Invasion probabilities, hitting times, and some fluctuation theory for the stochastic logistic process

Todd Parsons

Invasion Probabilities, Hitting Times, and Some Fluctuation Theory for the Stochastic Logistic Process

Todd L. Parsons[1]

[1]Laboratoire de Probabilités, Statistique et Modélisation, CNRS UMR 8001, Sorbonne Université, Paris, 75005, France.

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Abstract

We consider excursions for a class of stochastic processes describing a population of discrete individuals experiencing density-limited growth, such that the population has a finite carrying capacity and behaves qualitatively like the classical logistic model [20] when the carrying capacity is large. Being discrete and stochastic, however, our population nonetheless goes extinct in finite time. We present results concerning the maximum of the population prior to extinction in the large population limit, from which we obtain establishment probabilities and upper bounds for the process, as well as estimates for the waiting time to establishment and extinction. As a consequence, we show that conditional upon establishment, the stochastic logistic process will with high probability greatly exceed carrying capacity an arbitrary number of times prior to extinction.

1 Introduction

1.1 Population Variability Analysis, a Cautionary Tale

Consider the time series in Figure 1, which gives the number of individuals in a hypothetical population. Time is measured such that the expected age of first reproduction is one time unit, so the figure seems to show a population experiencing several generations (which might be years, or even decades, for large mammals *e.g.,* [19]) fluctuating about a carrying capacity of about seventy individuals, followed by a rapid decline. If this were a population of interest, how should we respond to such census data? Is this cause for alarm?

In this particular case, the answer is no. The above time series is actually showing a population returning to carrying capacity after an excursion caused only by demographic stochasticity. The prognosis for the population is actually quite good (Figure 2): it persists for hundreds of generations (the duration of the simulation), making even larger fluctuations above carrying capacity.

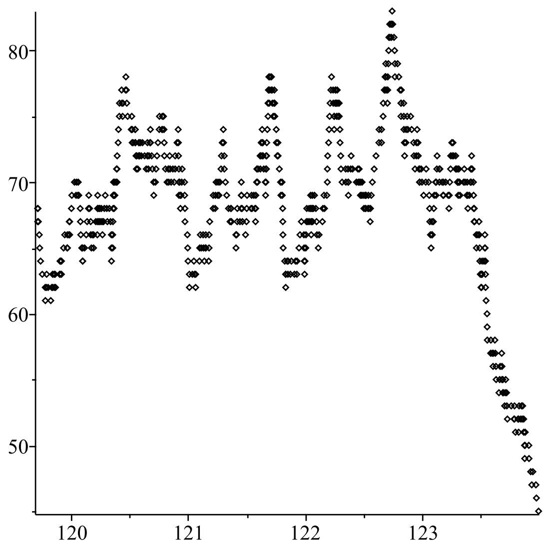

Figure 1: Census of a (simulated) population; one time unit is the expected first age of reproduction

This sort of behaviour is actually typical of trajectories of the simulated process, a birth and death process with density dependent mortality, and Markov transition rates

$$q^{(n)}_{k,k+1} = bk \quad \text{and} \quad q^{(n)}_{k,k-1} = d\left(1 + \tfrac{k}{n}\right) k, \tag{1}$$

which will either rapidly go extinct, or make many excursions well above carrying capacity prior to extinction. As such, to assess the viability of a population, we need to understand more than the mean behaviour of the population dynamics (and, as we discuss below, even the standard deviation can lead to misleading conclusions): we need to understand the stochastic fluctuations in long term.

1.2 Stochastic Logistic Processes

The process (1) is an example of the density-dependent population processes introduced in [15]. In what follows, we will investigate a class of density-dependent population processes that are "logistic-like", in the sense that in the large population limit, the process evolves in time according to Verhulst's logistic equation [20] or Kolmogorov's generalization thereof [14]. We'll call these *stochastic logistic processes.*

To be precise, we consider families of continuous time Markov chains, indexed by a "system-size" parameter $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, which is proportional to the carrying capacity – one can think of it as measuring the size of the habitat. These processes, $X^{(n)}$, have transition rates

$$q^{(n)}_{k,k+1} = \lambda^{(n)}_k = \lambda(\tfrac{k}{n})k \quad \text{and} \quad q^{(n)}_{k,k-1} = \mu^{(n)}_k = \mu(\tfrac{k}{n})k \tag{2}$$

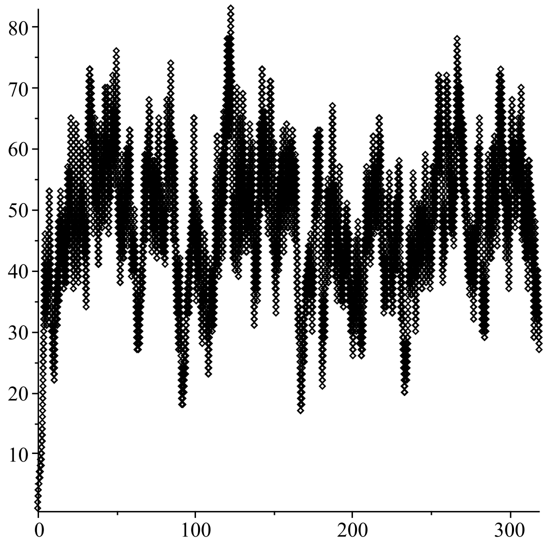

Figure 2: Census population size, simulated via the Markov Chain with rates (1), $b = 2$, $d = 1$, $n = 50$.

for non-negative, continuous functions $\lambda(x)$ and $\mu(x)$, which we assume to be continuously differentiable and bounded on compact sets. For ease of notation, we will suppress the exponents and write $\lambda_k^{(n)} = \lambda_k$ and $\mu_k^{(n)} = \mu_k$ in what follows.

In [15], it is shown that if $\frac{1}{n}X^{(n)}(0) \to x_0$ and $x(t, x_0)$ is the solution to

$$\dot{x} = (\lambda(x) - \mu(x))x \tag{3}$$

with initial condition $x(0, x_0) = x_0$, then the rescaled process $x^{(n)}(t) = \frac{1}{n}X^{(n)}(t)$ converges to $x(t, x_0)$. To be precise, for any fixed $T > 0$,

$$\lim_{n\to\infty} \sup_{t\leq T} |x^{(n)}(t) - x(t, x_0)| = 0 \quad \text{a.s.}$$

This implies that the process with rates (1) approaches – over finite time horizons – a deterministic limit satisfying the logistic equation,

$$\dot{x} = r\left(1 - \frac{x}{\kappa}\right)x$$

where $r = b - d$ and $\kappa = \frac{b}{d} - 1$. Moreover, provided $b > d$, the carrying capacity κ is a stable fixed point for the dynamics.

We want the deterministic process (3) to be competitive in the sense of [14], so the individual birth rate $\lambda(x)$ and death rate $\mu(x)$ will be required to be decreasing and increasing functions of the population density x, respectively[1]. Further, we want (3) to

[1] In [14], Kolmogorov actually makes the weaker assumption that the net per-capita growth rate

have bounded trajectories and a unique stable fixed point, and that $x = 0$ be a repeller for the dynamics. We thus assume there is a value $\kappa > 0$ such that $\lambda(x) - \mu(x) > 0$ for $0 < x < \kappa$, $\lambda(\kappa) = \mu(\kappa)$, and $\lambda(x) - \mu(x) < 0$ for $x > \kappa$. We will allow the possibility that $\lambda(\omega) = 0$ for some $\omega > 0$; by the above, we must have $\omega > \kappa$; *e.g.,* for a birth and death process with density dependent fecundity,

$$q^{(n)}_{k,k+1} = b\left(1 - \tfrac{k}{n}\right) k \quad \text{and} \quad q^{(n)}_{k,k-1} = dk,$$

we would have $\omega = 1$, whereas for the process with density dependent mortality, (1), $\omega = \infty$.

1.3 Fluctuations in Stochastic Logistic Processes

Based on Kurtz's theorem, we might thus reasonably assume that the paths of stochastic processes remain close to the corresponding unscaled carrying capacity, κn. Indeed, one can show [16, 3] that in the large n limit, the fluctuations $|x^{(n)}(t) - x(t, x_0)|$ may be described by a diffusion process which, as $t \to \infty$, relaxes to a quasi-stationary distribution that is approximately Gaussian with mean 0 and variance $\frac{1}{\sqrt{n}} \frac{\sigma(\kappa)}{2b'(\kappa)}$. The latter is asymptotically much smaller than the carrying capacity. Nonetheless, since 0 is the only absorbing state of the Markov chain, over a sufficiently long time horizon the stochastic logistic process will necessarily have at least one large fluctuation: it must eventually hit zero. We thus have our first indication that the first and second moments are insufficient to understand the long-term dynamics of a stochastic logistic process.

In what follows, we will develop an alternate approach, which allows us to give a more complete description (an approach to fixation probabilities in the same spirit as ours has recently appeared in [5], but their "continuous view" implicitly assumes an initial number of individuals that tends to infinity as $n \to \infty$). We shall demonstrate the perhaps counterintuitive result that, even starting from a single individual, in the limit as $n \to \infty$, a stochastic logistic process has a non-zero probability of greatly exceeding carrying capacity (*e.g.,* to more than double the carrying capacity for the model (1)). To be precise, there is a "potential barrier", $\eta > \kappa$, defined in the next section, such that as $n \to \infty$, the probability $x^{(n)}(t)$ reaches any $\iota < \eta$ tends to a non-zero limit that is independent of ι.

Moreover, once the population is established (which for present purposes, means reaching a population size of m_n individuals for *any* fixed sequence $\{m_n\}$ such that $m_n \to \infty$) then with high probability (*i.e.,* with probability approaching one as $n \to \infty$) it must have at least one fluctuation far above carrying capacity prior to extinction. Furthermore, having attained such a high value, it will, as $n \to \infty$, return there an arbitrary number of times. On the other hand, we shall also see that there is a sharp bound on such fluctuations: for any $\iota > \eta$, the probability that $x^{(n)}(t)$ reaches ι is exponentially small in n.

$\lambda(x) - \mu(x)$ is decreasing; we make this stronger assumption to ensure that $f(x) = \ln \frac{\mu(x)}{\lambda(x)}$ is increasing, which is crucial to our results. One can easily construct examples where the former holds, but not the latter, *e.g.,* $\lambda(x) = b + e^{x-\kappa}$, $\mu(x) = d(1 + x) + e^{x-\kappa}$, for $b > d > 0$ and $\kappa = \frac{b}{d} - 1$.

Having established that the process will greatly exceed carrying capacity many times, we address the natural question of how long this will take, how long the process will remain at or above some high level, and the time between returns to a high level. To understand these in context, we will also derive the expected time to first reach carrying capacity, and the time to extinction, conditioned on whether or not the process reaches carrying capacity first. The time to extinction has been addressed previously in varying degrees of rigour, but with conflicting conclusions [2, 10, 17]; the other results, are, to our knowledge, novel.

Our results also provide an elementary proof for the observation that the probability that a density-limited population successfully invades an unoccupied territory, in the limit as carrying capacity tends to infinity, is essentially the survival probability of a suitably chosen branching process. This has previously been asserted without proof and defended heuristically, *e.g.,* [11, 7, 21] or rigorously proven rigorously via coupling arguments, *e.g.,* [2, 6, 18].

While our approach lacks the intuition for the path-wise behaviour obtained via coupling with a branching process, in addition to its simplicity, it has the advantage of allowing us to make assertions about the behaviour of the logistic process for population sizes considerably larger than those for which the coupling remains exact (with high probability, the stochastic logistic process and the branching process will diverge once the population size has exceeded $\mathcal{O}(\sqrt{n})$ individuals, see [2]). As such, we can make observations about the maximum of the logistic process without having to resort to more technical large deviations arguments, can consider results over the lifetime of the population, rather than being limited to a compact time interval, and can avoid the "stitching-together" of various limits that coupling approaches demand.

2 Results

With the exception of our first result on invasion probabilities, where will introduce our mathematical framework and the definitions necessary to state subsequent results, all proofs are deferred to the appendix. We start by introducing the notation by which we express our results.

2.1 Notation

Let $X^{(n)}$ be a stochastic logistic process, as defined in the previous section. For any non-negative integer m, let

$$T_m^{(n)} = \inf\left\{t \geq 0 : X^{(n)}(t) = m\right\}$$

and

$$h_{a,b}^{(n)}(m) = \mathbb{P}_m\left\{T_a^{(n)} < T_b^{(n)}\right\},$$

where $\mathbb{P}_m$ indicates the probability conditional on $X^{(n)}(0) = m$ (similarly, we will write $\mathbb{E}_m$ for the expectation conditional on $X^{(n)}(0) = m$). Thus, $h_{a,b}^{(n)}(m)$ is the probability that, starting from m, the process hits a prior to hitting b. *A priori*, if $\lambda_m = 0$, then $T^{(n)}_{m+1} = \infty$ and $h^{(n)}_{a,b}(m) = 1$ for $b > m > a$.

More generally, for any set of nonnegative integers A, let

$$T^{(n)}_A = \inf\left\{t \geq 0 : X^{(n)}(t) \in A\right\},$$

be the first hitting time of A, and for any nonnegative m, let

$$T^{(n)}_{m+} = \inf\left\{t \geq T^{(n)}_{\mathbb{N}_0-\{m\}} : X^{(n)}(t) = m\right\}$$

be the first return time to m (note that for $X^{(n)}(0) \neq m$, $T^{(n)}_{m+} = T^{(n)}_m$). Let $N^{(n)}_m(t)$ be the number of visits of $X^{(n)}$ to m prior to time t, and let $S^{(n)}_m(t)$ be the total time spent in state m prior to time t.

Our results will be asymptotic in n, for which we will use Hardy-Vinogradov notation, so $f(n) \sim g(n)$ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1,$$

and $f(n) \lesssim g(n)$ if

$$\limsup_{n\to\infty} \frac{f(n)}{g(n)} \leq 1,$$

whereas $f(n) \ll g(n)$ if there exists a constant C such that

$$|f(n)| \leq C|g(n)|$$

for all n.

2.2 Invasion Probabilities

Given that any population is doomed to eventual extinction, we will define a successful invasion to be one in which starting from (a small number) m individuals, the population achieves some positive fraction of carrying capacity. As we will see below, this implies that it will with high probability also reach (and exceed) carrying capacity, and will have a species lifetime exponential in n (whereas species that fail to reach carrying capacity will on average go extinct in finite time $\ll 1$). The probability of invasion starting from m individuals is thus $1 - h^{(n)}_{0,\lfloor \iota n\rfloor}(m)$, for *any* $\iota \in [0, \kappa]$, and our first task is to understand the function $h^{(n)}_{a,b}(m)$.

By looking at the process $X^{(n)}(t)$ only at its jump times (*i.e.*, at the embedded Markov chain), it is clear that for $a < m < b$, the probabilities $h^{(n)}_{a,b}(m)$ satisfy a recurrence relation

$$h^{(n)}_{a,b}(m) = \frac{\lambda_m}{\lambda_m + \mu_m} h^{(n)}_{a,b}(m+1) + \frac{\mu_m}{\lambda_m + \mu_m} h^{(n)}_{a,b}(m-1)$$

with boundary conditions $h^{(n)}_{a,b}(a) = 1$ and $h^{(n)}_{a,b}(b) = 0$ ($\frac{\lambda_m}{\lambda_m+\mu_m}$ and $\frac{\mu_m}{\lambda_m+\mu_m}$ are the probability that, given there are m individuals prior to a given jump, that jump is a birth or death, respectively). This recurrence may be solved to yield

$$h^{(n)}_{a,b}(m) = \frac{\sum_{k=m}^{b-1} \prod_{j=1}^{k} \frac{\mu_j}{\lambda_j}}{\sum_{k=a}^{b-1} \prod_{j=1}^{k} \frac{\mu_j}{\lambda_j}}, \tag{4}$$

(we set $\sum_{k=b}^{b-1} \prod_{j=1}^{k} \frac{\mu_j}{\lambda_j} = 0$ and $\prod_{j=1}^{0} \frac{\mu_j}{\lambda_j} = 1$). A standard reference for such results is [12].

Now,

$$\prod_{j=1}^{k} \frac{\mu_j}{\lambda_j} = \prod_{j=1}^{k} \frac{\mu(\frac{j}{n})}{\lambda(\frac{j}{n})} = e^{\sum_{j=1}^{k} \ln \frac{\mu(\frac{j}{n})}{\lambda(\frac{j}{n})}},$$

so setting

$$V^{(n)}(m) := \begin{cases} \frac{1}{n} \sum_{k=1}^{m} \ln \frac{\mu(\frac{k}{n})}{\lambda(\frac{k}{n})} & \text{if } m > 0, \text{ and} \\ 0 & \text{if } m = 0, \end{cases}$$

we may write

$$h^{(n)}_{a,b}(m) = \frac{\sum_{k=m}^{b-1} e^{nV^{(n)}(k)}}{\sum_{k=a}^{b-1} e^{nV^{(n)}(k)}}. \tag{5}$$

Let $f(x) = \ln \frac{\mu(x)}{\lambda(x)}$, so f is an increasing function and $f(x) \le 0$ for $x \le \kappa$. For our choice of f we then have

$$V^{(n)}(k) = \frac{1}{n} \sum_{j=1}^{k} f\left(\frac{j}{n}\right),$$

and (see Lemma A.1),

$$\int_0^{\frac{k}{n}} f(x)\, dx \le V^{(n)}(k) \le \int_0^{\frac{k}{n}} f(x)\, dx + \frac{1}{n}\left(f\left(\frac{k}{n}\right) - f(0)\right).$$

In particular, for $\iota < \omega$, as $n \to \infty$,

$$V^{(n)}(\iota n) \to V(\iota) := \int_0^{\iota} f(x)\, dx,$$

and the latter has a unique minimum at κ.

Since f is increasing and $f(x) \ge 0$ for all $x \ge \kappa$, either $\omega < \infty$, or there exists $\kappa < \zeta < \infty$ such that

$$\int_0^{\zeta} f(x)\, dx = 0,$$

whereas $\int_0^{\iota} f(x)\, dx$ is positive for $\iota > \zeta$ and negative if $\iota < \zeta$. Let

$$\eta = \min\{\omega, \zeta\},$$

i.e., η is either the largest population density at which the birth rate is positive, or the least density at the same height in the potential well as zero, whichever is least. η will be a key quantity in our analysis.

We are now in a position to use the following discrete analogue to Laplace's method[2] to obtain asymptotic estimates of the infinite sums in in terms of the maximum of $V(x)$. Since $V(x)$ is concave, its maximum over any interval $[\alpha, \beta]$ $(0 \leq \alpha < \beta < \eta)$ occurs at one of the endpoints α or β, whereas the minimum at κ is the unique local (and thus global) minimum.

Proposition 1. *Let a_n and b_n be sequences of non-negative integers such that $\frac{a_n}{n} \to \alpha$ and $\frac{b_n}{n} \to \beta$, and suppose that $\psi(x)$ and $g(x)$ are, respectively, a continuously differentiable function and a continuous function on an open interval containing $[\alpha, \beta]$, and that ϵ_n is a sequence of functions on the set of integers $\{a_n, a_n+1, \ldots, b_n\}$, uniformly converging to 0. Suppose further that that there exists γ such that $\psi(\gamma) > \psi(x)$ for all $x \neq \gamma \in [\alpha, \beta]$.*

(i) If either $\gamma = \alpha$ and $\psi'(\alpha) < 0$, or $\gamma = \beta$ and $\psi'(\beta) > 0$, then setting $c_n = a_n$ or b_n, respectively, one has

$$\sum_{k=a_n}^{b_n-1} (1+\epsilon_n(k)) g\left(\frac{k}{n}\right) e^{n\psi\left(\frac{k}{n}\right)} \sim \frac{g\left(\frac{c_n}{n}\right) e^{n\psi\left(\frac{c_n}{n}\right)}}{1 - e^{-\psi'\left(\frac{c_n}{n}\right)}}$$

(ii) If ψ is twice continuously differentiable and $\psi'(\gamma) = 0$, and either $\gamma \in (\alpha, \beta)$ and $\psi''(\gamma) < 0$, or $\gamma \in \{\alpha, \beta\}$, $\psi''(\gamma) > 0$, and $\left|\frac{c_n}{n} - \gamma\right| \ll \frac{1}{\sqrt{n}}$ (c_n as previously), then

$$\sum_{k=a_n}^{b_n-1} (1+\epsilon_n(k)) g\left(\frac{k}{n}\right) e^{n\psi\left(\frac{k}{n}\right)}$$
$$\sim \begin{cases} g(\gamma) e^{n\psi(\gamma)} \sqrt{\frac{2n\pi}{|\psi''(\gamma)|}} & \text{if } \gamma \in (\alpha, \beta), \text{ and} \\ g(\gamma) e^{n\psi(\gamma)} \left(\frac{1}{2} + \sqrt{\frac{n\pi}{2|\psi''(\gamma)|}}\right) & \text{otherwise.} \end{cases}$$

Applying this with $\psi(x) = V(x)$, $g(x) = \sqrt{\frac{\mu(x)}{\lambda(x)} \frac{\lambda(0)}{\mu(0)}}$, and

$$\epsilon_n(i) = e^{nV^{(n)}(i) - nV\left(\frac{i}{n}\right) - \frac{1}{2}\left(f\left(\frac{i}{n}\right) - f(0)\right)} - 1$$

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[2] While it is appealing to observe that

$$\frac{1}{n} \sum_{k=a_n}^{b_n-1} (1+\epsilon_n(k)) g\left(\frac{k}{n}\right) e^{n\psi\left(\frac{k}{n}\right)}$$

is essentially the Riemann sum for

$$\int_\alpha^\beta g(x) e^{n\psi(x)}\, dx,$$

and then invoke the continuous form of Laplace's method, Proposition 1 shows that whilst the discrete and continuous results are identical for an interior maximum, they do not agree when ψ has its maximum at one of the endpoints, thus invalidating this "proof".

yields

Corollary 1. *Let a_n and b_n be as above. Then,*

(i) If $V(\beta) < V(\alpha)$, then

$$\sum_{k=a_n}^{b_n-1} e^{nV^{(n)}(k)} \sim \sqrt{\frac{\mu\left(\frac{a_n}{n}\right)}{\lambda\left(\frac{a_n}{n}\right)}\frac{\lambda(0)}{\mu(0)}}\frac{e^{nV\left(\frac{a_n}{n}\right)}}{1-\frac{\mu\left(\frac{a_n}{n}\right)}{\lambda\left(\frac{a_n}{n}\right)}}$$

(ii) If $V(\alpha) < V(\beta)$, then

$$\sum_{k=a_n}^{b_n-1} e^{nV^{(n)}(k)} \sim \sqrt{\frac{\mu\left(\frac{b_n}{n}\right)}{\lambda\left(\frac{b_n}{n}\right)}\frac{\lambda(0)}{\mu(0)}}\frac{e^{nV\left(\frac{b_n}{n}\right)}}{1-\frac{\lambda\left(\frac{b_n}{n}\right)}{\mu\left(\frac{b_n}{n}\right)}}.$$

(iii) In particular, if $a_n = o(n)$, then $\alpha = 0$ and, if $\beta < \eta$, then

$$\sum_{k=a_n}^{b_n-1} e^{nV^{(n)}(k)} \sim \frac{\left(\frac{\mu(0)}{\lambda(0)}\right)^{a_n}}{1-\frac{\mu(0)}{\lambda(0)}}.$$

Applying Corollary 1 to the numerator and denominator of (2.2) gives us our first result on invasion probabilities, namely that the large excursions of §1.1 are the rule, and not the exception, for density limited populations:

Proposition 2 (Invasion Probabilities)**.** *Let $\{m_n\}$ be a sequence of positive integers such that $m_n \ll n$ (we allow $m_n \equiv m$),*

$$h^{(n)}_{0,\lfloor \iota n\rfloor}(m_n) \sim \begin{cases} \left(\frac{\mu(0)}{\lambda(0)}\right)^{m_n} & \text{if } 0 < \iota < \eta, \text{ and} \\ 1-\sqrt{\frac{\lambda(\iota)}{\mu(\iota)}\frac{\mu(0)}{\lambda(0)}}\frac{1-\frac{\lambda(\iota)}{\mu(\iota)}}{1-\frac{\mu(0)}{\lambda(0)}}\left(1-\left(\frac{\mu(0)}{\lambda(0)}\right)^{m_n}\right)e^{-nV\left(\frac{\lfloor \iota n\rfloor}{n}\right)} & \text{if } \iota > \eta. \end{cases} \tag{6}$$

Remark 1. Unfortunately, while $\frac{\lfloor \iota n\rfloor}{n} \to \iota$ as $n \to \infty$, unless ι is integral, it is not generally the case that $e^{nV\left(\frac{\lfloor \iota n\rfloor}{n}\right)} \to e^{nV(\iota)}$. If V is continuously differentiable, then

$$n\left(V(\iota) - V\left(\frac{\lfloor \iota n\rfloor}{n}\right)\right) \sim -V'(\iota)(\iota - \lfloor \iota n\rfloor),$$

so $e^{n\left(V(\iota)-V\left(\frac{\lfloor \iota n\rfloor}{n}\right)\right)}$ is bounded. However, one can easily show that if ι is rational, say $\iota = \frac{p}{q}$ for p and q relatively prime, then $\iota - \lfloor \iota n\rfloor$ takes all values $\left\{0, \frac{1}{q}, \ldots, \frac{q-1}{q}\right\}$, whereas if ι is irrational, the set of points $\{\iota - \lfloor \iota n\rfloor : n \in \mathbb{N}_0\}$ is dense in $[0, 1]$.

Thus, for *any* $0 < \iota < \eta$, the hitting probability is independent of ι, and equal to the probability of extinction for the pure birth and death process with rates $\lambda_k = \lambda(0)k$ and $\mu_k = \mu(0)k$. Moreover, if $m_n \to \infty$ as $n \to \infty$, the process hits ι with probability approaching 1, whereas at $\iota = \eta$, this probability is reduced by a constant factor, while $X^{(n)}$ exceeds η with an exponentially small probability.

Example 1. In our example with density dependent mortality, (1), $\eta > 2\kappa$: for $x \geq 0$, $\ln(x) \leq x - 1$, so

$$f(x) \leq \frac{\mu(x)}{\lambda(x)} - 1 = \frac{\mu(x) - \lambda(x)}{\lambda(x)} = \frac{\mu}{\lambda}(x - \kappa),$$

with the inequality strict except at $x = \kappa$. Then, if $0 < \iota < 2\kappa$

$$\int_0^\iota f(x)\,dx < \frac{\mu}{\lambda}\int_0^\iota x - \kappa\,dx = \frac{\mu}{\lambda}\left(\frac{1}{2}\iota - \kappa\right)\iota.$$

Thus, the right hand side is less than zero and $\eta > 2\kappa$. Thus, depending on the model, fluctuations to twice carrying capacity are possible, though not generically: one can easily construct examples where $\eta < 2\kappa$, *e.g.,* take a process with density dependent fecundity, with birth and death rates $\lambda\left(1 - \frac{k}{n}\right)k$ and μk respectively.

An intuitive understanding of these results can be obtained by observing that the stochastic logistic process is essentially equivalent to a random walk on $\mathbb{N}_0$ with the potential $nV^{(n)}$, absorbed at 0. Recall that when in state k, the random walk in $nV^{(n)}$ increases by 1 with probability

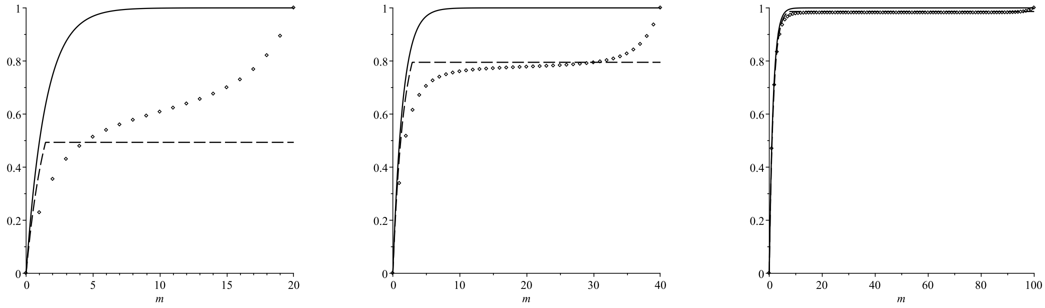
$$p_k := \frac{e^{-nV^{(n)}(k)}}{e^{-nV^{(n)}(k-1)} + e^{-nV^{(n)}(k)}}\left(= \frac{\lambda_k}{\lambda_k + \mu_k}\right),$$

and decreases by 1 with probability $q_k := 1 - p_k$; this is exactly the embedded Markov chain we used to compute $h_{a,b}^{(n)}(m)$. Because the potential is scaled by n, its walls become arbitrarily steeper as $n \to \infty$, and it becomes exponentially harder for the process to reach points of higher potential:

Proposition 3. *Suppose that* $\frac{m_n}{n} \to \nu$, $0 \leq \xi < \nu < \iota$, *and* $\xi, \nu, \iota \neq \kappa$. *Then,*

$$\mathbb{P}_{m_n}\left\{T_{\lfloor \xi n \rfloor} < T_{\lfloor \iota n \rfloor}\right\}$$
$$\sim \begin{cases} \sqrt{\frac{\mu(\nu)}{\lambda(\nu)}\frac{\lambda(\xi)}{\mu(\xi)}}\frac{1-\frac{\lambda(\xi)}{\mu(\xi)}}{1-\frac{\mu(\nu)}{\lambda(\nu)}}e^{n\left(V\left(\frac{\lfloor \nu n\rfloor}{n}\right)-V\left(\frac{\lfloor \xi n\rfloor}{n}\right)\right)} & \text{if } V(\xi)>V(\nu)>V(\iota), \\ \sqrt{\frac{\mu(\iota)}{\lambda(\iota)}\frac{\lambda(\xi)}{\mu(\xi)}}\frac{1-\frac{\lambda(\xi)}{\mu(\xi)}}{1-\frac{\mu(\iota)}{\lambda(\iota)}}e^{n\left(V\left(\frac{\lfloor \iota n\rfloor}{n}\right)-V\left(\frac{\lfloor \xi n\rfloor}{n}\right)\right)} & \text{if } V(\xi)>V(\iota)>V(\nu), \\ 1-\sqrt{\frac{\lambda(\iota)}{\mu(\iota)}\frac{\mu(\xi)}{\lambda(\xi)}}\frac{1-\frac{\mu(\iota)}{\lambda(\iota)}}{1-\frac{\lambda(\xi)}{\mu(\xi)}}e^{n\left(V\left(\frac{\lfloor \xi n\rfloor}{n}\right)-V\left(\frac{\lfloor \iota n\rfloor}{n}\right)\right)} & \text{if } V(\iota)>V(\xi)>V(\nu), \text{ and} \\ 1-\sqrt{\frac{\lambda(\iota)}{\mu(\iota)}\frac{\mu(\nu)}{\lambda(\nu)}}\frac{1-\frac{\mu(\iota)}{\lambda(\iota)}}{1-\frac{\lambda(\nu)}{\mu(\nu)}}e^{n\left(V\left(\frac{\lfloor \nu n\rfloor}{n}\right)-V\left(\frac{\lfloor \iota n\rfloor}{n}\right)\right)} & \text{if } V(\iota)>V(\nu)>V(\xi). \end{cases} \tag{7}$$

Remark 2. The ordering requires $V(\nu) < \max\{V(\xi), V(\iota)\}$.



(a) $n = 10$ (b) $n = 20$ (c) $n = 50$

Figure 3: The probability of hitting twice carrying capacity as a function of the initial number of individuals, m, for the stochastic logistic process with rates (1), $b = 2$, and $d = 1$ (so $\kappa = 1$ and carrying capacity is n). Diamonds are numerical evaluations of (4), the solid line is given by (6), and the dashed line is given by (7) with $\xi = 0$ and $\iota = 2\kappa$.

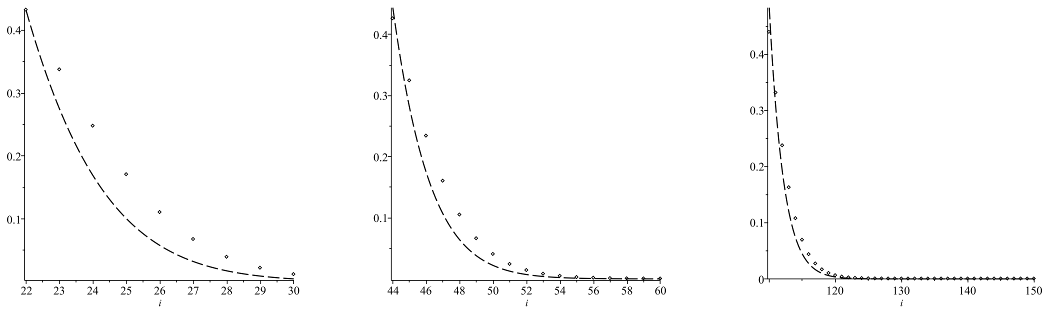
Remark 3. The cases when one of ξ, ν, ι is equal to κ may be obtained from Proposition 1, (ii), but we omit these for brevity.

Thus, despite our use of quotes in introducing it, η is truly a potential barrier for the process. Of necessity, the process will eventually make a large fluctuation down to zero, and numerous "failed attempts" en route. The fact that carrying capacity is the bottom of the well essentially ensures that it is equally likely to make large upward fluctuations as well.

We conclude this section with Figures 3 and 4, which illustrate the speed of convergence of (6) and (7). The former, which assumes small initial numbers, does well initially, but eventually overestimates the probability of reaching twice carrying capacity. The latter already shows good agreement for $n = 10$, whereas for the former, smaller order terms cause a substantial error for $n = 10$, whereas for $n = 20$, our asymptotic approximations are already qualitatively correct, and at $n = 50$, we have excellent agreement.

2.3 Some Fluctuation Theory

Having established that the process will greatly exceed carrying capacity, it is natural to ask what happens next. As the next result shows, it will almost certainly return towards carrying capacity before returning to the same or higher points in the potential:



(a) $n = 10$ (b) $n = 20$ (c) $n = 50$

Figure 4: The probability of attaining $i > \eta n$ individuals, starting from carrying capacity, for the stochastic logistic process with rates (1), $b = 2$, and $d = 1$ (so $\kappa = 1$ and carrying capacity is n). Diamonds are numerical evaluations of (4) and the dashed line is given by (7) with $\xi = 0$ and $\iota = \eta$.

Proposition 4. *Let* $\max\{\nu, \xi\} < \eta$. *Then,*

$$\mathbb{P}_{\lfloor \nu n \rfloor}\left\{T^{(n)}_{\lfloor \xi n \rfloor} < T^{(n)}_{\lfloor \nu n \rfloor +}\right\} \sim \begin{cases} \frac{\mu(\nu)}{\lambda(\nu)+\mu(\nu)}\left(1-\frac{\mu(\xi)}{\lambda(\xi)}\right)\sqrt{\frac{\lambda(\xi)}{\mu(\xi)}\frac{\mu(\nu)}{\lambda(\nu)}}e^{n\left(V\left(\frac{\lfloor \nu n\rfloor}{n}\right)-V\left(\frac{\lfloor \xi n\rfloor}{n}\right)\right)} & \text{if } V(\xi)>V(\nu) \text{ and } \xi<\nu, \\ \frac{|\lambda(\nu)-\mu(\nu)|}{\lambda(\nu)+\mu(\nu)} & \text{if } V(\xi)<V(\nu), \text{ and,} \\ \frac{\lambda(\nu)}{\lambda(\nu)+\mu(\nu)}\left(1-\frac{\lambda(\xi)}{\mu(\xi)}\right)\sqrt{\frac{\mu(\xi)}{\lambda(\xi)}\frac{\lambda(\nu)}{\mu(\nu)}}e^{n\left(V\left(\frac{\lfloor \nu n\rfloor}{n}\right)-V\left(\frac{\lfloor \xi n\rfloor}{n}\right)\right)} & \text{if } V(\xi)>V(\nu) \text{ and } \xi>\nu. \end{cases}$$

As an immediate consequence, we have the number of returns to $\lfloor \iota n \rfloor$. Since $\iota < \eta$, $V(\iota) < 0$, and this proposition tells us that the number of times the stochastic logistic process revisits a neighbourhood of $\lfloor \iota n \rfloor$ increases exponentially with n:

Proposition 5. *Fix* $\iota < \eta$. *Then,*

$$\mathbb{E}_{m_n}\left[N^{(n)}_{\lfloor \iota n \rfloor}(T^{(n)}_0)\middle| T^{(n)}_{\lfloor \iota n \rfloor} < T^{(n)}_0\right] \sim \frac{\lambda(\iota)+\mu(\iota)}{\mu(\iota)}\sqrt{\frac{\mu(0)}{\lambda(0)}\frac{\lambda(\iota)}{\mu(\iota)}}\frac{e^{-nV\left(\frac{\lfloor \iota n\rfloor}{n}\right)}}{1-\frac{\mu(0)}{\lambda(0)}}.$$

In particular, the ambiguous scenario of Figure 1 is not an anomalous event, but one that will happen many times over the lifetime of a population. In the next section, we will look at how long one has to wait for one of these events.

2.4 Hitting Times

In this section, we will look at the expected times to extinction, to carrying capacity, and to values far above. Unless stated otherwise, we assume that $\lambda(x)$ and $\mu(x)$ are twice continuously differentiable. We start with the expected time to carrying capacity, which is rapid:

Proposition 6. *Fix a positive integer m. Then,*

$$\mathbb{E}_m\left[T^{(n)}_{\lfloor\kappa n\rfloor}\middle|T^{(n)}_{\lfloor\kappa n\rfloor}<T^{(n)}_0\right]\sim\left(\frac{1}{\lambda(0)-\mu(0)}-\frac{1}{(\lambda'(\kappa)-\mu'(\kappa))\kappa}\right)\ln n.$$

Even more rapid is the extinction time conditioned on never reaching carrying capacity, which is of order one, so most populations that fail to reach carrying capacity will not persist long enough to be observed:

Proposition 7. *Let $\lambda(x)$ and $\mu(x)$ be continuous and fix $m<\lfloor\kappa n\rfloor$. Then,*

$$\lim_{n\to\infty}\mathbb{E}_m\left[T^{(n)}_0\middle|T^{(n)}_0<T^{(n)}_{\lfloor\kappa n\rfloor}\right]=\frac{1}{\mu(0)}\int_0^{\frac{\mu(0)}{\lambda(0)}}\frac{1-x^m}{(1-x)^2}\,dx$$

In particular,

$$\lim_{n\to\infty}\mathbb{E}_1\left[T^{(n)}_0\middle|T^{(n)}_0<T^{(n)}_{\lfloor\kappa n\rfloor}\right]=-\frac{1}{\mu(0)}\ln\left(1-\frac{\mu(0)}{\lambda(0)}\right).$$

whereas

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}_m\left[T^{(n)}_0\middle|T^{(n)}_0<T^{(n)}_{\lfloor\kappa n\rfloor}\right]=\frac{1}{\lambda(0)-\mu(0)}.$$

By contrast, a population that does reach carrying capacity will persist exponentially long:

Proposition 8. *Fix a positive integer m. Then,*

$$\mathbb{E}_m\left[T^{(n)}_0\right]\sim\sqrt{\frac{2\pi}{n\left(\frac{\mu'(\kappa)}{\mu(\kappa)}-\frac{\lambda'(\kappa)}{\lambda(\kappa)}\right)}}\frac{\mu(0)}{\lambda(0)}\frac{1-\left(\frac{\mu(0)}{\lambda(0)}\right)^m}{1-\left(\frac{\mu(0)}{\lambda(0)}\right)}\frac{e^{-nV(\kappa)}}{\mu(\kappa)\kappa\left(1-\frac{\lambda(0)}{\mu(0)}\right)}$$

The fact that $\mathbb{E}_m[T^{(n)}_0]$ is bounded in m also allows us to apply a result in Section 4.3 of [1] to conclude that the distribution of $T^{(n)}_0$ has exponential tails:

Corollary 2. *Independent of the initial state, m,*

$$\mathbb{P}_m\left\{T^{(n)}_0>t\right\}\lesssim e^{-\frac{t}{e}\sqrt{\frac{n\left(\frac{\mu'(\kappa)}{\mu(\kappa)}-\frac{\lambda'(\kappa)}{\lambda(\kappa)}\right)}{2\pi}}\frac{\lambda(0)}{\mu(0)}\mu(\kappa)\kappa\left(1-\frac{\lambda(0)}{\mu(0)}\right)e^{nV(\kappa)}}.$$

Using elements of the proof, we can also compute the expected time spent above any level above carrying capacity. Let $L^{(n)}_k(t)$ denote the total time spent in state k prior to time t (the *local time* at k). Then, applying Proposition 1 to the sum of the expected holding time in all states above $\lfloor\iota n\rfloor$, we have

Corollary 3. *Let $\kappa<\iota<\eta$. Then,*

$$\mathbb{E}_m\left[\sum_{k=\lfloor\iota n\rfloor+1}^{\infty}L^{(n)}_k(T^{(n)}_0)\right]\sim\frac{1-\left(\frac{\mu(0)}{\lambda(0)}\right)^m}{1-\frac{\mu(0)}{\lambda(0)}}\frac{e^{-nV\left(\frac{\lfloor\iota n\rfloor}{n}\right)}}{n\mu(\iota)\iota\sqrt{\frac{\mu(\iota)}{\lambda(\iota)}\frac{\lambda(0)}{\mu(0)}}},$$

We can also completely characterize the amount of time in any given state prior to extinction:

Corollary 4. *Let m be a positive integer.*

$$\mathbb{P}_m\left\{L_k^{(n)}(T_0^{(n)}) \leq t \middle| T_k^{(n)} < T_0^{(n)}\right\} \sim e^{-(\lambda_k+\mu_k)\left(1-\left(1-\frac{\mu(0)}{\lambda(0)}\right)\sqrt{\frac{\mu\left(\frac{k-1}{n}\right)}{\mu(0)}\frac{\lambda(0)}{\lambda\left(\frac{k-1}{n}\right)}}e^{nV\left(\frac{k-1}{n}\right)}\right)t}$$

i.e., conditioned upon hitting k prior to 0, the total time spent in k is exponentially distributed.

Together, these two corollaries tell us that the that the total of time spent in any state above carrying capacity is either exponentially long (below the barrier η), or exponentially short (above η), so from an absolute point of view, populations can be well above carrying capacity for long periods of time. From a relative point of view, however, Proposition 8 and Corollary 3, tell us the fraction of time spent above $\lfloor \iota n \rfloor$, for any $\iota > \kappa$, is of order $e^{n(V(\kappa)-V(\iota))}$, which for large populations is vanishingly small, so as the population grows large, it might be highly unlikely to observe such a fluctuation. Indeed, as the next two results show, one must also wait exponentially long to see a large fluctuation, whereas a typical excursion has a very short duration:

Proposition 9. *Let $\nu < \eta$. Then, for fixed $m \in \mathbb{N}$,*

$$\mathbb{E}_m\left[T_{\lfloor \nu n \rfloor}^{(n)} \middle| T_{\lfloor \nu n \rfloor}^{(n)} < T_0^{(n)}\right] \sim \sqrt{\frac{2\pi}{n\left(\frac{\mu'(\kappa)}{\mu(\kappa)} - \frac{\lambda'(\kappa)}{\lambda(\kappa)}\right)}\frac{\mu(\nu)}{\lambda(\nu)}}\frac{e^{n\left(V\left(\frac{\lfloor \nu n \rfloor}{n}\right)-V(\kappa)\right)}}{\lambda(\kappa)\kappa\left(1-\frac{\lambda(\nu)}{\mu(\nu)}\right)}$$

(as in Corollary 2, the tails of the distribution are exponential).

As one sees in the proof, the time spent in states k with $V\left(\frac{k}{n}\right) > V(\nu)$ is extremely short ($\ll \ln n$) compared to the time in states with $V\left(\frac{k}{n}\right) < V(\nu)$ this is because the process rapidly approaches the carrying capacity, but, prior to hitting $\lfloor \nu n \rfloor$, it makes many returns to each point such that $V\left(\frac{k}{n}\right) < V(\nu)$. These returns are quite rapid:

Proposition 10. *Let $\kappa < \nu < \eta$. Then,*

$$\lim_{n\to\infty} \mathbb{E}_{\lfloor \nu n \rfloor}\left[T_{\lfloor \kappa n \rfloor}^{(n)}\right] \sim -\frac{1}{(\lambda'(\kappa)-\mu'(\kappa))\kappa}\ln n$$

Finally, having established in Proposition 5 that the process will nonetheless reattain νn, it is natural to ask how long this will take. This leads us to:

Proposition 11. *Let $\kappa < \nu < \eta$. Then,*

$$\mathbb{E}_{\lfloor \nu n \rfloor}\left[T_{\lfloor \nu n \rfloor +}^{(n)} \middle| T_{\lfloor \nu n \rfloor +}^{(n)} < T_0^{(n)}\right] \sim \frac{\mu(\nu)}{\mu(\nu)+\lambda(\nu)}\sqrt{\frac{2\pi}{n\left(\frac{\mu'(\kappa)}{\mu(\kappa)} - \frac{\lambda'(\kappa)}{\lambda(\kappa)}\right)}\frac{\mu(\nu)}{\lambda(\nu)}}\frac{e^{n\left(V\left(\frac{\lfloor \nu n \rfloor}{n}\right)-V(\kappa)\right)}}{\lambda(\kappa)\kappa}.$$

3 Discussion

The results above provide a fairly complete picture of the long-term behaviour of the stochastic logistic process: an invading population either fails and disappears almost immediately, or rapidly attains carrying capacity. In the latter case, it persists exponentially long, eventually achieving every state of potential less than zero, returning to each such state an arbitrary number of times, but taking exponentially long to do so, and typically returning close to carrying capacity in time $\ll \ln n$. Together, the net effect is a standard deviation of order $\mathcal{O}(\sqrt{n})$, but the latter is deceptive, as it masks the rare but recurrent excursions.

To conclude, in addition to mathematical and theoretical interest, our results have practically important biological implications: depending on when we observe a population, it may in fact be far in excess of equilibrium, so that estimates of the population viability or Malthusian growth rate may be grossly inaccurate. Moreover, large increases or sudden decreases in population size, that might be interpreted as recovery or collapse, may be little more than demographic stochasticity. Some caveats are, of course, in order. First, as we observe in proposition 1, whilst significant fluctuations above carrying capacity (*i.e.,* of magnitude proportional to the population size) are highly likely to occur, the size of those fluctuations is nonetheless model-dependent, and in a more realistic model, many factors will limit population growth at high frequencies (*e.g.,* resource limitation, mate competition, disease, and predation, to name a few) so that the potential barrier will likely be closer to carrying capacity than in toy mathematical models. Similarly, including *e.g.,* Allee effects at small numbers would qualitatively change the shape of the potential, adding a second well at small numbers and a potential barrier to traverse to reach the carrying capacity, which would demand a more detailed analysis than is presented here, although one likely amenable to an analysis similar to that for diffusions with metastable states (*e.g.,* [4]).

More importantly, the expected waiting time for an excursion far above carrying capacity, despite being asymptotically smaller than the extinction time, is still exponentially large in n, so that one may have to wait an extremely long time to see such an excursion, even if arbitrarily many will occur prior to extinction. Moreover, the return time to equilibrium is considerably shorter than the time between excursions – so for large populations, we are extremely unlikely to observe the population far from its equilibrium size.

Nonetheless, when modelling populations *e.g.,* in performing viability analysis, we are often most interested in the smaller populations of uncommon species, which are at risk of extinction. In this case, large fluctuations may occur on the timescale at which we observe the population and confound our efforts to estimate that risk. In such cases, the existence of large magnitude stochastic fluctuations that we have demonstrated should serve as an important caution to the use of deterministic modelling in conservation biology, and leave us with the important question of how to distinguish collapsing populations from those reverting to the mean.

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Appendices

A Some Useful Lemmas

For the sake of completeness, we include some simple and well-known lemmas that we have used in the main text. The proofs are elementary and are thus omitted.

Lemma A.1. *(i) Let f be an non-decreasing function. Then,*

$$0 \leq \frac{1}{n} \sum_{j=a+1}^{b} f\left(\frac{j}{n}\right) - \int_{\frac{a}{n}}^{\frac{b}{n}} f(x)\,dx \leq \frac{1}{n}\left(f\left(\frac{b}{n}\right) - f\left(\frac{a}{n}\right)\right)$$

and

$$0 \leq \int_{\frac{a}{n}}^{\frac{b}{n}} f(x)\,dx - \frac{1}{n} \sum_{j=a}^{b-1} f\left(\frac{j}{n}\right) \leq \frac{1}{n}\left(f\left(\frac{b}{n}\right) - f\left(\frac{a}{n}\right)\right).$$

(ii) If f is differentiable and $|f'(x)|$ is bounded by M on $\left[\frac{a}{n}, \frac{b}{n}\right]$, then

$$\left| \int_{\frac{a}{n}}^{\frac{b}{n}} f(x)\,dx - \frac{1}{n} \sum_{j=a}^{b-1} f\left(\frac{j}{n}\right) \right| < \frac{M(b-a)}{2n^2}.$$

(iii) Further, if f is twice differentiable and $|f''(x)|$ is bounded by M on $\left[\frac{a}{n}, \frac{b}{n}\right]$, then

$$\left| \int_{\frac{a}{n}}^{\frac{b}{n}} f(x)\,dx - \frac{1}{n} \sum_{j=a}^{b-1} f\left(\frac{j}{n}\right) - \frac{1}{2n}\left(f\left(\frac{b}{n}\right) - f\left(\frac{a}{n}\right)\right) \right| < \frac{M(b-a)}{4n^3}.$$

Lemma A.2 (Dominated convergence theorem for series)**.** *Suppose that $a_{m,n}$ and b_m are sequences such that*

(i) $a_{m,n} \to a_m$ as $n \to \infty$,

(ii) $|a_{m,n}| \leq b_m$ for all n, and

(iii) $\sum_{m=0}^{\infty} b_m < \infty$.

Then,

$$\lim_{n\to\infty} \sum_{m=0}^{\infty} a_{m,n} = \sum_{m=0}^{\infty} a_m,$$

i.e., the sums on the left and right are convergent and one can interchange sum and limit.

Lemma A.3. *Suppose we have sequences $a_{n,k}, b_{n,k} \geq 0$ for all n, k, such that*

$$\lim_{k\to\infty} \sup_n \frac{a_{n,k}}{b_{n,k}} = 1,$$

and that

$$\sum_{k=1}^{\infty} b_{n,k} = \infty.$$

Then,

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} a_{n,k}}{\sum_{k=1}^{N} b_{n,k}} = 1.$$

B Proofs

B.1 Proofs for §2.2

Proposition 1. We will prove the result for $\gamma = \alpha$ and $\gamma \in (\alpha, \beta)$. The other cases follow similarly.

Fix $\varepsilon > 0$ such that $\psi'(\alpha) + \varepsilon < 0$. Using Taylor's theorem, we may write

$$\psi(x) = \psi(y) + \psi'(y)(x-y) + R(x,y)(x-y),$$

where $R(x,y) \to 0$ as $|x-y| \to 0$. Fix $\delta > 0$ such that

$$|\psi'(x) - \psi'(\alpha)| < \frac{\varepsilon}{2}$$

for all x such that $|x - \alpha| < \delta$ and

$$|R(x,\alpha)| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(x) - g(\alpha)| < \frac{\varepsilon}{2}$$

for all x such that $x - \alpha < 3\delta$, and choose $\eta > 0$ such that $\psi(x) < \psi(\alpha) - \eta$ for all x such that $x - \alpha \geq \delta$. Fix M and m such that $|g(x)| < M$ and $|h(x)| < m$ for all $x \in [\alpha - \delta, \beta + \delta]$ and all n. Since $\frac{a_n}{n} \to \alpha$ and $\frac{b_n}{n} \to \beta$, without loss of generality, we may assume that $|\frac{a_n}{n} - \alpha| < \delta$, $|\frac{b_n}{n} - \beta| < \delta$ and $|\epsilon_n| < \varepsilon$ for all n.

Then,

$$\begin{aligned}
&\sum_{k=a_n}^{b_n-1} (1+\epsilon_n(k)) g\left(\frac{k}{n}\right) e^{n\psi\left(\frac{k}{n}\right)} \\
&\qquad = e^{n\psi\left(\frac{a_n}{n}\right)} \sum_{k=a_n}^{b_n-1} (1+\epsilon_n(k)) g\left(\frac{k}{n}\right) e^{n\left(\psi\left(\frac{k}{n}\right) - \psi\left(\frac{a_n}{n}\right)\right)} \\
&\qquad = e^{n\psi\left(\frac{a_n}{n}\right)} \left(\sum_{k=a_n}^{a_n+\lceil 2n\delta\rceil-1} (1+\epsilon_n(k)) g\left(\frac{k}{n}\right) e^{n\left(\psi\left(\frac{k}{n}\right) - \psi\left(\frac{a_n}{n}\right)\right)} \right. \\
&\qquad\qquad \left. + \sum_{k=a_n+\lceil 2n\delta\rceil}^{b_n-1} (1+\epsilon_n(k)) g\left(\frac{k}{n}\right) e^{n\left(\psi\left(\frac{k}{n}\right) - \psi\left(\frac{a_n}{n}\right)\right)} \right),
\end{aligned}$$

and, provided $k \geq a_n + \lceil 2n\delta \rceil$, then $\frac{k}{n} \geq \alpha + \delta$, and

$$\left| \sum_{k=a_n+\lceil 2n\delta \rceil}^{b_n-1} (1+\epsilon_n(k)) g\left(\frac{k}{n}\right) e^{n\left(\psi\left(\frac{k}{n}\right)-\psi\left(\frac{a_n}{n}\right)\right)} \right| \leq (b_n - a_n) M (1+\varepsilon) e^{-n\eta} \to 0$$

as $n \to \infty$, whereas if $k < a_n + \lceil 2n\delta \rceil$, then $\frac{k}{n} < \alpha + 3\delta$, and

$$(1-\varepsilon)\left(g\left(\frac{a_n}{n}\right)-\varepsilon\right) \sum_{k=a_n}^{a_n+\lceil 2n\delta \rceil -1} e^{n\left(\psi'\left(\frac{a_n}{n}\right)-\frac{\varepsilon}{2}\right)\left(\frac{k}{n}-\frac{a_n}{n}\right)} \leq \sum_{k=a_n}^{a_n+\lceil 2n\delta \rceil -1} (1+\epsilon_n(k)) g\left(\frac{k}{n}\right) e^{n\left(\psi\left(\frac{k}{n}\right)-\psi\left(\frac{a_n}{n}\right)\right)}$$

$$\leq (1+\varepsilon)\left(g\left(\frac{a_n}{n}\right)+\varepsilon\right) \sum_{k=a_n}^{a_n+\lceil 2n\delta \rceil -1} e^{n\left(\psi'\left(\frac{a_n}{n}\right)+\frac{\varepsilon}{2}\right)\left(\frac{k}{n}-\frac{a_n}{n}\right)},$$

and

$$\sum_{k=a_n}^{a_n+\lceil 2n\delta \rceil -1} e^{n\left(\psi'\left(\frac{a_n}{n}\right)-\varepsilon\right)\left(\frac{k}{n}-\frac{a_n}{n}\right)} = \sum_{k=0}^{\lceil 2n\delta \rceil -1} e^{\left(\psi'\left(\frac{a_n}{n}\right)-\varepsilon\right)k} = \frac{e^{\left(\psi'\left(\frac{a_n}{n}\right)-\varepsilon\right)\lceil 2n\delta \rceil} - 1}{e^{\left(\psi'\left(\frac{a_n}{n}\right)-\varepsilon\right)} - 1}.$$

We now observe that $\left|\psi'\left(\frac{a_n}{n}\right) - \psi'(\alpha)\right| < \frac{\varepsilon}{2}$, so $\psi'\left(\frac{a_n}{n}\right) + \frac{\varepsilon}{2} < \psi'(\alpha) + \varepsilon < 0$ and

$$e^{\left(\psi'\left(\frac{a_n}{n}\right)+\frac{\varepsilon}{2}\right)\lceil 2n\delta \rceil - 1} \to 0$$

as $n \to \infty$. Proceeding similarly we obtain a lower bound. Since $\varepsilon > 0$ can be chosen arbitrarily small, the result follows.

To prove the case when $\psi'(\gamma) = 0$, we proceed as previously and write

$$\psi(x) = \psi(y) + \psi'(y)(x-y) + \left(\frac{1}{2}\psi''(y) + R(x,y)\right)(x-y)^2$$

where $R(x,y) \to 0$ as $|x-y| \to 0$. Fix $\varepsilon > 0$ sufficiently small that $\psi''(\gamma) + \varepsilon < 0$, and choose $\delta > 0$ sufficiently small that $|R(x,y)| < \varepsilon$ and $|g(x) - g(y)| < \varepsilon$ for all $|x-y| < 2\delta$. As before, suppose that $|g(x)| < M$ and $|h(x)| < m$ for $x \in [\alpha-\delta, \beta+\delta]$, that $\psi(\gamma) > \psi(x) + \eta$ for $|\gamma - x| > \delta$ and that $\epsilon_n < \frac{\varepsilon}{2m}$ for all n. Then, setting $c_n = \lfloor \gamma n \rfloor$

if $\gamma \in (\alpha, \beta)$, $c_n = a_n$ if $\gamma = \alpha$, and $c_n = b_n$ if $\gamma = \beta$,

$$\begin{aligned}
&\sum_{k=a_n}^{b_n-1} (1+\epsilon_n(k)) g\left(\frac{k}{n}\right) e^{n\psi\left(\frac{k}{n}\right)} \\
&\qquad = e^{n\psi\left(\frac{c_n}{n}\right)} \sum_{k=a_n}^{b_n-1} (1+\epsilon_n(k)) g\left(\frac{k}{n}\right) e^{n\left(\psi\left(\frac{k}{n}\right)-\psi\left(\frac{c_n}{n}\right)\right)} \\
&\qquad = e^{n\psi\left(\frac{c_n}{n}\right)} \left(\sum_{k=a_n}^{c_n-\lceil n\delta\rceil-1} (1+\epsilon_n(k)) g\left(\frac{k}{n}\right) e^{n\left(\psi\left(\frac{k}{n}\right)-\psi\left(\frac{c_n}{n}\right)\right)} \right. \\
&\qquad\quad + \sum_{k=c_n-\lceil n\delta\rceil}^{c_n+\lceil n\delta\rceil} (1+\epsilon_n(k)) g\left(\frac{k}{n}\right) e^{n\left(\psi\left(\frac{k}{n}\right)-\psi\left(\frac{c_n}{n}\right)\right)} \\
&\qquad\quad \left. + \sum_{k=c_n+\lceil n\delta\rceil+1}^{b_n-1} (1+\epsilon_n(k)) g\left(\frac{k}{n}\right) e^{n\left(\psi\left(\frac{k}{n}\right)-\psi\left(\frac{c_n}{n}\right)\right)} \right),
\end{aligned}$$

where, as before, the first and last sums are bounded above by $(b_n - a_n)(1+\varepsilon)Me^{-n\eta}$ and

$$\begin{aligned}
&(1-\varepsilon)\left(g\left(\frac{c_n}{n}\right)-\varepsilon\right) \sum_{k=c_n-\lceil n\delta\rceil}^{c_n+\lceil n\delta\rceil} e^{n\left(\psi'\left(\frac{c_n}{n}\right)\left(\frac{k}{n}-\frac{c_n}{n}\right)+\left(\frac{1}{2}\psi''\left(\frac{c_n}{n}\right)-\frac{\varepsilon}{2}\right)\left(\frac{k}{n}-\frac{c_n}{n}\right)^2\right)} \\
&\qquad \le \sum_{k=c_n-\lceil n\delta\rceil}^{c_n+\lceil n\delta\rceil} (1+\epsilon_n(k)) g\left(\frac{k}{n}\right) e^{n\left(\psi\left(\frac{k}{n}\right)-\psi\left(\frac{c_n}{n}\right)\right)} \\
&\qquad \le (1+\varepsilon)\left(g\left(\frac{c_n}{n}\right)+\varepsilon\right) \sum_{k=c_n-\lceil n\delta\rceil}^{c_n+\lceil n\delta\rceil} e^{n\left(\psi'\left(\frac{c_n}{n}\right)\left(\frac{k}{n}-\frac{c_n}{n}\right)+\left(\frac{1}{2}\psi''\left(\frac{c_n}{n}\right)+\frac{\varepsilon}{2}\right)\left(\frac{k}{n}-\frac{c_n}{n}\right)^2\right)},
\end{aligned}$$

As previously, we will show that for n sufficiently large, the upper sum has an upper bound arbitrarily close to $g(\gamma)e^{n\psi(\gamma)}\sqrt{\frac{2n\pi}{|\psi''(\gamma)|}}$. The lower sum is treated identically. Proceeding, we have

$$\begin{aligned}
&\sum_{k=c_n-\lceil 2n\delta\rceil}^{c_n+\lceil 2n\delta\rceil} e^{n\left(\psi'\left(\frac{c_n}{n}\right)\left(\frac{k}{n}-\frac{c_n}{n}\right)+\left(\frac{1}{2}\psi''\left(\frac{c_n}{n}\right)+\frac{\varepsilon}{2}\right)\left(\frac{k}{n}-\frac{c_n}{n}\right)^2\right)} \\
&\qquad = \sum_{k=-\lceil 2n\delta\rceil}^{\lceil 2n\delta\rceil} e^{\psi'\left(\frac{c_n}{n}\right)k+\frac{\psi''\left(\frac{c_n}{n}\right)+\frac{\varepsilon}{2}}{2n}k^2} = e^{z_n u_n^2} \sum_{k=-\lceil 2n\delta\rceil}^{\lceil 2n\delta\rceil} e^{-z_n(k-u_n)^2},
\end{aligned}$$

where $z_n = \left|\frac{\psi''\left(\frac{c_n}{n}\right)+\frac{\varepsilon}{2}}{2n}\right|$ and $u_n = \frac{\psi'\left(\frac{c_n}{n}\right)}{2z_n}$.

We analyze the latter sum via Poisson's summation formula [13], which tells us that for an integrable function f with Fourier transform $\hat{f}$,

$$\sum_{k=-\infty}^{\infty} f(k) = \sum_{k=-\infty}^{\infty} \hat{f}(k).$$

Applying this with $f(x) = e^{-z_n(x-u_n)^2}$ gives

$$\sum_{k=-\infty}^{\infty} e^{-z_n(k-u_n)^2} = \sum_{k=-\infty}^{\infty} \sqrt{\frac{\pi}{z_n}} e^{-iu_nk+\frac{k^2}{4z_n}}.$$

Now, for $k \neq 0$,

$$\lim_{n\to\infty} \left| \sqrt{\frac{\pi}{z_n}} e^{-iu_nk+\frac{k^2}{4z_n}} \right| = 0,$$

whereas, since ψ is twice continuously differentiable (and thus ψ' is Lipschitz), there exists a constant L such that

$$|z_n u_n^2| = \frac{4n|\psi'\left(\frac{c_n}{n}\right) - \psi'(\gamma)|}{\left|\psi''\left(\frac{c_n}{n}\right) + \frac{\varepsilon}{2}\right|} \leq 4Ln \left| \frac{c_n}{n} - \gamma \right|^2,$$

which thus vanishes as $n \to \infty$, provided $\left|\frac{c_n}{n} - \gamma\right| \ll \frac{1}{\sqrt{n}}$.

Since ε and δ may be chosen arbitrarily small, the result follows provided

$$\sum_{k=-\lfloor \delta n \rfloor}^{\lfloor \delta n \rfloor} e^{-z_n(k-u_n)^2} \sim \sum_{k=-\infty}^{\infty} e^{-z_n(k-u_n)^2}.$$

To see this, we first observe that

$$\begin{aligned} 0 \leq \sum_{k=\lfloor \delta n \rfloor+1}^{\infty} e^{-z_n(k-u_n)^2} &= \sum_{k=1}^{\infty} e^{-z_n\left(\lfloor \delta n \rfloor^2+(k-u_n)^2+2\lfloor \delta n \rfloor(k-u_n)\right)} \\ &< e^{-z_n\lfloor \delta n \rfloor^2+2z_nu_n\lfloor \delta n \rfloor} \sum_{k=1}^{\infty} e^{-z_n(k-u_n)^2} \end{aligned}$$

and, similarly,

$$0 \leq \sum_{k=-\infty}^{-\lfloor \delta n \rfloor-1} e^{-z_n(k-u_n)^2} < e^{-z_n\lfloor \delta n \rfloor^2-2z_nu_n\lfloor \delta n \rfloor} \sum_{k=-\infty}^{-1} e^{-z_n(k-u_n)^2}$$

Thus,

$$\begin{aligned} 0 \leq \sum_{k=-\infty}^{-\lfloor \delta n \rfloor-1} e^{-z_n(k-u_n)^2} &+ \sum_{k=\lfloor \delta n \rfloor+1}^{\infty} e^{-z_n(k-u_n)^2} \\ &< e^{-z_n\lfloor \delta n \rfloor^2+2z_n|u_n|\lfloor \delta n \rfloor} \left(\sum_{k=-\infty}^{\infty} e^{-z_n(k-u_n)^2} - 1 \right) \end{aligned}$$

so that

$$0 \le 1 - \frac{\sum_{k=-\lfloor \delta n \rfloor}^{\lfloor \delta n \rfloor} e^{-z_n(k-u_n)^2}}{\sum_{k=-\infty}^{\infty} e^{-z_n(k-u_n)^2}} < e^{-z_n\lfloor \delta n \rfloor^2 + 2z_n|u_n||\lfloor \delta n \rfloor} \left(1 - \frac{1}{\sum_{k=-\infty}^{\infty} e^{-z_n(k-u_n)^2}}\right),$$

and, since $z_n n^2 \ll n$, whereas $z_n|u_n|n = \ll n\left|\frac{c_n}{n} - \gamma\right| = o(\sqrt{n})$, the right hand side tends to 0 as $n \to \infty$. $\square$ $\square$

Remark 4. We note that provided $\alpha < \beta$ (resp. $\alpha < \gamma < \beta$) and the function g is bounded on some fixed interval containing $[\alpha, \beta]$, then the error in (i) and (ii) is independent of b_n and β or a_n and α respectively. Similarly, the bound in (iii) is independent of either endpoint.

Corollary 1. Let $\psi(x) = V(x)$, $g(x) = \sqrt{\frac{\mu(x)}{\lambda(x)}\frac{\lambda(0)}{\mu(0)}}$ and

$$\epsilon_n(k) = e^{nV^{(n)}(k) - nV\left(\frac{k}{n}\right) - \frac{1}{2}\left(f\left(\frac{k}{n}\right) - f(0)\right)} - 1.$$

Then $g(x)$ is continuous,

$$e^{nV^{(n)}(k)} = (1 + \epsilon_n(k)) g\left(\frac{k}{n}\right) e^{n\psi\left(\frac{k}{n}\right)},$$

and, from Lemma A.1, for any positive integers $a < b$,

$$|\epsilon_n(k)| < \frac{\sup_{x \in \left[\frac{a}{n}, \frac{b}{n}\right]} |f''(x)|(b-a)}{n^3}, \tag{8}$$

which we note is uniform in k. The first two assertions then follow from the corresponding parts of the Proposition.

The third statement follows immediately upon observing that

$$e^{nV'\left(\frac{a_n}{n}\right)} \sim e^{V'(0)a_n} = \left(\frac{\mu(0)}{\lambda(0)}\right)^{a_n}.$$

$\square$ $\square$

B.2 Proofs for §2.3

Proposition 4. Since the process can only change by increments of ± 1, for any m and j, we have

$$\begin{aligned}
\mathbb{P}_m\left\{T_j^{(n)} < T_{m+}^{(n)}\right\} &= \begin{cases} \frac{\mu_m}{\lambda_m + \mu_m} \mathbb{P}_{m-1}\left\{T_j^{(n)} < T_m^{(n)}\right\} & \text{if } j < m, \text{ and} \\ \frac{\lambda_m}{\lambda_m + \mu_m} \mathbb{P}_{m+1}\left\{T_j^{(n)} < T_m^{(n)}\right\} & \text{if } j > m. \end{cases} \\
&= \begin{cases} \frac{\mu_m}{\lambda_m + \mu_m} \frac{e^{nV^{(n)}(m-1)}}{\sum_{k=j}^{m-1} e^{nV^{(n)}(k)}} & \text{if } j < m, \text{ and} \\ \frac{\lambda_m}{\lambda_m + \mu_m} \frac{e^{nV^{(n)}(m)}}{\sum_{k=m}^{m-1} e^{nV^{(n)}(k)}} & \text{if } j > m. \end{cases}
\end{aligned}$$

Taking $m = \lfloor \nu n \rfloor$ and $j = \lfloor \xi n \rfloor$, we are thus left with the task of estimating the sums

$$\sum_{k=\lfloor \xi n \rfloor}^{\lfloor \nu n \rfloor - 1} e^{n(V^{(n)}(k) - V^{(n)}(\lfloor \nu n \rfloor - 1))} \quad \text{and} \quad \sum_{k=\lfloor \nu n \rfloor}^{\lfloor \xi n \rfloor - 1} e^{n(V^{(n)}(k) - V^{(n)}(\lfloor \nu n \rfloor - 1))},$$

using Corollary 1, where $V(x) - V(\nu)$ finds its maximum at either ξ or ν , and this maximum occurs at either the right or left side of the interval of interest, $[\xi, \nu]$ or $[\nu, \xi]$, depending on where ξ lies.

Now, given that $V(x)$ is convex with a minimum at κ, and $V(0) = 0$, there exists a unique $\nu' \neq \nu$ such that $V(\nu') = V(\nu)$. If $\xi < \nu < \kappa < \nu'$ or $\xi < \nu' < \kappa < \nu$, the interval is $[\xi, \nu]$, the maximum occurs at $x = \xi$ and

$$\sum_{k=\lfloor \xi n \rfloor}^{\lfloor \nu n \rfloor - 1} e^{n(V^{(n)}(k) - V^{(n)}(\lfloor \nu n \rfloor - 1))} \sim \frac{\sqrt{\frac{\mu(\xi)}{\lambda(\xi)} \frac{\lambda(\nu)}{\mu(\nu)}} e^{n\left(V\left(\frac{\lfloor \xi \rfloor}{n}\right) - V\left(\frac{\lfloor \nu \rfloor}{n}\right)\right)}}{1 - \frac{\mu(\xi)}{\lambda(\xi)}},$$

whereas if $\nu < \kappa < \nu' < \xi$ or $\nu' < \kappa < \nu < \xi$, the interval is $[\nu, \xi]$, the maximum occurs at $x = \xi$ and

$$\sum_{k=\lfloor \nu n \rfloor}^{\lfloor \xi n \rfloor - 1} e^{n(V^{(n)}(k) - V^{(n)}(\lfloor \nu n \rfloor - 1))} \sim \frac{\sqrt{\frac{\mu(\xi)}{\lambda(\xi)} \frac{\lambda(\nu)}{\mu(\nu)}} e^{n\left(V\left(\frac{\lfloor \xi \rfloor}{n}\right) - V\left(\frac{\lfloor \nu \rfloor}{n}\right)\right)}}{1 - \frac{\lambda(\xi)}{\mu(\xi)}}.$$

If if $\nu < \xi < \nu'$ or $\nu' < \xi < \nu$, the maximum is at $x = \nu$ whereas the interval is $[\nu, \xi]$ or $[\xi, \nu]$ respectively, and one has

$$\sum_{k=\lfloor \xi n \rfloor}^{\lfloor \nu n \rfloor - 1} e^{n(V^{(n)}(k) - V^{(n)}(\lfloor \nu n \rfloor - 1))} \sim \frac{1}{1 - \frac{\mu(\nu)}{\lambda(\nu)}},$$

and

$$\sum_{k=\lfloor \nu n \rfloor}^{\lfloor \xi n \rfloor - 1} e^{n(V^{(n)}(k) - V^{(n)}(\lfloor \nu n \rfloor - 1))} \sim \frac{1}{1 - \frac{\lambda(\nu)}{\mu(\nu)}},$$

respectively. □ □

Proposition 5. Since the process $X^{(n)}(t)$ is Markov, for any integer k, each excursion from k is an independent renewal, and thus the number of returns prior to hitting zero has a geometric distribution with success parameter $\mathbb{P}_k \left\{ T_0^{(n)} < T_{k+}^{(n)} \right\}$:

$$\mathbb{P}_m \left\{ N_k^{(n)}(T_0^{(n)}) = l \middle| T_{\lfloor \nu n \rfloor}^{(n)} < T_0^{(n)} \right\} = \mathbb{P}_k \left\{ T_0^{(n)} < T_{k+}^{(n)} \right\} \left(1 - \mathbb{P}_k \left\{ T_0^{(n)} < T_{k+}^{(n)} \right\} \right)^{l-1}$$

with mean

$$\mathbb{E}_m \left[N_k^{(n)}(T_0^{(n)}) \middle| T_{\lfloor \nu n \rfloor}^{(n)} < T_0^{(n)} \right] = \frac{1}{\mathbb{P}_k \left\{ T_0^{(n)} < T_{k+}^{(n)} \right\}}.$$

The result follows taking $k = \lfloor \nu n \rfloor$, and using the asymptotic for $\mathbb{P}_{\lfloor \nu n \rfloor} \left\{ T_0^{(n)} < T_{k+}^{(n)} \right\}$ from the previous proposition with $\xi = 0$, recalling that $V(0) = 0$. □ □

B.3 Proofs for §2.4

Proposition 6. The proof presented here is based upon the treatment given for the Moran model in [9]. We first observe that the hitting time of 0 or $\lfloor \kappa n \rfloor$ is the sum of the time spent in all in-between states prior to $T^{(n)}_{\lfloor \kappa n \rfloor}$, so that

$$\begin{aligned}\mathbb{E}_m\left[T^{(n)}_{\lfloor \kappa n \rfloor}\middle|T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_0\right] &= \sum_{k=1}^{\lfloor \kappa n \rfloor - 1} \mathbb{E}_m\left[S^{(n)}_k(T^{(n)}_{\lfloor \kappa n \rfloor})\middle|T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_0\right] \\ &= \frac{1}{\lambda_k+\mu_k}\mathbb{E}_m\left[N^{(n)}_k(T^{(n)}_{\lfloor \kappa n \rfloor})\middle|T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_0\right],\end{aligned}$$

as $\frac{1}{\lambda_k+\mu_k}$ is the expected time spent in state k, which is exponentially distributed with parameter $\lambda_k+\mu_k$.

Now, $N^{(n)}_m(T^{(n)}_{\lfloor \kappa n \rfloor})$ has a modified geometric distribution:

$$\begin{aligned}&\mathbb{P}_m\left\{N^{(n)}_k(T^{(n)}_{\lfloor \kappa n \rfloor}) = l\middle|T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_0\right\} \\ &= \begin{cases}\mathbb{P}_m\left\{T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_k\middle|T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_0\right\} & \text{if } l=0, \text{ and} \\ \mathbb{P}_m\left\{T^{(n)}_k < T^{(n)}_{\lfloor \kappa n \rfloor}\middle|T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_0\right\} & \\ \quad\times \mathbb{P}_k\left\{T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_{k+}\middle|T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_0\right\} & \text{if } l \geq 1, \\ \quad\times\left(1-\mathbb{P}_k\left\{T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_{k+}\middle|T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_0\right\}\right)^{l-1} & \end{cases}\end{aligned}$$

which has mean

$$\frac{\mathbb{P}_m\left\{T^{(n)}_k < T^{(n)}_{\lfloor \kappa n \rfloor}\middle|T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_0\right\}}{\mathbb{P}_k\left\{T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_{k+}\middle|T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_0\right\}}.$$

Now, if we specialize to the case when $m=1$, then the process must pass through k en route to $\lfloor \kappa n \rfloor$, so

$$\mathbb{P}_1\left\{T^{(n)}_k < T^{(n)}_{\lfloor \kappa n \rfloor}\middle|T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_0\right\} = 1$$

Moreover, conditional on $T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_0$, starting from k, $T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_{k+}$ if and only if a birth occurs *and* the process hits $\lfloor \kappa n \rfloor$ prior to k:

$$\mathbb{P}_k\left\{T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_{k+}\middle|T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_0\right\} = \frac{\lambda_k}{\lambda_k+\mu_k}\mathbb{P}_{k+1}\left\{T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_k\middle|T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_0\right\}$$

so that

$$\mathbb{E}_1\left[T^{(n)}_{\lfloor \kappa n \rfloor}\middle|T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_0\right] = \sum_{k=1}^{\lfloor \kappa n \rfloor - 1} \frac{1}{\lambda_k \mathbb{P}_{k+1}\left\{T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_k\middle|T^{(n)}_{\lfloor \kappa n \rfloor} < T^{(n)}_0\right\}}$$

whereas

$$\mathbb{P}_{k+1}\left\{T^{(n)}_{\lfloor\kappa n\rfloor}<T^{(n)}_{k}\middle|T^{(n)}_{\lfloor\kappa n\rfloor}<T^{(n)}_{0}\right\}=\frac{\mathbb{P}_{k+1}\left\{T^{(n)}_{\lfloor\kappa n\rfloor}<T^{(n)}_{k}\right\}}{\mathbb{P}_{k+1}\left\{T^{(n)}_{\lfloor\kappa n\rfloor}<T^{(n)}_{0}\right\}},$$

since $\left\{T^{(n)}_{\lfloor\kappa n\rfloor}<T^{(n)}_{k}\right\}\subseteq\left\{T^{(n)}_{\lfloor\kappa n\rfloor}<T^{(n)}_{0}\right\}$, as to reach 0 from $k+1$, the process must pass via k.

Now, Proposition 2 and its proof tell us that $\mathbb{P}_{k+1}\left\{T^{(n)}_{\lfloor\kappa n\rfloor}<T^{(n)}_{0}\right\}$ tends to 1 as $k\to\infty$, and, moreover, that this convergence is uniform in n. We may thus apply Lemma A.3, to conclude that

$$\mathbb{E}_1\left[T^{(n)}_{\lfloor\kappa n\rfloor}\middle|T^{(n)}_{\lfloor\kappa n\rfloor}<T^{(n)}_{0}\right]\sim\sum_{k=1}^{\lfloor\kappa n\rfloor-1}\frac{1}{\lambda_k\mathbb{P}_{k+1}\left\{T^{(n)}_{\lfloor\kappa n\rfloor}<T^{(n)}_{k}\right\}}$$

We now observe that

$$\mathbb{P}_{k+1}\left\{T^{(n)}_{\lfloor\kappa n\rfloor}<T^{(n)}_{k}\right\}=h^{(n)}_{\lfloor\kappa n\rfloor,k}(k+1)=\frac{e^{nV^{(n)}(k)}}{\sum_{j=k}^{\lfloor\kappa n\rfloor-1}e^{nV^{(n)}(j)}},$$

which, by Corollary 1 is asymptotically equivalent to $1-\frac{\mu\left(\frac{k}{n}\right)}{\lambda\left(\frac{k}{n}\right)}$, so recalling that $\lambda_k=\lambda\left(\frac{k}{n}\right)k$,

$$\sum_{k=1}^{\lfloor\kappa n\rfloor-1}\frac{1}{\lambda_k\mathbb{P}_{k+1}\left\{T^{(n)}_{\lfloor\kappa n\rfloor}<T^{(n)}_{k}\right\}}\sim\sum_{k=1}^{\lfloor\kappa n\rfloor-1}\frac{1}{\left(\lambda\left(\frac{k}{n}\right)-\mu\left(\frac{k}{n}\right)\right)k}.$$

The latter is the Riemann sum for the integral of $\frac{1}{(\lambda(x)-\mu(x))x}$ over $[0,\kappa]$, but this integral diverges at both endpoints. To deal with this, first observe that, using Taylor's theorem, we may write

$$\lambda(x)-\mu(x)=\lambda(0)-\mu(0)+(\lambda'(0)-\mu'(0)+r(x))x,$$

where $r(x)\to 0$ as $x\to 0$ and

$$\lambda(x)-\mu(x)=(\lambda'(\kappa)-\mu'(\kappa))(x-\kappa)+(\lambda''(\kappa)-\mu''(\kappa)+R(x))(x-\kappa)^2,$$

for a continuous function $R(x)$ such that $R(x)\to 0$ as $x\to\kappa$. Then, for arbitrary $\varepsilon>0$, we can choose n sufficiently large that

$$\lambda(0)-\mu(0)<\lambda\left(\tfrac{k}{n}\right)-\mu\left(\tfrac{k}{n}\right)<\lambda(0)-\mu(0)+\varepsilon$$

for all $k\leq\frac{n}{\ln n}$ and

$$(\lambda'(\kappa)-\mu'(\kappa))\left(\tfrac{k}{n}-\kappa\right)-\varepsilon<\lambda\left(\tfrac{k}{n}\right)-\mu\left(\tfrac{k}{n}\right)<(\lambda'(\kappa)-\mu'(\kappa))\left(\tfrac{k}{n}-\kappa\right)+\varepsilon$$

for all $\lfloor \kappa n \rfloor - \left\lfloor \frac{n}{\ln n} \right\rfloor \leq k < \lfloor \kappa n \rfloor$, and split the sum in three:

$$\sum_{k=1}^{\left\lfloor \frac{n}{\ln n} \right\rfloor} \frac{1}{\left(\lambda\left(\frac{k}{n}\right) - \mu\left(\frac{k}{n}\right)\right) k} + \sum_{k=\left\lfloor \frac{n}{\ln n} \right\rfloor+1}^{\lfloor \kappa n \rfloor - \left\lfloor \frac{n}{\ln n} \right\rfloor - 1} \frac{1}{\left(\lambda\left(\frac{k}{n}\right) - \mu\left(\frac{k}{n}\right)\right) k} + \sum_{k=\lfloor \kappa n \rfloor - \left\lfloor \frac{n}{\ln n} \right\rfloor}^{\lfloor \kappa n \rfloor - 1} \frac{1}{\left(\lambda\left(\frac{k}{n}\right) - \mu\left(\frac{k}{n}\right)\right) k}.$$

For the first sum, we have that

$$\frac{1}{\lambda(0) - \mu(0) + \varepsilon} \sum_{k=1}^{\left\lfloor \frac{n}{\ln n} \right\rfloor} \frac{1}{k} \leq \sum_{k=1}^{\left\lfloor \frac{n}{\ln n} \right\rfloor} \frac{1}{\left(\lambda\left(\frac{k}{n}\right) - \mu\left(\frac{k}{n}\right)\right) k} \leq \frac{1}{\lambda(0) - \mu(0)} \sum_{k=1}^{\left\lfloor \frac{n}{\ln n} \right\rfloor} \frac{1}{k},$$

whereas

$$\sum_{k=1}^{\left\lfloor \frac{n}{\ln n} \right\rfloor} \frac{1}{k} = \ln \left\lfloor \frac{n}{\ln n} \right\rfloor + \gamma + \epsilon_{\left\lfloor \frac{n}{\ln n} \right\rfloor},$$

where γ is the Euler-Mascheroni constant and $\epsilon_{\left\lfloor \frac{n}{\ln n} \right\rfloor} \sim \frac{1}{2\left\lfloor \frac{n}{\ln n} \right\rfloor}$.

Similarly,

$$\frac{1}{(\lambda'(\kappa) - \mu'(\kappa)) + \varepsilon)\kappa} \sum_{k=\lfloor \kappa n \rfloor - \left\lfloor \frac{n}{\ln n} \right\rfloor}^{\lfloor \kappa n \rfloor - 1} \frac{1}{k - \kappa n} \leq \sum_{k=\lfloor \kappa n \rfloor - \left\lfloor \frac{n}{\ln n} \right\rfloor}^{\lfloor \kappa n \rfloor - 1} \frac{1}{\left(\lambda\left(\frac{k}{n}\right) - \mu\left(\frac{k}{n}\right)\right) k}$$
$$\leq \frac{1}{(\lambda'(\kappa) - \mu'(\kappa)) - \varepsilon)(\kappa - \delta)} \sum_{k=\lfloor \kappa n \rfloor - \left\lfloor \frac{n}{\ln n} \right\rfloor}^{\lfloor \kappa n \rfloor - 1} \frac{1}{k - \kappa n}$$

and,

$$\sum_{k=\lfloor \kappa n \rfloor - \left\lfloor \frac{n}{\ln n} \right\rfloor}^{\lfloor \kappa n \rfloor - 1} \frac{1}{k - \kappa n} = - \sum_{k=1}^{\left\lfloor \frac{n}{\ln n} \right\rfloor} \frac{1}{k + \kappa n - \lfloor \kappa n \rfloor}.$$

Since $0 \leq \kappa n - \lfloor \kappa n \rfloor < 1$,

$$\sum_{k=1}^{\left\lfloor \frac{n}{\ln n} \right\rfloor} \frac{1}{k+1} < \sum_{k=1}^{\left\lfloor \frac{n}{\ln n} \right\rfloor} \frac{1}{k + \kappa n - \lfloor \kappa n \rfloor} \leq \sum_{k=1}^{\left\lfloor \frac{n}{\ln n} \right\rfloor} \frac{1}{k}$$

Finally, to deal with the middle sum, we first note that

$$\frac{d}{dx} \frac{1}{(\lambda(x) - \mu(x))x} = - \frac{(\lambda'(x) - \mu'(x))x + \lambda(x) - \mu(x)}{(\lambda(x) - \mu(x))^2 x^2}$$

is bounded on any closed interval in $(0, \kappa)$ and tends to $+\infty$ at 0, where it is decreasing, and at κ, where it is increasing; in particular, on $\left[\frac{1}{\ln n}, \kappa - \frac{1}{\ln n}\right]$ the derivative is bounded above by its values at the endpoints, which are bounded above by

$$\frac{\sup_{x \in [0,\kappa]} -(\lambda'(x) - \mu'(x))x}{\min\{\lambda(0) - \mu(0), (\lambda'(\kappa) - \mu'(\kappa))\kappa\}} (\ln n)^2.$$

Thus, applying Lemma A.1, we have that

$$\left| \sum_{k=\lfloor \frac{n}{\ln n} \rfloor +1}^{\lfloor \kappa n \rfloor - \lfloor \frac{n}{\ln n} \rfloor -1} \frac{1}{\left(\lambda\left(\frac{k}{n}\right) - \mu\left(\frac{k}{n}\right)\right) k} - \int_{\frac{1}{n}\left(\lfloor \frac{n}{\ln n} \rfloor +1\right)}^{\frac{1}{n}\left(\lfloor \kappa n \rfloor - \lfloor \frac{n}{\ln n} \rfloor\right)} \frac{dx}{(\lambda(x)-\mu(x))x} \right|$$
$$\leq \frac{\sup_{x\in[0,\kappa]} -(\lambda'(x)-\mu'(x))x}{\min\{\lambda(0)-\mu(0), (\lambda'(\kappa)-\mu'(\kappa))\kappa\}} \frac{(\ln n)^2}{2n}$$

Moreover,

$$0 \leq \int_{\frac{1}{n}\left(\lfloor \frac{n}{\ln n} \rfloor +1\right)}^{\frac{1}{n}\left(\lfloor \kappa n \rfloor - \lfloor \frac{n}{\ln n} \rfloor\right)} \frac{dx}{(\lambda(x)-\mu(x))x} \leq \int_{\frac{1}{\ln n}}^{\kappa - \frac{1}{\ln n}} \frac{dx}{(\lambda(x)-\mu(x))x},$$

and, since $r(x)$ and $R(x)$ are continuous, and thus bounded on $[0,\kappa]$,

$$\int_{\frac{1}{\ln n}}^{\frac{\kappa}{2}} \frac{dx}{(\lambda(x)-\mu(x))x} - \int_{\frac{1}{\ln n}}^{\frac{\kappa}{2}} \frac{dx}{(\lambda(0)-\mu(0))x} = \int_{\frac{1}{\ln n}}^{\frac{\kappa}{2}} \frac{\lambda'(0)-\mu'(0)+r(x)}{(\lambda(0)-\mu(0))(\lambda(x)-\mu(x))}\,dx$$

and

$$\int_{\frac{\kappa}{2}}^{\kappa-\frac{1}{\ln n}} \frac{dx}{(\lambda(x)-\mu(x))x} - \int_{\frac{\kappa}{2}}^{\kappa-\frac{1}{\ln n}} \frac{dx}{(\lambda'(\kappa)-\mu'(\kappa))\kappa(x-\kappa)}$$
$$= \int_{\frac{1}{\ln n}}^{\frac{\kappa}{2}} \frac{\lambda''(\kappa)-\mu''(\kappa)+R(x)}{(\lambda'(\kappa)-\mu'(\kappa))\kappa h(x)}\,dx$$

are bounded, where

$$h(x) = \begin{cases} \frac{(\lambda(x)-\mu(x))x}{(x-\kappa)} & \text{for } x \neq \kappa, \text{ and} \\ (\lambda'(\kappa)-\mu'(\kappa))\kappa & \text{for } x = \kappa. \end{cases}$$

Finally, we observe that

$$\int_{\frac{1}{\ln n}}^{\frac{\kappa}{2}} \frac{dx}{(\lambda(0)-\mu(0))x} + \frac{1}{\lambda(0)-\mu(0)}\left(\ln\frac{\kappa}{2} - \ln\left(\frac{1}{\ln n}\right)\right)$$

and

$$\int_{\frac{\kappa}{2}}^{\kappa-\frac{1}{\ln n}} \frac{dx}{(\lambda'(\kappa)-\mu'(\kappa))\kappa(x-\kappa)} = \frac{1}{(\lambda'(\kappa)-\mu'(\kappa))\kappa}\left(\ln\left(\frac{1}{\ln n}\right) - \ln\frac{\kappa}{2}\right),$$

so that the middle sum is $\ll \ln\ln n$.

Since the choice of ε is arbitrary, the result follows. □ □

Proposition 7. We begin with a pair of lemmas:

Lemma B.1. *The logistic process conditioned on the event* $T_0^{(n)} < T_M^{(n)}$ *is a Markov birth and death process with transition rates*

$$\tilde{\lambda}_k^{(n)} = \lambda_k^{(n)} \frac{h_{0,M}^{(n)}(k+1)}{h_{0,M}^{(n)}(k)} \quad \text{and} \quad \tilde{\mu}_k^{(n)} = \mu_k^{(n)} \frac{h_{0,M}^{(n)}(k-1)}{h_{0,M}^{(n)}(k)},$$

In particular, taking $M = \lfloor \nu n \rfloor$ *for* $\kappa < \nu < \eta$*, we have that*

$$\lim_{n\to\infty} \tilde{\lambda}_k^{(n)} = \mu(0)k \quad \text{and} \quad \lim_{n\to\infty} \tilde{\mu}_k^{(n)} = \lambda(0)k$$

Proof. This is a special case of Doob's h-transform [8]. □ □

Lemma B.2. *Let*

$$\tau_M^{(n)}(m) = \mathbb{E}_m \left[T_0^{(n)} \middle| T_0^{(n)} < T_M^{(n)} \right].$$

Then,

$$\tau_M^{(n)}(m) = \sum_{k=1}^{m} \sum_{j=k}^{M-1} \frac{1}{\tilde{\lambda}_j^{(n)}} \prod_{l=k}^{j} \frac{\tilde{\lambda}_l^{(n)}}{\tilde{\mu}_l^{(n)}}$$

Proof. For $m < M$ the function $\tau_M^{(n)}$ satisfies the recurrence relation

$$\tau_M^{(n)}(m) = \frac{1}{\tilde{\lambda}_m^{(n)} + \tilde{\mu}_m^{(n)}} + \frac{\tilde{\lambda}_m^{(n)}}{\tilde{\lambda}_m^{(n)} + \tilde{\mu}_m^{(n)}} \tau_M^{(n)}(m+1) + \frac{\tilde{\mu}_m^{(n)}}{\tilde{\lambda}_m^{(n)} + \tilde{\mu}_m^{(n)}} \tau_M^{(n)}(m-1),$$

with boundary $\tau_M^{(n)}(0) = 0$, whilst

$$\tau_M^{(n)}(M-1) = \frac{1}{\tilde{\mu}_{M-1}^{(n)}} + \tau_M^{(n)}(M-2).$$

Solving the recurrence equation gives the result. As previously, we refer to [12] for a detailed treatment. □ □

The proof consists in showing that the sum

$$\sum_{j=i}^{\lfloor \kappa n \rfloor - 1} \frac{1}{\tilde{\lambda}_j^{(n)}} \prod_{k=i}^{j} \frac{\tilde{\lambda}_k^{(n)}}{\tilde{\mu}_k^{(n)}}$$

is uniformly bounded in n, so that we can apply Lemma A.2 to interchange sum and limit to obtain

$$\lim_{n\to\infty} \tau_{\lfloor \kappa n \rfloor}^{(n)}(m) = \sum_{k=1}^{m} \sum_{j=k}^{\infty} \frac{1}{\mu(0)j} \left(\frac{\mu(0)}{\lambda(0)} \right)^j = \frac{1}{\mu(0)} \sum_{k=1}^{m} \sum_{j=k}^{\infty} \int_0^{\frac{\mu(0)}{\lambda(0)}} x^{j-1}\, dx$$

$$= \frac{1}{\mu(0)} \sum_{k=1}^{m} \int_0^{\frac{\mu(0)}{\lambda(0)}} \frac{x^{k-1}}{1-x}\, dx = \frac{1}{\mu(0)} \int_0^{\frac{\mu(0)}{\lambda(0)}} \frac{1-x^m}{(1-x)^2}\, dx.$$

First,

$$\prod_{l=k}^{j}\frac{\tilde{\lambda}_l^{(n)}}{\tilde{\mu}_l^{(n)}} = \prod_{l=k}^{j}\frac{\lambda_l^{(n)}}{\mu_l^{(n)}}\prod_{l=k}^{j}\frac{h_{0,\lfloor\kappa n\rfloor}^{(n)}(l+1)}{h_{0,\lfloor\kappa n\rfloor}^{(n)}(k-1)}$$
$$= \prod_{l=1}^{k-1}\frac{\mu_l^{(n)}}{\lambda_l^{(n)}}\prod_{l=1}^{j}\frac{\lambda_l^{(n)}}{\mu_l^{(n)}}\frac{h_{0,\lfloor\kappa n\rfloor}^{(n)}(j+1)h_{0,\lfloor\kappa n\rfloor}^{(n)}(j)}{h_{0,\lfloor\kappa n\rfloor}^{(n)}(k-1)h_{0,\lfloor\kappa n\rfloor}^{(n)}(k-1)},$$

so, since $i \le m$, we can ignore terms in i and consider only the sum

$$\sum_{j=1}^{\lfloor\kappa n\rfloor-1}\frac{h_{0,\lfloor\kappa n\rfloor}^{(n)}(j+1)h_{0,\lfloor\kappa n\rfloor}^{(n)}(j)}{\tilde{\lambda}_j^{(n)}}\prod_{k=1}^{j}\frac{\lambda_k^{(n)}}{\mu_k^{(n)}} = \sum_{j=1}^{\lfloor\kappa n\rfloor-1}\frac{(h_{0,\lfloor\kappa n\rfloor}^{(n)}(j))^2}{\lambda_j^{(n)}}e^{-nV^{(n)}(j)}$$
$$\le \frac{2}{\left(\sum_{k=0}^{\lfloor\kappa n\rfloor-1}e^{nV^{(n)}(k)}\right)^2}\sum_{j=1}^{\lfloor\kappa n\rfloor-1}\frac{1}{\lambda_j^{(n)}}\sum_{k=j}^{\lfloor\kappa n\rfloor-1}e^{n(2V^{(n)}(k)-V^{(n)}(j))}$$
$$\le \frac{2}{\lambda(0)\left(\sum_{k=0}^{\lfloor\kappa n\rfloor-1}e^{nV^{(n)}(k)}\right)^2}\sum_{j=1}^{\lfloor\kappa n\rfloor-1}\sum_{k=j}^{\lfloor\kappa n\rfloor-1}e^{nV^{(n)}(k)}.$$

Now, by Lemma A.1, $nV^{(n)}(k) \le n\int_0^{\frac{k}{n}} f(x)\,dx + f\left(\frac{k}{n}\right) - f(0)$, whereas, by the intermediate value theorem for integrals, we have

$$n\int_0^{\frac{k}{n}} f(x)\,dx = f(z_{k,n})i$$

for some $z_{k,n} \in [0, \frac{k}{n}]$. Now, fix $0 < \varepsilon < \kappa$. Provided $k \le \lfloor\nu n\rfloor$ for $0 < \nu < \eta$, either $\frac{k}{n} < \varepsilon$, in which case $f(z_{k,n}) < f(\varepsilon) < 0$, or

$$f(z_{k,n})\varepsilon < f(z_{k,n})\frac{k}{n} = \int_0^{\frac{k}{n}} f(x)\,dx < \min\left\{\int_0^{\varepsilon} f(x)\,dx, \int_0^{\nu} f(x)\,dx\right\} < 0,$$

and thus, $\rho := \sup_n f(z_{k,n}) < 0$.

Now $0 \le f\left(\frac{k}{n}\right) - f(0) \le f(\nu) - f(0)$, so if

$$a_{n,k} = \begin{cases}\prod_{j=1}^{k}\frac{\mu_j}{\lambda_j} & \text{if } k \le \lfloor\nu n\rfloor, \text{ and}\\ 0 & \text{otherwise}\end{cases},$$

then $a_{n,k} \le e^{f(\nu)-f(0)}e^{\rho i}$, and $e^{\rho} < 1$, so

$$1 \le \sum_{k=j}^{\lfloor\nu n\rfloor-1} e^{nV^{(n)}(k)} \le \sum_{k=j}^{\infty} e^{f(\nu)-f(0)}e^{\rho k} = \frac{e^{f(\nu)-f(0)}}{1-e^{\rho}}e^{\rho j},$$

and the sum above is bounded, independently of n. □ □

Proposition 8. We first observe that the time to extinction is simply the time spent in all states $k > 0$:

$$\mathbb{E}_m\left[T_0^{(n)}\right] = \sum_{k=1}^{\infty} \mathbb{E}_m\left[S_k^{(n)}(T_0^{(n)})\right] = \sum_{k=1}^{\infty} \frac{1}{\lambda_k + \mu_k} \mathbb{E}_m\left[N_k^{(n)}(T_0^{(n)})\right],$$

as $\frac{1}{\lambda_k+\mu_k}$ is the expected time spent in state k per visit, and, by definition, $N_k^{(n)}(T_0^{(n)})$ is the total number of visits to k prior to extinction. As before, $N_k^{(n)}(T_0^{(n)})$ has a modified geometric distribution with mean

$$\frac{\mathbb{P}_m\left\{T_k^{(n)} < T_0^{(n)}\right\}}{\mathbb{P}_k\left\{T_0^{(n)} < T_{k+}^{(n)}\right\}}.$$

For the denominator, the process can only fail to return to k if the next event is a death and the process hits 0 prior to hitting k:

$$\mathbb{P}_k\left\{T_0^{(n)} < T_{k+}^{(n)}\right\} = \frac{\mu_k}{\lambda_k + \mu_k} \mathbb{P}_{k-1}\left\{T_0^{(n)} < T_k^{(n)}\right\} = \frac{\mu_k}{\lambda_k + \mu_k} \frac{e^{nV^{(n)}(k-1)}}{\sum_{j=0}^{k-1} e^{nV^{(n)}(j)}}.$$

We first note that in light of (8), there is a bound ϵ_n that tends to 0 as $n \to \infty$ such that

$$\left| \frac{e^{nV^{(n)}(k-1)}}{\sqrt{\frac{\mu\left(\frac{k-1}{n}\right)}{\mu(0)} \frac{\lambda(0)}{\lambda\left(\frac{k-1}{n}\right)}} e^{nV\left(\frac{k-1}{n}\right)}} - 1 \right| < \epsilon_n$$

uniformly in k.

Now, fix some small $\delta > 0$ such that $V(\delta) > V(\kappa)$. We consider the sum in k in two parts, $k \le \lfloor \delta n \rfloor$, and $k > \lfloor \delta n \rfloor$. We first consider the latter.

As we observed in Remark 4, since $V(0) > V(\frac{k-1}{n})$ and $k > \lfloor \delta n \rfloor$,

$$\left| \frac{\sum_{j=0}^{k-1} e^{nV^{(n)}(j)}}{\frac{1}{1-\frac{\mu(0)}{\lambda(0)}}} - 1 \right| < \eta_n,$$

where $\eta_n \to 0$ as $n \to \infty$, independently of k. Thus,

$$\mathbb{P}_k\left\{T_0^{(n)} < T_{k+}^{(n)}\right\} \sim \left(1 - \frac{\mu(0)}{\lambda(0)}\right) \sqrt{\frac{\mu\left(\frac{k-1}{n}\right)}{\mu(0)} \frac{\lambda(0)}{\lambda\left(\frac{k-1}{n}\right)}} e^{nV\left(\frac{k-1}{n}\right)}$$

uniformly in i.

For the numerator, from Proposition 2, we know that

$$\mathbb{P}_m\left\{T_k^{(n)} < T_0^{(n)}\right\} \sim 1 - \left(\frac{\mu(0)}{\lambda(0)}\right)^m$$

if $m < \eta n$, whereas

$$\lim_{n\to\infty} \mathbb{P}_m \left\{ T_k^{(n)} < T_0^{(n)} \right\} = 0$$

otherwise, again uniformly in k.

Thus, since $\mu_k = n\mu\left(\frac{k}{n}\right)\frac{k}{n}$, $-V(x)$ is maximized at $x=\kappa$, and $\mu(\kappa)=\lambda(\kappa)$, from Proposition 1 we have

$$\sum_{k=\lfloor \delta n \rfloor+1}^{\infty} \frac{1}{\lambda_k+\mu_k}\mathbb{E}_m\left[N_k^{(n)}(T_0^{(n)})\right]$$

$$\sim \frac{1-\left(\frac{\mu(0)}{\lambda(0)}\right)^m}{1-\frac{\mu(0)}{\lambda(0)}} \sum_{k=\lfloor \delta n\rfloor+1}^{\lfloor \eta n\rfloor} \frac{e^{-nV\left(\frac{k-1}{n}\right)}}{n\mu\left(\frac{k}{n}\right)\frac{k}{n}\sqrt{\frac{\mu\left(\frac{k-1}{n}\right)}{\lambda\left(\frac{k-1}{n}\right)}\frac{\lambda(0)}{\mu(0)}}}$$

$$\sim \sqrt{\frac{2\pi}{n\left(\frac{\mu'(\kappa)}{\mu(\kappa)}-\frac{\lambda'(\kappa)}{\lambda(\kappa)}\right)}\frac{\mu(0)}{\lambda(0)}}\frac{1-\left(\frac{\mu(0)}{\lambda(0)}\right)^m}{1-\left(\frac{\mu(0)}{\lambda(0)}\right)}\frac{e^{-nV(\kappa)}}{\mu(\kappa)\kappa\left(1-\frac{\lambda(0)}{\mu(0)}\right)}.$$

Finally, we observe that for $k \le \lfloor \delta n\rfloor$,

$$\mathbb{P}_m\left\{T_k^{(n)} < T_0^{(n)}\right\} \le 1,$$

whereas, since $V^{(n)}(j) < 0$,

$$\sum_{j=0}^{k-1} e^{nV^{(n)}(j)} \le k-1 \le \delta n.$$

Moreover, since λ and μ are, respectively, decreasing and increasing,

$$e^{nV^{(n)}(k-1)} \ge \sqrt{\frac{\mu\left(\frac{k-1}{n}\right)}{\mu(0)}\frac{\lambda(0)}{\lambda\left(\frac{k-1}{n}\right)}}e^{nV\left(\frac{k-1}{n}\right)}(1-\epsilon_n) \ge e^{nV(\delta)}(1-\epsilon_n),$$

so that

$$\sum_{k=1}^{\lfloor \delta n\rfloor} \frac{1}{\lambda_k+\mu_k}\mathbb{E}_m\left[N_k^{(n)}(T_0^{(n)})\right] \le \frac{(\delta n)^2}{\mu(0)(1-\epsilon_n)}e^{-nV(\delta)},$$

which is asymptotically smaller than the sum over $k > \lfloor \delta n\rfloor$. □ □

Corollary 4. By the strong Markov property, each excursion starting from state k is independent. Thus, conditional on n_k visits to k, the time spent in k after each return is a sum of n_k independent exponentially distributed random variables with rate $\lambda_k+\mu_k$ *i.e.,* a gamma-distributed with shape and rate parameters n_k and $\lambda_k+\mu_k$: the probability that the total time is in $[t, t+dt)$ is

$$\int_0^t \int_0^{t-t_1} \cdots \int_0^{t-t_1-t_2-\cdots-t_{n_k-2}} \prod_{j=1}^{n_k-1} (\lambda_k+\mu_k) e^{-(\lambda_k+\mu_k)t_j}$$
$$\times (\lambda_k+\mu_k) e^{-(\lambda_k+\mu_k)(t-t_1-t_2-\cdots-t_{n_k-1})}\, dt_1 dt_2 \cdots dt_{n_k-1}$$
$$= \frac{(\lambda_k+\mu_k)^{n_k}}{(n_k-1)!} t^{n_k-1} e^{-(\lambda_k+\mu_k)t}.$$

Now, we observed above that the number of visits to k prior to extinction, $N_k^{(n)}(T_0^{(n)})$, has a modified geometric distribution, with probability $\mathbb{P}_m\left\{T_k^{(n)} < T_0^{(n)}\right\}$ of reaching k prior to extinction, and return probability $\mathbb{P}_k\left\{T_0^{(n)} < T_{k+}^{(n)}\right\}$. The former gives the probability that $L_k^{(n)}(T_0^{(n)}) > 0$, whereas summing over the distribution of $N_k^{(n)}(T_0^{(n)})$ the probability that $L_k^{(n)}(T_0^{(n)}) \in [t, t+dt)$ is

$$\mathbb{P}_m\left\{T_k^{(n)} < T_0^{(n)}\right\}\left(1-\mathbb{P}_k\left\{T_0^{(n)} < T_{k+}^{(n)}\right\}\right)$$
$$\times \sum_{n_k=1}^{\infty} \frac{(\lambda_k+\mu_k)^{n_k}}{(n_k-1)!} t^{n_k-1} e^{-(\lambda_k+\mu_k)t} \mathbb{P}_k\left\{T_0^{(n)} < T_{k+}^{(n)}\right\}^{n_k-1}$$
$$= \mathbb{P}_m\left\{T_k^{(n)} < T_0^{(n)}\right\}(\lambda_k+\mu_k)\left(1-\mathbb{P}_k\left\{T_0^{(n)} < T_{k+}^{(n)}\right\}\right) e^{-(\lambda_k+\mu_k)\left(1-\mathbb{P}_k\left\{T_0^{(n)}<T_{k+}^{(n)}\right\}\right)t}.$$

The result then follows using the asymptotic approximations of the previous proof. □

□

Proposition 9. Proceeding as previously, we have that

$$\mathbb{E}_1\left[T_{\lfloor \nu n\rfloor}^{(n)} \middle| T_{\lfloor \nu n\rfloor}^{(n)} < T_0^{(n)}\right] \sim \sum_{k=1}^{\lfloor \nu n\rfloor-1} \frac{1}{\lambda_k \mathbb{P}_{k+1}\left\{T_{\lfloor \nu n\rfloor}^{(n)} < T_k^{(n)}\right\}}$$

and

$$\mathbb{P}_{k+1}\left\{T_{\lfloor \nu n\rfloor}^{(n)} < T_k^{(n)}\right\} = \frac{1}{\sum_{j=k}^{\lfloor \nu n\rfloor-1} e^{n(V^{(n)}(j)-V^{(n)}(k))}}.$$

Recall, $\nu' \neq \nu$ is the unique value such that $V(\nu') = V(\nu)$. Then, for $k < \lfloor \nu' n\rfloor$, $V\left(\frac{j}{n}\right) - V\left(\frac{k}{n}\right)$ is maximized at $j = k$, whereas for $\lfloor \nu' n\rfloor < k < \lfloor \nu n\rfloor$, it is maximized at $j = \lfloor \nu n\rfloor - 1$.

We thus have

$$\mathbb{P}_{k+1}\left\{T_{\lfloor \nu n\rfloor}^{(n)} < T_k^{(n)}\right\} \sim 1 - \frac{\mu\left(\frac{k}{n}\right)}{\lambda\left(\frac{k}{n}\right)}$$

for $k < \lfloor \nu' n \rfloor$, whereas for $\lfloor \nu' n \rfloor < k < \lfloor \nu n \rfloor$,

$$\frac{1}{\mathbb{P}_{k+1}\left\{T_{\lfloor \nu n \rfloor}^{(n)} < T_k^{(n)}\right\}} \sim \frac{\sqrt{\frac{\mu(\nu)\lambda\left(\frac{k}{n}\right)}{\lambda(\nu)\mu\left(\frac{k}{n}\right)}} e^{n\left(V\left(\frac{\lfloor \nu n \rfloor}{n}\right) - V\left(\frac{k}{n}\right)\right)}}{1 - \frac{\lambda(\nu)}{\mu(\nu)}}$$

We now split the sum over k at $\lfloor \nu' n \rfloor$. Then,

$$\sum_{k=1}^{\lfloor \nu' n \rfloor - 1} \frac{1}{\lambda_k \mathbb{P}_{k+1}\left\{T_{\lfloor \nu n \rfloor}^{(n)} < T_k^{(n)}\right\}} \sim \sum_{k=1}^{\lfloor \nu' n \rfloor - 1} \frac{1}{\left(\lambda\left(\frac{k}{n}\right) - \mu\left(\frac{k}{n}\right)\right) i},$$

whereas

$$\lambda(0) - \mu(0) \leq \lambda\left(\frac{k}{n}\right) - \mu\left(\frac{k}{n}\right) \leq \lambda(\nu) - \mu(\nu),$$

so, as previously,

$$\begin{aligned} \frac{1}{\lambda(\nu) - \mu(\nu)} &\leq \liminf_{n\to\infty} \frac{1}{\ln(\nu' n)} \sum_{k=1}^{\lfloor \nu' n \rfloor - 1} \frac{1}{\lambda_k \mathbb{P}_{k+1}\left\{T_{\lfloor \nu n \rfloor}^{(n)} < T_k^{(n)}\right\}} \\ &\leq \limsup_{n\to\infty} \frac{1}{\ln(\nu' n)} \sum_{k=1}^{\lfloor \nu' n \rfloor - 1} \frac{1}{\lambda_k \mathbb{P}_{k+1}\left\{T_{\lfloor \nu n \rfloor}^{(n)} < T_k^{(n)}\right\}} \leq \frac{1}{\lambda(0) - \mu(0)}. \end{aligned}$$

On the other hand, we observe that for $x \in [\nu', \nu]$, $V(\nu) - V(x)$ is maximized at $x = \kappa$, so that, applying Proposition 1, we have

$$\begin{aligned} \sum_{k=\lfloor \nu' n \rfloor}^{\lfloor \nu n \rfloor - 1} \frac{1}{\lambda_k \mathbb{P}_{k+1}\left\{T_{\lfloor \nu n \rfloor}^{(n)} < T_k^{(n)}\right\}} &\sim \sum_{k=\lfloor \nu' n \rfloor}^{\lfloor \nu n \rfloor - 1} \frac{e^{n\left(V\left(\frac{\lfloor \nu n \rfloor}{n}\right) - V\left(\frac{k}{n}\right)\right)}}{n\lambda\left(\frac{k}{n}\right)\left(\frac{k}{n}\right)\left(1 - \frac{\lambda(\nu)}{\mu(\nu)}\right)} \\ &\sim \sqrt{\frac{2\pi}{n\left(\frac{\mu'(\kappa)}{\mu(\kappa)} - \frac{\lambda'(\kappa)}{\lambda(\kappa)}\right)}} \frac{\sqrt{\frac{\mu(\nu)}{\lambda(\nu)}\frac{\lambda(\kappa)}{\mu(\kappa)}} e^{n\left(V\left(\frac{\lfloor \nu n \rfloor}{n}\right) - V(\kappa)\right)}}{\lambda(\kappa)\kappa\left(1 - \frac{\lambda(\nu)}{\mu(\nu)}\right)}. \end{aligned}$$

The result follows on observing that $\lambda(\kappa) = \mu(\kappa)$. □ □

Proposition 10. As previously, we have that

$$\mathbb{E}_{\lfloor \nu n \rfloor}\left[T_{\lfloor \kappa n \rfloor}^{(n)}\right] = \sum_{k=\lfloor \kappa n \rfloor + 1}^{\infty} \frac{1}{n(\lambda_k + \mu_k)} \frac{\mathbb{P}_{\lfloor \nu n \rfloor}\left\{T_k^{(n)} < T_{\lfloor \kappa n \rfloor}^{(n)}\right\}}{\mathbb{P}_k\left\{T_{\lfloor \kappa n \rfloor}^{(n)} < T_{k+}^{(n)}\right\}}.$$

Now,

$$\mathbb{P}_{\lfloor \nu n \rfloor}\left\{T_k^{(n)} < T_{\lfloor \kappa n \rfloor}^{(n)}\right\} = \begin{cases} 1 & \text{if } \lfloor \kappa n \rfloor < k \leq \lfloor \nu n \rfloor \text{, and} \\ \frac{\sum_{j=\lfloor \kappa n \rfloor}^{\lfloor \nu n \rfloor - 1} e^{nV^{(n)}(j)}}{\sum_{j=\lfloor \kappa n \rfloor}^{k-1} e^{nV^{(n)}(j)}} & \text{if } \lfloor \nu n \rfloor < k, \end{cases}$$

whereas

$$\mathbb{P}_k\left\{T^{(n)}_{\lfloor\kappa n\rfloor} < T^{(n)}_{k+}\right\} = \frac{\mu_k}{\lambda_k+\mu_k}\mathbb{P}_{k-1}\left\{T^{(n)}_{\lfloor\kappa n\rfloor} < T^{(n)}_{k}\right\}$$

and

$$\mathbb{P}_{k-1}\left\{T^{(n)}_{\lfloor\kappa n\rfloor} < T^{(n)}_{k}\right\} = \frac{e^{nV^{(n)}(k-1)}}{\sum_{j=\lfloor\kappa n\rfloor}^{k-1} e^{nV^{(n)}(j)}}.$$

Thus, for $k > \lfloor \nu n\rfloor$,

$$\frac{1}{n(\lambda_k+\mu_k)}\frac{\mathbb{P}_{\lfloor\nu n\rfloor}\left\{T^{(n)}_{k} < T^{(n)}_{\lfloor\kappa n\rfloor}\right\}}{\mathbb{P}_k\left\{T^{(n)}_{\lfloor\kappa n\rfloor} < T^{(n)}_{k+}\right\}} = \frac{\sum_{j=\lfloor\kappa n\rfloor}^{\lfloor\nu n\rfloor-1} e^{nV^{(n)}(j)}}{n\mu_k e^{nV^{(n)}(k-1)}} \sim \frac{\sqrt{\frac{\mu(\nu)\lambda\left(\frac{k-1}{n}\right)}{\lambda(\nu)\mu\left(\frac{k-1}{n}\right)}}e^{n\left(V\left(\frac{\lfloor\nu n\rfloor}{n}\right)-V\left(\frac{k-1}{n}\right)\right)}}{\mu\left(\frac{k-1}{n}\right)k\left(1-\frac{\lambda(\nu)}{\mu(\nu)}\right)}$$

since V is minimized at κ. Moreover, as $\mu(x)$ and $\lambda(x)$ are, respectively, increasing and decreasing, the latter is bounded above by

$$\frac{e^{n\left(V\left(\frac{\lfloor\nu n\rfloor}{n}\right)-V\left(\frac{k-1}{n}\right)\right)}}{(\mu(\nu)-\lambda(\nu))\nu}.$$

Further,

$$V\left(\frac{\lfloor\nu n\rfloor}{n}\right) - V\left(\frac{k-1}{n}\right) = -V'(z)\left(\frac{k-1}{n}-\frac{\lfloor\nu n\rfloor}{n}\right) < -V'\left(\frac{\lfloor\nu n\rfloor}{n}\right)\left(\frac{k-1}{n}-\frac{\lfloor\nu n\rfloor}{n}L\right)$$

for some $z \in \left[\frac{\lfloor\nu n\rfloor}{n}, \frac{k-1}{n}\right]$; the inequality follows since $V''(x) > 0$ for all x. Thus,

$$\sum_{k=\lfloor\nu n\rfloor}^{\infty} e^{n\left(V\left(\frac{\lfloor\nu n\rfloor}{n}\right)-V\left(\frac{k-1}{n}\right)\right)} < e^{V'\left(\frac{\lfloor\nu n\rfloor}{n}\right)}\sum_{k=0}^{\infty} e^{-V'\left(\frac{\lfloor\nu n\rfloor}{n}\right)k} = \frac{e^{V'\left(\frac{\lfloor\nu n\rfloor}{n}\right)}}{1-e^{-V'\left(\frac{\lfloor\nu n\rfloor}{n}\right)}},$$

since $\nu > \kappa$ and $V'(\kappa) = 0$. In particular, we see that the sum

$$\sum_{k=\lfloor\nu n\rfloor+1}^{\infty}\frac{1}{n(\lambda_k+\mu_k)}\frac{\mathbb{P}_{\lfloor\nu n\rfloor}\left\{T^{(n)}_{k} < T^{(n)}_{\lfloor\kappa n\rfloor}\right\}}{\mathbb{P}_k\left\{T^{(n)}_{\lfloor\kappa n\rfloor} < T^{(n)}_{k+}\right\}}$$

is bounded above.

Finally, consider

$$\sum_{k=\lfloor\kappa n\rfloor+1}^{\lfloor\nu n\rfloor}\frac{1}{n\mu_k\mathbb{P}_{k-1}\left\{T^{(n)}_{\lfloor\kappa n\rfloor} < T^{(n)}_{k}\right\}}.$$

Arguing as above,

$$\mathbb{P}_{k-1}\left\{T^{(n)}_{\lfloor\kappa n\rfloor} < T^{(n)}_{k}\right\} \sim \frac{1}{1-\frac{\lambda\left(\frac{k-1}{n}\right)}{\mu\left(\frac{k-1}{n}\right)}} \sim \frac{1}{1-\frac{\lambda\left(\frac{k}{n}\right)}{\mu\left(\frac{k}{n}\right)}},$$

and, proceeding as in Proposition 6, one can show that

$$\sum_{k=\lfloor \kappa n\rfloor+1}^{\lfloor \nu n\rfloor} \frac{1}{\left(\lambda\left(\frac{k}{n}\right)-\mu\left(\frac{k}{n}\right)\right)k} \sim -\frac{1}{(\lambda'(\kappa)-\mu'(\kappa))\kappa}\ln n.$$

□ □

Proposition 11. To begin, we decompose the expectation according to whether, starting from $\lfloor \nu n\rfloor$, the next event is a birth or a death:

$$\begin{aligned}
\mathbb{E}_{\lfloor \nu n\rfloor}\left[T^{(n)}_{\lfloor \nu n\rfloor+}\middle| T^{(n)}_{\lfloor \nu n\rfloor+} < T^{(n)}_0\right] &= \frac{\lambda_{\lfloor \nu n\rfloor}}{\lambda_{\lfloor \nu n\rfloor}+\mu_{\lfloor \nu n\rfloor}}\mathbb{E}_{\lfloor \nu n\rfloor+1}\left[T^{(n)}_{\lfloor \nu n\rfloor}\middle| T^{(n)}_{\lfloor \nu n\rfloor} < T^{(n)}_0\right] \\
&\quad + \frac{\mu_{\lfloor \nu n\rfloor}}{\lambda_{\lfloor \nu n\rfloor}+\mu_{\lfloor \nu n\rfloor}}\mathbb{E}_{\lfloor \nu n\rfloor-1}\left[T^{(n)}_{\lfloor \nu n\rfloor}\middle| T^{(n)}_{\lfloor \nu n\rfloor} < T^{(n)}_0\right] \\
&= \frac{\lambda_{\lfloor \nu n\rfloor}}{\lambda_{\lfloor \nu n\rfloor}+\mu_{\lfloor \nu n\rfloor}}\sum_{k=\lfloor \nu n\rfloor+1}^{\infty}\frac{\mathbb{P}_{\lfloor \nu n\rfloor+1}\left\{T^{(n)}_k < T^{(n)}_{\lfloor \nu n\rfloor}\middle| T^{(n)}_{\lfloor \nu n\rfloor} < T^{(n)}_0\right\}}{n\mu_k\mathbb{P}_{k-1}\left\{T^{(n)}_{\lfloor \nu n\rfloor} < T^{(n)}_k\middle| T^{(n)}_{\lfloor \nu n\rfloor} < T^{(n)}_0\right\}} \\
&\quad + \frac{\mu_{\lfloor \nu n\rfloor}}{\lambda_{\lfloor \nu n\rfloor}+\mu_{\lfloor \nu n\rfloor}}\sum_{k=1}^{\lfloor \nu n\rfloor-1}\frac{\mathbb{P}_{\lfloor \nu n\rfloor-1}\left\{T^{(n)}_k < T^{(n)}_{\lfloor \nu n\rfloor}\middle| T^{(n)}_{\lfloor \nu n\rfloor} < T^{(n)}_0\right\}}{n\lambda_k\mathbb{P}_{k+1}\left\{T^{(n)}_{\lfloor \nu n\rfloor} < T^{(n)}_k\middle| T^{(n)}_{\lfloor \nu n\rfloor} < T^{(n)}_0\right\}}.
\end{aligned}$$

For the first sum, we observe that for any $k \geq \lfloor \nu n\rfloor$,

$$\mathbb{P}_k\left\{T^{(n)}_{\lfloor \nu n\rfloor} < T^{(n)}_0\right\} = 1,$$

and we may thus replace the conditional probabilities with the unconditional ones. Then, using (2.2),

$$\frac{\mathbb{P}_{\lfloor \nu n\rfloor+1}\left\{T^{(n)}_k < T^{(n)}_{\lfloor \nu n\rfloor}\right\}}{\mathbb{P}_{k-1}\left\{T^{(n)}_{\lfloor \nu n\rfloor} < T^{(n)}_k\right\}} = e^{n\left(V^{(n)}(\lfloor \nu n\rfloor)-V^{(n)}(k)\right)},$$

so that, using Lemma 1, the first sum is asymptotic to

$$\frac{\mu(\nu)}{\mu(\nu)+\lambda(\nu)}\frac{1}{(\mu(\nu)-\lambda(\nu))\nu}.$$

For the second sum, we observe that

$$\left\{T^{(n)}_{\lfloor \nu n\rfloor} < T^{(n)}_k\right\} \cap \left\{T^{(n)}_{\lfloor \nu n\rfloor} < T^{(n)}_0\right\} = \left\{T^{(n)}_{\lfloor \nu n\rfloor} < T^{(n)}_k\right\},$$

whereas

$$\mathbb{P}_{\lfloor \nu n\rfloor-1}\left\{T^{(n)}_k < T^{(n)}_{\lfloor \nu n\rfloor}, T^{(n)}_{\lfloor \nu n\rfloor} < T^{(n)}_0\right\} = \mathbb{P}_{\lfloor \nu n\rfloor-1}\left\{T^{(n)}_k < T^{(n)}_{\lfloor \nu n\rfloor}\right\}\mathbb{P}_k\left\{T^{(n)}_{\lfloor \nu n\rfloor} < T^{(n)}_0\right\},$$

so that, applying Bayes' theorem,

$$\frac{\mathbb{P}_{\lfloor \nu n \rfloor - 1}\left\{T_k^{(n)} < T_{\lfloor \nu n \rfloor}^{(n)} \middle| T_{\lfloor \nu n \rfloor}^{(n)} < T_0^{(n)}\right\}}{\mathbb{P}_{k+1}\left\{T_{\lfloor \nu n \rfloor}^{(n)} < T_k^{(n)} \middle| T_{\lfloor \nu n \rfloor}^{(n)} < T_0^{(n)}\right\}} = \frac{\mathbb{P}_{\lfloor \nu n \rfloor - 1}\left\{T_k^{(n)} < T_{\lfloor \nu n \rfloor}^{(n)}\right\} \mathbb{P}_k\left\{T_{\lfloor \nu n \rfloor}^{(n)} < T_0^{(n)}\right\}}{\mathbb{P}_{k+1}\left\{T_{\lfloor \nu n \rfloor}^{(n)} < T_k^{(n)}\right\}}.$$

Again, from (2.2), we see that

$$\frac{\mathbb{P}_{\lfloor \nu n \rfloor - 1}\left\{T_k^{(n)} < T_{\lfloor \nu n \rfloor}^{(n)}\right\}}{\mathbb{P}_{k+1}\left\{T_{\lfloor \nu n \rfloor}^{(n)} < T_k^{(n)}\right\}} = e^{nV^{(n)}(\lfloor \nu n \rfloor - 1) - V^{(n)}(k))},$$

so this sum reduces to

$$\sum_{k=1}^{\lfloor \nu n \rfloor - 1} \frac{\mathbb{P}_k\left\{T_{\lfloor \nu n \rfloor}^{(n)} < T_0^{(n)}\right\}}{n\lambda_k} e^{nV^{(n)}(\lfloor \nu n \rfloor - 1) - V^{(n)}(k))}.$$

To evaluate the sum, it is useful to consider it in two pieces. To do so, we first re-introduce $\nu' < \kappa$ such that $V(\nu') = V(\nu)$, and then consider

$$\sum_{k=1}^{\lfloor \nu' n \rfloor - 1} \frac{\mathbb{P}_k\left\{T_{\lfloor \nu n \rfloor}^{(n)} < T_0^{(n)}\right\}}{n\lambda_k} e^{nV^{(n)}(\lfloor \nu n \rfloor - 1) - V^{(n)}(k))} + \sum_{k=\lfloor \nu' n \rfloor}^{\lfloor \nu n \rfloor - 1} \frac{\mathbb{P}_k\left\{T_{\lfloor \nu n \rfloor}^{(n)} < T_0^{(n)}\right\}}{n\lambda_k} e^{nV^{(n)}(\lfloor \nu n \rfloor - 1) - V^{(n)}(k))}.$$

For the former, $V^{(n)}(\lfloor \nu n \rfloor - 1) - V^{(n)}(k))$ is maximized at $k = \lfloor \nu' n \rfloor - 1$, whereas $\mathbb{P}_k\left\{T_{\lfloor \nu n \rfloor}^{(n)} < T_0^{(n)}\right\}$ us bounded above by 1. Using Lemma 1, the first sum is asymptotically bounded above by

$$\frac{1}{\left(\lambda\left(\frac{\lfloor \nu' n \rfloor - 1}{n}\right) - \mu\left(\frac{\lfloor \nu' n \rfloor - 1}{n}\right)\right)\left(\lfloor \nu' n \rfloor - 1\right)} \sqrt{\frac{\mu\left(\frac{\lfloor \nu n \rfloor - 1}{n}\right)\lambda\left(\frac{\lfloor \nu' n \rfloor - 1}{n}\right)}{\lambda\left(\frac{\lfloor \nu n \rfloor - 1}{n}\right)\mu\left(\frac{\lfloor \nu' n \rfloor - 1}{n}\right)}}.$$

For the second piece, we note that for $\lfloor \nu' n \rfloor \leq k < \lfloor \nu' n \rfloor$, $\mathbb{P}_k\left\{T_{\lfloor \nu n \rfloor}^{(n)} < T_0^{(n)}\right\} \sim 1$, whereas $V^{(n)}(\lfloor \nu n \rfloor - 1) - V^{(n)}(k))$ is maximized at $\lfloor \kappa n \rfloor$, so appealing to Lemma 1, it is asymptotically equivalent to

$$\sqrt{\frac{2\pi}{n\left(\frac{\mu'(\kappa)}{\mu(\kappa)} - \frac{\lambda'(\kappa)}{\lambda(\kappa)}\right)}} \frac{\mu(\nu)}{\lambda(\nu)} \frac{e^{n\left(V\left(\frac{\lfloor \nu n \rfloor}{n}\right) - V(\kappa)\right)}}{\lambda(\kappa)\kappa}$$

The result follows. □ □

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