and the set Γ_0 is connected, *there exists a unique solution*

$$\boldsymbol{e} \in \mathbb{V}(\Omega)$$

to the variational formulation of the intrinsic 3d-*equations of linearized elasticity.* More specifically, it was shown in [7] that in this case, *the continuous linear operator*

$$\boldsymbol{\mathcal{F}} : \boldsymbol{v} = (v_i) \in \boldsymbol{V}(\Omega) \to \boldsymbol{\mathcal{F}}(\boldsymbol{v}) = (\mathcal{F}_{ij}(\boldsymbol{v})) := \Big(\frac{1}{2}(\partial_j v_i + \partial_i v_j)\Big) \in \mathbb{V}(\Omega)$$

is one-to-one and onto, so that the inverse operator

$$\mathcal{G} = (\mathcal{G}_i) : \mathbb{V}(\Omega) \to \boldsymbol{V}(\Omega)$$

is also continuous (both spaces $\boldsymbol{V}(\Omega)$ and $\mathbb{V}(\Omega)$ are Hilbert spaces) and that *there exists a unique solution* $\boldsymbol{e} = (e_{ij}) \in \mathbb{V}(\Omega)$ *to the intrinsic variational equations*

$$\int_\Omega A^{ijk\ell} e_{k\ell} t_{ij} \, \mathrm{d}x = \int_\Omega f^i \mathcal{G}_i(\boldsymbol{t}) \, \mathrm{d}x \text{ for all } \boldsymbol{t} = (t_{ij}) \in \mathbb{V}(\Omega),$$

which constitute the *variational formulation of the intrinsic* 3d-*equations of linearized elasticity* listed earlier.

The objective of the present paper is to study the special case where the elastic body is a *linearly elastic plate* of *thickness* $2\varepsilon > 0$ *clamped along a portion of its lateral face*, i.e., when

$$\Omega = \omega \times \,]-\varepsilon, \varepsilon[\text{ and } \Gamma_0 = \gamma_0 \times \,]-\varepsilon, \varepsilon[\,,$$

where ω is a domain in $\mathbb{R}^2$ and γ_0 is a non-empty relatively open subset of $\partial\omega$ that is of class $\mathcal{C}^4$.

First, we *explicitly compute* the corresponding functions $\gamma^\sharp_{\alpha\beta}(\boldsymbol{e})$ and $\rho^\sharp_{\alpha\beta}(\boldsymbol{e})$ found in this case in the intrinsic 3d-boundary condition of place when the components $e_{ij} : \overline{\Omega} \to \mathbb{R}$ of the matrix field $\boldsymbol{e}$ are smooth enough functions (see Lemma 3.1 and 3.2). Doing so thus provides an interesting example of *actual computation of an intrinsic* 3d-*boundary condition of place.*

Second, we perform an *asymptotic analysis as* $\varepsilon \to 0$ *of the intrinsic variational equations* corresponding to this special case (see Theorem 4.1). As expected, our analysis relies on the well-known asymptotic analysis as $\varepsilon \to 0$ of the variational formulation of the 3d-equations of the classical approach, i.e., with the displacement vector field as the unknown; cf. Chapter 1 in [8].

In so doing, we retrieve the *intrinsic* 2d-*equations of a linearly elastic plate* recently identified by Ciarlet & Mardare [9] by means of a completely different approach, which directly considers the linearized change of metric and change of curvature tensors of the middle surface $\overline{\omega}$ appearing in the 2d-equations of the Kirchhoff-Love theory of a linearly elastic plate as the unknown, instead of the displacement field of $\overline{\omega}$ in the classical formulation. These “limit” equations, the somewhat lengthy expressions of which are given at the end of Sect. 4, include in particular explicit expressions of the *intrinsic* 2d-*boundary conditions that correspond to the classical* 2d-*boundary conditions of clamping.*

2. General expression of an intrinsic 3d-boundary condition of place

As a preparation to the explicit computation of an intrinsic boundary condition of place along the lateral face of a plate (cf. Sect. 3), we briefly review the formulation, due to [7], of a *"general" intrinsic* 3d-*boundary condition of place* along a relatively open subset Γ_0 of the boundary Γ of a general domain in $\mathbb{R}^3$. Here and subsequently, it is assumed that Γ_0 *is of class* $\mathcal{C}^4$ and that Γ_0 *can be represented by means of a single local chart* $\boldsymbol{\theta} : u \to \mathbb{E}^3$ (for simplicity only; the extension to the case where *several* overlapping local charts are needed to cover Γ_0, as in [7], offers no difficulty other than notational; it simply requires an additional index) in such a way that there exist an open subset $u \subset \mathbb{R}^2$, an immersion $\boldsymbol{\theta} \in \mathcal{C}^4(u; \mathbb{E}^3)$, and $\delta > 0$ with the following properties: first,

$$\Gamma_0 = \boldsymbol{\theta}(u);$$

second, the mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(U; \mathbb{E}^3)$ defined by

$$\boldsymbol{\Theta}((y_i)) = \boldsymbol{\theta}((y_\alpha)) + y_3 \boldsymbol{n}((y_\alpha))$$

at each point

$$(y_i) = ((y_\alpha), y_3) \in U := u \times \left]-\delta, \delta\right[,$$

where $\boldsymbol{n}((y_\alpha))$ denotes the unit inner normal vector at each point $\boldsymbol{\theta}((y_\alpha))$ of the subset Γ_0 of the boundary of Ω, is a $\mathcal{C}^3$-diffeomorphism onto its image; and finally,

$$\boldsymbol{\Theta}\left(u \times \left]0, \delta\right[\right) \subset \Omega.$$

Let $\tilde{\partial}_i := \partial / \partial y_i$ (recall that the notation ∂_i designates the partial derivative with respect to each Cartesian coordinate x_i; cf. Sect 1). Then one classically defines the vectors

$$\boldsymbol{g}_i := \tilde{\partial}_i \boldsymbol{\Theta} \in \mathcal{C}^2(U; \mathbb{E}^3)$$

of the covariant bases, the vectors $\boldsymbol{g}^j \in \mathcal{C}^2(U; \mathbb{E}^3)$ of the contravariant bases by means of the relations

$$\boldsymbol{g}^j \cdot \boldsymbol{g}_i = \delta_i^j,$$

and the Christoffel symbols

$$\Gamma_{ij}^k := \partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}^k \in \mathcal{C}^1(U).$$

The covariant derivatives of a smooth enough vector field $\tilde{v}_i \boldsymbol{g}^i : U \to \mathbb{R}^3$, and of a smooth enough tensor field $\tilde{e}_{ij} \boldsymbol{g}^i \otimes \boldsymbol{g}^j$, both given by means of their covariant components $\tilde{v}_i$ and $\tilde{e}_{ij}$, are then respectively given by

$$\begin{aligned}
\tilde{v}_{i\|j} &:= \tilde{\partial}_j \tilde{v}_i - \Gamma_{ij}^k \tilde{v}_k, \\
\tilde{e}_{ij\|k} &:= \tilde{\partial}_k \tilde{e}_{ij} - \Gamma_{ki}^\ell \tilde{e}_{\ell j} - \Gamma_{kj}^\ell \tilde{e}_{i\ell}.
\end{aligned}$$

With any tensor $e_{ij}\boldsymbol{e}^i \otimes \boldsymbol{e}^j$ expressed in terms of the Cartesian coordinates x_i of the set $\boldsymbol{\Theta}(U)$ is then associated a tensor $\tilde{e}_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j$ expressed in terms of the curvilinear coordinates y_k of the set U by means of the defining relation

$$e_{ij}(x)\boldsymbol{e}^i \otimes \boldsymbol{e}^j = \tilde{e}_{ij}(y)\boldsymbol{g}^i(y) \otimes \boldsymbol{g}^j(y) \text{ for all } x = \boldsymbol{\Theta}(y),\, y \in U.$$

Note that the mappings

$$\boldsymbol{v} = (v_i) \to \varepsilon_{ij}(\boldsymbol{v}) := \frac{1}{2}(\partial_j v_i + \partial_i v_j) \text{ in } \Omega,$$

can be also written in matrix form as

$$\boldsymbol{v} \to (\varepsilon_{ij}(\boldsymbol{v})) := \boldsymbol{\nabla}_s \boldsymbol{v} \text{ in } \Omega,$$

where $\boldsymbol{\nabla}_s$ denotes the symmetrized gradient operator in Cartesian coordinates. Recall in this respect that

$$\begin{aligned}\operatorname{Ker} \boldsymbol{\nabla}_s = \{\boldsymbol{v} : \Omega \to \mathbb{E}^3;\ &\boldsymbol{v}(x) = \boldsymbol{a} + \boldsymbol{B}x,\ x \in \mathbb{E}^3,\\ &\text{for some } \boldsymbol{a} \in \mathbb{E}^3 \text{ and } \boldsymbol{B} \in \mathbb{A}^3\},\end{aligned}$$

and that the elements of $\operatorname{Ker} \boldsymbol{\nabla}_s$ are called *infinitesimal rigid displacements.*

Define the spaces

$$\operatorname{Im} \boldsymbol{\nabla}_s := \{\boldsymbol{\nabla}_s \boldsymbol{v};\ \boldsymbol{v} \in \boldsymbol{\mathcal{C}}^2(\overline{\Omega})\} \subset \mathbb{C}^1(\overline{\Omega}) \subset \mathbb{L}^2(\Omega).$$

Then one can show (cf. [7]) that, if Ω is a domain in $\mathbb{R}^3$, the closure $\overline{\operatorname{Im} \boldsymbol{\nabla}_s}$ of $\operatorname{Im} \boldsymbol{\nabla}_s$ in the space $\mathbb{L}^2(\Omega)$ is given by

$$\overline{\operatorname{Im} \boldsymbol{\nabla}_s} = \{\boldsymbol{\nabla}_s \boldsymbol{v};\ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega)\} \subset \mathbb{L}^2(\Omega).$$

It thus follows that, if Ω is a simply-connected domain, one also has

$$\overline{\operatorname{Im} \boldsymbol{\nabla}_s} = \mathbb{E}(\Omega).$$

Using the various notations defined above, we are now in a position to gather in Theorem 2.1 below the main results of [7] (viz., Theorems 4.1 and 6.1 in *ibid.*), to which we also refer the reader for the definition of the spaces $\mathcal{C}^1(\Gamma_0)$, $H^{-1}(\Gamma_0)$, and $H^{-2}(\Gamma_0)$.

In what follows, a notation such as $\tilde{e}_{\alpha\beta}(\cdot, 0)$ designates the function $(y_\alpha) \in u \to \tilde{e}_{\alpha\beta}((y_\alpha), 0)$. Given a tensor field $\boldsymbol{e}$ with Cartesian coordinates e_{ij}, the notation $\tilde{e}_{ij}$ designates the components of the same tensor field, but this time expressed in terms of the curvilinear coordinates associated with the $\mathcal{C}^3$-diffeomorphism $\boldsymbol{\Theta}$ associated as above with the local chart $\boldsymbol{\theta}$.

Theorem 2.1. *Let Ω be a domain in $\mathbb{E}^3$, let Γ_0 be a non-empty, connected, relatively open subset of $\partial\Omega$ of class $\mathcal{C}^4$ that can be represented by a single local chart $\boldsymbol{\theta}$. For each tensor field $\boldsymbol{e} \in \operatorname{Im} \boldsymbol{\nabla}_s$, let*

$$\begin{aligned}\gamma^\sharp_{\alpha\beta}(\boldsymbol{e}) &:= \tilde{e}_{\alpha\beta}(\cdot, 0) \text{ in } u,\\ \rho^\sharp_{\alpha\beta}(\boldsymbol{e}) &:= (\tilde{e}_{\alpha 3\|\beta} + \tilde{e}_{\beta 3\|\alpha} - \tilde{e}_{\alpha\beta\|3} + \Gamma^3_{\alpha\beta}\tilde{e}_{33})(\cdot, 0) \text{ in } u.\end{aligned}$$

Then the linear operators

$$\gamma^\sharp_{\alpha\beta} : \operatorname{Im} \boldsymbol{\nabla}_s \to \mathcal{C}^1(\Gamma_0) \text{ and } \rho^\sharp_{\alpha\beta} : \operatorname{Im} \boldsymbol{\nabla}_s \to \mathcal{C}^0(\Gamma_0)$$

defined in this fashion admit unique continuous linear extensions

$$\tilde{\gamma}^\sharp_{\alpha\beta} \in \mathcal{L}(\overline{\operatorname{Im} \boldsymbol{\nabla}_s};\ H^{-1}(\Gamma_0) \text{ and } \tilde{\rho}^\sharp_{\alpha\beta} \in \mathcal{L}(\overline{\operatorname{Im} \boldsymbol{\nabla}_s};\ H^{-2}(\Gamma_0)).$$

Besides, these extensions possess the following properties: Given a vector field $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, *let*

$$\boldsymbol{e} := \boldsymbol{\nabla}_s \boldsymbol{u} \in \mathbb{L}^2(\Omega).$$

Then

$$\tilde{\gamma}^\sharp_{\alpha\beta}(\boldsymbol{e}) = 0 \text{ in } H^{-1}(\Gamma_0) \text{ and } \tilde{\rho}^\sharp_{\alpha\beta}(\boldsymbol{e}) = 0 \text{ in } H^{-2}(\Gamma_0)$$

if and only if

$$\boldsymbol{u} + \boldsymbol{r} = \boldsymbol{0} \text{ on } \Gamma_0 \text{ for some } \boldsymbol{r} \in \operatorname{Ker} \boldsymbol{\nabla}_s.$$

Remark 2.2. In the particular case where Γ_0 is a portion of the plane $\{(x_i);\ x_3 = 0\}$ and $\boldsymbol{e} = \boldsymbol{\nabla}_s \boldsymbol{u}$ with $\boldsymbol{u} = (u_i) \in \boldsymbol{\mathcal{C}}^2(\overline{\Omega})$, the boundary conditions

$$\tilde{\gamma}^\sharp_{\alpha\beta}(\boldsymbol{e}) = 0 \text{ in } H^{-1}(\Gamma_0), \text{ resp. } \tilde{\rho}^\sharp_{\alpha\beta}(\boldsymbol{e}) = 0 \text{ in } H^{-2}(\Gamma_0),$$

are equivalent to the boundary conditions

$$\partial_\alpha u_\beta + \partial_\beta u_\alpha = 0 \text{ on } \Gamma_0, , \text{ resp. } \partial_{\alpha\beta} u_3 = 0 \text{ on } \Gamma_0,$$

since in this case

$$\frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha)(\cdot, 0) = e_{\alpha\beta}(\cdot, 0) = \gamma^\sharp_{\alpha\beta}(\boldsymbol{e}) = \tilde{\gamma}^\sharp_{\alpha\beta}(\boldsymbol{e}) \text{ in } u$$

and

$$\partial_{\alpha\beta} u_3(\cdot, 0) = (\partial_\alpha e_{\beta 3} + \partial_\beta e_{\alpha 3} - \partial_3 e_{\alpha\beta})(\cdot, 0) = \rho^\sharp_{\alpha\beta}(\boldsymbol{e}) = \tilde{\rho}^\sharp_{\alpha\beta}(\boldsymbol{e}) \text{ in } u.$$

3. Explicit computation of an intrinsic boundary condition of place along the lateral face of a plate

We assume throughout this section that $\overline{\Omega}$ is the reference configuration of a *partially clamped plate with constant thickness* $2\varepsilon > 0$. This means that $\Omega = \omega \times]-\varepsilon, \varepsilon[$, where ω is a domain in $\mathbb{R}^2$, and that Γ_0 is a portion of the lateral face $\partial\omega \times]-\varepsilon, \varepsilon[$ of the form $\Gamma_0 = \gamma_0 \times]-\varepsilon, \varepsilon[$, where γ_0 is a portion of $\partial\omega$ assumed to be parametrized in terms of its curvilinear abscissa s by means of a mapping $\boldsymbol{f} = \begin{pmatrix} f^1 \\ f^2 \end{pmatrix} : I \to \mathbb{R}^2$ defined and of class $\mathcal{C}^4$ on an open interval I of $\mathbb{R}$. Note that $\boldsymbol{f} : I \to \mathbb{R}^2$ is then an immersion, since in this case (in what follows, differentiation with respect to s is denoted by a prime)

$$\left|\boldsymbol{f}'(s)\right| = 1 \text{ for all } s \in I.$$

At each point $\boldsymbol{f}(s)$, $s \in I$, of γ_0, the vector

$$\boldsymbol{t}(s) = \begin{pmatrix} t^1(s) \\ t^2(s) \end{pmatrix} := \boldsymbol{f}'(s)$$

is thus a unit tangent vector to γ_0, while the vector

$$\boldsymbol{n}(s) = \begin{pmatrix} n^1(s) \\ n^2(s) \end{pmatrix} := \begin{pmatrix} -t^2(s) \\ t^1(s) \end{pmatrix}$$

is a unit normal vector to γ_0. Without loss of generality, we assume that $\boldsymbol{n}(s)$ is the unit inner normal vector to the boundary of the set $\overline{\Omega}$ at the same point (otherwise it suffices to replace the parameter s by $-s$). Then the *curvature* of γ_0 at each point $\boldsymbol{f}(s)$, $s \in I$, is given by

$$\kappa(s) := \boldsymbol{t}'(s) \cdot \boldsymbol{n}(s).$$

Besides, the following well-known *Frenet formulas for a planar curve* hold:

$$\boldsymbol{t}'(s) = \kappa(s)\boldsymbol{n}(s) \text{ and } \boldsymbol{n}'(s) = -\kappa(s)\boldsymbol{t}(s) \text{ at each } s \in I.$$

In order to be in a setting analogous to that of Sect. 2, we then let

$$u :=]-\varepsilon, \varepsilon[\times I \text{ and } U := u \times]-\delta, \delta[$$

for some small enough $\delta > 0$, and we denote by

$$x_3 \in]-\varepsilon, \varepsilon[\,, \ s \in I, \text{ and } t \in]-\delta, \delta[$$

the three corresponding *curvilinear coordinates*; in other words, we have in this case

$$y = (y_1, y_2, y_3) := (x_3, s, t), \text{ and } \tilde{\partial}_1 := \partial/\partial x_3, \ \tilde{\partial}_2 := \partial/\partial s, \ \tilde{\partial}_3 := \partial/\partial t,$$

the notations to the left of the equality signs being those of Sect. 2. This means that the corresponding mappings $\boldsymbol{\theta} : u \to \mathbb{E}^3$ and $\boldsymbol{\Theta} : U \to \mathbb{E}^3$ of Sect. 2 are respectively given in this case by

$$\boldsymbol{\theta}(x_3, s) = \begin{pmatrix} f^1(s) \\ f^2(s) \\ x_3 \end{pmatrix} \text{ at each } (x_3, s) \in u,$$

and

$$\boldsymbol{\Theta}(x_3, s, t) = \begin{pmatrix} f^1(s) \\ f^2(s) \\ x_3 \end{pmatrix} + t \begin{pmatrix} n^1(s) \\ n^2(s) \\ 0 \end{pmatrix} \text{ at each } (x_3, s, t) \in U.$$

The next lemma provides explicit relations expressing an *intrinsic boundary condition of place* in the special case considered in this section. Note that, as expected, these relations are independent of the "transverse" variable $x_3 \in]-\varepsilon, \varepsilon[$.

Recall that

$$\partial_i = \partial/\partial x_i$$

denote the partial derivative with respect to the Cartesian coordinate x_i of the points $x \in \overline{\Omega}$.

Lemma 3.1. *The assumptions on the sets Ω and Γ_0 and the various notations being those indicated above, let there be given a tensor field*

$$\boldsymbol{e} = (e_{ij}) \in \operatorname{Im} \boldsymbol{\nabla}_s \subset \mathbb{C}^1(\overline{\Omega}),$$

so that

$$\tilde{\gamma}^\sharp_{\alpha\beta}(\boldsymbol{e}) = \gamma^\sharp_{\alpha\beta}(\boldsymbol{e}) \text{ and } \tilde{\rho}^\sharp_{\alpha\beta}(\boldsymbol{e}) = \rho^\sharp_{\alpha\beta}(\boldsymbol{e})$$

in this case (Theorem 2.1). *Then:*

(a) *The boundary conditions*

$$\gamma^\sharp_{\alpha\beta}(\boldsymbol{e}) = 0 \text{ on } \Gamma_0$$

are equivalent to the relations:

$$\begin{aligned} e_{33}(x) &= 0, \\ e_{3\alpha}(x)t^\alpha(s) &= 0, \\ e_{\alpha\beta}(x)t^\alpha(s)t^\beta(s) &= 0, \end{aligned}$$

at each point $x = \boldsymbol{\Theta}(x_3, s, 0) \in \Gamma_0$, $(x_3, s) \in\,]-\varepsilon, \varepsilon[\, \times I$.

(b) *The boundary conditions*

$$\rho^\sharp_{\alpha\beta}(\boldsymbol{e}) = 0 \text{ on } \Gamma_0$$

are equivalent to the relations:

$$\begin{aligned} \partial_\alpha e_{33}(x)n^\alpha(s) - 2\partial_3 e_{3\alpha}(x)n^\alpha(s) = 0, \\ \partial_\beta e_{3\alpha}(x)(n^\beta(s)t^\alpha(s) - t^\beta(s)n^\alpha(s)) - \partial_3 e_{\alpha\beta}(x)t^\alpha(s)n^\beta(s) = 0, \\ \partial_\sigma e_{\alpha\beta}(x)t^\alpha(s)\Big(n^\sigma(s)t^\beta(s) - 2t^\sigma(s)n^\beta(s)\Big) - \kappa(s)e_{\alpha\beta}(x)n^\alpha(s)n^\beta(s) = 0, \end{aligned}$$

at each point $x = \boldsymbol{\Theta}(x_3, s, 0) \in \Gamma_0$, $(x_3, s) \in\,]-\varepsilon, \varepsilon[\, \times I$.

Proof. (i) To begin with, we compute the expressions of the vector fields $\boldsymbol{g}_i$ and $\boldsymbol{g}^j$, of the Christoffel symbols Γ^k_{ij}, etc., at each point $(x_3, s, t) \in U$, using the formulas recalled in Sect. 2. For the sake of brevity, the explicit dependence on $(x_3, s, t) \in U$ is only provided in the right-hand sides of these expressions, however. In what follows, we assume without loss of generality that $\delta > 0$ is chosen small enough so that $(1 - t\kappa(s))$ is > 0 for all $(s, t) \in I \times\,]-\delta, \delta[$.

First, the vectors of the covariant and contravariant bases are respectively given by

$$\boldsymbol{g}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{g}_2 = (1 - t\kappa(s)) \begin{pmatrix} t^1(s) \\ t^2(s) \\ 0 \end{pmatrix}, \quad \boldsymbol{g}_3 = \begin{pmatrix} n^1(s) \\ n^2(s) \\ 0 \end{pmatrix},$$

$$\boldsymbol{g}^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{g}^2 = \frac{1}{(1 - t\kappa(s))} \begin{pmatrix} t^1(s) \\ t^2(s) \\ 0 \end{pmatrix}, \quad \boldsymbol{g}^3 = \begin{pmatrix} n^1(s) \\ n^2(s) \\ 0 \end{pmatrix},$$

so that the Christoffel symbols $\Gamma_{ij}^k = \partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}^k$ are given by

$$\Gamma_{ij}^1 = 0,\ \Gamma_{ij}^2 = 0 \text{ if } (i,j) \notin \{(2,2),(2.3),(3.2)\},\ \Gamma_{ij}^3 = 0 \text{ if } (i,j) \neq (2,2),$$
$$\Gamma_{22}^2 = -\frac{t\kappa'(s)}{(1-t\kappa(s))},\ \Gamma_{23}^2 = \Gamma_{32}^2 = -\frac{\kappa(s)}{1-t\kappa(s)},\ \Gamma_{22}^3 = \kappa(s)(1-t\kappa(s)).$$

Second, each covariant derivative of the covariant components of the tensor $\boldsymbol{e}$, viz.,

$$\tilde{e}_{ij\|k} = \tilde{\partial}_k \tilde{e}_{ij} - \Gamma_{ki}^\ell \tilde{e}_{\ell j} - \Gamma_{kj}^\ell \tilde{e}_{i\ell},$$

is then computed using the above expressions of the Christoffel symbols.

Third, the covariant components $\tilde{e}_{ij}$ of the tensor $\boldsymbol{e}$ are computed in terms of the Cartesian components e_{ij} of the same tensor $\boldsymbol{e}$ by means of the classical formulas (k, resp. ℓ, designates the row, resp. column, index)

$$\tilde{e}_{ij}(y) = e_{k\ell}(x)\left[\boldsymbol{g}_i(y) \otimes \boldsymbol{g}_j(y)\right]^{k\ell} \text{ at each point } x = \boldsymbol{\Theta}(y),\ y \in U,$$

with

$$\boldsymbol{g}_1 \otimes \boldsymbol{g}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}(s),$$
$$\boldsymbol{g}_1 \otimes \boldsymbol{g}_2 = (1-t\kappa(s))\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t^1 & t^2 & 0 \end{pmatrix}(s),$$
$$\boldsymbol{g}_2 \otimes \boldsymbol{g}_1 = (\boldsymbol{g}_1 \otimes \boldsymbol{g}_2)^T,$$
$$\boldsymbol{g}_2 \otimes \boldsymbol{g}_2 = (1-t\kappa(s))^2\begin{pmatrix} t^1t^1 & t^1t^2 & 0 \\ t^2t^1 & t^2t^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}(s),$$
$$\boldsymbol{g}_1 \otimes \boldsymbol{g}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ n^1 & n^2 & 0 \end{pmatrix}(s),$$
$$\boldsymbol{g}_3 \otimes \boldsymbol{g}_1 = (\boldsymbol{g}_1 \otimes \boldsymbol{g}_3)^T,$$
$$\boldsymbol{g}_2 \otimes \boldsymbol{g}_3 = (1-t\kappa(s))\begin{pmatrix} t^1n^1 & t^1n^2 & 0 \\ t^2n^1 & t^2n^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}(s),$$
$$\boldsymbol{g}_3 \otimes \boldsymbol{g}_2 = (\boldsymbol{g}_2 \otimes \boldsymbol{g}_3)^T,$$
$$\boldsymbol{g}_3 \otimes \boldsymbol{g}_3 = \begin{pmatrix} n^1n^1 & n^1n^2 & 0 \\ n^2n^1 & n^2n^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}(s).$$

This gives

$$
\begin{aligned}
\tilde{e}_{11} &= e_{33}(x),\\
\tilde{e}_{12} = \tilde{e}_{21} &= (1 - t\kappa(s))e_{3\alpha}(x)t^\alpha(s),\\
\tilde{e}_{13} = \tilde{e}_{31} &= e_{3\alpha}(x)n^\alpha(s),\\
\tilde{e}_{22} &= (1 - t\kappa(s))^2 e_{\alpha\beta}(x)t^\alpha(s)t^\beta(s),\\
\tilde{e}_{23} = \tilde{e}_{32} &= (1 - t\kappa(s))e_{\alpha\beta}(x)t^\alpha(s)n^\beta(s),\\
\tilde{e}_{33} &= e_{\alpha\beta}(x)n^\alpha(s)n^\beta(s).
\end{aligned}
$$

Finally, using the chain rule and the above expressions, we compute those partial derivatives $\tilde{\partial}_k\tilde{e}_{ij}$ that appear in the covariant derivatives $\tilde{e}_{ij\|k}$ found in the expressions $\rho^\sharp_{\alpha\beta}(\boldsymbol{e})$. This gives:

$$
\begin{aligned}
\tilde{\partial}_3\tilde{e}_{11} &= \partial_\alpha e_{33}(x)n^\alpha(s),\\
\tilde{\partial}_3\tilde{e}_{12} &= -\kappa(s)e_{3\alpha}(x)t^\alpha(s) + (1 - t\kappa(s))\partial_\beta e_{3\alpha}(x)n^\beta(s)t^\alpha(s),\\
\tilde{\partial}_1\tilde{e}_{31} &= \partial_3 e_{3\alpha}(x)n^\alpha(s),\\
\tilde{\partial}_2\tilde{e}_{31} &= \partial_\beta e_{3\alpha}(x)t^\beta(s)n^\alpha(s) - \kappa(s)e_{3\alpha}(x)t^\alpha(s),\\
\tilde{\partial}_3\tilde{e}_{22} &= -2\kappa(s)(1 - t\kappa(s))e_{\alpha\beta}(x)t^\alpha(s)t^\beta(s)\\
&\quad + (1 - t\kappa(s))^2\partial_\sigma e_{\alpha\beta}(x)n^\sigma(s)t^\alpha(s)t^\beta(s),\\
\tilde{\partial}_1\tilde{e}_{32} &= (1 - t\kappa(s))\partial_3 e_{\alpha\beta}(x)t^\alpha(s)n^\beta(s),\\
\tilde{\partial}_2\tilde{e}_{32} &= -t\kappa'(s)e_{\alpha\beta}(x)t^\alpha(s)n^\beta(s) + (1 - t\kappa(s))\partial_\sigma e_{\alpha\beta}(x)t^\sigma(s)t^\alpha(s)n^\beta(s)\\
&\quad + \kappa(s)(1 - t\kappa(s))e_{\alpha\beta}(x)(n^\alpha(s)n^\beta(s) - t^\alpha(s)t^\beta(s))
\end{aligned}
$$

(the last expression uses the Frenet formulas for a planar curve).

(ii) It is then easily seen that, thanks to the computations carried out in (i), the relations (cf. Theorem 2.1)

$$
\begin{aligned}
\gamma^\sharp_{\alpha\beta}(\boldsymbol{e}) &:= \tilde{e}_{\alpha\beta}(\cdot, 0) \text{ in } u,\\
\rho^\sharp_{\alpha\beta}(\boldsymbol{e}) &:= (\tilde{e}_{\alpha3\|\beta} + \tilde{e}_{\beta3\|\alpha} - \tilde{e}_{\alpha\beta\|3} + \Gamma^3_{\alpha\beta}\tilde{e}_{33})(\cdot, 0) \text{ in } u,
\end{aligned}
$$

are indeed equivalent to those given in the statement of the theorem. □

Interestingly, the intrinsic boundary condition of place of Lemma 3.1 can be equivalently expressed in the following matrix form:

Lemma 3.2. *The assumptions and notations are the same as in* Lemma 3.1. *Then the boundary conditions of place*

$$\gamma^\sharp_{\alpha\beta}(\boldsymbol{e}) = 0 \text{ and } \rho^\sharp_{\alpha\beta}(\boldsymbol{e}) = 0 \text{ on } \Gamma_0$$

are equivalent to the following two relations between 2×2 *symmetric matrices:*

$$
\begin{pmatrix} e_{\alpha\beta}(x)t^\alpha(s)t^\beta(s) & e_{3\alpha}(x)t^\alpha(s)\\ e_{3\alpha}(x)t^\alpha(s) & e_{33}(x) \end{pmatrix} = \begin{pmatrix} 0 & 0\\ 0 & 0 \end{pmatrix}
$$

and

$$\frac{\partial}{\partial t}\begin{pmatrix} e_{\alpha\beta}(x)t^\alpha(s)t^\beta(s) & e_{3\alpha}(x)t^\alpha(s) \\ e_{3\alpha}(x)t^\alpha(s) & e_{33}(x) \end{pmatrix}$$

$$= \frac{\partial}{\partial s}\begin{pmatrix} 2e_{\alpha\beta}(x)t^\alpha(s)n^\beta(s) & e_{3\alpha}(x)n^\alpha(s) \\ e_{3\alpha}(x)n^\alpha(s) & 0 \end{pmatrix}$$

$$+ \frac{\partial}{\partial x_3}\begin{pmatrix} 0 & e_{\alpha\beta}(x)t^\alpha(s)n^\beta(s) \\ e_{\alpha\beta}(x)t^\alpha(s)n^\beta(s) & 2e_{3\alpha}(x)n^\alpha(s) \end{pmatrix}$$

$$- \kappa(s)\begin{pmatrix} e_{\alpha\beta}(x)n^\alpha(s)n^\beta(s) & 0 \\ 0 & 0 \end{pmatrix}$$

at each point $x = \mathbf{\Theta}(x_3, s, 0) \in \Gamma_0$, $(x_3, s) \in \,]-\varepsilon, \varepsilon[\, \times I$.

Proof. The first matrix relation is simply a re-statement of part (a) of Lemma 3.1 in matrix form.

Noting that

$$\tilde{\partial}_1 = \partial/\partial x_3 = \partial_3,\ \tilde{\partial}_2 = \partial/\partial s = t^\alpha(s)\partial_\alpha,\ \tilde{\partial}_3 = \partial/\partial t = n^\alpha(s)\partial_\alpha,$$

the equality at the first row and first column of the second matrix relation, viz.,

$$\frac{\partial}{\partial t}\Big(e_{\alpha\beta}(x)t^\alpha(s)t^\beta(s)\Big) = 2\frac{\partial}{\partial s}\Big(e_{\alpha\beta}(x)t^\alpha(s)n^\beta(s)\Big) - \kappa(s)e_{\alpha\beta}(x)n^\alpha(s)n^\beta(s),$$

becomes (recall that $\boldsymbol{t}'(s) = \kappa(s)\boldsymbol{n}(s)$ and $\boldsymbol{n}'(s) = -\kappa(s)\boldsymbol{t}(s)$):

$$\begin{aligned}(\partial_\sigma e_{\alpha\beta}(x))n^\sigma(s)t^\alpha(s)t^\beta(s) = {}& 2\partial_\sigma e_{\alpha\beta}(x)t^\sigma(s)t^\alpha(s)n^\beta(s) \\ & + 2\kappa(s)e_{\alpha\beta}(x)\Big(n^\alpha(s)n^\beta(s) - t^\alpha(s)t^\beta(s)\Big) \\ & - \kappa(s)e_{\alpha\beta}(x)n^\alpha(s)n^\beta(s).\end{aligned}$$

But, by the first matrix relation, $e_{\alpha\beta}(x)t^\alpha(s)t^\beta(s) = 0$, so we are left with

$$\partial_\sigma e_{\alpha\beta}(x)t^\alpha(s)\Big(n^\sigma(s)t^\beta(s) - 2t^\sigma(s)n^\beta(s)\Big) = \kappa(s)e_{\alpha\beta}(x)n^\alpha(s)n^\beta(s),$$

which is precisely the third relation in Lemma 3.1(b).

Likewise, the equality at the first row and second column of the second matrix relation, viz.,

$$\frac{\partial}{\partial t}(e_{3\alpha}(x)t^\alpha(s)) = \frac{\partial}{\partial s}(e_{3\alpha}(x)n^\alpha(s)) + \frac{\partial}{\partial x_3}\Big(e_{\alpha\beta}(x)t^\alpha(s)n^\beta(s)\Big),$$

becomes

$$\begin{aligned}\partial_\beta e_{3\alpha}(x)n^\beta(s)t^\alpha(s) = {}& \partial_\beta e_{3\alpha}(x)t^\beta(s)n^\alpha(s) \\ & - \kappa(s)e_{3\alpha}(x)t^\alpha(s) + \partial_3 e_{\alpha\beta}(x)t^\alpha(s)n^\beta(s).\end{aligned}$$

But, by the first matrix relation, $e_{3\alpha}(x)t^\alpha(s) = 0$; so we are left with

$$\partial_\beta e_{3\alpha}(x)\left(n^\beta(s)t^\alpha(s) - t^\beta(s)n^\alpha(s)\right) - \partial_3 e_{\alpha\beta}(x)t^\alpha(s)n^\beta(s) = 0,$$

which is precisely the second relation in Lemma 3.1(b).

Finally, the equality at the second row and second column of the second matrix relation becomes

$$\partial_\alpha e_{33}(x)n^\alpha(s) = 2\partial_3 e_{3\alpha}(x)n^\alpha(s),$$

which is precisely the first relation in Lemma 3.1(b). □

The well-known *Kirchhoff-Love theory* of a linearly elastic plate (cf., e.g., Chapter 1 in [8]) clamped over a portion $\Gamma_0 = \gamma_0 \times]-\varepsilon, \varepsilon[$ of the boundary of its reference configuration $\overline{\Omega} = \overline{\omega} \times [-\varepsilon, \varepsilon]$ asserts that the displacement vector field $\boldsymbol{u} = (u_i) \in \boldsymbol{H}^1(\Omega)$ is a *Kirchhoff-Love displacement field*, in the sense that its components are of the form

$$u_\alpha(\cdot, x_3) = \zeta_\alpha - x_3\partial_\alpha\zeta_3 \text{ and } u_3(\cdot, x_3) = \zeta_3,$$

with

$$\zeta_\alpha \in H^1(\omega) \text{ and } \zeta_\alpha = 0 \text{ on } \gamma_0, \text{ and } \zeta_3 \in H^2(\omega) \text{ with } \zeta_3 = \partial_\nu\zeta_3 = 0 \text{ on } \gamma_0.$$

Hence in this case, the Cartesian components of the corresponding strain tensor $(e_{ij}) \in \mathbb{L}^2(\Omega)$ are of the form

$$e_{\alpha\beta}(x) = c_{\alpha\beta}(x') - x_3 r_{\alpha\beta}(x') \text{ and } e_{i3}(x) = 0$$

at each

$$x = (x', x_3) \in \overline{\omega} \times [-\varepsilon, \varepsilon],$$

where

$$c_{\alpha\beta} = \frac{1}{2}(\partial_\beta\zeta_\alpha + \partial_\alpha\zeta_\beta) \text{ and } r_{\alpha\beta} = \partial_{\alpha\beta}\zeta_3.$$

Assume that such a Kirchhoff-Love displacement field is in the space $\boldsymbol{C}^2(\overline{\Omega})$ so that $\boldsymbol{e} \in \mathbb{C}^1(\overline{\Omega})$. Then Lemma 3.1 can be applied, showing that the boundary condition $\gamma^\sharp_{\alpha\beta}(\boldsymbol{e}) = 0$ on Γ_0 and $\rho^\sharp_{\alpha\beta}(\boldsymbol{e}) = 0$ on Γ_0 are respectively equivalent in this case to the four relations:

$$c_{\alpha\beta}(x')t^\alpha(s)t^\beta(s) = 0,$$
$$r_{\alpha\beta}(x')t^\alpha(s)t^\beta(s) = 0,$$

and

$$r_{\alpha\beta}(x')t^\alpha(s)n^\beta(s) = 0,$$
$$\partial_\sigma c_{\alpha\beta}(x')t^\alpha(s)\left(n^\sigma(s)t^\beta(s) - 2t^\sigma(s)n^\beta(s)\right) = \kappa(s)c_{\alpha\beta}(x')n^\alpha(s)n^\beta(s),$$

at each point $x' = \boldsymbol{f}(s) \in \gamma_0$, $s \in I$; note that the last relation that can be derived from Lemma 3.1(b), viz.,

$$\partial_\sigma r_{\alpha\beta}(x')t^\alpha(s)\left(n^\sigma(s)t^\beta(s) - 2t^\sigma(s)n^\beta(s)\right) = \kappa(s)r_{\alpha\beta}(x')n^\alpha(s)n^\beta(s),$$

at each point $x' = \boldsymbol{f}(s) \in \gamma_0$, $s \in I$, is superfluous, as it is implied by the first and second relations and the observation that $r_{\alpha\beta} = \partial_{\alpha\beta}\zeta_3$ implies that $\partial_\sigma r_{\alpha\beta} = \partial_\beta r_{\alpha\sigma}$ in $\overline{\omega}$.

Remarkably, the above four *intrinsic* 2d-*boundary conditions*, derived here under the *a priori assumption* that the displacement field inside the plate is a Kirchhoff-Love one, can be justified rigorously by means of an *asymptotic analysis* of the intrinsic 3d-equations when the thickness of the plate approaches zero.

The objective of the next section consists in carrying out such an asymptotic analysis (cf. in particular Theorem 4.3 below).

4. Asymptotic analysis as the thickness of a plate approaches zero

Let ω be a domain in $\mathbb{R}^2$ and let γ_0 be a non-empty relatively open subset of $\partial\omega$. For each $\varepsilon > 0$, let

$$\Omega^\varepsilon := \omega \times]-\varepsilon, \varepsilon[\,,\ \Gamma_0^\varepsilon := \gamma_0 \times]-\varepsilon, \varepsilon[\,,$$

let $x^\varepsilon = (x_i^\varepsilon)$ denote a generic point in the set $\overline{\Omega}^\varepsilon$, let

$$\partial_i^\varepsilon := \partial/\partial x_i^\varepsilon,\ \partial_{ij}^\varepsilon := \partial^2/\partial x_i^\varepsilon \partial x_j^\varepsilon,$$

and, given a smooth enough vector field $\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) : \overline{\Omega}^\varepsilon \to \mathbb{E}^3$, define the tensor field

$$\boldsymbol{\nabla}_s^\varepsilon \boldsymbol{v}^\varepsilon = \left(\frac{1}{2}(\partial_i^\varepsilon v_j^\varepsilon + \partial_j^\varepsilon v_i^\varepsilon)\right) : \overline{\Omega}^\varepsilon \to \mathbb{S}^3.$$

In this section, we consider a *family* of linearly elastic plates, with $\overline{\Omega}^\varepsilon$ as their reference configuration, clamped over the portion Γ_0^ε of their lateral face, and subjected to applied body forces of density $(f^{i,\varepsilon}) : \Omega^\varepsilon \to \mathbb{R}^3$, for each $\varepsilon > 0$. We assume that all the plates are made of the same constituting material, characterized by two Lamé constants $\lambda \geq 0$ and $\mu > 0$.

Our objective is to perform an *asymptotic analysis of the intrinsic* 3d-*equations that model such plates as ε approaches zero*, and in this fashion, to recover "in the limit" the *intrinsic* 2d-*equations of a linearly elastic plate*, directly obtained in [9] from the classical 2d-equations of such a plate.

To begin with, we show that *intrinsic* 3d-*equations* similar to, but more general than, those of [7] hold under weaker smoothness assumptions. Note that the next theorem (applied here to a linearly elastic clamped plate) holds as well if Ω^ε, resp. Γ_0^ε, is replaced by any domain in $\mathbb{E}^3$, resp. by any non-empty relatively open subset of $\partial\Omega^\varepsilon$. Also, note that the space $\mathbb{V}(\Omega^\varepsilon)$ as defined in Theorem 4.1(a) below coincides with the space $\mathbb{V}(\Omega^\varepsilon)$ as defined in the introduction under the additional assumption that ω is simply-connected and γ_0 is of class $\mathcal{C}^4$; this is why it is licit to designate it by the same notation.

Theorem 4.1. (a) *Given any $\varepsilon > 0$, let the sets Ω^ε and Γ_0^ε be defined as above. Define the spaces*

$$\begin{aligned}\boldsymbol{V}(\Omega^\varepsilon) &:= \{\boldsymbol{v}^\varepsilon \in \boldsymbol{H}^1(\Omega^\varepsilon);\ \boldsymbol{v}^\varepsilon = \boldsymbol{0} \text{ on } \Gamma_0^\varepsilon\}\\ \mathbb{V}(\Omega^\varepsilon) &:= \{\boldsymbol{t}^\varepsilon \in \mathbb{L}^2(\Omega^\varepsilon);\ \text{there exists } \boldsymbol{v}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon) \text{ such that } \boldsymbol{t}^\varepsilon = \boldsymbol{\nabla}_s^\varepsilon \boldsymbol{v}^\varepsilon\}.\end{aligned}$$

Then the space $(\mathbb{V}(\Omega^\varepsilon); \|\cdot\|_{\mathbb{L}^2(\Omega^\varepsilon)})$ is a Hilbert space, and the mapping

$$\boldsymbol{\mathcal{F}}^\varepsilon : \boldsymbol{v}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon) \to \boldsymbol{\mathcal{F}}^\varepsilon(\boldsymbol{v}^\varepsilon) := \boldsymbol{\nabla}_s^\varepsilon v^\varepsilon \in \mathbb{V}(\Omega^\varepsilon)$$

is an isomorphism.

(b) *Let*

$$\boldsymbol{\mathcal{G}}^\varepsilon = (\mathcal{G}_i^\varepsilon) := (\boldsymbol{\mathcal{F}}^\varepsilon)^{-1} : \mathbb{V}(\Omega^\varepsilon) \to \boldsymbol{V}(\Omega^\varepsilon).$$

Let

$$A^{ijk\ell} = \lambda\delta^{ij}\delta^{k\ell} + \mu(\delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk}),$$

and let functions $(f^{i,\varepsilon}) \in \boldsymbol{L}^2(\Omega^\varepsilon)$ be given. Then the variational equations $\mathbb{P}(\Omega^\varepsilon)$, viz.,

$$\int_{\Omega^\varepsilon} A^{ijk\ell} e_{k\ell}^\varepsilon t_{ij}^\varepsilon \, \mathrm{d}x^\varepsilon = \int_{\Omega^\varepsilon} f^{i,\varepsilon} \mathcal{G}_i^\varepsilon(\boldsymbol{t}^\varepsilon) \, \mathrm{d}x^\varepsilon \text{ for all } \boldsymbol{t}^\varepsilon = (t_{ij}^\varepsilon) \in \mathbb{V}(\Omega^\varepsilon),$$

have a unique solution $\boldsymbol{e}^\varepsilon = (e_{ij}^\varepsilon) \in \mathbb{V}(\Omega^\varepsilon)$. Besides,

$$\boldsymbol{e}^\varepsilon = \boldsymbol{\nabla}_s^\varepsilon \boldsymbol{u}^\varepsilon,$$

where $\boldsymbol{u}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon)$ is the unique solution to the variational equations

$$\int_{\Omega^\varepsilon} A^{ijk\ell} (\boldsymbol{\nabla}_s^\varepsilon \boldsymbol{u}^\varepsilon)_{k\ell} (\boldsymbol{\nabla}_s^\varepsilon \boldsymbol{v}^\varepsilon)_{ij} \, \mathrm{d}x^\varepsilon = \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \, \mathrm{d}x^\varepsilon \text{ for all } \boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \boldsymbol{V}(\Omega^\varepsilon).$$

(c) *If the solution $\boldsymbol{e}^\varepsilon$ to the variational equations $\mathbb{P}(\Omega^\varepsilon)$ is smooth enough, it satisfies the following intrinsic* 3d*-equations:*

$$\begin{aligned} -\partial_j^\varepsilon(A^{ijk\ell} \boldsymbol{e}_{k\ell}^\varepsilon) &= f^{i,\varepsilon} \quad \text{in } \Omega^\varepsilon,\\ \partial_{j\ell}^\varepsilon e_{ki}^\varepsilon + \partial_{ik}^\varepsilon e_{\ell j}^\varepsilon - \partial_{i\ell}^\varepsilon e_{kj}^\varepsilon - \partial_{jk}^\varepsilon e_{\ell i}^\varepsilon &= 0 \quad \text{in } \Omega^\varepsilon,\\ \mathcal{G}_i^\varepsilon(\boldsymbol{e}^\varepsilon) &= 0 \quad \text{on } \Gamma_0^\varepsilon,\\ A^{ijk\ell} e_{k\ell}^\varepsilon n_j^\varepsilon &= 0 \quad \text{on } \partial\Omega^\varepsilon - \Gamma_0^\varepsilon,\end{aligned}$$

where (n_j^ε) denotes the unit outer normal vector along $\partial\Omega^\varepsilon$.

Proof. The mapping $\boldsymbol{\mathcal{F}}^\varepsilon : \boldsymbol{V}(\Omega^\varepsilon) \to \mathbb{V}(\Omega^\varepsilon)$ is clearly continuous, onto by definition of the space $\mathbb{V}(\Omega^\varepsilon)$, and one-to-one, since

$$\boldsymbol{v}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon) \text{ and } \boldsymbol{\mathcal{F}}^\varepsilon(\boldsymbol{v}^\varepsilon) = \boldsymbol{0} \text{ implies } \frac{1}{2}(\partial_i^\varepsilon v_j^\varepsilon + \partial_j^\varepsilon v_i^\varepsilon) = 0 \text{ in } \Omega,$$

and thus $\boldsymbol{v}^\varepsilon = \boldsymbol{0}$ since $\boldsymbol{v}^\varepsilon = \boldsymbol{0}$ on a subset Γ_0 of $\partial\Omega^\varepsilon$, whose area is > 0.

Let $(\boldsymbol{e}_n^\varepsilon)_{n=0}^\infty$ be a Cauchy sequence in the space $(\mathbb{V}(\Omega^\varepsilon); \|\cdot\|_{\mathbb{L}^2(\Omega^\varepsilon)})$. Then the sequence $(\boldsymbol{v}_n^\varepsilon)$, where $\boldsymbol{v}_n^\varepsilon := \boldsymbol{\mathcal{G}}^\varepsilon(\boldsymbol{e}_n^\varepsilon) \in \boldsymbol{V}(\Omega^\varepsilon)$, is a Cauchy sequence, since, by *Korn's inequality*, there exists a constant C^ε such that, for all $m, n \geq 0$,

$$\|\boldsymbol{v}_m^\varepsilon - \boldsymbol{v}_n^\varepsilon\|_{\boldsymbol{H}^1(\Omega)} \leq C^\varepsilon \|\boldsymbol{\nabla}_s^\varepsilon \boldsymbol{v}_m^\varepsilon - \boldsymbol{\nabla}_s^\varepsilon \boldsymbol{v}_n^\varepsilon\|_{\mathbb{L}^2(\Omega^\varepsilon)} = C^\varepsilon \|\boldsymbol{e}_m^\varepsilon - \boldsymbol{e}_n^\varepsilon\|_{\mathbb{L}^2(\Omega^\varepsilon)}.$$

Since the space $(\boldsymbol{V}(\Omega^\varepsilon), \|\cdot\|_{\boldsymbol{H}^1(\Omega^\varepsilon)})$ is complete, there exists $\boldsymbol{v}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon)$ such that

$$\boldsymbol{v}_n^\varepsilon \to \boldsymbol{v} \text{ in } \boldsymbol{H}^1(\Omega^\varepsilon) \text{ as } n \to \infty.$$

Therefore,

$$\boldsymbol{e}_n^\varepsilon = \boldsymbol{\mathcal{F}}^\varepsilon(\boldsymbol{v}_n^\varepsilon) \to \boldsymbol{\mathcal{F}}^\varepsilon(\boldsymbol{v}) \text{ in } \mathbb{L}^2(\Omega^\varepsilon) \text{ as } n \to \infty,$$

since the mapping $\boldsymbol{\mathcal{F}}^\varepsilon$ is continuous. This shows that the space $(\mathbb{V}(\Omega^\varepsilon), \|\cdot\|_{\mathbb{L}^2(\Omega^\varepsilon)})$ is complete; consequently, the mapping $\boldsymbol{\mathcal{G}}^\varepsilon := (\boldsymbol{\mathcal{F}}^\varepsilon)^{-1}$ is also continuous, by Banach open mapping theorem. This proves (a).

It is well-known that, thanks to the assumptions $\lambda \geq 0$ and $\mu > 0$, the fourth-order tensor $(A^{ijk\ell})$ is positive-definite, i.e., there exists a constant $C > 0$ such that

$$A^{ijk\ell} t_{k\ell} t_{ij} \geq C \sum_{i,j} |t_{ij}|^2 \text{ for all } (t_{ij}) \in \mathbb{S}^3.$$

Besides, the linear form

$$\boldsymbol{t}^\varepsilon \in \mathbb{V}(\Omega^\varepsilon) \to \int_{\Omega^\varepsilon} f^{i,\varepsilon} \mathcal{G}_i^\varepsilon(\boldsymbol{t}^\varepsilon) \, \mathrm{d}x^\varepsilon \in \mathbb{R}$$

is continuous since $\boldsymbol{\mathcal{G}}^\varepsilon = (\mathcal{G}_i^\varepsilon) \in \mathcal{L}(\mathbb{V}(\Omega^\varepsilon); \boldsymbol{V}(\Omega^\varepsilon))$. Therefore the variational equations of (b) have a unique solution $\boldsymbol{e}^\varepsilon = (e_{ij}^\varepsilon) \in \mathbb{V}(\Omega^\varepsilon)$. That $\boldsymbol{\nabla}_s^\varepsilon \boldsymbol{u}^\varepsilon = \boldsymbol{e}^\varepsilon$ is clear. This in turn implies that the components e_{ij}^ε of $\boldsymbol{e}^\varepsilon$ satisfy the Saint-Venant compatibility conditions

$$\partial_{j\ell}^\varepsilon e_{ki}^\varepsilon + \partial_{ik}^\varepsilon e_{\ell j}^\varepsilon - \partial_{i\ell}^\varepsilon e_{kj}^\varepsilon - \partial_{jk}^\varepsilon e_{\ell i}^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon;$$

see, e.g., [6].

The variational equations of (b) are equivalent to the variational equations

$$\int_{\Omega^\varepsilon} A^{ijk\ell} e_{k\ell}^\varepsilon \Big(\frac{1}{2} (\partial_j^\varepsilon v_i^\varepsilon + \partial_i^\varepsilon v_j^\varepsilon) \Big) \, \mathrm{d}x^\varepsilon = \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \, \mathrm{d}x^\varepsilon \text{ for all } (v_i^\varepsilon) \in \boldsymbol{V}(\Omega^\varepsilon).$$

Since

$$A^{ijk\ell} e_{k\ell}^\varepsilon \Big(\frac{1}{2} (\partial_j^\varepsilon v_i^\varepsilon + \partial_i^\varepsilon v_j^\varepsilon) \Big) = A^{ijk\ell} e_{k\ell}^\varepsilon \partial_j v_i^\varepsilon \text{ for all } (v_i^\varepsilon) \in \boldsymbol{V}(\Omega^\varepsilon),$$

the Green formula shows that, if the tensor field $\boldsymbol{e}^\varepsilon$ is smooth enough, it satisfies the intrinsic 3d-equations of (c). $\square$

We next transform the variational equations $\mathbb{P}(\Omega^\varepsilon)$ posed over each domain Ω^ε, $\varepsilon > 0$, into variational equations, denoted $\mathbb{P}(\varepsilon; \Omega)$ in the next theorem, posed over a *fixed* domain Ω. To this end, we make appropriate *scalings on the unknowns* (the components e_{ij}^ε of the tensor field $\boldsymbol{e}^\varepsilon$) and *assumptions on the data* (the components $f^{i,\varepsilon}$ of the applied body force density and the Lamé constants), following in this fashion a well-known procedure in linear plate theory (cf. Chapter 1 in [8]). More specifically, we let

$$\Omega := \omega \times]{-1}, 1[\,, \ \Gamma_0 := \gamma_0 \times]{-1}, 1[\,, \ \partial_i := \partial / \partial x_i, \ \partial_{ij} := \partial^2 / \partial x_i \partial x_j,$$

where $x = (x_i)$ denotes a generic point in the set Ω, and, given a smooth enough vector field $\boldsymbol{v} = (v_i) \in \overline{\Omega} \to \mathbb{E}^3$, we define the tensor field

$$\boldsymbol{\nabla}_s \boldsymbol{v} := \Big(\frac{1}{2}(\partial_i v_j + \partial_j v_i)\Big) : \overline{\Omega} \to \mathbb{S}^3.$$

Then, for each $\varepsilon > 0$, we define the mapping

$$\pi^\varepsilon : x = (x_i) = (x_1, x_2, x_3) \in \overline{\Omega} \\ \to \pi^\varepsilon x = x^\varepsilon = (x_i^\varepsilon) := (x_1, x_2, \varepsilon x_3) \in \overline{\Omega^\varepsilon},$$

so that $x_\alpha^\varepsilon = x_\alpha$, $x_3^\varepsilon = \varepsilon x_3$, $\partial_\alpha^\varepsilon = \partial_\alpha$, $\partial_3^\varepsilon = \dfrac{1}{\varepsilon}\partial_3$, and we *assume* that there exist functions $f^i \in L^2(\Omega)$ such that

$$f^{\alpha,\varepsilon} \circ \pi^\varepsilon = \varepsilon^2 f^\alpha \text{ and } f^{3,\varepsilon} \circ \pi^\varepsilon = \varepsilon^3 f^3 \text{ for all } \varepsilon > 0,$$

and that the Lamé constants $\lambda \geq 0$ and $\mu > 0$ are independent of ε.

The following result will be the point of departure of our asymptotic analysis.

Lemma 4.2. *Define the spaces*

$$\begin{aligned} \boldsymbol{V}(\Omega) &:= \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0\}, \\ \mathbb{V}(\Omega) &:= \{\boldsymbol{t} \in \mathbb{L}^2(\Omega);\ \text{there exists } \boldsymbol{v} \in \boldsymbol{V}(\Omega) \text{ such that } \boldsymbol{t} = \boldsymbol{\nabla}_s \boldsymbol{v}\}. \end{aligned}$$

Then the space $(\mathbb{V}(\Omega);\ \|\cdot\|_{\mathbb{L}^2(\Omega)})$ *is a Hilbert space, and the mapping*

$$\boldsymbol{\mathcal{F}} : \boldsymbol{v} \in \boldsymbol{V}(\Omega) \to \boldsymbol{\mathcal{F}}(\boldsymbol{v}) := \boldsymbol{\nabla}_s \boldsymbol{v} \in \mathbb{V}(\Omega)$$

is an isomorphism. With the tensor fields $\boldsymbol{e}^\varepsilon = (e_{ij}^\varepsilon) \in \mathbb{V}(\Omega)$ *and* $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon) \in \boldsymbol{V}(\Omega^\varepsilon)$ *that satisfy the variational equations of* Theorem 4.1(b), *we associate for each* $\varepsilon > 0$ *the scaled tensor field*

$$\boldsymbol{\kappa}(\varepsilon) = (\kappa_{ij}(\varepsilon)) \in \mathbb{V}(\Omega) \text{ defined by } \kappa_{ij}(\varepsilon) := \frac{1}{\varepsilon^2} e_{ij}^\varepsilon \circ \pi^\varepsilon,$$

and the scaled vector field

$$\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) \in \boldsymbol{V}(\Omega)$$

defined by

$$u_\alpha(\varepsilon) := \frac{1}{\varepsilon^2} u_\alpha^\varepsilon \circ \pi^\varepsilon \text{ and } u_3(\varepsilon) := \frac{1}{\varepsilon} u_3^\varepsilon \circ \pi^\varepsilon.$$

Then the scaled tensor $\boldsymbol{\kappa}(\varepsilon) \in \mathbb{V}(\Omega)$ *satisfies the variational equations* $(\mathbb{P}(\varepsilon;\Omega))$, *viz.,*

$$\int_\Omega A^{ijk\ell} \kappa_{k\ell}(\varepsilon) \chi_{ij}\, \mathrm{d}x = \int_\Omega f^i \mathcal{G}_i(\boldsymbol{\chi})\, \mathrm{d}x \text{ for all } \boldsymbol{\chi} = (\chi_{ij}) \in \mathbb{V}(\Omega),$$

where

$$\boldsymbol{\mathcal{G}} = (\mathcal{G}_i) := \boldsymbol{\mathcal{F}}^{-1} : \mathbb{V}(\Omega) \to \boldsymbol{V}(\Omega).$$

Besides,

$$\kappa_{\alpha\beta}(\varepsilon) = (\boldsymbol{\nabla}_s \boldsymbol{u}(\varepsilon))_{\alpha\beta},\ \kappa_{\alpha 3}(\varepsilon) = \frac{1}{\varepsilon}(\boldsymbol{\nabla}_s \boldsymbol{u}(\varepsilon))_{\alpha 3},\ \kappa_{33}(\varepsilon) = \frac{1}{\varepsilon^2}(\boldsymbol{\nabla}_s \boldsymbol{u}(\varepsilon))_{33}.$$

Proof. The variational equations $\mathbb{P}(\varepsilon;\Omega)$ simply constitute a re-writing of the variational equations $\mathbb{P}(\Omega^\varepsilon)$ after the above scalings and assumptions are taken into account. That $(\mathbb{V}(\Omega); \|\cdot\|_{\mathbb{L}^2(\Omega)})$ is a Hilbert space and that $\boldsymbol{\mathcal{F}} : \boldsymbol{V}(\Omega) \to \mathbb{V}(\Omega)$ is an isomorphism is established in Theorem 4.1 (with $\varepsilon = 1$). $\square$

The next theorem constitutes the main result of this section. It shows that, *as* $\varepsilon \to 0$, *the solutions* $\boldsymbol{\kappa}(\varepsilon)$ *to the variational equations* $\mathbb{P}(\varepsilon;\Omega)$ *of* Lemma 4.2 *converge in* $\mathbb{L}^2(\Omega)$ *to a "two-dimensional limit"* $\boldsymbol{\kappa}$. This abus de language means that $\boldsymbol{\kappa}$ can be entirely recovered from the solution of 2d-variational equations (denoted $((c_{\alpha\beta}),(r_{\alpha\beta}))$ and $\mathbb{P}(\omega)$ in Theorem 4.3 below), which constitute the *scaled intrinsic* 2d-*equations of a linearly elastic plate* (the corresponding "de-scaled" equations are briefly discussed at the end of this section).

In what follows, $(\nu_\alpha) = (\nu^\alpha)$ denotes the unit inner normal vector field along $\partial\omega$ and $(\tau_\alpha) = (\tau^\alpha)$ where $\tau_1 := -\nu_2$ and $\tau_2 := \nu_1$ denotes a unit tangential vector field along $\partial\omega$ (in Sect. 3, the same vector fields were respectively denoted $\boldsymbol{n}(s)$ and $\boldsymbol{t}(s)$, $s \in I$, along a portion of $\partial\omega$ parametrized in terms of its curvilinear abscissa s); the associated normal and tangential derivative operators along $\partial\omega$ are denoted $\partial_\nu := \nu^\alpha\partial_\alpha$ and $\partial_\tau := \tau^\alpha\partial_\alpha$; the function $\kappa : \partial\omega \to \mathbb{R}$ denotes the signed curvature along $\partial\omega$; and finally, $d\omega := dx_1 dx_2$.

Theorem 4.3. (a) *Define the spaces*

$$\boldsymbol{V}(\omega) := \{\boldsymbol{\eta} = (\eta_i) = ((\eta_\alpha), \eta_3) \in \boldsymbol{H}^1(\omega) \times H^2(\omega);\\ \eta_i = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0\},$$

$$\mathbb{V}(\omega) := \{((d_{\alpha\beta}),(s_{\alpha\beta})) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega);\ \text{there exists } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega)\\ \text{such that } d_{\alpha\beta} = \frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta) \text{ and } s_{\alpha\beta} = \partial_{\alpha\beta}\eta_3\}.$$

Then the space $(\mathbb{V}(\omega), \|\cdot\|_{\mathbb{L}^2(\omega)\times\mathbb{L}^2(\omega)})$ *is a Hilbert space, and the mapping*

$$\boldsymbol{\varphi} : \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega) \to \boldsymbol{\varphi}(\boldsymbol{\eta}) = \Big(\frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta), (\partial_{\alpha\beta}\eta_3)\Big) \in \mathbb{V}(\omega)$$

is an isomorphism.

(b) *Let* $\boldsymbol{\kappa}(\varepsilon) = (\kappa_{ij}(\varepsilon)) \in \mathbb{V}(\Omega)$ *denote for each* $\varepsilon > 0$ *the unique solution to the variational equations* $\mathbb{P}(\varepsilon;\Omega)$ *of* Lemma 4.2. *Then, as* $\varepsilon \to 0$, *the family* $(\boldsymbol{\kappa}(\varepsilon))$ *converges in the space* $(\mathbb{V}(\Omega), \|\cdot\|_{\mathbb{L}^2(\Omega)})$ *towards a limit* $\boldsymbol{\kappa} = (\kappa_{ij})$ *of the form*

$$\kappa_{\alpha\beta} = c_{\alpha\beta} - x_3 r_{\alpha\beta},\ \kappa_{\alpha 3} = 0,\ \kappa_{33} = -\frac{\lambda}{\lambda + 2\mu}\kappa_{\sigma\sigma},$$

where

$$((c_{\alpha\beta}),(r_{\alpha\beta})) \in \mathbb{V}(\omega)$$

is the unique solution to the variational equations $\mathbb{P}(\omega)$, *viz.,*

$$\begin{aligned}&\int_\omega a^{\alpha\beta\sigma\tau} c_{\sigma\tau} d_{\alpha\beta}\, \mathrm{d}\omega + \frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau} r_{\sigma\tau} s_{\alpha\beta}\, \mathrm{d}\omega \\ &= \int_\omega p^i \psi_i((d_{\alpha\beta}), (s_{\alpha\beta)}))\, \mathrm{d}\omega \text{ for all } ((d_{\alpha\beta}), (s_{\alpha\beta)})) \in \mathbb{V}(\omega),\end{aligned}$$

where

$$\begin{aligned} a^{\alpha\beta\sigma\tau} &:= \frac{4\lambda\mu}{\lambda+2\mu}\delta^{\alpha\beta}\delta^{\sigma\tau} + 2\mu(\delta^{\alpha\sigma}\delta^{\beta\tau} + \delta^{\alpha\tau}\delta^{\beta\sigma}),\\ p^i &:= \int_{-1}^1 f^i\, \mathrm{d}x_3,\\ \boldsymbol{\psi} = (\psi_i) &:= \boldsymbol{\varphi}^{-1} : \mathbb{V}(\omega) \to \boldsymbol{V}(\omega).\end{aligned}$$

(c) *Assume that the boundary of* ω *is of class* $\mathcal{C}^2$ *and that the solution* $((c_{\alpha\beta}), (r_{\alpha\beta}))$ *to the variational equations* $\mathbb{P}(\omega)$ *is smooth enough. Then the tensor field* $(c_{\alpha\beta})$ *satisfies the following* (scaled) *intrinsic* 2d*-equations:*

$$\begin{aligned} -\partial_\beta(a^{\alpha\beta\sigma\tau}c_{\sigma\tau}) &= p^\alpha \quad \text{in } \omega,\\ \partial_{\tau\alpha}c_{\beta\sigma} + \partial_{\sigma\beta}c_{\alpha\tau} - \partial_{\sigma\alpha}c_{\beta\tau} - \partial_{\tau\beta}c_{\alpha\sigma} &= 0 \quad \text{in } \omega,\\ c_{\alpha\beta}\tau^\alpha\tau^\beta &= 0 \quad \text{on } \gamma_0,\\ \partial_\sigma c_{\alpha\beta}\tau^\alpha(\tau^\beta\nu^\sigma - 2\tau^\sigma\nu^\beta) - \kappa c_{\alpha\beta}\nu^\alpha\nu^\beta &= 0 \quad \text{on } \gamma_0,\\ a^{\alpha\beta\sigma\tau}c_{\sigma\tau}\nu_\beta &= 0 \quad \text{on } \gamma_1 = \partial\omega - \gamma_0,\end{aligned}$$

and the tensor field $(r_{\alpha\beta})$ *satisfies the following* (scaled) *intrinsic* 2d*-equations:*

$$\begin{aligned} \partial_{\alpha\beta}\Big(\frac{1}{3}a^{\alpha\beta\sigma\rho}r_{\sigma\rho}\Big) &= p^3 \quad \text{in } \omega,\\ \partial_\alpha r_{\beta\sigma} - \partial_\beta r_{\alpha\sigma} &= 0 \quad \text{in } \omega,\\ r_{\alpha\beta}\tau^\alpha\tau^\beta &= 0 \quad \text{on } \gamma_0,\\ r_{\alpha\beta}\tau^\alpha\nu^\beta &= 0 \quad \text{on } \gamma_0,\\ a^{\alpha\beta\sigma\rho}r_{\sigma\rho}\nu_\alpha\nu_\beta &= 0 \quad \text{on } \gamma_1,\\ (\partial_\alpha(a^{\alpha\beta\sigma\rho}r_{\sigma\rho}))\nu_\beta + \partial_\tau(a^{\alpha\beta\sigma\rho}r_{\sigma\rho}\nu_\alpha\tau_\beta) &= 0 \quad \text{on } \gamma_1.\end{aligned}$$

Proof. For clarity, the proof is broken into five steps, numbered (i) to (v).

(i) *Define the spaces* (the subscript "KL" reminds that the vector fields in the space $\boldsymbol{V}_{KL}(\Omega)$ are "scaled Kirchhoff-Love displacement fields"; cf. Theorem 1.4-4 in [8])

$$\begin{aligned}\boldsymbol{V}_{KL}(\Omega) := \{&(v_i) \in \boldsymbol{H}^1(\Omega);\ v_i = 0 \text{ on } \Gamma_0,\\ &\frac{1}{2}(\partial_i v_3 + \partial_3 v_i) = 0 \text{ in } \Omega\},\\ \mathbb{V}_{KL}(\Omega) := \{&(\kappa_{\alpha\beta}) \in \mathbb{L}^2(\Omega);\ \text{there exists } (v_i) \in \boldsymbol{V}_{KL}(\Omega) \text{ such that}\\ &\kappa_{\alpha\beta} = \frac{1}{2}(\partial_\beta v_\alpha + \partial_\alpha v_\beta) \text{ in } \Omega\}.\end{aligned}$$

Then the space $(\mathbb{V}_{KL}(\Omega); \|\cdot\|_{\mathbb{L}^2(\Omega)})$ *is a Hilbert space, and the mapping*

$$\boldsymbol{\Phi} : \boldsymbol{v} = (v_i) \in \boldsymbol{V}_{KL}(\Omega) \to \boldsymbol{\Phi}(\boldsymbol{v}) := \Big(\frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha)\Big) \in \mathbb{V}_{KL}(\Omega)$$

is an isomorphism.

The mapping $\boldsymbol{\Phi} : \boldsymbol{V}_{KL}(\Omega) \to \mathbb{V}_{KL}(\Omega)$ is clearly continuous, onto (by definition of the space $\mathbb{V}_{KL}(\Omega)$), and one-to-one, since

$$\boldsymbol{v} \in \boldsymbol{V}_{KL}(\Omega) \text{ and } \boldsymbol{\Phi}(\boldsymbol{v}) = \boldsymbol{0} \text{ implies } \frac{1}{2}(\partial_i v_j + \partial_j v_i) = 0 \text{ in } \Omega.$$

Hence $\boldsymbol{v} = \boldsymbol{0}$ in Ω since $\boldsymbol{v} = \boldsymbol{0}$ on Γ_0.

To show that $\mathbb{V}_{KL}(\Omega)$ is complete, let $(\boldsymbol{\kappa}^n)_{n=0}^\infty$ be a Cauchy sequence in $\mathbb{V}_{KL}(\Omega)$, and let $\boldsymbol{v}^n := \boldsymbol{\Phi}^{-1}(\boldsymbol{\kappa}^n) \in \boldsymbol{V}_{KL}(\Omega)$, $n \geq 0$. Since then

$$\begin{aligned}
\Big\|\frac{1}{2}(\partial_\beta v_\alpha^m + \partial_\alpha v_\beta^m) - \frac{1}{2}(\partial_\beta v_\alpha^n + \partial_\alpha v_\beta^n)\Big\|_{L^2(\Omega)} &\to 0 \text{ as } m, n \to \infty,\\
\Big\|\frac{1}{2}(\partial_i v_3^m + \partial_3 v_i^m) - \frac{1}{2}(\partial_i v_3^n + \partial_3 v_i^n)\Big\|_{L^2(\Omega)} &= 0 \text{ for all } m, n \geq 0,
\end{aligned}$$

the 3d-*Korn's inequality* implies that the sequence $(\boldsymbol{v}^n)_{n=0}^\infty$ is a Cauchy sequence in $\boldsymbol{V}(\Omega)$. Consequently,

$$\boldsymbol{\kappa}^n \to \boldsymbol{\Phi}(\boldsymbol{v}) \text{ as } n \to \infty,$$

where $\boldsymbol{v} = \lim_{n\to\infty} \boldsymbol{v}_n \in \boldsymbol{V}_{KL}(\Omega)$. Hence the space $\mathbb{V}_{KL}(\Omega)$ is complete, which in turn shows that $\boldsymbol{\Phi} : \boldsymbol{V}_{KL}(\Omega) \to \mathbb{V}_{KL}(\Omega)$ is an isomorphism.

(ii) Let the spaces $\boldsymbol{V}(\omega)$ and $\mathbb{V}(\omega)$ and the mapping $\boldsymbol{\varphi} : \boldsymbol{V}(\omega) \to \mathbb{V}(\omega)$ be defined as in the statement of the theorem. Then the mapping $\boldsymbol{\varphi}$ is clearly continuous, onto (by definition of the space $\mathbb{V}(\omega)$), and one-to-one since $\boldsymbol{\eta} = ((\eta_\alpha), \eta_3) \in \boldsymbol{V}(\omega)$ and $\boldsymbol{\varphi}(\boldsymbol{\eta}) = \boldsymbol{0}$ implies $\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) = 0$ and $\partial_{\alpha\beta}\eta_3 = 0$ in ω. Hence $\eta_\alpha = 0$ since $\eta_\alpha = 0$ on γ_0 and $\eta_3 = 0$ since $\eta_3 = \partial_\nu \eta_3 = 0$ on γ_0.

To show that the space $\mathbb{V}(\omega)$ is complete, let $((d_{\alpha\beta}^n, (s_{\alpha\beta}^n))_{n=0}^\infty$ be a Cauchy sequence in the space $\mathbb{V}(\omega)$, and let $\boldsymbol{\eta}^n := \boldsymbol{\varphi}^{-1}((d_{\alpha\beta}^n), (s_{\alpha\beta}^n)) \in \boldsymbol{V}(\omega)$, $n \geq 0$. Since then

$$\begin{aligned}
\|\frac{1}{2}(\partial_\beta \eta_\alpha^m + \partial_\alpha \eta_\beta^m) - \frac{1}{2}(\partial_\beta \eta_\alpha^n + \partial_\alpha \eta_\beta^n)\|_{L^2(\omega)} &\to 0 \text{ as } m, n \to \infty,\\
\|\partial_{\alpha\beta}\eta_3^m - \partial_{\alpha\beta}\eta_3^n\|_{L^2(\omega)} &\to 0 \text{ as } m, n \to \infty,
\end{aligned}$$

the 2d-*Korn inequality* implies that the sequence $((\eta_\alpha^n))_{n=0}^\infty$ converges in the space $\{(\eta_\alpha) \in \boldsymbol{H}^1(\omega); \eta_\alpha = 0 \text{ on } \gamma_0\}$, and the equivalence of the norm $\|\cdot\|_{H^2(\omega)}$ with the semi-norm $\eta \to \sum_{\alpha,\beta} \|\partial_{\alpha\beta}\eta\|_{L^2(\omega)}$ over the space $\{\eta \in H^2(\omega); \eta = \partial_\nu \eta = 0 \text{ on } \gamma_0\}$ implies that the sequence $(\eta_3^n)_{n=0}^\infty$ converges in this space. Consequently,

$$((d_{\alpha\beta}^n), (s_{\alpha\beta}^n)) \to \boldsymbol{\varphi}(\boldsymbol{\eta}) \text{ where } \boldsymbol{\eta} = ((\eta_\alpha), \eta_3) := \lim_{n\to\infty}((\eta_\alpha^n), n_3^n) \in \boldsymbol{V}(\omega).$$

Hence the space $\mathbb{V}(\omega)$ is complete, which in turn shows that $\boldsymbol{\varphi} : \boldsymbol{V}(\omega) \to \mathbb{V}(\omega)$ is an isomorphism. *This proves* (a).

(iii) *The family* $(\boldsymbol{\kappa}(\varepsilon))_{\varepsilon>0}$ *converges in the space* $\mathbb{V}(\Omega)$ *as* $\varepsilon \to 0$ *to a limit* $\boldsymbol{\kappa} = (\kappa_{ij})$, *where*

$$\kappa_{\alpha 3} = 0 \text{ and } \kappa_{33} = -\frac{\lambda}{\lambda + 2\mu}\kappa_{\sigma\sigma},$$

and the tensor $(\kappa_{\alpha\beta})$ *belongs to the space* $\mathbb{V}_{KL}(\Omega)$ *and is the unique solution to the variational equations* $\mathbb{P}_{KL}(\Omega)$, *viz.*,

$$\frac{1}{2}\int_\Omega a^{\alpha\beta\sigma\tau}\kappa_{\sigma\tau}\chi_{\alpha\beta}\,\mathrm{d}x = \int_\Omega f^i\Psi_i(\boldsymbol{\chi})\,\mathrm{d}x \text{ for all } \boldsymbol{\chi} = (\chi_{\alpha\beta}) \in \mathbb{V}_{KL}(\Omega),$$

where

$$\begin{aligned} a^{\alpha\beta\sigma\tau} &:= \frac{4\lambda\mu}{\lambda+2\mu}\delta^{\alpha\beta}\delta^{\sigma\tau} + 2\mu(\delta^{\alpha\sigma}\delta^{\beta\tau} + \delta^{\alpha\tau}\delta^{\beta\sigma}) \\ (\Psi_i) &:= \boldsymbol{\Phi}^{-1} : \mathbb{V}_{KL}(\Omega) \to \boldsymbol{V}_{KL}(\Omega). \end{aligned}$$

That the family $(\boldsymbol{\kappa}(\varepsilon))_{\varepsilon>0}$ converges with respect to the norm $\|\cdot\|_{\mathbb{L}^2(\Omega)}$, hence in the space $\mathbb{V}_{KL}(\Omega)$ which is closed in $\mathbb{L}^2(\Omega)$, to a limit $\boldsymbol{\kappa}$ of the form indicated above is established in the proof of Theorem 1.4-1 of [8].

It is well-known that, thanks to the assumptions $\lambda \geq 0$ and $\mu > 0$, the fourth-order tensor $(a^{\alpha\beta\sigma\tau})$ is positive-definite, i.e., there exists a constant $c > 0$ such that

$$a^{\alpha\beta\sigma\tau}t_{\sigma\tau}t_{\alpha\beta} \geq c\sum_{\alpha,\beta}|t_{\alpha\beta}|^2 \text{ for all } (t_{\alpha\beta}) \in \mathbb{S}^2,$$

where $\mathbb{S}^2$ denotes the set of all 2×2 symmetric matrices.

Besides, the linear form

$$\boldsymbol{\chi} = (\chi_{\alpha\beta}) \in \mathbb{V}_{KL}(\Omega) \to \int_\Omega f^i\Psi_i(\boldsymbol{\chi})\,\mathrm{d}x$$

is continuous since $\boldsymbol{\Psi} = (\Psi_i) \in \mathcal{L}(\mathbb{V}_{KL}(\Omega); \boldsymbol{V}_{KL}(\Omega))$ by (i). Therefore the variational equations $\mathbb{P}_{KL}(\Omega)$ have a unique solution $(\kappa_{\alpha\beta}) \in \mathbb{V}_{KL}(\Omega)$.

(iv) It also follows from the proof of Theorem 1.4-1 in [8] that the functions $\kappa_{\alpha\beta}$ are of the form

$$\kappa_{\alpha\beta} = \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha),$$

where the functions u_α are of the form

$$u_\alpha(\cdot, x_3) = \zeta_\alpha - x_3\partial_\alpha\zeta_3,$$

the vector field $\boldsymbol{\zeta} = (\zeta_i) = ((\zeta_\alpha), \zeta_3) \in \boldsymbol{V}(\omega)$ (the space $\boldsymbol{V}(\omega)$ is defined in the statement of the theorem) being the unique solution to the variational equations

$$\int_\omega a^{\alpha\beta\sigma\tau}\frac{1}{2}(\partial_\tau\zeta_\sigma + \partial_\sigma\zeta_\tau)\frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta)\,\mathrm{d}\omega + \frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\zeta_3\partial_{\alpha\beta}\eta_3\,\mathrm{d}\omega$$

$$= \int_\omega p^i\eta_i\,\mathrm{d}\omega \text{ for all } ((\eta_\alpha), \eta_3) \in \boldsymbol{V}(\omega),$$

where

$$p^i := \int_{-1}^{1} f^i \, \mathrm{d}x_3.$$

Letting

$$c_{\alpha\beta} := \frac{1}{2}(\partial_\beta \zeta_\alpha + \partial_\alpha \zeta_\beta) \text{ and } r_{\alpha\beta} := \partial_{\alpha\beta}\zeta_3,$$

$$d_{\alpha\beta} := \frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) \text{ and } s_{\alpha\beta} := \partial_{\alpha\beta}\eta_3 \quad \text{for each } ((\eta_\alpha), \eta_3) \in \boldsymbol{V}(\omega),$$

then shows that $((c_{\alpha\beta}), (r_{\alpha\beta})) \in \mathbb{V}(\omega)$ is the unique solution to the variational equations $\mathbb{P}(\omega)$. *This proves* (b).

(v) Assume that the solution $((c_{\alpha\beta}), (r_{\alpha\beta}))$ to the variational equations $\mathbb{P}(\omega)$ is smooth enough. Since these equations are equivalent to the variational equations

$$\int_\omega a^{\alpha\beta\sigma\tau} c_{\sigma\tau} \frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) \, \mathrm{d}\omega + \frac{1}{3} \int_\omega a^{\alpha\beta\sigma\tau} r_{\sigma\tau} \partial_{\alpha\beta}\eta_3 \, \mathrm{d}\omega = \int_\omega p^i \eta_i \, \mathrm{d}\omega$$

for all $((\eta_\alpha), \eta_3) \in \boldsymbol{V}(\omega)$, and since

$$a^{\alpha\beta\sigma\tau} c_{\sigma\tau} \frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) = a^{\alpha\beta\sigma\tau} c_{\sigma\tau} \partial_\beta \eta_\alpha,$$

the Green formula and the definition of the mapping $(\psi_i) := \boldsymbol{\varphi}^{-1} : \mathbb{V}(\omega) \to \boldsymbol{V}(\omega)$ together show that $((c_{\alpha\beta}), (r_{\alpha\beta}))$ satisfies the following boundary value problem (by construction, the components ψ_α depend only on $(c_{\alpha\beta})$, while the component ψ_3 depends only on $(r_{\alpha\beta})$):

$$\begin{aligned}
-\partial_\beta(a^{\alpha\beta\sigma\tau} c_{\sigma\tau}) &= \; p^\alpha && \text{in } \omega,\\
\partial_{\tau\alpha} c_{\beta\sigma} + \partial_{\sigma\beta} c_{\alpha\tau} - \partial_{\sigma\alpha} c_{\beta\tau} - \partial_{\tau\beta} c_{\alpha\sigma} &= \; 0 && \text{in } \omega,\\
\psi_\sigma((c_{\alpha\beta})) &= \; 0 && \text{on } \gamma_0,\\
a^{\alpha\beta\sigma\tau} c_{\sigma\tau} \nu_\beta &= \; 0 && \text{on } \gamma_1,\\
\partial_{\alpha\beta}(a^{\alpha\beta\sigma\tau} r_{\sigma\tau}) &= \; p^3 && \text{in } \omega,\\
\partial_\alpha r_{\beta\sigma} - \partial_\beta r_{\alpha\sigma} &= \; 0 && \text{in } \omega,\\
\psi_3((r_{\alpha\beta})) = \partial_\nu \psi_3((r_{\alpha\beta})) &= \; 0 && \text{on } \gamma_0,\\
a^{\alpha\beta\sigma\tau} r_{\sigma\tau} \nu_\alpha \nu_\beta &= \; 0 && \text{on } \gamma_1,\\
(\partial_\alpha(a^{\alpha\beta\sigma\tau} r_{\sigma\tau}))\nu_\beta + \partial_\tau(a^{\alpha\beta\sigma\rho} r_{\sigma\rho} \nu_\alpha \nu_\beta) &= \; 0 && \text{on } \gamma_1.
\end{aligned}$$

Note that the second and sixth equations of the above boundary value problem are *necessarily* satisfied by any element $((c_{\alpha\beta}), (c_{\alpha\beta}))$ of the space $\mathbb{V}(\omega)$; cf. [11].

It was shown in [9] that, if $\partial\omega$ is of class $\mathcal{C}^2$, the boundary conditions

$$\psi_\sigma((c_{\alpha\beta})) = 0 \quad \text{on } \gamma_0,$$

or equivalently $\zeta_\alpha = 0$ on γ_0, are equivalent to the boundary conditions

$$\begin{aligned} c_{\alpha\beta}\tau^\alpha\tau^\beta &= 0 \quad \text{on } \gamma_0, \\ \partial_\sigma c_{\alpha\beta}\tau^\alpha(\tau^\beta\nu^\sigma - 2\tau^\sigma\nu^\beta) - \kappa c_{\alpha\beta}\nu^\alpha\nu^\beta &= 0 \quad \text{on } \gamma_0, \end{aligned}$$

and that the boundary conditions

$$\psi_3((r_{\alpha\beta})) = \partial_\nu\psi_3((r_{\alpha\beta})) = 0 \text{ on } \gamma_0,$$

or equivalently $\zeta_3 = \partial_\nu\zeta_3 = 0$ on γ_0, are equivalent to the boundary conditions

$$\begin{aligned} r_{\alpha\beta}\tau^\alpha\tau^\beta &= 0 \ \text{ on } \gamma_0, \\ r_{\alpha\beta}\tau^\alpha\nu^\beta &= 0 \ \text{ on } \gamma_0. \end{aligned}$$

This proves (c). □

In order to retrieve physically significant unknowns and equations, it remains to "de-scale" the unknowns and equations found in Theorem 4.3. More specifically, let

$$\begin{aligned} c^\varepsilon_{\alpha\beta} &:= \varepsilon^2 c_{\alpha\beta} \quad \text{and} \quad r^\varepsilon_{\alpha\beta} := \varepsilon r_{\alpha\beta}, \\ n^{\alpha\beta,\varepsilon} &:= \varepsilon a^{\alpha\beta\sigma\tau} c^\varepsilon_{\sigma\tau} \quad \text{and} \quad m^{\alpha\beta,\varepsilon} := \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau} r^\varepsilon_{\sigma\tau}, \end{aligned}$$

where $((c_{\alpha\beta}), (r_{\alpha\beta})) \in \mathbb{V}(\omega)$ denotes the unique solution to the variational equations $\mathbb{P}(\omega)$ found in Theorem 4.3(b), and let

$$p^{i,\varepsilon} := \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}\,\mathrm{d}x_3^\varepsilon,$$

where the functions $f^{i,\varepsilon}$ are those appearing in Theorem 4.1. Then it immediately follows from Theorem 4.3(c) that, if $((c_{\alpha\beta}), (r_{\alpha\beta}))$ is smooth enough, the following *intrinsic* 2d-*equations of a linearly elastic plate* are satisfied:

$$\begin{aligned} -\partial^\varepsilon_\beta n^{\alpha\beta,\varepsilon} &= p^{\alpha,\varepsilon} \ \text{ in } \omega, \\ \partial^\varepsilon_{\tau\alpha}c^\varepsilon_{\beta\sigma} + \partial^\varepsilon_{\sigma\beta}c^\varepsilon_{\alpha\tau} - \partial^\varepsilon_{\sigma\alpha}c^\varepsilon_{\beta\tau} - \partial^\varepsilon_{\tau\beta}c^\varepsilon_{\alpha\sigma} &= 0 \quad \text{in } \omega, \\ c^\varepsilon_{\alpha\beta}\tau^\alpha\tau^\beta &= 0 \quad \text{on } \gamma_0, \\ \partial_\sigma c^\varepsilon_{\alpha\beta}\tau^\alpha(\tau^\beta\nu^\sigma - 2\tau^\sigma\nu^\beta) - \kappa c^\varepsilon_{\alpha\beta}\nu^\alpha\nu^\beta &= 0 \quad \text{on } \gamma_0, \\ n^{\alpha\beta,\varepsilon}\nu_\beta &= 0 \quad \text{on } \gamma_1, \\ \partial^\varepsilon_{\alpha\beta}m^{\alpha\beta,\varepsilon} &= p^{3,\varepsilon} \ \text{ in } \omega, \\ \partial^\varepsilon_\alpha r^\varepsilon_{\beta\sigma} - \partial^\varepsilon_\beta r^\varepsilon_{\alpha\sigma} &= 0 \quad \text{in } \omega, \\ r^\varepsilon_{\alpha\beta}\tau^\alpha\tau^\beta &= 0 \quad \text{on } \gamma_0, \\ r^\varepsilon_{\alpha\beta}\tau^\alpha\nu^\beta &= 0 \quad \text{on } \gamma_0, \\ m^{\alpha\beta,\varepsilon}\nu_\alpha\nu_\beta &= 0 \quad \text{on } \gamma_1, \\ (\partial_\alpha m^{\alpha\beta,\varepsilon})\nu_\beta + \partial_\tau(\partial_\alpha m^{\alpha\beta,\varepsilon}\nu_\alpha\tau_\beta) &= 0 \quad \text{on } \gamma_1. \end{aligned}$$

We have thus retrieved the intrinsic 2d-equations found in [9] through a completely different approach (based on an asymptotic analysis of the intrinsic 3d-equations).

The functions $c^\varepsilon_{\alpha\beta}$, resp. $r^\varepsilon_{\alpha\beta}$, represent the *linearized change of metric*, resp., *change of curvature tensors*, of the middle surface $\overline{\omega}$ of the plate, while the functions $n^{\alpha\beta,\varepsilon}$, resp. $m^{\alpha\beta,\varepsilon}$, represent the *stress resultants*, resp. the *bending moments*, inside the plate. The boundary value problem satisfied by the tensor field $(n^{\alpha\beta,\varepsilon})$, resp. the tensor field $(m^{\alpha\beta,\varepsilon})$, constitutes the *intrinsic membrane*, resp. *flexural, equations of a linearly elastic plate.*

5. Concluding remarks

Consider again a linearly elastic plate with $\overline{\Omega}^\varepsilon = \overline{\omega} \times [-\varepsilon, \varepsilon]$, $\varepsilon > 0$, as its reference configuration and clamped over the portion $\Gamma^\varepsilon_0 = \gamma_0 \times \,]-\varepsilon, \varepsilon[$ of its lateral face, where ω is a simply-connected domain in $\mathbb{R}^2$ and γ_0 is a non-empty relatively open connected subset of class $\mathcal{C}^4$ of the boundary $\partial\omega$. As recalled in the Introduction, it follows from [7] that the space

$$\mathbb{V}(\Omega^\varepsilon) = \{\boldsymbol{e}^\varepsilon \in \mathbb{L}^2(\Omega^\varepsilon);\ \text{there exists } \boldsymbol{v}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon) \\ \text{such that } \boldsymbol{e}^\varepsilon = \boldsymbol{\nabla}^\varepsilon_s \boldsymbol{v}^\varepsilon\}$$

as defined in Theorem 4.1(a) can be given another equivalent definition in this case, viz.,

$$\mathbb{V}(\Omega^\varepsilon) = \{\boldsymbol{t}^\varepsilon \in \mathbb{E}^2(\Omega^\varepsilon);\ \tilde{\gamma}^\sharp_{\alpha\beta}(\boldsymbol{t}^\varepsilon) = 0 \ \text{ in } H^{-1}(\Gamma^\varepsilon_0) \\ \text{and } \tilde{\rho}^\sharp_{\alpha\beta}(\boldsymbol{t}^\varepsilon) = 0 \text{ in } H^{-2}(\Gamma^\varepsilon_0)\},$$

where

$$\mathbb{E}(\Omega^\varepsilon) := \{\boldsymbol{t}^\varepsilon = (t^\varepsilon_{ij}) \in \mathbb{L}^2(\Omega^\varepsilon); \\ \partial^\varepsilon_{j\ell} t^\varepsilon_{ki} + \partial^\varepsilon_{ik} t^\varepsilon_{\ell j} - \partial^\varepsilon_{i\ell} t^\varepsilon_{kj} - \partial^\varepsilon_{jk} t^\varepsilon_{\ell i} = 0 \text{ in } H^{-2}(\Omega^\varepsilon)\}$$

and the operators

$$(\tilde{\gamma}^\sharp_{\alpha\beta}) \in \mathcal{L}(\mathbb{E}(\Omega^\varepsilon); \mathbb{H}^{-1}(\Gamma^\varepsilon_0)) \ \text{ and } \ (\tilde{\rho}^\sharp_{\alpha\beta}) \in \mathcal{L}(\mathbb{E}(\Omega^\varepsilon); \mathbb{H}^{-2}(\Gamma^\varepsilon_0))$$

are defined as recalled in Sect. 2. More specifically, the *Saint-Venant compatibility conditions*

$$\mathcal{S}^\varepsilon_{ijk\ell}(\boldsymbol{t}^\varepsilon) := \partial^\varepsilon_{j\ell} t^\varepsilon_{ki} + \partial^\varepsilon_{ik} t^\varepsilon_{\ell j} - \partial^\varepsilon_{i\ell} t^\varepsilon_{kj} - \partial^\varepsilon_{jk} t^\varepsilon_{\ell i} = 0 \text{ in } H^{-2}(\Omega^\varepsilon)$$

are necessary, and sufficient under the *additional assumptions that the domain Ω^ε is simply-connected* (cf. [6]), for a tensor field $\boldsymbol{t}^\varepsilon = (t^\varepsilon_{ij}) \in \mathbb{L}^2(\Omega^\varepsilon)$ to be of the form

$$\boldsymbol{t}^\varepsilon = \boldsymbol{\nabla}^\varepsilon_s \boldsymbol{v}^\varepsilon \text{ for some } \boldsymbol{v}^\varepsilon \in \boldsymbol{H}^1(\Omega)^\varepsilon,$$

and that the definition of the operators $\tilde{\gamma}^\sharp_{\alpha\beta}$ and $\tilde{\rho}^\sharp_{\alpha\beta}$ as given in [7] hinges essentially on the *additional assumption that γ_0 is of class $\mathcal{C}^4$.*

We briefly discuss in this section the effect of the asymptotic analysis carried out in Sect. 4 from the above perspective. To begin with, we consider the above *compatibility conditions* $\mathcal{S}^{\varepsilon}_{ijk\ell}(\boldsymbol{t}^{\varepsilon}) = 0$ *in* $H^{-2}(\Omega^{\varepsilon})$.

The notations used below are those of Sect. 4. Under the scalings performed at the beginning of Sect. 4, these compatibility conditions become

$$\begin{aligned}
\partial_{\tau\alpha}\kappa_{\beta\sigma}(\varepsilon) + \partial_{\sigma\beta}\kappa_{\alpha\tau}(\varepsilon) - \partial_{\sigma\alpha}\kappa_{\beta\tau}(\varepsilon) - \partial_{\tau\beta}\kappa_{\alpha\sigma}(\varepsilon) = 0 \ \text{ in } H^{-2}(\Omega),\\
\varepsilon(\partial_{3\alpha}\kappa_{3\beta}(\varepsilon) + \partial_{\beta 3}\kappa_{\alpha 3}(\varepsilon)) - \varepsilon^2\partial_{\alpha\beta}\kappa_{33}(\varepsilon) - \partial_{33}\kappa_{\alpha\beta}(\varepsilon)) = 0 \ \text{ in } H^{-2}(\Omega),\\
\partial_{3\alpha}\kappa_{\beta\sigma}(\varepsilon) + \varepsilon(\partial_{\sigma\beta}\kappa_{\alpha 3}(\varepsilon) - \partial_{\sigma\alpha}\kappa_{\beta 3}(\varepsilon)) - \partial_{3\beta}\kappa_{\alpha\sigma}(\varepsilon) = 0 \ \text{ in } H^{-2}(\Omega).
\end{aligned}$$

Then the convergence $\boldsymbol{\kappa}(\varepsilon) = (\kappa_{ij}(\varepsilon)) \to \boldsymbol{\kappa} = (\kappa_{ij})$ in $\mathbb{L}^2(\Omega)$ established in Theorem 4.3(b) implies that (simply by taking the limits as $\varepsilon \to 0$ in the above relations):

$$\begin{aligned}
\partial_{\tau\alpha}\kappa_{\beta\sigma} + \partial_{\sigma\beta}\kappa_{\alpha\tau} - \partial_{\sigma\alpha}\kappa_{\beta\tau} - \partial_{\tau\beta}\kappa_{\alpha\sigma} &= 0 \ \text{ in } H^{-2}(\Omega),\\
-\partial_{33}\kappa_{\alpha\beta} &= 0 \ \text{ in } H^{-2}(\Omega),\\
\partial_{3\alpha}\kappa_{\beta\sigma} - \partial_{3\beta}\kappa_{\alpha\sigma} &= 0 \ \text{ in } H^{-2}(\Omega).
\end{aligned}$$

It is then an easy matter to show that, together, the above "limit relations" are necessary and sufficient for a tensor field

$$(\kappa_{ij}) \in \mathbb{L}^2(\Omega) \text{ such that } \kappa_{i3} = 0 \ \text{ in } \Omega$$

to be such that there exists a vector field

$$\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega) \text{ such that } \kappa_{\alpha\beta} = (\boldsymbol{\nabla}_s\boldsymbol{v})_{\alpha\beta} \text{ and } \kappa_{i3} = (\boldsymbol{\nabla}_s\boldsymbol{v})_{i3} \ \text{ in } \Omega.$$

Note that the limit relations

$$\partial_{33}\kappa_{\alpha\beta} = 0 \ \text{ in } H^{-2}(\Omega)$$

satisfied by the 2×2 tensor field $(\kappa_{\alpha\beta}) \in \mathbb{L}^2(\Omega)$ implies that there exist functions $c_{\alpha\beta} \in L^2(\omega)$ and $r_{\alpha\beta} \in L^2(\omega)$ such that

$$\kappa_{\alpha\beta}(\cdot, x_3) = c_{\alpha\beta} - x_3 r_{\alpha\beta} \ \text{ in } \Omega$$

(cf., e.g., [10]), a conclusion that was also reached, but through a different means, in the course of the proof of Theorem 4.3.

Inserted into the limit equations $\partial_{3\alpha}\kappa_{\beta\sigma} - \partial_{3\beta}\kappa_{\alpha\sigma} = 0$, the above specific form of the functions $\kappa_{\alpha\beta}$ implies that

$$\partial_\alpha r_{\beta\sigma} = \partial_\beta r_{\alpha\sigma} \ \text{ in } H^{-1}(\omega),$$

while, inserted into the limit equations $\partial_{\tau\alpha}\kappa_{\beta\sigma} + \partial_{\sigma\beta}\kappa_{\alpha\tau} - \partial_{\sigma\alpha}\kappa_{\beta\tau} - \partial_{\tau\beta}\kappa_{\alpha\sigma} = 0$, it implies that

$$\partial_{\tau\alpha}c_{\beta\sigma} + \partial_{\sigma\beta}c_{\alpha\tau} - \partial_{\sigma\alpha}c_{\beta\tau} - \partial_{\tau\beta}c_{\alpha\sigma} = 0 \ \text{ in } H^{-2}(\omega)$$

(the factors of x_3 in the resulting equations vanish thanks to the already established relations $\partial_\alpha r_{\beta\sigma} = \partial_\beta r_{\alpha\sigma}$, which imply that $\partial_{\alpha\tau} r_{\beta\sigma} = \partial_{\beta\tau} r_{\alpha\sigma}$ and $\partial_{\sigma\beta} r_{\alpha\tau} = \partial_{\alpha\sigma} r_{\beta\tau}$).

Interestingly, the above compatibility conditions, which are *necessarily* satisfied by the 2×2 tensor fields

$$\boldsymbol{r} = (r_{\alpha\beta}) \in \mathbb{L}^2(\omega) \text{ and } \boldsymbol{c} = (c_{\alpha\beta}) \in \mathbb{L}^2(\omega)$$

become also *sufficient*, if the domain ω is *simply-connected*, for the existence of functions $\zeta_3 \in H^2(\omega)$ and vector fields $(\zeta_\alpha) \in \boldsymbol{H}^1(\omega)$ such that

$$r_{\alpha\beta} = \partial_{\alpha\beta}\zeta_3 \text{ and } c_{\alpha\beta} = \frac{1}{2}(\partial_\beta\zeta_\alpha + \partial_\alpha\zeta_\beta)$$

(cf. Theorem 2.4 in [11]).

Under the additional assumption that the subset γ_0 of $\partial\omega$ is of class $\mathcal{C}^2$ and connected, it was shown in Theorem 4.1 of [9] (and already mentioned at the end of Sect. 3) that the boundary conditions on γ_0 appearing in Theorem 4.3(c), viz.,

$$\begin{aligned} c_{\alpha\beta}\tau^\alpha\tau^\beta &= 0 \text{ on } \gamma_0, \\ \partial_\sigma c_{\alpha\beta}\tau^\alpha(\tau^\beta\nu^\sigma - 2\tau^\sigma\nu^\beta) - \kappa c_{\alpha\beta}\nu^\alpha\nu^\beta &= 0 \text{ on } \gamma_0, \end{aligned}$$

and

$$\begin{aligned} r_{\alpha\beta}\tau^\alpha\tau^\beta &= 0 \text{ on } \gamma_0, \\ r_{\alpha\beta}\tau^\alpha\nu^\beta &= 0 \text{ on } \gamma_0, \end{aligned}$$

are respectively equivalent to the boundary conditions

$$\zeta_\alpha + \eta_\alpha = 0 \text{ on } \gamma_0 \text{ and } \zeta_3 + \eta_3 = \partial_\nu(\zeta_3 + \eta_3) = 0 \text{ on } \gamma_0,$$

for some infinitesimal rigid displacement $\boldsymbol{\eta} = ((\eta_\alpha), \eta_3) : \omega \to \mathbb{E}^3$ of the form

$$\begin{aligned} \eta_1(x_1, x_2) = a_1 - b_1 x_2, \quad \eta_2(x_1, x_2) = a_2 + b_1 x_1, \quad \text{and} \\ \eta_3(x_1, x_2) = a_3 + b_2 x_1 + b_3 x_2, \quad (x_1, x_2) \in \omega, \end{aligned}$$

for some constants $a_i \in \mathbb{R}$ and $b_i \in \mathbb{R}$.

The above boundary conditions, which are thus expressed in terms of the restrictions to γ_0 of the tensor fields $\boldsymbol{c} = (c_{\alpha\beta})$ and $\boldsymbol{r} = (r_{\alpha\beta})$ thus play the same role for the intrinsic 2d-equations of Theorem 4.3(d) as that played by the boundary conditions

$$\tilde{\gamma}^\sharp_{\alpha\beta}(\boldsymbol{e}) = 0 \text{ and } \tilde{\rho}^\sharp_{\alpha\beta}(\boldsymbol{e}) = 0 \text{ on } \Gamma_0$$

for the intrinsic 3d-equations (Sects. 2 and 3). This observation is the basis for the following result (for simplicity, only scaled equations are considered here).

Theorem 5.1. *Let ω be a simply-connected domain in $\mathbb{R}^2$, and let γ_0 be a nonempty relatively open subset of class $\mathcal{C}^2$ of the boundary $\partial\omega$. Define the spaces*

$$\begin{aligned} \mathbb{X}_H(\omega) &:= \Big\{ \Big(\frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta)\Big) \in \mathbb{C}^1(\overline{\omega});\ (\eta_\alpha) \in \boldsymbol{\mathcal{C}}^2(\overline{\omega}) \Big\}, \\ \mathbb{X}_3(\omega) &:= \{ (\partial_{\alpha\beta}\eta_3) \in \mathbb{C}^0(\overline{\omega});\ \eta_3 \in \mathcal{C}^2(\overline{\omega}) \}. \end{aligned}$$

Then the linear operators

$$\begin{aligned} \gamma_H^\flat : \mathbb{X}_H(\omega) &\to \mathcal{C}^1(\gamma_0) \times \mathcal{C}^0(\gamma_0), \\ \gamma_3^\flat : \mathbb{X}_3(\omega) &\to \mathcal{C}^0(\gamma_0) \times \mathcal{C}^0(\gamma_0), \end{aligned}$$

defined by

$$\begin{aligned} \boldsymbol{\gamma}_H^\flat(d_{\alpha\beta}) &= ((d_{\alpha\beta}\tau^\alpha\tau^\beta)|_{\gamma_0}, (\partial_\sigma d_{\alpha\beta}\tau^\alpha(\tau^\beta\nu^\sigma - 2\tau^\sigma\nu^\beta) - \kappa d_{\alpha\beta}\nu^\alpha\nu^\beta)|_{\gamma_0}), \\ \boldsymbol{\gamma}_3^\flat(s_{\alpha\beta}) &= ((s_{\alpha\beta}\tau^\alpha\tau^\beta)|_{\gamma_0}, (s_{\alpha\beta}\tau^\alpha\nu^\beta)|_{\gamma_0}), \end{aligned}$$

admit unique continuous linear extensions

$$\begin{aligned} \tilde{\boldsymbol{\gamma}}_H^\flat : \mathbb{E}_H(\omega) &\to H^{-1}(\gamma_0) \times H^{-2}(\gamma_0), \\ \tilde{\boldsymbol{\gamma}}_3^\flat : \mathbb{E}_3(\omega) &\to H^{-1}(\gamma_0) \times H^{-1}(\gamma_0), \end{aligned}$$

over the spaces

$$\begin{aligned} \mathbb{E}_H(\omega) := \{(d_{\alpha\beta}) \in \mathbb{L}^2(\omega);\ &\partial_{\tau\alpha}d_{\beta\sigma} + \partial_{\sigma\beta}d_{\alpha\tau} \\ &- \partial_{\sigma\alpha}d_{\beta\tau} - \partial_{\tau\beta}d_{\alpha\sigma} = 0 \in H^{-2}(\omega)\}, \\ \mathbb{E}_3(\omega) := \{(s_{\alpha\beta}) \in \mathbb{L}^2(\omega);\ &\partial_\alpha s_{\beta\sigma} - \partial_\beta s_{\alpha\sigma} = 0 \in H^{-1}(\omega)\}. \end{aligned}$$

Besides, the space

$$\begin{aligned} \mathbb{V}(\omega) := \{((d_{\alpha\beta}), (s_{\alpha\beta})) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega);\ \text{there exists} \\ \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega) \text{ such that } d_{\alpha\beta} = \tfrac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta) \text{ and } s_{\alpha\beta} = \partial_{\alpha\beta}\eta_3\} \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{V}(\omega) := \{\boldsymbol{\eta} = (\eta_i) = ((\eta_\alpha), \eta_3) \in \boldsymbol{H}^1(\omega) \times H^2(\omega); \\ \eta_i = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0\}, \end{aligned}$$

appearing in Theorem 4.3 *can be equivalently defined in this case as*

$$\begin{aligned} \mathbb{V}(\omega) := \{((d_{\alpha\beta}), (s_{\alpha\beta})) \in \mathbb{E}_H(\omega) \times \mathbb{E}_3(\omega); \\ \tilde{\boldsymbol{\gamma}}_H^\flat((d_{\alpha\beta})) = 0 \ \text{ in } H^{-1}(\gamma_0) \times H^{-2}(\gamma_0), \\ \tilde{\boldsymbol{\gamma}}_3^\flat((s_{\alpha\beta})) = 0 \ \text{ in } H^{-1}(\gamma_0) \times H^{-1}(\gamma_0)\}. \end{aligned}$$

Proof. The proof, long and technical, is otherwise similar to that of Theorems 3.2 and 4.1 in [7]; for this reason, it is omitted. □

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