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A novel treatment of Greenwood-Johnson's mechanism of transformation plasticity - Case of spherical growth of nuclei of daughter-phase

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Abstract

This paper proposes an innovative treatment of Greenwood and Johnson (1965)'s mechanism of transformation plasticity of metals and alloys, based on the disregard of elasticity and the powerful kinematic method of limit-analysis. In the new approach the spherical representative unit cell considered in the homogenization process includes only the mother-phase surrounding a growing nucleus of daughter-phase, but the external loading arising from the macroscopic stress applied is supplemented with an internal one arising from the volumetric transformation strain of the enclosed nucleus. The treatment brings considerable improvement to the classical one of Leblond *et al.* (1989), not only by eliminating the need for *ad hoc* hypotheses of limited validity, but more importantly by extending its results to more general situations involving large external stresses, comparable in magnitude to the yield stress of the weaker, mother-phase. The theoretical results are compared to other theories, experiments, and finite element micromechanical simulations considering a representative volume of shape identical to that in the theory. In addition, the methodology presented paves the way to incorporation, in a future work, of the effect of anisotropies of morphologic type (tied for instance to growth of nuclei of daughter-phase elongated in a certain privileged direction) upon transformation plasticity; this will be done through consideration of representative unit cells of more complex shape.

Keywords : Transformation plasticity; Greenwood-Johnson's mechanism; spherical phase growth; limit-analysis; finite element micromechanical simulations

1 Introduction

The wording *transformation plasticity* was coined to denote the anomalous plastic behavior of metals and alloys during solid-solid phase transformations. Such transformations occur notably during the cooling phase of thermomechanical treatments such as welding and quenching. The major impact of transformation plasticity upon residual stresses and distortions resulting from such treatments has been underlined by various authors, see

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e.g. Miyao *et al.* (1986); Fukumoto *et al.* (2001); Taleb *et al.* (2004), among many others. This impact constitutes the major motivation for developing good models of transformation plasticity, to be implemented in finite element programmes and used for accurate predictions of residual stresses and distortions produced by thermomechanical treatments.

It is widely accepted nowadays that transformation plasticity arises from two distinct physical mechanisms, which were proposed by Greenwood and Johnson (1965) and Magee (1966) respectively:

- In Greenwood and Johnson (1965)’s interpretation, transformation plasticity is due to standard, dislocation-induced plasticity at the microscopic scale in the weaker, mother-phase (that prevailing at high temperatures, with a generally much lower yield stress). Microscopic plasticity arises from the difference of specific volume between the phases coexisting during the transformation (volumetric part of the transformation strain), which generates internal stresses of sufficient magnitude to induce plasticity in the weaker one. The effect occurs even in the absence of external stress; but such a stress, when present, takes advantage of the internal “transformation-induced weakness” of the material to deform it plastically.
- In contrast, in Magee (1966)’s interpretation, transformation plasticity is not due to standard plasticity but to the deviatoric part of the transformation strain of the transforming regions. (In this interpretation the expression “transformation *plasticity*” is somewhat inappropriate, since the transformation strain responsible for the effect is of essentially pseudo-elastic, reversible nature; but the wording is classical). In Magee’s view, when the external stress is zero the deviatoric part of the transformation strain varies randomly in direction in the material, and therefore averages out to zero; but when the external stress is nonzero, it “orients” this deviatoric part and the net result is a nonzero macroscopic strain.

In this paper, we concentrate on Greenwood and Johnson (1965)’s mechanism. (Some comments on this point will be offered below). The most classical - and still widely used, despite its age - constitutive model of this mechanism is due to Leblond *et al.* (1986a, 1989). The basis of this model was an additive decomposition of microscopic strains, followed by homogenization and detailed analysis of the decomposition of macroscopic strains (Leblond *et al.*, 1986a); this led to an expression of the “transformation plastic strain rate” as an average value over the domain occupied by the mother-phase, which was finally evaluated explicitly using a simple model of growth of a spherical nucleus of daughter-phase within a concentric spherical shell of mother-phase, considered as a typical representative volume element (RVE) in the transforming material (Leblond *et al.*, 1989).

The explanation of the popularity of this model probably lies in the fact that the expression of the transformation plastic strain rate it proposes involves only standard mechanical parameters of the two phases, the yield stress of the weaker phase and the volumetric transformation strain, in the absence of any *ad hoc* parameter to be determined experimentally. Such a feature is essential for applications to the numerical simulation of thermomechanical treatments, since most of the time no experimental data concerning transformation plasticity are available for the specific material considered.¹

¹ Even data pertaining to the yield stress of the weaker phase at the temperatures of interest,

Leblond *et al.* (1989)'s model however suffered from serious drawbacks, which have long been noted. First, it relied on some technical hypotheses of doubtful applicability (recalled below). Also, the disregard of elasticity and the hypothesis of instantaneous total “plasticification” of the mother-phase it relied upon, led to an infinite value of the transformation plastic strain rate at the very beginning of the transformation, which has repeatedly been criticized. But the most important drawback was that the treatment was by nature limited to small external stresses only (much smaller in magnitude than the yield stress of the weaker phase); as a consequence it failed to predict the well-documented nonlinear dependence of the transformation plastic strain rate upon the magnitude of the external stress tensor (see e.g. Figs. 3 to 6 in the experimental section of Greenwood and Johnson (1965)'s paper).

The model was improved in later years through refinement of the description of the mechanical behaviour of the mother-phase; see the works of Diani *et al.* (1995); Cherkaoui *et al.* (2000); Taleb and Sidoroff (2003); Vincent *et al.* (2003); Fischlschweiger *et al.* (2012); Weisz-Paltrault (2017), among others. For instance, the issue of the infinite initial value of the transformation plastic strain rate was settled by Taleb and Sidoroff (2003) and Weisz-Paltrault (2017) by accounting for elasticity and discarding the hypothesis of instantaneous complete plasticification of the mother-phase. The very author of the model and coworkers contributed to the effort by introducing viscous effects, which are often important at the temperatures of interest (Vincent *et al.*, 2003). Weisz-Paltrault (2017) extended the model by accounting for the possibility of consecutive or simultaneous formation of several phases. It seems fair to say, however, that in spite of these efforts the model has remained essentially the same over the years and still suffers from serious shortcomings, most notably its lack of generality.

The aim of this paper is to propose an enhanced theoretical framework for the analysis of Greenwood and Johnson (1965)'s mechanism. This framework will be based on the Mandel (1964) - Hill (1967) theory of homogenization like before; but Leblond *et al.* (1989)'s approximate treatment, based on specific and disputable hypotheses, of the problem of a spherical volume of mother-phase containing a growing spherical core of daughter-phase, will be replaced by a much more rigorous and efficient treatment of the same problem, based on the powerful “kinematic approach” of limit-analysis. (See e.g. Leblond *et al.* (2018) for a very recent summary and review of the theory). The idea of using limit-analysis is logical in view of the assumption - in line with Leblond *et al.* (1989)'s hypotheses - of negligible effect of elasticity. The main difficulty raised by use of this theory, in the context of transformation plasticity, is the presence within the material of some “internal active mechanism” - the growth, within the matrix of mother-phase, of a daughter-phase with a different specific volume - not accounted for in limit-analysis, which considers only “passive” materials. This difficulty will be settled by including only, in the volume considered, the domain - a hollow sphere - occupied by the plastically deforming mother-phase; the internal mechanism just mentioned will then simply become an additional loading condition imposed on the inner boundary of this volume. The presence of this extra loading will constitute the sole difference with elementary cells and loadings considered in theoretical analyses of porous plastic materials (with traction-free voids); and this will permit to use, with some adaptations, the results of some previous works on the ductile

necessary to apply the model, are often scarce.

rupture of metals (Gurson, 1977; Monchiet *et al.*, 2011).

The benefits to be expected from the new approach are three-fold:

- Use of the variational framework of limit-analysis will permit to dispense with some of Leblond *et al.* (1989)'s most criticizable hypotheses. Also, the greater rigour of the treatment will be seen to lead to a finite value of the transformation plastic strain rate at the onset of the transformation without demanding incorporation of the effect of elasticity.
- More importantly, the wider generality of the treatment will permit to address not only small external stresses like before, but also stresses comparable in magnitude to the yield stress of the weaker mother-phase. The experimentally well-known nonlinear dependence of the transformation plastic strain rate upon the external stress tensor will thus be given a theoretical interpretation, apparently the first of its kind.
- But perhaps the most important benefit will not lie in the improvement of the treatment of the problem of spherical growth of nuclei of daughter-phase, but in the perspectives opened, thanks to the versatility of the method, for future extensions to more complex situations. Such situations could include for instance growth of spheroidal rather than spherical nuclei of daughter-phase; the treatment of such a problem could permit to study and model effects arising from anisotropies of morphological type (tied to development of domains of daughter-phase of anisotropic, for instance elongated shape).

The paper is organized as follows:

- As a necessary prerequisite, Section 2 recalls some elements of Leblond's general theory of transformation plasticity (Leblond *et al.*, 1986a) and approximate treatment of the model problem of growth of a spherical nucleus of daughter-phase within a spherical RVE of mother-phase (Leblond *et al.*, 1989).
- Section 3 then presents the limit-analysis of a spherical cell subjected to conditions of homogeneous boundary stress (Mandel, 1964; Hill, 1967) on both its outer and inner boundaries. The treatment makes an extensive use of Monchiet *et al.* (2011)'s generalization - based on trial velocity fields of Eshelby (1957)'s type - of Gurson (1977)'s classical analysis of the same problem but in the absence of internal loading.
- Section 4 explains how the results derived may be used to predict the value of the transformation plastic strain rate, assumed to arise only from Greenwood and Johnson (1965)'s mechanism. An analytical, albeit somewhat complex expression of this strain rate is derived in general, and shown to reduce to simple forms in two special cases: (i) small external stresses; (ii) beginning of the transformation, whatever the external stresses.
- Finally Section 5 compares the new theoretical predictions to (i) those of Leblond *et al.* (1989)'s former theory; (ii) some experimental results of Desalos (1981); Coret *et al.* (2002, 2004) for the A533 steel; and (iii) the results of some new finite element micromechanical simulations of the mechanism studied, considering the same typical spherical geometry as the theory.

2 Elements of Leblond's classical theory of transformation plasticity

2.1 Generalities

We consider, within some metal or alloy, a RVE Ω containing “mother-” and “daughter-” phases occupying respectively the domains Ω_M and Ω_D , separated by an interface \mathcal{I} (Fig. 1). The volume fraction f of the daughter-phase is defined as

$$f \equiv \frac{\text{vol}(\Omega_D)}{\text{vol}(\Omega)} \quad (1)$$

where the symbol “vol” denotes the volume of a domain. The mother-phase may transform into the daughter-phase through motion of the interface \mathcal{I} in the direction of Ω_M .

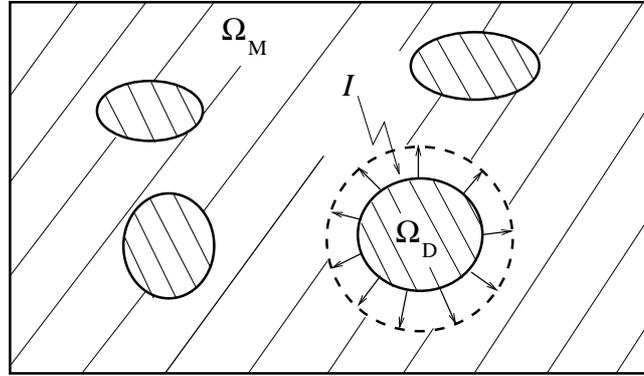


Fig. 1. The general RVE considered.

Use is made of the Mandel (1964) - Hill (1967) homogenization theory assuming conditions of homogeneous boundary stress or strain.

2.2 Decomposition of microscopic strains

The behaviour of both phases is thermo-elastic-plastic. The microscopic strain ϵ_k in phase k ($k = M, D$) thus splits into thermal, elastic and plastic parts:

$$\epsilon_k = \epsilon_k^{\text{th}} + \epsilon_k^{\text{e}} + \epsilon_k^{\text{p}} \quad (k = M, D). \quad (2)$$

- The thermal strain ϵ_k^{th} of phase k , depending on the temperature T , is defined as the “free strain” which would exist in a small, unstressed volume of this phase at this temperature, measured in reference to a given phase - in dilatometric diagrams generally the daughter - and a given reference temperature T_0 . It is of the form

$$\epsilon_k^{\text{th}}(T) = [\alpha_k(T - T_0) + \beta_k] \mathbf{1} \quad (k = M, D), \quad (3)$$

where α_k is the thermal expansion coefficient of phase k and β_k is related to its free strain at the temperature T_0 . The relative change of specific volume $\frac{\Delta V}{V}$ from the mother-

to the daughter-phase (volumetric part of the transformation strain) depends on the temperature T and is given by

$$\frac{\Delta V}{V}(T) \equiv \text{tr} [\boldsymbol{\epsilon}_D^{\text{th}}(T) - \boldsymbol{\epsilon}_M^{\text{th}}(T)] = 3 [(\alpha_D - \alpha_M)(T - T_0) + \beta_M - \beta_D]. \quad (4)$$

Further comments on these strains and their macroscopic counterpart \mathbf{E}^{thm} (see below) are provided in Appendix A.

- The elastic strain $\boldsymbol{\epsilon}_k^e$ of phase k is related to the local stress $\boldsymbol{\sigma}_k$ in this phase through the relation

$$\boldsymbol{\epsilon}_k^e = \mathbf{S} : \boldsymbol{\sigma}_k \quad (5)$$

where \mathbf{S} is the (isotropic) elastic compliance tensor. No distinction is made here between the compliance tensor of the mother-phase, that of the daughter-phase and the overall compliance tensor; the idea being that the major factors influencing transformation plasticity are the standard plastic strain in the weaker mother-phase (Greenwood and Johnson (1965)'s mechanism), and the transformation strain (Magee (1966)'s mechanism); not the difference of elastic moduli between the phases nor the anisotropic nature of their crystal elasticity.

- The plastic strains $\boldsymbol{\epsilon}_k^p$ ($k = M, D$) of the phases conventionally include, in Leblond *et al.* (1986a)'s definition, not only standard, dislocation-induced plastic strains (arising from Greenwood and Johnson (1965)'s mechanism), but also the deviatoric part of the transformation strain, possibly oriented by the external stress applied (Magee (1966)'s mechanism). (The inclusion of the latter strain in the *plastic* strain is admittedly illogical because of its different, pseudo-elastic nature; but it is in line with the equally illogical but widely accepted preservation of the wording “transformation *plasticity*” to designate Magee (1966)'s pseudo-elastic mechanism). With this definition, the plastic strain is discontinuous across the interface \mathcal{I} separating the phases, its discontinuity

$$[[\boldsymbol{\epsilon}^p]] \equiv \boldsymbol{\epsilon}_D^p - \boldsymbol{\epsilon}_M^p \quad (6)$$

being identical to the deviatoric part of the transformation strain.

2.3 Decomposition of macroscopic strains

With the hypothesis of identical elastic compliance tensors of the phases, the Mandel (1964) - Hill (1967) “localization tensors” appearing in the formulae connecting the microscopic thermal, elastic and plastic strains to their macroscopic counterparts, are identical to the unit tensor. The macroscopic strain tensor \mathbf{E} is thus given by

$$\mathbf{E} \equiv \langle \boldsymbol{\epsilon} \rangle_\Omega = \mathbf{E}^{\text{thm}} + \mathbf{E}^e + \mathbf{E}^p, \quad \begin{cases} \mathbf{E}^{\text{thm}} \equiv \langle \boldsymbol{\epsilon}^{\text{th}} \rangle_\Omega = (1 - f)\boldsymbol{\epsilon}_M^{\text{th}}(T) + f\boldsymbol{\epsilon}_D^{\text{th}}(T) \\ \mathbf{E}^e \equiv \langle \boldsymbol{\epsilon}^e \rangle_\Omega = \mathbf{S} : \boldsymbol{\Sigma} \\ \mathbf{E}^p \equiv \langle \boldsymbol{\epsilon}^p \rangle_\Omega, \end{cases} \quad (7)$$

where

$$\boldsymbol{\Sigma} \equiv \langle \boldsymbol{\sigma} \rangle_\Omega \quad (8)$$

is the macroscopic stress tensor (the symbol $\langle \cdot \rangle_\Omega$ denotes an average value over Ω).

The symbol “thm” in \mathbf{E}^{thm} stands for “thermo-metallurgical”; the word “metallurgical” is added to emphasize the fact that this strain does not only depend on the temperature T , but also on the volume fraction f of the daughter-phase, and hence on the “metallurgical structure”. Some additional comments on this strain are provided in Appendix A.

Time-differentiation of the expression (7)₄ of the overall plastic strain \mathbf{E}^{P} yields, account being taken of the discontinuity of $\boldsymbol{\epsilon}^{\text{P}}$ across the interface \mathcal{I} :

$$\dot{\mathbf{E}}^{\text{P}} = \langle \dot{\boldsymbol{\epsilon}}^{\text{P}} \rangle_\Omega + \frac{1}{\text{vol}(\Omega)} \int_{\mathcal{I}} [[\boldsymbol{\epsilon}^{\text{P}}]] V_n dS = \langle \dot{\boldsymbol{\epsilon}}^{\text{P}} \rangle_\Omega + \dot{f} \langle [[\boldsymbol{\epsilon}^{\text{P}}]] \rangle_{\mathcal{I}/V_n}, \quad (9)$$

where

$$\langle [[\boldsymbol{\epsilon}^{\text{P}}]] \rangle_{\mathcal{I}/V_n} \equiv \frac{\int_{\mathcal{I}} [[\boldsymbol{\epsilon}^{\text{P}}]] V_n dS}{\int_{\mathcal{I}} V_n dS} = \frac{\int_{\mathcal{I}} [[\boldsymbol{\epsilon}^{\text{P}}]] V_n dS}{f \text{vol}(\Omega)} \quad (10)$$

denotes the average value of the discontinuity $[[\boldsymbol{\epsilon}^{\text{P}}]]$ over the interface \mathcal{I} , with a weight identical to the normal velocity V_n of this interface.

The next step consists in distinguishing, in the local plastic strain rate $\dot{\boldsymbol{\epsilon}}^{\text{P}}$, those parts due to variations of the macroscopic parameters $\boldsymbol{\Sigma}$, T and f :

$$\dot{\boldsymbol{\epsilon}}^{\text{P}} = \frac{\delta \boldsymbol{\epsilon}^{\text{P}}}{\delta \boldsymbol{\Sigma}} : \dot{\boldsymbol{\Sigma}} + \frac{\delta \boldsymbol{\epsilon}^{\text{P}}}{\delta T} \dot{T} + \frac{\delta \boldsymbol{\epsilon}^{\text{P}}}{\delta f} \dot{f} \quad (11)$$

where the notations are self-evident.² The unusual term proportional to \dot{f} here arises from the fact that even when the overall stress $\boldsymbol{\Sigma}$ and the temperature T are fixed, the progress of the transformation induces a local plastic flow through Greenwood and Johnson (1965)’s mechanism. Inserting the expression (11) of $\dot{\boldsymbol{\epsilon}}^{\text{P}}$ in equation (9), we get

$$\dot{\mathbf{E}}^{\text{P}} = \dot{\mathbf{E}}^{\text{cp}} + \dot{\mathbf{E}}^{\text{tp}}, \quad \begin{cases} \dot{\mathbf{E}}^{\text{cp}} \equiv \langle \frac{\delta \boldsymbol{\epsilon}^{\text{P}}}{\delta \boldsymbol{\Sigma}} \rangle_\Omega : \dot{\boldsymbol{\Sigma}} + \langle \frac{\delta \boldsymbol{\epsilon}^{\text{P}}}{\delta T} \rangle_\Omega \dot{T} \\ \dot{\mathbf{E}}^{\text{tp}} \equiv \left[\langle \frac{\delta \boldsymbol{\epsilon}^{\text{P}}}{\delta f} \rangle_\Omega + \langle [[\boldsymbol{\epsilon}^{\text{P}}]] \rangle_{\mathcal{I}/V_n} \right] \dot{f}. \end{cases} \quad (12)$$

The strain rate $\dot{\mathbf{E}}^{\text{cp}}$ here may be interpreted as corresponding to “classical plasticity”, insofar as it arises from the sole variations of the external stress and the temperature. On the other hand the strain rate $\dot{\mathbf{E}}^{\text{tp}}$ due to the sole variation of the fraction of the daughter-phase, that is to the progress of the transformation, may be interpreted as corresponding to “transformation plasticity”. Furthermore the latter strain rate naturally appears as a sum of two terms, $\langle \frac{\delta \boldsymbol{\epsilon}^{\text{P}}}{\delta f} \rangle_\Omega \dot{f}$ and $\langle [[\boldsymbol{\epsilon}^{\text{P}}]] \rangle_{\mathcal{I}/V_n} \dot{f}$. The first corresponds to Greenwood and Johnson (1965)’s mechanism since it arises from the standard (dislocation-induced) plastic flow generated by the progress of the transformation, and the second to Magee (1966)’s mechanism since it arises from the “plastic strain discontinuity” between the two phases, that is the deviatoric part of the transformation strain. (The dependence of $[[\boldsymbol{\epsilon}^{\text{P}}]]$ upon the external stress is implicit in equation (12)₃).

² The rates \dot{T} and \dot{f} are formally considered as independent here. In reality these rates are tied through the kinetics of the transformation and latent heat effects; but it is not the object of this paper to discuss such aspects.

2.4 Additional hypotheses

Although the “classical plastic strain rate” $\dot{\mathbf{E}}^{\text{cp}}$ has been studied in detail by Leblond *et al.* (1986b), we shall focus here on the sole “transformation plastic strain rate” $\dot{\mathbf{E}}^{\text{tp}}$. The reason is that numerous numerical calculations performed since Leblond *et al.* (1986a,b)’s work have shown that $\dot{\mathbf{E}}^{\text{cp}}$ becomes much smaller than $\dot{\mathbf{E}}^{\text{tp}}$ as soon as the transformation begins: in the former strain rate, the effect of the term $\langle \frac{\delta \epsilon^{\text{p}}}{\delta \Sigma} \rangle_{\Omega} : \dot{\Sigma}$ roughly amounts to a slight modification of the value of Young’s modulus (Leblond *et al.*, 1986b), of minor importance; and the other term $\langle \frac{\delta \epsilon^{\text{p}}}{\delta T} \rangle_{\Omega} \dot{T}$ is dominated by the term $\langle \frac{\delta \epsilon^{\text{p}}}{\delta f} \rangle_{\Omega} \dot{f}$ in $\dot{\mathbf{E}}^{\text{tp}}$, because during the transformation, the difference in the thermal contractions of the phases is small compared to the volume expansion due to the transformation (see Fig. A.1 of Appendix A).

Also, for the detailed calculation of $\dot{\mathbf{E}}^{\text{tp}}$, Leblond *et al.* (1989) considered only Greenwood and Johnson (1965)’s mechanism, and we shall follow the same path here. Two points of view are possible with regard to such a simplification.

- One may consider that the paper of Leblond *et al.* (1989) and the present one only provide a partial treatment of the problem, which should be completed by an additional independent modelling of Magee (1966)’s mechanism. (It must be stressed, however, that the theoretical treatment of this mechanism is much more difficult than that of Greenwood and Johnson (1965)’s mechanism; indeed in contrast to Greenwood and Johnson (1965)’s mechanism which is of purely “mechanical” nature, that of Magee (1966) involves complex couplings between metallurgical and mechanical aspects - so it seems impossible, when modelling it, to avoid introduction of some adjustable *ad hoc* parameter(s), thus impairing the practical application of the model, as explained above).
- One may also consider that Leblond *et al.* (1989)’s paper and the present one apply only to materials in which Greenwood and Johnson (1965)’s mechanism dominates. The relative importance of the two mechanisms, according to the type of material and situation considered, and their possible experimental separation, are discussed in Appendix B; with the conclusion that although Magee (1966)’s mechanism is present in different materials and situations, and may even completely dominate over that of Greenwood and Johnson (1965), in many cases the converse is true and Magee (1966)’s mechanism may safely be disregarded.

2.5 The spherical growth model and its approximate treatment

To evaluate the contribution of Greenwood and Johnson (1965)’s mechanism to the expression (12)₃ of the transformation plastic strain rate $\dot{\mathbf{E}}^{\text{tp}}$, Leblond *et al.* (1989) consider the simplest possible model of growth of a spherical nucleus of daughter-phase within a concentric spherical matrix of mother-phase, subjected to some overall stress tensor Σ . Figure 2 shows the RVE; the external radius is denoted R , and that of the central core of daughter-phase r_f , where the lower index underlines the connection with the fraction

f of this phase:

$$f = \frac{r_f^3}{R^3}. \quad (13)$$

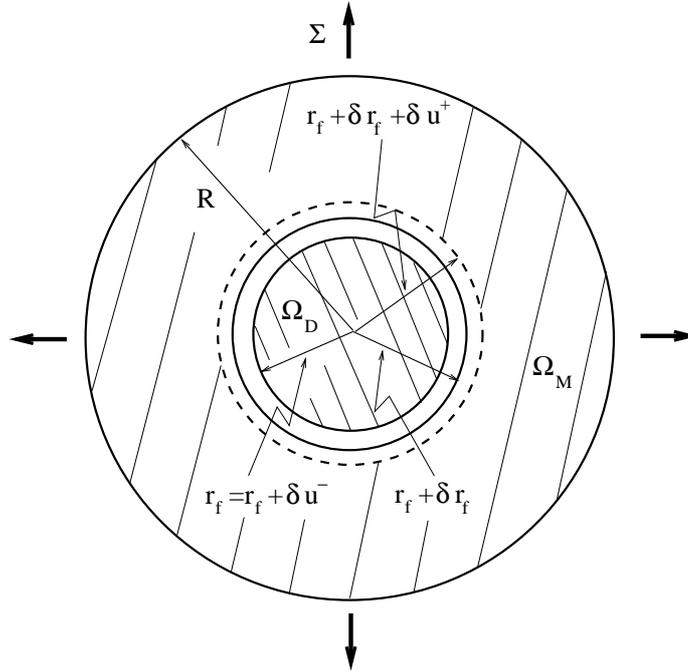


Fig. 2. The spherical RVE considered by Leblond *et al.* (1989).

Between times t and $t + \delta t$, the spherical layer lying between the radii r_f and $r_f + \delta r_f$ transforms from the mother- to the daughter-phase; the accompanying expansion generates increments of radial displacement δu^- , δu^+ of the inner and outer boundaries of this layer. These displacements induce a plastic strain rate distribution within the RVE, which must be evaluated.

Leaving aside the contribution of Magee (1966)'s mechanism, and introducing the simplifying hypotheses of complete plastification of the weaker mother-phase and absence of plasticity in the stronger daughter-phase (justified by the large difference between their respective yield stresses), we transform the expression (12)₃ of $\dot{\mathbf{E}}^{\text{tp}}$ into

$$\dot{\mathbf{E}}^{\text{tp}} = (1 - f) \left\langle \frac{\delta \epsilon^{\text{p}}}{\delta f} \right\rangle_{\Omega_M} \dot{f}, \quad (14)$$

which reduces the problem to evaluating the average value $\left\langle \frac{\delta \epsilon^{\text{p}}}{\delta f} \right\rangle_{\Omega_M}$. The task is made more difficult by presence of the load Σ , which destroys spherical symmetry.

To obviate this difficulty, Leblond *et al.* (1989) write the local flow rule (for the von Mises criterion) in the classical form $\frac{\delta \epsilon^{\text{p}}}{\delta f} = \frac{3}{2} \frac{\delta \epsilon_{\text{eq}}}{\delta f} \frac{\boldsymbol{\sigma}'}{\bar{\sigma}_M}$ where $\delta \epsilon_{\text{eq}} \equiv (\frac{2}{3} \delta \epsilon^{\text{p}} : \delta \epsilon^{\text{p}})^{1/2}$ denotes the equivalent increment of plastic strain, $\boldsymbol{\sigma}' \equiv \boldsymbol{\sigma} - \frac{1}{3}(\text{tr } \boldsymbol{\sigma})\mathbf{1}$ the local stress deviator and $\bar{\sigma}_M$ the yield stress of the mother-phase, assumed to be ideal-plastic. They also introduce some approximations:

- *Approximation (1)*: correlations between the quantities $\frac{\delta \epsilon_{\text{eq}}}{\delta f}$ and $\boldsymbol{\sigma}'$ are negligible so that

$\langle \frac{\delta \epsilon_{eq}}{\delta f} \boldsymbol{\sigma}' \rangle_{\Omega_M} \simeq \langle \frac{\delta \epsilon_{eq}}{\delta f} \rangle_{\Omega_M} \langle \boldsymbol{\sigma}' \rangle_{\Omega_M}$ in equation (14);

- *Approximation (2)*: the average value of $\boldsymbol{\sigma}'$ over the mother-phase differs little from that over the entire RVE, so that $\langle \boldsymbol{\sigma}' \rangle_{\Omega_M} \simeq \boldsymbol{\Sigma}'$ where $\boldsymbol{\Sigma}' \equiv \boldsymbol{\Sigma} - \frac{1}{3}(\text{tr } \boldsymbol{\Sigma})\mathbf{1}$ is the overall stress deviator.

Equation (14) then becomes

$$\dot{\mathbf{E}}^{\text{tp}} = \frac{3}{2} \frac{1-f}{\bar{\sigma}_M} \langle \frac{\delta \epsilon_{eq}}{\delta f} \rangle_{\Omega_M} \boldsymbol{\Sigma}' \dot{f}. \quad (15)$$

Whereas the *tensorial* average value $\langle \frac{\delta \epsilon^{\text{p}}}{\delta f} \rangle_{\Omega_M}$ appearing in equation (14) vanished with $\boldsymbol{\Sigma}$, the *scalar* average value $\langle \frac{\delta \epsilon_{eq}}{\delta f} \rangle_{\Omega_M}$ appearing in equation (15) does not. Thus a good estimate of this new average value, *applicable to small overall stresses*, may be derived by taking $\boldsymbol{\Sigma} = \mathbf{0}$ - thus restoring spherical symmetry.

Elasticity being neglected, the radial increments of displacement δu^- , δu^+ of the bounding surfaces of the transforming shell are given by

$$\delta u^- = 0 \quad ; \quad \delta u^+ = \frac{\Delta V}{V} \delta r_f \quad (16)$$

where the first value results from complete rigidity (no elasticity, no plasticity) of the core of daughter-phase, and the second from the known variation of volume of the transforming shell. Then, with $\boldsymbol{\Sigma} = \mathbf{0}$ and spherical symmetry, plastic incompressibility in the mother-phase implies (in the absence of elasticity) that the purely radial increment of displacement there is given by

$$\delta u_r(r) = \delta u^+ \frac{r_f^2}{r^2} = \frac{\Delta V}{V} \frac{r_f^2}{r^2} \delta r_f. \quad (17)$$

It is then easy to calculate the local value of $\delta \epsilon_{eq}$ and the average value $\langle \frac{\delta \epsilon_{eq}}{\delta f} \rangle_{\Omega_M}$ through integration over r , and the result reads

$$\langle \frac{\delta \epsilon_{eq}}{\delta f} \rangle_{\Omega_M} = -\frac{2}{3} \frac{\Delta V}{V} \frac{\ln f}{1-f}. \quad (18)$$

The final expression of $\dot{\mathbf{E}}^{\text{tp}}$ then follows by equation (15):³

$$\dot{\mathbf{E}}^{\text{tp}} = -\frac{1}{\bar{\sigma}_M} \frac{\Delta V}{V} (\ln f) \boldsymbol{\Sigma}' \dot{f} \quad [\text{Leblond } et \text{ al. (1989)}]. \quad (19)$$

Remarks

- (1) Equation (19) predicts an infinite “slope” $d\mathbf{E}^{\text{tp}}/df$ at the beginning of the transformation ($f = 0$). This prediction does not seem to be confirmed by experiments (see e.g. Desalos (1981), among many other works), and has been repeatedly criticized (see e.g. Taleb and Sidoroff (2003) and Weisz-Paltrault (2017)), on the grounds

³ Since various formulae for $\dot{\mathbf{E}}^{\text{tp}}$ will be given, an indication of the authors or context of application of each one will be provided between brackets.

that it is a consequence of the over-simplifying hypothesis of instantaneous complete plastification of the mother-phase, tied to the disregard of elasticity.

- (2) Leblond *et al.* (1989) have assessed Approximations (1) and (2) above through finite element micromechanical simulations. They found that whereas Approximation (1) is reasonable (see their Fig. 6), Approximation (2) gives a good value of $\langle \boldsymbol{\sigma}' \rangle_{\Omega_M}$ only during the first half of the transformation, the approximation $\langle \boldsymbol{\sigma}' \rangle_{\Omega_M} \simeq (1 - f)\boldsymbol{\Sigma}'$ being in fact better (see their Fig. 7). With this improved approximation the expression (19) of $\dot{\mathbf{E}}^{\text{tp}}$ becomes

$$\dot{\mathbf{E}}^{\text{tp}} = -\frac{1}{\bar{\sigma}_M} \frac{\Delta V}{V} (1 - f)(\ln f) \boldsymbol{\Sigma}' \dot{f} \quad [\text{Leblond } et al. (1989), \text{ improved}]. \quad (20)$$

3 Limit-analysis of a hollow sphere subjected to external and internal loadings

3.1 Preliminary remarks

Since Leblond *et al.* (1989)'s approach of Greenwood and Johnson (1965)'s mechanism is based on disregard of elasticity, it seems natural to use the theory of limit-analysis - in which elasticity plays no role either, see e.g. Leblond *et al.* (2018)'s review - for an alternative treatment.⁴ But this raises two difficulties.

- In Greenwood and Johnson (1965)'s interpretation, the “driving force” of transformation plasticity is an “internal active mechanism” (the gradual growth of nuclei of some new phase with a different specific volume), in practice due to the variation of the temperature. But limit-analysis considers only “passive” materials devoid of such mechanisms.
- An ancillary difficulty is that the object of limit-analysis is to define the set of limit-loads. Such loads are not arbitrary but lie on some hypersurface in their defining space. In contrast, when looking for an expression of the transformation plastic strain rate, we want the macroscopic stress to be arbitrary; how the two points of view can be reconciled is not *a priori* clear.

In this work the first, major difficulty will be obviated by excluding the growing core of daughter-phase from the volume considered. The internal mechanism just mentioned will then become an inner boundary condition (some prescribed displacement arising from the expansion of the embedded daughter-phase), perfectly accounted for by limit-analysis. The same trick will be seen to also solve the second difficulty, albeit in a more subtle way.

The aim of the limit-analysis will be to derive the explicit equation of the set of limit-loads of the RVE considered, together with the associated overall flow rule, that is the expression

⁴ In essence, the idea of employing such a theory could be found, in a sketchy form, in Abrassart (1972)'s thesis. Unfortunately rough approximations made in the choice of typical geometry and trial displacement fields led to large errors in the final result, the transformation plastic strain rate being underestimated by a factor of the order of 3.

of the various components of its overall deformation rate. How these results can be applied to the modelling of Greenwood and Johnson (1965)'s mechanism of transformation plasticity will be explained in Section 4 below.

A final remark pertains to some important difference between the former treatment of the problem, presented in Subsection 2.5, and the new one to be discussed now. The old treatment relied on an assumption of small external stresses, since the average value $\langle \frac{\delta \epsilon_{eq}}{\delta f} \rangle_{\Omega_M}$ was evaluated at order 0 in Σ , that is for $\Sigma = \mathbf{0}$. In contrast the new treatment will not make any hypothesis on the magnitude of the applied stress tensor Σ . This is the keypoint that will permit to treat the case of external stresses of the order of the yield stress of the weaker, mother-phase.

3.2 Generalities

We thus consider (Fig. 3) a hollow sphere made exclusively of mother-phase. In line with the notations of Subsection 2.5, the outer and inner radii are denoted R and r_f , respectively (with f given by equation (13)); the whole domain, Ω ; the outer spherical shell of mother-phase, Ω_M ; and the inner *empty* sphere, Ω_D . For commodity in the calculations three sets of coordinates: Cartesian (x_1, x_2, x_3) with origin at the centre O of the sphere; cylindrical (ρ, ϕ, z) with $z \equiv x_3$; spherical (r, θ, ϕ) , will be used, together with the naturally associated orthonormal bases $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$; $(\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z)$; $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$. On the outer boundary $\partial\Omega$, a single unit normal vector oriented outwards, $\mathbf{n} \equiv \mathbf{e}_r$, is considered. On the inner boundary $\partial\Omega_D$, two unit normal vectors are considered, $\mathbf{n}^+ \equiv \mathbf{e}_r$ oriented toward the exterior of Ω_D , and $\mathbf{n}^- \equiv -\mathbf{e}_r$ oriented toward the exterior of Ω_M .

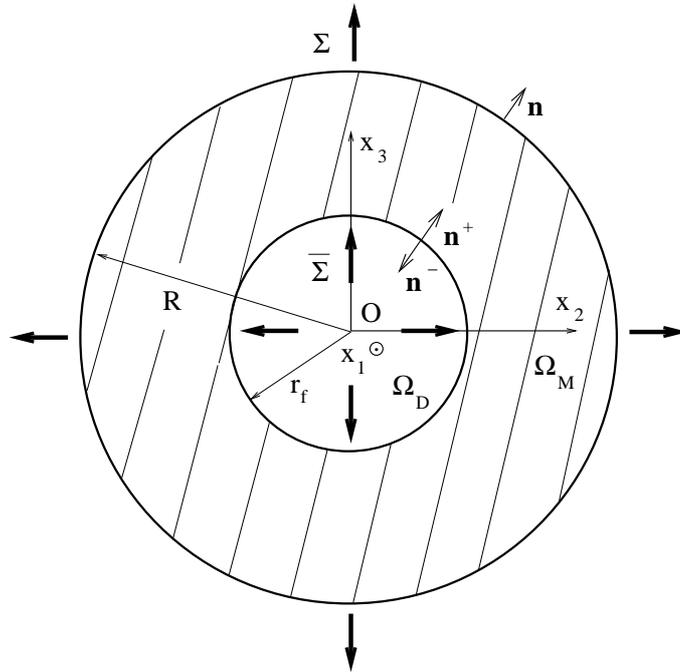


Fig. 3. The spherical RVE considered here.

The outer and inner boundaries of the RVE Ω_M , identical to $\partial\Omega$ and $\partial\Omega_D$, are subjected

to conditions of homogeneous stress (Mandel, 1964; Hill, 1967):

$$\begin{cases} \boldsymbol{\sigma}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \boldsymbol{\Sigma} \cdot \mathbf{n}(\mathbf{x}) & \text{on } \partial\Omega \quad (r = R) \\ \boldsymbol{\sigma}(\mathbf{x}) \cdot \mathbf{n}^-(\mathbf{x}) = -\bar{\boldsymbol{\Sigma}} \cdot \mathbf{n}^-(\mathbf{x}) & \text{on } \partial\Omega_D \quad (r = r_f) \end{cases} \quad (21)$$

where $\boldsymbol{\Sigma}$ and $\bar{\boldsymbol{\Sigma}}$ are symmetric second-rank tensors, the first representing the macroscopic stress applied onto the volume considered and the second some internal loading. The minus sign in equation (21)₂ is introduced conventionally so as to produce a more natural-looking expression of the virtual power of external forces, see equation (24) below.

It should be noted that equations (21) do not define the only possible choice of boundary conditions. It would be possible for instance to use, on the inner boundary, conditions of homogeneous boundary strain rate (Mandel, 1964; Hill, 1967), compatible with the kinematic conditions imposed in the transformation plasticity problem by the expansion of the inner core of daughter-phase. The limit-analysis based on such boundary conditions would be somewhat more involved but ultimately lead to the same results. The choice of conditions of homogeneous boundary stresses is therefore made here for the sake of simplicity, without any detrimental consequences on the results.

For any displacement field $\mathbf{u}(\mathbf{x})$ defined over the domain of mother-phase Ω_M and extended arbitrarily (but smoothly) over the void Ω_D , we define some overall external and internal strain tensors \mathbf{E} , $\bar{\mathbf{E}}$ through the natural formulae

$$\mathbf{E} \equiv \langle \boldsymbol{\epsilon}(\mathbf{u}) \rangle_{\Omega} \quad ; \quad \bar{\mathbf{E}} \equiv \langle \boldsymbol{\epsilon}(\mathbf{u}) \rangle_{\Omega_D}, \quad (22)$$

or equivalently by Green's theorem:

$$E_{ij} = \frac{1}{\text{vol}(\Omega)} \int_{\partial\Omega} \frac{1}{2} (u_i n_j + u_j n_i) dS \quad ; \quad \bar{E}_{ij} = \frac{1}{\text{vol}(\Omega_D)} \int_{\partial\Omega_D} \frac{1}{2} (u_i n_j^+ + u_j n_i^+) dS. \quad (23)$$

Equations (23) make it clear that \mathbf{E} and $\bar{\mathbf{E}}$ are in fact independent of the arbitrary extension of the displacement field over Ω_D .

With the boundary conditions (21), the virtual power \mathcal{P}_e of external forces exerted on the boundary of the RVE considered, that is the domain Ω_M occupied by the sole mother-phase, is given by

$$\begin{aligned} \mathcal{P}_e &\equiv \int_{\partial\Omega} \sigma_{ij} n_j \dot{u}_i dS + \int_{\partial\Omega_D} \sigma_{ij} n_j^- \dot{u}_i dS = \int_{\partial\Omega} \Sigma_{ij} n_j \dot{u}_i dS + \int_{\partial\Omega_D} \bar{\Sigma}_{ij} n_j^+ \dot{u}_i dS \\ &= \Sigma_{ij} \int_{\partial\Omega} \frac{1}{2} (\dot{u}_i n_j + \dot{u}_j n_i) dS + \bar{\Sigma}_{ij} \int_{\partial\Omega_D} \frac{1}{2} (\dot{u}_i n_j^+ + \dot{u}_j n_i^+) dS, \end{aligned}$$

that is by equations (23) and (13):

$$\mathcal{P}_e = \text{vol}(\Omega) \Sigma_{ij} \dot{E}_{ij} + \text{vol}(\Omega_D) \bar{\Sigma}_{ij} \dot{\bar{E}}_{ij} = \text{vol}(\Omega) \left(\boldsymbol{\Sigma} : \dot{\mathbf{E}} + f \bar{\boldsymbol{\Sigma}} : \dot{\bar{\mathbf{E}}} \right). \quad (24)$$

3.3 Trial displacement fields and load parameters

The limit-analysis problem can in principle tolerate any macroscopic stress and strain tensors Σ , $\bar{\Sigma}$, \mathbf{E} , $\bar{\mathbf{E}}$. However in the problem of transformation plasticity of interest here, overall geometrical and material isotropy entails identity of the principal directions of all 4 tensors. Indeed the prescribed quantities in this problem consist of the external stress tensor Σ , which introduces 3 orthogonal planes of symmetry defined by its principal directions, plus the internal strain $\bar{\mathbf{E}}$, a multiple of the unit tensor (see equation (45) below) which does not modify the group of symmetries. The “responses” to these prescribed quantities, consisting of the internal stress tensor $\bar{\Sigma}$ plus the external strain tensor \mathbf{E} , must respect the planes of symmetry defined by Σ , and therefore possess the same principal directions.

This means that we may choose the x_1 -, x_2 - and x_3 -directions parallel to the common principal directions of the tensors Σ , $\bar{\Sigma}$, \mathbf{E} , $\bar{\mathbf{E}}$; they are then all diagonal in the Cartesian basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, and their diagonal components may be denoted with a single index, Σ_i , $\bar{\Sigma}_i$, E_i , \bar{E}_i ($i = 1, 2, 3$).

This being said, the first issue to be addressed is that of the minimum number of incompressible trial displacement fields needed. The family of fields considered must account for all possible values of the kinematic parameters E_1 , E_2 , E_3 , \bar{E}_1 , \bar{E}_2 , \bar{E}_3 . But these 6 parameters are tied by the relation

$$\text{tr } \mathbf{E} = E_1 + E_2 + E_3 = f \text{tr } \bar{\mathbf{E}} = f (\bar{E}_1 + \bar{E}_2 + \bar{E}_3) \quad (25)$$

resulting from incompressibility of the mother-phase. (This equation means that the volumetric strain of the RVE arises exclusively from the expansion of the core of daughter-phase). Thus there are only 5 independent kinematic parameters, so that a minimum of 5 independent incompressible trial displacement fields is required.

We shall exploit here an analogy with the problem of a spherical RVE containing a concentric spherical *traction-free* void; this problem has been considered many times, starting with the pioneering work of Gurson (1977), for the derivation of models for porous plastic materials, in the context of prediction of ductile rupture. The more complex boundary conditions considered here, with the additional internal loading, are of no consequence upon the space of kinematically admissible, incompressible displacement fields adapted to the problem. We shall therefore use the elementary fields introduced by Gurson (1977), completed by the more complex Eshelby (1957)-type fields proposed by Monchiet *et al.*

(2011), which read as follows:

$$\left\{ \begin{array}{l} \mathbf{u}^{(1)}(\mathbf{x}) = \frac{R^3}{r^2} \mathbf{e}_r \\ \mathbf{u}^{(2)}(\mathbf{x}) = -\frac{x_1}{2} \mathbf{e}_1 - \frac{x_2}{2} \mathbf{e}_2 + x_3 \mathbf{e}_3 = -\frac{\rho}{2} \mathbf{e}_\rho + z \mathbf{e}_z \\ \mathbf{u}^{(3)}(\mathbf{x}) = \frac{2}{5f} \frac{r_f^5}{r^4} \left(-\frac{\sin \theta}{2} \mathbf{e}_\rho + \cos \theta \mathbf{e}_z \right) + \frac{1}{2f} \frac{r_f^3}{r^2} \left(1 - \frac{r_f^2}{r^2} \right) (3 \cos^2 \theta - 1) \mathbf{e}_r \\ \mathbf{u}^{(4)}(\mathbf{x}) = x_1 \mathbf{e}_1 - x_2 \mathbf{e}_2 \\ \mathbf{u}^{(5)}(\mathbf{x}) = \frac{2}{5f} \frac{r_f^5}{r^4} \sin \theta (\cos \phi \mathbf{e}_1 - \sin \phi \mathbf{e}_2) + \frac{1}{f} \frac{r_f^3}{r^2} \left(1 - \frac{r_f^2}{r^2} \right) \sin^2 \theta \cos(2\phi) \mathbf{e}_r. \end{array} \right. \quad (26)$$

The first three fields are axisymmetric: the first and second, proposed by Gurson (1977), correspond to a radial incompressible expansion and a uniform deviatoric straining of the spherical shell, respectively; the third, introduced by Monchiet *et al.* (2011), is of Eshelby (1957)-type. The fourth and fifth fields are non-axisymmetric: again the fourth, proposed by Gurson (1977), corresponds to a uniform deviatoric straining of the shell, whereas the fifth, introduced by Monchiet *et al.* (2011), is of Eshelby (1957)-type.⁵ Using Cartesian, cylindrical or spherical coordinates to calculate the integrals of equations (23) providing the overall external and internal strain tensors corresponding to these 5 fields,⁶ we get with obvious notations:

$$\left\{ \begin{array}{l} E_1^{(1)} = E_2^{(1)} = E_3^{(1)} = 1 \quad ; \quad \overline{E}_1^{(1)} = \overline{E}_2^{(1)} = \overline{E}_3^{(1)} = \frac{1}{f} \\ E_1^{(2)} = E_2^{(2)} = -\frac{1}{2}, \quad E_3^{(2)} = 1 \quad ; \quad \overline{E}_1^{(2)} = \overline{E}_2^{(2)} = -\frac{1}{2}, \quad \overline{E}_3^{(2)} = 1 \\ E_1^{(3)} = E_2^{(3)} = -\frac{1}{5}, \quad E_3^{(3)} = \frac{2}{5} \quad ; \quad \overline{E}_1^{(3)} = \overline{E}_2^{(3)} = -\frac{1}{5f}, \quad \overline{E}_3^{(3)} = \frac{2}{5f} \\ E_1^{(4)} = 1, \quad E_2^{(4)} = -1, \quad E_3^{(4)} = 0 \quad ; \quad \overline{E}_1^{(4)} = 1, \quad \overline{E}_2^{(4)} = -1, \quad \overline{E}_3^{(4)} = 0 \\ E_1^{(5)} = \frac{2}{5}, \quad E_2^{(5)} = -\frac{2}{5}, \quad E_3^{(5)} = 0 \quad ; \quad \overline{E}_1^{(5)} = \frac{2}{5f}, \quad \overline{E}_2^{(5)} = -\frac{2}{5f}, \quad \overline{E}_3^{(5)} = 0. \end{array} \right. \quad (27)$$

Note that all fields except $\mathbf{u}^{(1)}$ have $\text{tr} \mathbf{E}^{(i)} = 0$; that is, the volumetric strain of the RVE is to be described through a single field, $\mathbf{u}^{(1)}$.

We now consider an incompressible trial displacement field of the form

$$\mathbf{u}(\mathbf{x}) \equiv \sum_{i=1}^5 q_i \mathbf{u}^{(i)}(\mathbf{x}) \quad (28)$$

⁵ The third field is obtained from Monchiet *et al.* (2011)'s expressions by taking, with their notations, $d_{11}^* = d_{22}^* = -\frac{1}{2f}$, $d_{33}^* = \frac{1}{f}$, $d_{12}^* = d_{13}^* = d_{23}^* = 0$; the fifth, by taking $d_{11}^* = -d_{22}^* = \frac{1}{f}$, $d_{33}^* = d_{12}^* = d_{13}^* = d_{23}^* = 0$.

⁶ The use of equations (23) rather than (22) permits to circumvent the problem of defining smooth extensions of the fields $\mathbf{u}^{(i)}$ ($i = 1, \dots, 5$) over the void domain Ω_D ; a non-trivial task in view of the singularity of some of these fields at the origin.

where the q_i 's ($i = 1, \dots, 5$) are kinematic parameters. (Note that although any prescribed values of the kinematic parameters $E_1, E_2, E_3, \bar{E}_1, \bar{E}_2, \bar{E}_3$ obeying relation (25) may be matched through some suitable choice of the q_i 's, this does not mean that the fields $\mathbf{u}^{(i)}$ ($i = 1, \dots, 5$) define a basis of the space of general admissible fields; equation (28) just defines the projection of such fields onto the subspace generated by the fields $\mathbf{u}^{(i)}$ ($i = 1, \dots, 5$)).

The components of the corresponding overall external and internal strain tensors read, by equations (27):

$$\begin{cases} E_1 = q_1 - \frac{q_2}{2} - \frac{q_3}{5} + q_4 + \frac{2q_5}{5} \\ E_2 = q_1 - \frac{q_2}{2} - \frac{q_3}{5} - q_4 - \frac{2q_5}{5} \\ E_3 = q_1 + q_2 + \frac{2q_3}{5} \\ \bar{E}_1 = \frac{q_1}{f} - \frac{q_2}{2} - \frac{q_3}{5f} + q_4 + \frac{2q_5}{5f} \\ \bar{E}_2 = \frac{q_1}{f} - \frac{q_2}{2} - \frac{q_3}{5f} - q_4 - \frac{2q_5}{5f} \\ \bar{E}_3 = \frac{q_1}{f} + q_2 + \frac{2q_3}{5f}. \end{cases} \quad (29)$$

The load parameters Q_i ($i = 1, \dots, 5$) “conjugate” to the kinematic parameters q_i are defined *via* the expression of the virtual power of external forces, $\mathcal{P}_e \equiv \sum_{i=1}^5 Q_i \dot{q}_i$, applicable to arbitrary values of the virtual velocities \dot{q}_i . By equations (24) and (29), they are given by

$$\begin{cases} Q_1 \equiv \text{vol}(\Omega) (\Sigma_1 + \Sigma_2 + \Sigma_3 + \bar{\Sigma}_1 + \bar{\Sigma}_2 + \bar{\Sigma}_3) \\ Q_2 \equiv \text{vol}(\Omega) \left[\Sigma_3 - \frac{\Sigma_1}{2} - \frac{\Sigma_2}{2} + f \left(\bar{\Sigma}_3 - \frac{\bar{\Sigma}_1}{2} - \frac{\bar{\Sigma}_2}{2} \right) \right] \\ Q_3 \equiv \text{vol}(\Omega) \cdot \frac{2}{5} \left(\Sigma_3 - \frac{\Sigma_1}{2} - \frac{\Sigma_2}{2} + \bar{\Sigma}_3 - \frac{\bar{\Sigma}_1}{2} - \frac{\bar{\Sigma}_2}{2} \right) \\ Q_4 \equiv \text{vol}(\Omega) \left[\Sigma_1 - \Sigma_2 + f (\bar{\Sigma}_1 - \bar{\Sigma}_2) \right] \\ Q_5 \equiv \text{vol}(\Omega) \cdot \frac{2}{5} (\Sigma_1 - \Sigma_2 + \bar{\Sigma}_1 - \bar{\Sigma}_2). \end{cases} \quad (30)$$

Remark. Although the family of trial displacement fields considered is directly borrowed from the work of Monchiet *et al.* (2011), the fields $\mathbf{u}^{(3)}, \mathbf{u}^{(5)}$ of Eshelby (1957)-type play different roles in the previous work and here. For a traction-free void, as considered by Monchiet *et al.* (2011), there are only 3 independent kinematic parameters, E_1, E_2, E_3 , so that no more than 3 displacement fields are strictly necessary; the fields $\mathbf{u}^{(3)}, \mathbf{u}^{(5)}$ were added by Monchiet *et al.* (2011) to Gurson (1977)'s fields $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(4)}$ merely as an *improvement*, to get a better approximation of the overall yield criterion and flow rule (at the expense of an additional minimization of the overall dissipation). With the internal loading considered here, the fields $\mathbf{u}^{(3)}, \mathbf{u}^{(5)}$ become *indispensable* since there are 5 independent kinematic parameters and therefore a minimum of 5 fields is required.

3.4 Overall plastic dissipation

The local plastic dissipation in the mother-phase (rigid-ideal-plastic material obeying the von Mises criterion with yield stress $\bar{\sigma}_M$) is defined as the product $\bar{\sigma}_M \dot{\epsilon}_{eq}$ where $\dot{\epsilon}_{eq} \equiv (\frac{2}{3} \dot{\epsilon} : \dot{\epsilon})^{1/2}$ is the equivalent strain rate. For any incompressible trial velocity field $\dot{\mathbf{u}}(\mathbf{x})$ defined by the rates \dot{q}_i ($i = 1, \dots, 5$), identical to the time-derivative of a trial displacement field $\mathbf{u}(\mathbf{x})$ of type (28), the *overall plastic dissipation* \mathcal{D} is then defined as

$$\mathcal{D}(\dot{q}_1, \dots, \dot{q}_5) \equiv \int_{\Omega_M} \bar{\sigma}_M \dot{\epsilon}_{eq} d\Omega. \quad (31)$$

This dissipation is independent of the value of the tensor $\bar{\Sigma}$ characterizing the internal loading. It follows that its value is exactly the same as in the case considered by Monchiet *et al.* (2011) of a traction-free void. We shall therefore content ourselves with providing their final, approximate result for this value, referring the interested reader to their paper for details of their approximations and calculations:

$$\mathcal{D}(\dot{q}_1, \dots, \dot{q}_5) = \text{vol}(\Omega) \bar{\sigma}_M \int_1^{1/f} \left\{ \dot{q}_2^2 + \frac{4}{3} \dot{q}_4^2 + \left[4\dot{q}_1^2 + \frac{6g(f)}{25} \left(\dot{q}_3^2 + \frac{4}{3} \dot{q}_5^2 \right) \right] x^2 \right\}^{1/2} \frac{dx}{x^2} \quad (32)$$

where the change of variable $x \equiv R^3/r^3$ has been used, and $g(f)$ is the positive function defined by the formula

$$g(f) \equiv 1 - 4f \frac{(1 - f^{2/3})^2}{1 - f}. \quad (33)$$

(The integral in equation (32) may be calculated in closed form but the result will not be needed here).

3.5 Overall yield criterion

The fundamental inequality of the kinematic approach of limit-analysis reads (Hill, 1951)

$$\sum_{i=1}^5 Q_i \dot{q}_i \leq \mathcal{D}(\dot{q}_1, \dots, \dot{q}_5) \quad (34)$$

for all quintuplets of sustainable loads⁷ (Q_1, \dots, Q_5) and rates of kinematic parameters $(\dot{q}_1, \dots, \dot{q}_5)$. The set of limit-loads, that is the *overall yield locus*, is the boundary of the domain defined in the space of quintuplets (Q_1, \dots, Q_5) by inequalities (34) for all values of the quintuplet $(\dot{q}_1, \dots, \dot{q}_5)$; that is the envelope of the family of hyperplanes of equation $\sum_{i=1}^5 Q_i \dot{q}_i = \mathcal{D}(\dot{q}_1, \dots, \dot{q}_5)$ parametrized by the quintuplet $(\dot{q}_1, \dots, \dot{q}_5)$. The parametric equations of this envelope read

$$Q_i = \frac{\partial \mathcal{D}}{\partial \dot{q}_i}(\dot{q}_1, \dots, \dot{q}_5) \quad (i = 1, \dots, 5) \quad (35)$$

⁷ This expression designates loads that may be applied to the structure without inducing plastic collapse.

where the \dot{q}_i 's act as parameters; in these equations the quintuplets (Q_1, \dots, Q_5) and $(\dot{q}_1, \dots, \dot{q}_5)$ are no longer independent like in inequality (34), but tied through the macroscopic constitutive law - that is, the quintuplet of kinematic rates $(\dot{q}_1, \dots, \dot{q}_5)$ is that actually generated by the load (Q_1, \dots, Q_5) .

In order to get the explicit equation of the overall yield locus, we must eliminate the parameters \dot{q}_i ($i = 1, \dots, 5$) in equations (35). This may be done by using *Gurson's lemma*, stated and proved with maximum clarity and economy by Madou and Leblond (2012):

Gurson's lemma. *Let $I(\alpha, \beta)$ denote the integral defined by*

$$I(\alpha, \beta) \equiv \int_{x_1}^{x_2} \sqrt{\alpha^2 + \beta^2 x^2} \frac{dx}{x^2} \quad (36)$$

where x_1 and x_2 are fixed positive bounds. Then the partial derivatives of I are connected through the following relation independent of α and β :

$$\left(\frac{\partial I}{\partial \alpha} \right)^2 + \frac{2}{x_1 x_2} \cosh \left(\frac{\partial I}{\partial \beta} \right) - \frac{1}{x_1^2} - \frac{1}{x_2^2} = 0. \quad (37)$$

This lemma, applied here to equation (32) with

$$x_1 \equiv 1, \quad x_2 \equiv \frac{1}{f}, \quad \alpha \equiv \sqrt{\dot{q}_2^2 + \frac{4}{3} \dot{q}_4^2}, \quad \beta \equiv \sqrt{4\dot{q}_1^2 + \frac{6g(f)}{25} \left(\dot{q}_3^2 + \frac{4}{3} \dot{q}_5^2 \right)}, \quad I(\alpha, \beta) \equiv \frac{\mathcal{D}}{\text{vol}(\Omega) \bar{\sigma}_M}, \quad (38)$$

yields

$$\left(\frac{\partial \mathcal{D} / \partial \alpha}{\text{vol}(\Omega) \bar{\sigma}_M} \right)^2 + 2f \cosh \left(\frac{\partial \mathcal{D} / \partial \beta}{\text{vol}(\Omega) \bar{\sigma}_M} \right) - 1 - f^2 = 0. \quad (39)$$

But the definitions (38)_{3,4} of α and β imply that

$$\left\{ \begin{array}{l} \frac{\partial \mathcal{D}}{\partial \dot{q}_1} = \frac{4\dot{q}_1}{\beta} \frac{\partial \mathcal{D}}{\partial \beta} \\ \frac{\partial \mathcal{D}}{\partial \dot{q}_3} = \frac{6g(f)\dot{q}_3}{25\beta} \frac{\partial \mathcal{D}}{\partial \beta} \\ \frac{\partial \mathcal{D}}{\partial \dot{q}_5} = \frac{8g(f)\dot{q}_5}{25\beta} \frac{\partial \mathcal{D}}{\partial \beta} \end{array} \right. ; \quad \left\{ \begin{array}{l} \frac{\partial \mathcal{D}}{\partial \dot{q}_2} = \frac{\dot{q}_2}{\alpha} \frac{\partial \mathcal{D}}{\partial \alpha} \\ \frac{\partial \mathcal{D}}{\partial \dot{q}_4} = \frac{4\dot{q}_4}{3\alpha} \frac{\partial \mathcal{D}}{\partial \alpha} \end{array} \right. ,$$

and it follows, using again these definitions, that

$$\left\{ \begin{array}{l} \frac{1}{4} \left(\frac{\partial \mathcal{D}}{\partial \dot{q}_1} \right)^2 + \frac{25}{6g(f)} \left(\frac{\partial \mathcal{D}}{\partial \dot{q}_3} \right)^2 + \frac{25}{8g(f)} \left(\frac{\partial \mathcal{D}}{\partial \dot{q}_5} \right)^2 = \left(\frac{\partial \mathcal{D}}{\partial \beta} \right)^2 \\ \left(\frac{\partial \mathcal{D}}{\partial \dot{q}_2} \right)^2 + \frac{3}{4} \left(\frac{\partial \mathcal{D}}{\partial \dot{q}_4} \right)^2 = \left(\frac{\partial \mathcal{D}}{\partial \alpha} \right)^2 \end{array} \right. .$$

Inserting these expressions into equation (39), then using equation (35) and the definitions

(30) of the load parameters Q_i , we get the equation of the overall yield locus in the form

$$\begin{aligned} & \frac{1}{\bar{\sigma}_M^2} \left(T_3 - \frac{T_1}{2} - \frac{T_2}{2} \right)^2 + \frac{3}{4\bar{\sigma}_M^2} (T_1 - T_2)^2 \\ & + 2f \cosh \left\{ \frac{1}{\bar{\sigma}_M} \left[\frac{(S_1 + S_2 + S_3)^2}{4} + \frac{2}{3g(f)} \left(S_3 - \frac{S_1}{2} - \frac{S_2}{2} \right)^2 + \frac{(S_1 - S_2)^2}{2g(f)} \right]^{1/2} \right\} \\ & - 1 - f^2 = 0 \end{aligned} \quad (40)$$

where $S_1, S_2, S_3, T_1, T_2, T_3$ are the diagonal components of the tensors \mathbf{S} and \mathbf{T} defined by

$$\mathbf{S} \equiv \boldsymbol{\Sigma} + \bar{\boldsymbol{\Sigma}} \quad ; \quad \mathbf{T} \equiv \boldsymbol{\Sigma} + f\bar{\boldsymbol{\Sigma}}. \quad (41)$$

Re-expressing quantities in equation (40) in terms of the invariants of \mathbf{S} and \mathbf{T} , we get the final form of the overall yield criterion:

$$\Phi(\boldsymbol{\Sigma}, \bar{\boldsymbol{\Sigma}}, f) \equiv \frac{T_{eq}^2}{\bar{\sigma}_M^2} + 2f \cosh \left(\frac{S_H}{\bar{\sigma}_M} \right) - 1 - f^2 = 0, \quad S_H \equiv \left[\frac{1}{4} (\text{tr } \mathbf{S})^2 + \frac{2}{3g(f)} S_{eq}^2 \right]^{1/2} \quad (42)$$

where $X_{eq} \equiv \left(\frac{3}{2} \mathbf{X}' : \mathbf{X}' \right)^{1/2}$ denotes the von Mises norm of a given stress tensor \mathbf{X} (\mathbf{X}' is the deviator of \mathbf{X}); $\Phi(\boldsymbol{\Sigma}, \bar{\boldsymbol{\Sigma}}, f)$ is the *overall yield function*.

In the absence of an internal loading ($\bar{\boldsymbol{\Sigma}} = \mathbf{0}$), S_{eq} and T_{eq} become identical to Σ_{eq} , $\text{tr } \mathbf{S}$ to $\text{tr } \boldsymbol{\Sigma}$ and the criterion (42) reduces to that of Monchiet *et al.* (2011) (their equation (41)). In the general case, however, it is new.

3.6 Overall flow rule

We follow here Hill (1951)'s classical approach, adapted to the present situation. Let \mathcal{E} denote the space of pairs $(\mathbf{X}, \bar{\mathbf{X}})$ of symmetric second-rank tensors, equipped with the scalar product \bullet defined by $(\mathbf{X}, \bar{\mathbf{X}}) \bullet (\mathbf{Y}, \bar{\mathbf{Y}}) \equiv \mathbf{X} : \mathbf{Y} + \bar{\mathbf{X}} : \bar{\mathbf{Y}}$. Consider, within \mathcal{E} :

- a pair $(\boldsymbol{\Sigma}, \bar{\boldsymbol{\Sigma}})$ of overall stress tensors defining a limit-load;
- the pair $(\dot{\mathbf{E}}, \dot{\bar{\mathbf{E}}})$ of overall external and internal strain rates generated by this limit-load;
- another pair $(\boldsymbol{\Sigma}^*, \bar{\boldsymbol{\Sigma}}^*)$ of overall stress tensors defining an arbitrary sustainable load.

Consider also the following fields:

- the stress field $\boldsymbol{\sigma}(\mathbf{x})$ resulting from application of the limit-load $(\boldsymbol{\Sigma}, \bar{\boldsymbol{\Sigma}})$;
- the strain rate field $\dot{\boldsymbol{\epsilon}}(\mathbf{x})$ generated by this stress field through the local plastic flow rule;
- a stress field $\boldsymbol{\sigma}^*(\mathbf{x})$ statically admissible with the pair $(\boldsymbol{\Sigma}^*, \bar{\boldsymbol{\Sigma}}^*)$ and plastically admissible (respecting the inequality $\sigma_{eq}^* \leq \bar{\sigma}_M$ at every point) - the existence of such a field being guaranteed by the very definition of a sustainable load.

Then, by the expression (24) of the virtual power of external forces and the principle of virtual work,

$$\text{vol}(\Omega) \left[(\boldsymbol{\Sigma} - \boldsymbol{\Sigma}^*) : \dot{\mathbf{E}} + f(\bar{\boldsymbol{\Sigma}} - \bar{\boldsymbol{\Sigma}}^*) : \dot{\bar{\mathbf{E}}} \right] = \int_{\Omega_M} (\boldsymbol{\sigma} - \boldsymbol{\sigma}^*) : \dot{\boldsymbol{\epsilon}} \, d\Omega.$$

The integrand here is non-negative as a consequence of the local normality rule. It follows that

$$(\boldsymbol{\Sigma} - \boldsymbol{\Sigma}^*) : \dot{\mathbf{E}} + (\bar{\boldsymbol{\Sigma}} - \bar{\boldsymbol{\Sigma}}^*) : f\dot{\bar{\mathbf{E}}} = (\boldsymbol{\Sigma} - \boldsymbol{\Sigma}^*, \bar{\boldsymbol{\Sigma}} - \bar{\boldsymbol{\Sigma}}^*) \bullet (\dot{\mathbf{E}}, f\dot{\bar{\mathbf{E}}}) \geq 0$$

for every sustainable load $(\boldsymbol{\Sigma}^*, \bar{\boldsymbol{\Sigma}}^*)$.

It results from there, following a classical reasoning, that the pair $(\dot{\mathbf{E}}, f\dot{\bar{\mathbf{E}}})$ is positively collinear in the space \mathcal{E} to the outer normal vector to the set of limit-loads, that is the overall yield locus of equation $\Phi(\boldsymbol{\Sigma}, \bar{\boldsymbol{\Sigma}}, f) = 0$; thus there exists a non-negative scalar $\dot{\Lambda}$ - the *overall plastic multiplier* - such that

$$\dot{\mathbf{E}} = \dot{\Lambda} \frac{\partial \Phi}{\partial \boldsymbol{\Sigma}}(\boldsymbol{\Sigma}, \bar{\boldsymbol{\Sigma}}, f) \quad ; \quad f\dot{\bar{\mathbf{E}}} = \dot{\Lambda} \frac{\partial \Phi}{\partial \bar{\boldsymbol{\Sigma}}}(\boldsymbol{\Sigma}, \bar{\boldsymbol{\Sigma}}, f). \quad (43)$$

Calculating the derivatives here, we get the final form of the two-part overall flow rule:

$$\begin{cases} \dot{\mathbf{E}} = \frac{\dot{\Lambda}}{\bar{\sigma}_M^2} \left\{ 3\mathbf{T}' + f \frac{\sinh(S_H/\bar{\sigma}_M)}{S_H/\bar{\sigma}_M} \left[\frac{1}{2} (\text{tr } \mathbf{S}) \mathbf{1} + \frac{2}{g(f)} \mathbf{S}' \right] \right\} \\ \dot{\bar{\mathbf{E}}} = \frac{\dot{\Lambda}}{\bar{\sigma}_M^2} \left\{ 3\mathbf{T}' + \frac{\sinh(S_H/\bar{\sigma}_M)}{S_H/\bar{\sigma}_M} \left[\frac{1}{2} (\text{tr } \mathbf{S}) \mathbf{1} + \frac{2}{g(f)} \mathbf{S}' \right] \right\} \end{cases}, \quad \dot{\Lambda} \geq 0 \quad (44)$$

where again a prime applied to a tensor denotes its deviatoric part.

Note that equations (44) are compatible with relation (25) connecting the traces of the outer and inner strain tensors.

4 Application to Greenwood-Johnson's mechanism of transformation plasticity

4.1 Principle and preliminaries

An elementary volume in a metal or alloy undergoing a phase transformation under stress is represented by the RVE defined in Subsection 3.2, subjected to a prescribed external stress $\boldsymbol{\Sigma}$ plus an internal strain rate $\dot{\bar{\mathbf{E}}}$ resulting from the expansion of the gradually growing core of daughter-phase. The elements of the problem are as follows:

- The data include the tensors $\boldsymbol{\Sigma}$ and $\dot{\bar{\mathbf{E}}}$ - the calculation of the second is presented just below.
- The unknowns include the internal stress tensor $\bar{\boldsymbol{\Sigma}}$, the overall strain rate $\dot{\mathbf{E}}$ and the plastic multiplier $\dot{\Lambda}$. What is looked for is $\dot{\mathbf{E}}$; $\bar{\boldsymbol{\Sigma}}$ and $\dot{\Lambda}$ are ancillary unknowns of no intrinsic interest.

- The equations available to determine the unknowns from the data are the overall yield criterion (42) and two-part flow rule (44).

Note that this summary elucidates the second difficulty mentioned in Subsection 3.1, in that it shows how an expression of the transformation plastic rate may be derived for arbitrary values of the external stress from limit-analysis, even though this theory concentrates only on special (limit-) loads. The keypoint is that the overall yield criterion ties the external and internal stress tensors together but *does not constrain their values individually*, so that the value of the former may still be chosen at will - the value of the latter will simply “adjust” through the criterion.

The first, preliminary task is to determine the value of the internal strain rate $\dot{\bar{\mathbf{E}}}$ resulting from the transformation. Consider the situation of Subsection 2.5 wherein the spherical layer lying between r_f and $r_f + \delta r_f$ transforms from the mother- to the daughter-phase within a small time-interval $[t, t + \delta t]$, generating a radial increment of displacement δu^+ on its outer boundary. The diagonal components of the increment of the internal strain tensor are then obviously

$$\delta \bar{E}_1 = \delta \bar{E}_2 = \delta \bar{E}_3 = \frac{\delta u^+}{r_f} = \frac{\Delta V}{V} \frac{\delta r_f}{r_f} = \frac{1}{3} \frac{\Delta V}{V} \frac{\delta f}{f}$$

where equations (13) and (16) have been used; the value of the internal strain rate tensor follows:

$$\dot{\bar{\mathbf{E}}} = \frac{1}{3} \frac{\Delta V}{V} \dot{f} \mathbf{1}. \quad (45)$$

A straightforward consequence of equation (45) is that the deviatoric part of the internal strain rate tensor $\dot{\bar{\mathbf{E}}}$ is zero. By the flow rule (44)₂ combined with the definitions (41) of the tensors \mathbf{S} and \mathbf{T} , this implies that

$$\dot{\bar{\Sigma}}' = - \frac{3 + \frac{2}{g(f)} \frac{\sinh(S_H/\bar{\sigma}_M)}{S_H/\bar{\sigma}_M}}{3f + \frac{2}{g(f)} \frac{\sinh(S_H/\bar{\sigma}_M)}{S_H/\bar{\sigma}_M}} \dot{\Sigma}'. \quad (46)$$

A second preliminary task is to determine the relation between the transformation plastic strain rate $\dot{\mathbf{E}}^{\text{tp}}$ and the overall strain rate $\dot{\bar{\mathbf{E}}}$ resulting from the limit-analysis. This limit-analysis being carried out in the absence of elasticity and without any variations of Σ and T , equations (7)₁ in rate form plus (12)₁ yield $\dot{\bar{\mathbf{E}}} = \dot{\mathbf{E}}^{\text{thm}} + \dot{\mathbf{E}}^{\text{tp}}$. But $\dot{\mathbf{E}}^{\text{thm}}$ is a multiple of the unit tensor by equation (7)₂, and $\dot{\mathbf{E}}^{\text{tp}}$ is traceless by equation (12)₃. It follows that $\dot{\mathbf{E}}^{\text{tp}}$ identifies with the deviatoric part of the overall strain rate $\dot{\bar{\mathbf{E}}}$ obtained in the limit-analysis:⁸

$$\dot{\mathbf{E}}^{\text{tp}} \equiv \dot{\mathbf{E}}' = \frac{\dot{\Lambda}}{\bar{\sigma}_M^2} \left[3\mathbf{T}' + \frac{2f}{g(f)} \frac{\sinh(S_H/\bar{\sigma}_M)}{S_H/\bar{\sigma}_M} \mathbf{S}' \right] \quad (47)$$

where the flow rule (44)₁ has been used.

⁸ And, of course, $\dot{\mathbf{E}}^{\text{thm}}$ identifies with the hydrostatic part of $\dot{\bar{\mathbf{E}}}$; this is confirmed by equations (25) and (45) which yield the value $\frac{1}{3} \frac{\Delta V}{V} \dot{f} \mathbf{1}$ for this hydrostatic part.

4.2 Special case: small external stresses

In a first step, we shall consider the case of small applied stresses ($\Sigma_{eq} \ll \bar{\sigma}_M$), in which a simple and explicit formula for the transformation plastic strain rate may be obtained; $\dot{\mathbf{E}}^{tp}$ will be formally calculated at order 1 in Σ' . Preliminary calculations of some quantities at order 0 will however be necessary.

Calculations at order 0. At this order, Σ' is considered as nil. It then follows from equation (46) that $\bar{\Sigma}'$ is also zero, so that by equation (42)₂, $S_H = \frac{1}{2}|\text{tr } \mathbf{S}|$. The criterion (42)₁ then yields

$$2f \cosh\left(\frac{S_H}{\bar{\sigma}_M}\right) - 1 - f^2 = 0 \quad \Rightarrow \quad \cosh\left(\frac{S_H}{\bar{\sigma}_M}\right) = \frac{1}{2}\left(f + \frac{1}{f}\right) \quad \Rightarrow \quad \frac{S_H}{\bar{\sigma}_M} = -\ln f$$

where we have discarded the negative solution $\frac{S_H}{\bar{\sigma}_M} = \ln f$ since S_H is by definition non-negative. From there follows that

$$\frac{\sinh(S_H/\bar{\sigma}_M)}{S_H/\bar{\sigma}_M} = \frac{1 - f^2}{-2f \ln f}. \quad (48)$$

Combination of the flow rule (44)₂ and the expression (45) of $\dot{\bar{\mathbf{E}}}$ then yields

$$\frac{\dot{\Lambda}}{\bar{\sigma}_M^2} \frac{1 - f^2}{-4f \ln f} \text{tr } \mathbf{S} = \frac{1}{3} \frac{\Delta V}{V} \frac{\dot{f}}{f}.$$

Since $\dot{\Lambda}$ is positive and in practice so is also $\frac{\Delta V}{V}$ (see Fig. A.1 of Appendix A), this equation entails that $\text{tr } \mathbf{S} > 0$ and therefore, by what precedes, $\text{tr } \mathbf{S} = 2S_H = -2\bar{\sigma}_M \ln f$; the value of the plastic multiplier follows:

$$\frac{\dot{\Lambda}}{\bar{\sigma}_M^2} = \frac{2}{3\bar{\sigma}_M} \frac{\Delta V}{V} \frac{\dot{f}}{1 - f^2}. \quad (49)$$

Calculations at order 1. Equation (46) shows that $\bar{\Sigma}'$ is of order 1 in Σ' ; more precisely, using the value of $\frac{\sinh(S_H/\bar{\sigma}_M)}{S_H/\bar{\sigma}_M}$ at order 0, equation (48), we get

$$\bar{\Sigma}' = -\frac{1 - f^2 - 3fg(f) \ln f}{1 - f^2 - 3f^2g(f) \ln f} \Sigma'$$

and therefore by the definitions (41) of \mathbf{S} and \mathbf{T} ,

$$\mathbf{S}' = \frac{3f(1-f)g(f) \ln f}{1 - f^2 - 3f^2g(f) \ln f} \Sigma' \quad ; \quad \mathbf{T}' = \frac{(1-f)^2(1+f)}{1 - f^2 - 3f^2g(f) \ln f} \Sigma'.$$

We then see that in the expression (47) of $\dot{\mathbf{E}}^{tp}$, we only need the values of $\frac{\sinh(S_H/\bar{\sigma}_M)}{S_H/\bar{\sigma}_M}$ and $\frac{\dot{\Lambda}}{\bar{\sigma}_M^2}$ at order 0; using equations (48) and (49), we finally get the desired value of $\dot{\mathbf{E}}^{tp}$ at order 1:

$$\dot{\mathbf{E}}^{tp} = \frac{2}{\bar{\sigma}_M} \frac{\Delta V}{V} \frac{(1-f)^2}{1 - f^2 - 3f^2g(f) \ln f} \Sigma' \dot{f} \quad [\Sigma_{eq} \ll \bar{\sigma}_M]. \quad (50)$$

Comments on this result will be made below, but it is worth noting immediately that *the “slope” $d\mathbf{E}^{\text{tp}}/df$ at the onset of the transformation is no longer infinite like in Leblond et al. (1989)’s work*, but amounts to

$$\frac{d\mathbf{E}^{\text{tp}}}{df}(f=0) = \frac{2}{\bar{\sigma}_M} \frac{\Delta V}{V} \boldsymbol{\Sigma}' \quad [\Sigma_{eq} \ll \bar{\sigma}_M]. \quad (51)$$

This shows that the infinite slope obtained in the previous work was primarily due to the inadequacy of the “uncontrolled” Approximations (1) and (2) it relied upon, see Subsection 2.5. (The disregard of elasticity however also contributed to this incorrect result, see the works of Taleb and Sidoroff (2003) and Weisz-Paltrault (2017)).

4.3 General case: arbitrary external stresses

We shall now see that for arbitrary external stresses (Σ_{eq} comparable to $\bar{\sigma}_M$), the transformation plastic strain rate $\dot{\mathbf{E}}^{\text{tp}}$ still admits an analytical expression, but now involving a quantity which must be evaluated numerically by solving a transcendental equation.

Let us define the following dimensionless quantities:

$$X \equiv \frac{\Sigma_{eq}}{\bar{\sigma}_M} \quad ; \quad \bar{X} \equiv \frac{\bar{\Sigma}_{eq}}{\bar{\sigma}_M} \quad ; \quad Y \equiv \frac{\frac{1}{3} \text{tr} \mathbf{S}}{\bar{\sigma}_M} \quad ; \quad Z \equiv \frac{S_H}{\bar{\sigma}_M}, \quad (52)$$

the first of which is supposedly known whereas the other three are not.

Let us first show how the values of Y and Z may be deduced from that of \bar{X} . Equation (46) implies that the tensors $\boldsymbol{\Sigma}'$ and $\bar{\boldsymbol{\Sigma}}'$ are negatively collinear so that

$$\bar{\boldsymbol{\Sigma}}' = -\frac{\bar{\Sigma}_{eq}}{\Sigma_{eq}} \boldsymbol{\Sigma}' = -\frac{\bar{X}}{X} \boldsymbol{\Sigma}'. \quad (53)$$

It then follows from the definition of \mathbf{S} , equation (41)₁, that $\mathbf{S}' = (1 - \bar{X}/X)\boldsymbol{\Sigma}'$ so that $S_{eq}^2 = (1 - \bar{X}/X)^2 \Sigma_{eq}^2 = \bar{\sigma}_M^2 (\bar{X} - X)^2$. The definition (42)₂ of S_H then implies that

$$Z = \left[\frac{9}{4} Y^2 + \frac{2}{3g(f)} (\bar{X} - X)^2 \right]^{1/2} \quad \Rightarrow \quad Y = \frac{2}{3} \left[Z^2 - \frac{2}{3g(f)} (\bar{X} - X)^2 \right]^{1/2} \quad (54)$$

(the positivity of Y is justified below). Also, it follows from the definition of \mathbf{T} , equation (41)₂, and equation (53) that $\mathbf{T}' = (1 - f\bar{X}/X)\boldsymbol{\Sigma}'$ so that $T_{eq}^2 = (1 - f\bar{X}/X)^2 \Sigma_{eq}^2 = \bar{\sigma}_M^2 (X - f\bar{X})^2$. The criterion (42) then yields

$$(X - f\bar{X})^2 + 2f \cosh Z - 1 - f^2 = 0 \quad \Rightarrow$$

$$Z = F(X - f\bar{X}; f) \quad \text{where} \quad F(x; f) \equiv \arg \cosh \left(\frac{1 + f^2 - x^2}{2f} \right). \quad (55)$$

Thus, if the value of \bar{X} is known, those of Z and Y follow from equations (55) and (54)₂.

To now derive an equation on \bar{X} , combine equations (46) and (53) to get

$$\frac{3 + \frac{2}{g(f)} \frac{\sinh Z}{Z}}{3f + \frac{2}{g(f)} \frac{\sinh Z}{Z}} = \frac{\bar{X}}{X} \quad \Rightarrow$$

$$3(X - f\bar{X}) + \frac{2}{g(f)} \frac{\sinh[F(X - f\bar{X}); f]}{F(X - f\bar{X}; f)} (X - \bar{X}) = 0 \quad (56)$$

where equation (55) has been used. The transcendental equation (56) on \bar{X} permits to deduce its value from that of X at least numerically.

Let us finally show how to calculate $\dot{\mathbf{E}}^{\text{tp}}$ once \bar{X} , Y and Z are known. It follows from the deviatoric part of the flow rule (44)₂ and the fact that $\dot{\bar{\mathbf{E}}} = \mathbf{0}$ that $3\mathbf{T}' + \frac{2}{g(f)} \frac{\sinh Z}{Z} \mathbf{S}' = \mathbf{0}$, so that the expression (47) of $\dot{\mathbf{E}}^{\text{tp}}$ simplifies into

$$\dot{\mathbf{E}}^{\text{tp}} = -\frac{\dot{\Lambda}}{\bar{\sigma}_M^2} \frac{2(1-f)}{g(f)} \frac{\sinh Z}{Z} \mathbf{S}'.$$

Now combination of the hydrostatic part of the flow rule (44)₂ and the expression (45) of $\dot{\bar{\mathbf{E}}}$ implies that

$$\frac{\dot{\Lambda}}{\bar{\sigma}_M^2} \frac{\sinh Z}{2Z} \text{tr} \mathbf{S} = \frac{1}{3} \frac{\Delta V}{V} \frac{\dot{f}}{f}.$$

(Incidentally, since $\dot{\Lambda}$ and $\frac{\Delta V}{V}$ are positive, this equation shows that $\text{tr} \mathbf{S}$ and therefore Y are positive, as anticipated above). The preceding expression of $\dot{\mathbf{E}}^{\text{tp}}$ then becomes

$$\dot{\mathbf{E}}^{\text{tp}} = -\frac{4(1-f)}{3fg(f)} \frac{\Delta V}{V} \dot{f} \frac{\mathbf{S}'}{\text{tr} \mathbf{S}}.$$

Using finally the relations $\text{tr} \mathbf{S} = 3Y\bar{\sigma}_M$ and $\mathbf{S}' = (1 - \bar{X}/X)\mathbf{\Sigma}'$, we get from there the final expression of $\dot{\mathbf{E}}^{\text{tp}}$:

$$\dot{\mathbf{E}}^{\text{tp}} = \frac{4}{9\bar{\sigma}_M} \frac{1-f}{fg(f)} \frac{\Delta V}{V} \frac{\bar{X}/X - 1}{Y} \mathbf{\Sigma}' \dot{f} \quad [\text{General formula}]. \quad (57)$$

To summarize, the procedure to get the transformation plastic strain rate is as follows: get \bar{X} by solving the transcendental equation (56); then get Z and Y from equations (55) and (54)₂; finally get $\dot{\mathbf{E}}^{\text{tp}}$ from equation (57).

4.4 Explicit formula for the beginning of the transformation

As a particular case of formula (57), an explicit, remarkably simple expression of the transformation plastic strain rate at the beginning of the transformation ($f \rightarrow 0^+$) may be derived, even for arbitrary external stresses. The keypoint here is that provided that

the quantity $f\bar{X}$ is assumed to be of order $O(f)$ (which will be verified *a posteriori*), the value of Z may be calculated directly from equation (55) independently of that of \bar{X} :

$$Z = \arg \cosh \left[\frac{1 - X^2}{f} (1 + O(f)) \right] = \ln \left(\frac{1 - X^2}{f} \right) + O(f),$$

implying that

$$\frac{\sinh Z}{Z} = \frac{1 - X^2}{2f \ln \left(\frac{1 - X^2}{f} \right)} (1 + O(f)).$$

One then gets from equations (56) (upon division by X) and (54)₂:

$$\frac{\bar{X}}{X} - 1 = \frac{3f \ln \left(\frac{1 - X^2}{f} \right)}{1 - X^2} (1 + O(f)) \quad ; \quad Y = \frac{2}{3} \ln \left(\frac{1 - X^2}{f} \right) + O(f).$$

(The first equation shows that $\bar{X} = O(1)$ and therefore $f\bar{X} = O(f)$, as announced). Equation (57) then finally yields

$$\dot{\mathbf{E}}^{\text{tp}} = \frac{2}{\bar{\sigma}_M} \frac{\Delta V}{V} \frac{\Sigma'}{1 - \Sigma_{eq}^2 / \bar{\sigma}_M^2} (1 + O(f)) \dot{f} \quad [f \rightarrow 0^+]. \quad (58)$$

This formula evidences the amplifying effect of the factor $\frac{1}{1 - \Sigma_{eq}^2 / \bar{\sigma}_M^2}$ upon the transformation plastic strain rate $\dot{\mathbf{E}}^{\text{tp}}$, when the von Mises equivalent stress Σ_{eq} becomes comparable to the yield stress $\bar{\sigma}_M$ of the mother-phase.

4.5 A composite approximation

Use of the general formula (57) for $\dot{\mathbf{E}}^{\text{tp}}$ is somewhat hampered by the necessity of solving equation (56) numerically to get the value of \bar{X} . It is therefore tempting to propose the following simple ‘‘composite approximation’’ of $\dot{\mathbf{E}}^{\text{tp}}$, based on a heuristic mix of formulae (50), applicable for $\Sigma_{eq} \ll \bar{\sigma}_M$ but arbitrary f , and (58), applicable for arbitrary Σ_{eq} but $f \ll 1$:

$$\dot{\mathbf{E}}^{\text{tp}} = \frac{2}{\bar{\sigma}_M} \frac{\Delta V}{V} \frac{(1 - f)^2}{1 - f^2 - 3f^2 g(f) \ln f} \frac{\Sigma'}{1 - \Sigma_{eq}^2 / \bar{\sigma}_M^2} \dot{f} \quad [\text{Approximation}]. \quad (59)$$

5 Comparison with other theories, experiments, and micromechanical simulations

5.1 Elements of comparison - Generalities

The comparisons presented in this section are based on the following theoretical, experimental and numerical elements:

- Leblond *et al.* (1989)'s theoretical formula (19) for $\dot{\mathbf{E}}^{\text{tp}}$ and its supposedly improved variant (20); our general formula (57) and its approximation (59);
- Desalos (1981)'s old, but high-quality experimental study of transformation plasticity in the A533 steel, completed more recently with respect to multiaxial aspects by Coret *et al.* (2002, 2004);
- new finite element micromechanical simulations of Greenwood and Johnson (1965)'s mechanism, performed with the commercial programme SYSWELD[®] developed by ESI-Group (SYSWELD, 2017).

Although it is not the object of this paper to present and discuss these new micromechanical simulations in detail, a few comments on them are in order here. In principle, they are analogous to many previous ones (see the pioneering work of Leblond *et al.* (1989) and those of Ganghoffer *et al.* (1993); Barbe *et al.* (2007, 2008); Barbe and Quey (2011); Otsuka *et al.* (2018), to cite just a few of its many successors). However they differ from them through consideration of a RVE of spherical shape, exactly identical to that used in the theoretical homogenization process; the aim being to provide the “cleanest” possible assessment of the model predictions by eliminating all sources of discrepancies tied to differences in the RVEs considered. A spherical volume made of elastic-plastic material is thus meshed in concentric spherical layers, which are “transformed” one after another from the centre to the external surface by artificially modifying the values of their specific volume and yield stress from these of the mother-phase to those of the daughter-phase. This volume is loaded externally⁹ through conditions of homogeneous boundary stress (HBStress) or homogeneous boundary strain (HBStrain) - the aim being to complete the theoretical analysis based exclusively on HBStress conditions, by studying the effect of boundary conditions.

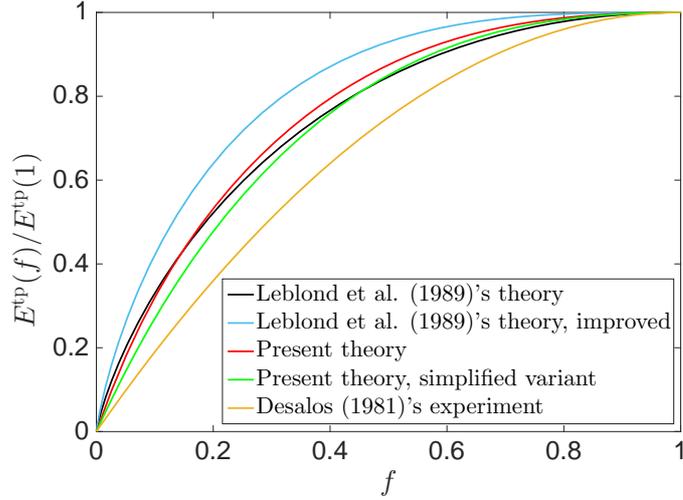
The physical constants used correspond to the γ - and α - phases of the A533 steel employed in the experimental studies of Desalos (1981) and Coret *et al.* (2002, 2004), and are as follows: Young's modulus and Poisson's ratio, $E = 182,000$ MPa and $\nu = 0.3$ (identical values for both phases); yield stress, $\bar{\sigma}_M = 145$ MPa for the mother-phase, $\bar{\sigma}_D = 950$ MPa for the daughter-phase; relative change of specific volume from the mother- to the daughter-phase, $\frac{\Delta V}{V} = 0.0252$.

In the cases considered for comparison in Subsections 5.2 and 5.3, the RVE is subjected to a *uniaxial and constant* stress denoted Σ , and the resulting transformation plastic strain in the direction of the load is denoted $E^{\text{tp}}(f)$. (The other components are of no interest since they are either zero or trivially tied to $E^{\text{tp}}(f)$ through incompressibility).

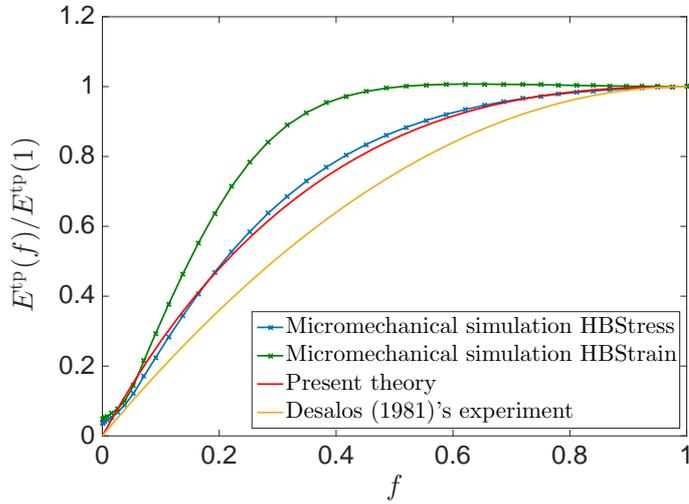
5.2 Comparison of evolutions of the transformation plastic strain

In a first step we focus on the evolution of the transformation plastic strain as the transformation proceeds, characterized by the ratio $E^{\text{tp}}(f)/E^{\text{tp}}(1)$ which increases from 0 to 1. Figure 4 shows this ratio as a function of the volume fraction f of the daughter-phase.

⁹ There is no internal boundary - and therefore no internal loading - since the core of daughter-phase is included in the mesh.



(a) Theories and experiment.



(b) Theory and micromechanical simulations.

Fig. 4. Comparison of evolutions of the transformation plastic strain.

Subfigure 4(a) compares the predictions of Leblond *et al.* (1989)'s original formula (19); its supposedly improved variant (20); our general formula (57); its approximation (59); and Desalos (1981)'s phenomenological formula $E^{\text{tp}}(f)/E^{\text{tp}}(1) \simeq f(2-f)$, which he found applicable to all his experimental results for the A533 steel irrespective of the value of the stress applied. The predictions of equations (19), (20) and (59) are independent of the value of the von Mises equivalent stress Σ_{eq} , but those of equation (57) are not; for this equation the curve shown corresponds to the value $\Sigma_{eq} \equiv \Sigma = 100$ MPa (to be compared to the yield stress of the mother-phase, $\bar{\sigma}_M = 145$ MPa). Several points are noteworthy.

- All theoretical curves, except that corresponding to the supposedly improved variant (20) of Leblond *et al.* (1989)'s original formula (19), provide reasonable representations of Desalos (1981)'s experimental results. However in all cases the increase of the transformation plastic strain they predict is a bit fast during the first half of the transformation.
- The infinite slope dE^{tp}/df at the onset of the transformation ($f = 0$) predicted by both

Leblond *et al.* (1989)’s original formula (19) and its supposedly improved variant (20), is in clear contradiction with Desalos (1981)’s experimental results.

- In contrast our new formulae (57) and (59) both predict a finite slope at $f = 0$, in better agreement with Desalos (1981)’s results. Nevertheless the predicted slopes are still a bit too large. The discrepancy may be due to the total disregard of elasticity in the theoretical treatment, which leads to an overestimation of the plastic strains in the deforming mother-phase; see the works of Taleb and Sidoroff (2003) and Weisz-Paltrault (2017) on this topic.
- Use, instead of Leblond *et al.* (1989)’s original formula (19), of its supposedly improved variant (20), in fact results in a degradation of the agreement of model predictions and experimental results. This observation is a bit surprising since formula (20) is based on a better estimate of the mean value of the stress deviator in the mother-phase, and underlines the limitations of Leblond *et al.* (1989)’s original approach based on the specific Approximations (1) and (2), see Subsection 2.5.
- There is little difference between the predictions of our general formula (57) and its simplified variant (59), in spite of the fact that the value of the ratio $\Sigma_{eq}/\bar{\sigma}_M$ is high (0.69).

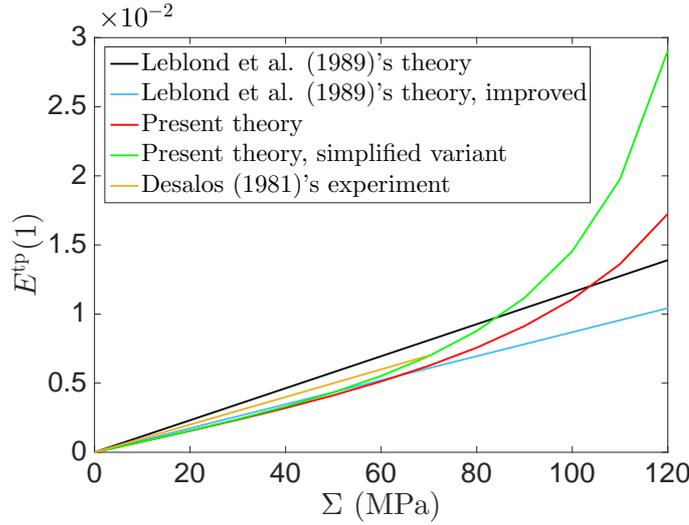
Subfigure 4(b) compares the predictions of our general formula (57) to the results of micromechanical simulations obtained with HBStress and HBStrain conditions, this time for a low stress, $\Sigma = 20$ MPa. (The curve corresponding to Desalos (1981)’s heuristic formula mentioned above, applicable to all values of the stress he considered, is also again provided for reference).

- There is a non-negligible gap between the numerical results obtained with HBStress and HBStrain conditions. This emphasizes the influence of boundary conditions, and evidences the limitations of an approach based on such a simple and small RVE as a spherical volume of mother-phase containing a single growing core of daughter-phase.
- The theoretical predictions agree extremely well with these numerical results obtained with HBStress conditions, much less so with those obtained with HBStrain conditions.
- Desalos (1981)’s experimental results serving as a reference, the numerical simulation based on HBStrain conditions predicts an evolution of the transformation plastic strain which is too fast during the first half of the transformation, and too slow during the second half.

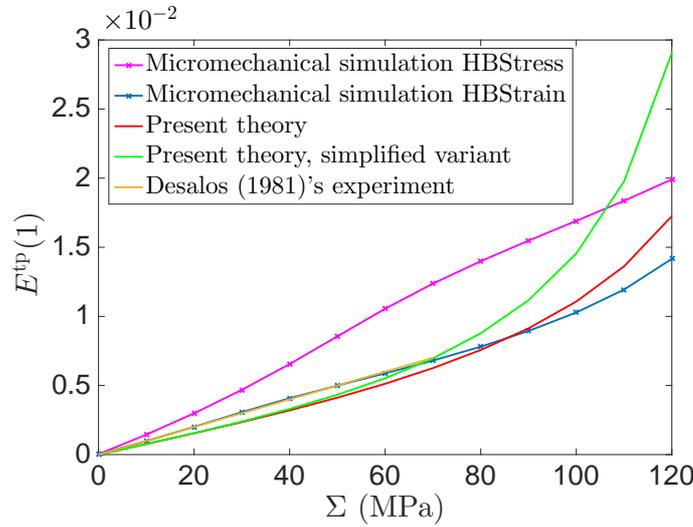
5.3 Comparison of final values of the transformation plastic strain

We now study the “amplitude” of transformation plasticity, characterized by the value of the transformation plastic strain after complete transformation, $E^{tp}(1)$. Figure 5 shows this quantity as a function of the overall stress applied Σ .

Subfigure 5(a) compares the predictions of Leblond *et al.* (1989)’s original formula (19); its variant (20); our general formula (57); its approximation (59); and Desalos (1981)’s experimental results for the A533 steel up to a stress of 70 MPa, which he found to be well reproduced by the heuristic formula $E^{tp}(1) \simeq 10^{-4} \Sigma$ (with Σ in MPa).



(a) Theories and experiment.



(b) Theory and micromechanical simulations.

Fig. 5. Comparison of transformation plastic strains after complete transformation.

- All theoretical curves provide acceptable representations of Desalos (1981)'s experimental results. In particular, the predictions of Leblond *et al.* (1989)'s original formula (19) and its variant (20) are of comparable quality.
- For large stresses, there is a large difference between model predictions obtained with our general formula (57) and its approximation (59). But Desalos (1981)'s experimental results limited to moderate stress levels do not permit to decide which is the better one.

Subfigure 5(b) compares the predictions of our general formula (57) to the results of micromechanical simulations obtained with HBStress and HBStrain conditions (Desalos (1981)'s experimental results being again provided for reference).

- The numerical results obtained with HBStress and HBStrain conditions are again markedly different. But the situation is opposite to that for the ratio $E^{tp}(f)/E^{tp}(1)$, see Subsection 5.2 above: the results corresponding to HBStrain conditions are here better than those corresponding to HBStress conditions, Desalos (1981)'s experimen-

tal results being taken as a reference. HBStress conditions clearly seem to lead to an overestimation of the amplitude of transformation plasticity.

- The predictions of our general formula (57) agree quite well with the numerical results obtained with HBStrain conditions;¹⁰ they reproduce in particular the tendency of numerical values of $E^{tp}(1)$ to increase nonlinearly with the stress applied.
- Numerical simulations permit to settle the question of the compared quality of our general formula (57) and its simplified variant (59): the former formula is definitely better, the latter leading to a clear overestimation of the transformation plastic strain after complete transformation.

5.4 Comparison with multiaxial experiments and micromechanical simulations

Another interesting aspect deserving to be checked versus experiments and micromechanical simulations is the prediction of our general formula (57) - in line with that of Leblond *et al.* (1989)'s earlier one (19) - of collinearity of the transformation plastic strain rate and the overall stress deviator. It will just be mentioned here that the said collinearity has been approximately verified experimentally by Coret *et al.* (2002, 2004) for the A533 steel, and is also approximately confirmed by extra finite element micromechanical simulations, analogous to those discussed above but with various multiaxial loadings instead of uniaxial ones.

6 Synthesis and perspectives

The aim of this paper was to propose a new theoretical treatment of Greenwood and Johnson (1965)'s mechanism of transformation plasticity of metals and alloys, based on limit-analysis. The new approach rests upon the same basic elements as Leblond *et al.* (1989)'s classical one, including in particular an approximate solution to the model problem of growth of a spherical core of daughter-phase within a concentric spherical volume of mother-phase subjected to a general 3D loading. But use of the powerful variational framework of the kinematic approach of limit-analysis makes it both more general and versatile.

In a first step, we have developed an approximate limit-analysis of a hollow sphere subjected to both external and internal loadings through classical conditions of homogeneous boundary stress (Mandel, 1964; Hill, 1967). The internal loading is intended to represent the effect, upon the external shell of mother-phase, of the expansion of the core of daughter-phase due to the volumetric part of the transformation strain. The treatment is based on an extension of a work of Monchiet *et al.* (2011) on plastic porous materials - containing traction-free voids - which itself stood as an extension of the classical

¹⁰ This is maybe not as surprising as it may seem. Indeed although the limit-analysis presented in Section 3 was, strictly speaking, developed for HBStress conditions, it is probably also appropriate, in some approximate sense, for HBStrain conditions insofar as the trial displacement fields used satisfy the latter conditions in an average sense, if not exactly.

work of Gurson (1977) supplementing this author’s elementary trial velocity fields with more complex, Eshelby (1957)-type ones. The output consists of some approximate overall yield locus and associated plastic flow rule obeying the normality property, extending those found by Monchiet *et al.* (2011) through incorporation of some additional internal loading.

In a second step, we have shown how to apply these results to the treatment of Greenwood and Johnson (1965)’s mechanism of transformation plasticity. An analytical formula for the transformation plastic strain rate has been found for fully general 3D loading conditions; this formula unfortunately contains an auxiliary unknown which must be determined by numerically solving a transcendental equation. However it reduces to simpler ones free of such complexities, in two distinct special cases: (i) small external stresses; (ii) beginning of the transformation (whatever the magnitude of the external stress tensor). All these theoretical results have been compared, with globally satisfying results, to experiments performed on the A533 steel (Desalos, 1981; Coret *et al.*, 2002, 2004), and also fresh finite element micromechanical simulations of the mechanism investigated, based on exactly the same spherical RVE as the theoretical analysis.

There are several advantages to the new approach. A minor one consists in dispensing with some *ad hoc* technical hypotheses made in Leblond *et al.* (1989)’s classical treatment, among which one of highly questionable validity. A more important one is that the value of the transformation plastic strain rate at the onset of the transformation is no longer infinite like in the previous treatment - a non-physical prediction which did not fail to raise numerous criticisms. A third one lies in the possibility to consider external stresses of arbitrary magnitude instead of small ones only; the new theory predicts a non-linear increase of the transformation plastic strain with the stress applied, in line with experimental observations (Greenwood and Johnson, 1965).

However the major advantage of the new approach probably lies in the new perspectives it opens, thanks to the power and efficiency of the kinematic method of limit-analysis.

- Refining the approximate overall yield criterion and plastic flow rule by including extra trial velocity fields, for instance, would not raise major difficulties.
- In this work the hypothesis was made, in line with Greenwood and Johnson (1965)’s ideas, that internal stresses generating microplasticity in the weaker mother-phase essentially arise from the difference of specific volume between the phases, that is the volumetric part of the transformation strain. However in reality this transformation strain is not a multiple of the unit tensor, and it is perfectly conceivable that its deviatoric part also contributes to the microplasticity of the mother-phase. This effect could be accounted for by modifying the expression of the overall strain rate imposed on the inner boundary of the domain of mother-phase, without making the limit-analysis significantly more complex.¹¹
- Another interesting extension would consist of considering growth of a spheroidal, instead of spherical, core of daughter-phase, using Monchiet *et al.* (2014)’s extension of

¹¹ Such an extension would however raise the difficult issue of accounting for the *self-accommodation* of the transforming regions, that is the partial or complete compensation of the deviatoric parts of the transformation strains from one region to another.

their earlier work on plastic materials containing spherical voids (Monchiet *et al.*, 2014) to such a pore geometry. The spheroidal geometry would permit to represent, in an approximate way, growing nuclei of daughter-phase of elongated shape; such nuclei have been observed in rolled plates and are known to result in a strong anisotropy of transformation plasticity (see e.g. Desalos (1981), among other authors) - for instance, the transformation plastic strain is no longer zero in the absence of external stress.

In addition, the notable gap found between the results of finite element micromechanical simulations performed with conditions of homogeneous boundary strain and stress, underlines the influence of boundary conditions for such a small and simplistic RVE as a spherical volume containing a single growing core of daughter-phase; and suggests to perform more realistic simulations on much larger RVEs containing many cores of daughter-phase, subjected to rigorous periodic boundary conditions. In addition to settling the issue of the effect of boundary conditions, these simulations would permit to study the respective influences of nucleation and growth of nuclei of daughter-phase, unlike those discussed here which automatically ruled out nucleation of multiple nuclei through consideration of a single one. Such simulations will be performed in the near future using Moulinec and Suquet (1998)'s FFT-based method of numerical homogenization, much better adapted to calculations of this type than the standard finite element method.

References

- Abrassart F. (1972). *Influence des transformations martensitiques sur les propriétés mécaniques des alliages du système Fe-Ni-Cr-C*. Ph.D. thesis, Université de Nancy I.
- Barbe F., Quey R., Taleb L. (2007). Numerical modelling of the plasticity induced during diffusive transformation. Case of a cubic array of nuclei. *Eur. J. Mech. A/Solids*, **26**, 611-625.
- Barbe F., Quey R., Taleb L., Souza de Cursi E. (2008). Numerical modelling of the plasticity induced during diffusive transformation. An ensemble averaging approach for the case of random arrays of nuclei. *Eur. J. Mech. A/Solids*, **27**, 1121-1139.
- Barbe F., Quey R. (2011). A numerical modelling of 3D polycrystal-to-polycrystal diffusive phase transformations involving crystal plasticity. *Int. J. Plasticity*, **27**, 823-840.
- Cherkaoui M., Berveiller M., Lemoine X. (2000). Couplings between plasticity and martensitic phase transformation : overall behavior of polycrystalline TRIP steels. *Int. J. Plasticity*, **16**, 1215-1241.
- Coret M., Calloch S., Combescure A. (2002). Experimental study of the phase transformation plasticity of 16MND5 low carbon steel under multiaxial loading. *Int. J. Plasticity*, **18**, 1707-1727.
- Coret M., Calloch S., Combescure A. (2004). Experimental study of the phase transformation plasticity of 16MND5 low carbon steel induced by proportional and nonproportional biaxial loading paths. *Eur. J. Mech. A/Solids*, **23**, 823-842.
- Desalos Y. (1981). Comportement dilatométrique et mécanique de l'austénite métastable d'un acier A533. IRSID Report 95349401.
- Diani J.M., Sabar H., Berveiller M. (1995). Micromechanical modelling of the transformation induced plasticity (TRIP) phenomenon in steels. *Int. J. Engng. Sci.*, **33**, 1921-1934.

- Eshelby J. (1957). The determination of the elastic field of an ellipsoidal inclusion, and related problems. *Proc. Roy. Soc. London A*, **241**, 376-396.
- Fischlschweiger M., Cailletaud G., Antretter T. (2012). A mean field model for transformation induced plasticity including backstress effects for non-proportional loadings. *Int. J. Plasticity*, **37**, 53-71.
- Fukumoto M., Yoshizaki M., Imataka H., Okamura K. (2001). Three-dimensional FEM analysis of helical gear subjected to the carburized quenching process. *J. Soc. Mater. Sci. Japan*, **50**, 598-605.
- Ganghoffer J.F., Denis S., Gautier E., Simon A. (1993). Finite element calculation of the micromechanics of a diffusional transformation. *Eur. J. Mech. A/Solids*, **12**, 21-32.
- Gigou D. (1985). *Plasticité de transformation : cas de la transformation martensitique*. Mémoire de DEA, Université Paris 11.
- Greenwood G.W., Johnson R.H. (1965). The deformation of metals under small stresses during phase transformations. *Proc. Roy. Soc. London A*, **283**, 403-422.
- Gurson A.L. (1977). Continuum theory of ductile rupture by void nucleation and growth: Part I - Yield criteria and flow rules for porous ductile media. *ASME J. Engng. Mater. Technol.*, **99**, 2-15.
- Hill R. (1951). On the state of stress in a plastic-rigid body at the yield point, *Phil. Mag.*, **42**, 868-875.
- Hill R. (1967). The essential structure of constitutive laws of metal composites and polycrystals. *J. Mech. Phys. Solids*, **15**, 79-95.
- Leblond J.B., Devaux J., Devaux J.C. (1989). Mathematical modelling of transformation plasticity in steels I: Case of ideal-plastic phases. *Int. J. Plasticity*, **5**, 551-572.
- Leblond J.B., Kondo D., Morin L., Remmal A. (2018). Classical and sequential limit-analysis revisited. *C.R. Mécanique*, **346**, 336-349.
- Leblond J.B., Mottet G., Devaux J.C. (1986a). A theoretical and numerical approach to the plastic behaviour of steels during phase transformations - I. Derivation of general relations. *J. Mech. Phys. Solids*, **34**, 395-409.
- Leblond J.B., Mottet G., Devaux J.C. (1986b). A theoretical and numerical approach to the plastic behaviour of steels during phase transformations - II. Study of classical plasticity for ideal-plastic phases. *J. Mech. Phys. Solids*, **34**, 411-432.
- Madou K., Leblond J.B. (2012). A Gurson-type criterion for porous ductile solids containing arbitrary ellipsoidal voids - I: Limit-analysis of some representative cell. *J. Mech. Phys. Solids*, **60**, 1020-1036.
- Magee C.L. (1966). *Transformation kinetics, microplasticity and ageing of martensite in Fe-31 Ni*. Ph.D. thesis, Carnegie-Mellon University, Pittsburgh.
- Mandel J. (1964). Contribution théorique à l'étude de l'écoulement et des lois de l'écoulement plastique. In : *Proc. 11th Int. Congr. on Applied Mechanics*, Munich, pp. 502-509.
- Miyao K., Wang Z.G., Inoue T. (1986). Analysis of temperature, stress and metallic structure in carburized-quenched gear considering transformation plasticity. *J. Soc. Mater. Sci. Japan*, **35**, 1352-1357.
- Monchiet V., Charkaluk E., Kondo D. (2011). A micromechanics-based modification of the Gurson criterion by using Eshelby-like velocity fields. *Eur. J. Mech. A/Solids*, **30**, 940-949.
- Monchiet V., Charkaluk E., Kondo D. (2014). Macroscopic yield criteria for ductile materials containing spheroidal voids: An Eshelby-like velocity fields approach. *Mech. Mater.*,

72, 1-18.

- Moulinec H., Suquet P. (1998). A numerical method for computing the overall response of nonlinear composites with complex microstructures. *Comput. Meth. Appl. Mech. Engng.*, **157**, 69-94.
- ESI-Group (2017). *SYSWELD User's Manual*.
- Otsuka T., Brenner R., Bacroix B. (2018). FFT-based modelling of transformation plasticity in polycrystalline materials during diffusive phase transformation. *Int. J. Engng. Sci.*, **127**, 92-113.
- Taleb T., Petit S. (2006). New investigations on transformation induced plasticity and its interaction with classical plasticity. *Int. J. Plasticity*, **22**, 110-130.
- Taleb T., Petit S., Jullien J.F. (2004). Prediction of residual stresses in the heat-affected zone. *J. Physique IV*, **120**, 705-712.
- Taleb L., Sidoroff F. (2003). A micromechanical modeling of the Greenwood-Johnson mechanism in transformation induced plasticity. *Int. J. Plasticity*, **19**, 1821-1842.
- Vincent Y., Bergheau J.M., Leblond J.B. (2003). Viscoplastic behavior of steels during phase transformations. *C.R. Mécanique*, **331**, 587-594.
- Weisz-Paltrault (2017). Multiphase model for transformation induced plasticity. Extended Leblond's model. *J. Mech. Phys. Solids*, **106**, 152-175.

A Appendix : additional considerations on the thermal strains of individual phases and the overall thermometallurgical strain

As a typical example, Fig. A.1 show the dilatometric diagram (free strain vs. temperature) of the A533 steel heated at 30°C s^{-1} and cooled at -2°C s^{-1} , determined experimentally by Desalos (1981). The diagram evidences both transformations from the α -phase (ferrite, bainite or martensite) to the γ -phase (austenite) during heating, and from the γ -phase to the α -phase during cooling; but our focus in the present paper is on the latter transformation - the effect of transformation plasticity during heating being erased by the subsequent vanishing of the stresses at high temperatures. Hence the mother- and daughter-phases are considered to be the γ - and α -phases respectively. Note that the observed strains are defined in reference to the daughter (α) phase at a temperature of 20°C . The thermal expansions of the two phases, with different coefficients α_M and α_D , are conspicuous here; and the temperature-dependent relative difference of specific volume $\frac{\Delta V}{V}(T)$ between them, given by equation (4), is apparent in the vertical gap between the two straight lines corresponding to the pure phases.

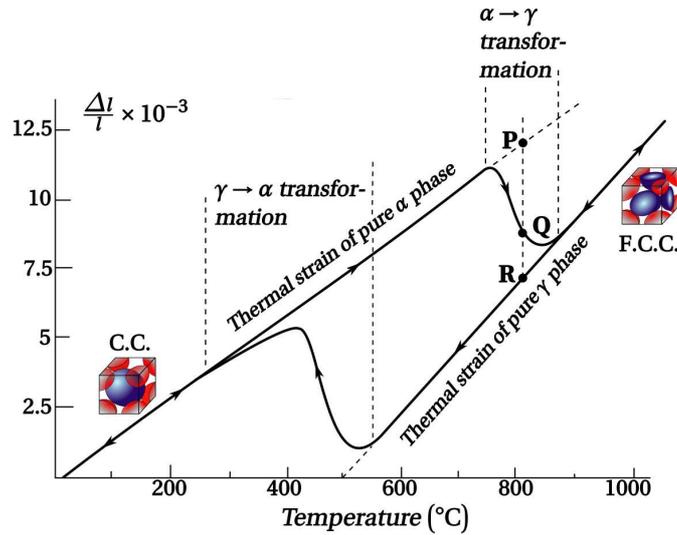


Fig. A.1. The dilatometric diagram of the A533 steel (after Desalos (1981)).

During the transformations the two phases coexist so that the overall strain observed lies between the thermal strains of individual phases at the temperature considered. Now by definition the dilatometric diagram is obtained in the absence of any external load so that the overall elastic and plastic strains \mathbf{E}^e , \mathbf{E}^p are zero; under such conditions the overall strain coincides with the thermo-metallurgical strain \mathbf{E}^{thm} defined by equation (7)₂. The geometrical interpretation of the simple “linear mixture rule” giving this strain in terms of the thermal strains of individual phases is that the volume fraction f of the daughter-phase may be obtained from the dilatometric diagram by a simple “lever rule”, $f \equiv \frac{QR}{PR}$ at a temperature of 800°C in Fig. A.1.

A final remark is that as a consequence of equation (7)₂, the rate $\dot{\mathbf{E}}^{\text{thm}}$ of the thermometallurgical strain naturally splits into a “thermal part” proportional to \dot{T} and a

“transformation part” proportional to \dot{f} :

$$\dot{\mathbf{E}}^{\text{thm}} = [(1 - f)\alpha_M \mathbf{1} + f\alpha_D \mathbf{1}] \dot{T} + [\epsilon_D^{\text{th}}(T) - \epsilon_M^{\text{th}}(T)] \dot{f}.$$

But it must be stressed that the thermometallurgical strain \mathbf{E}^{thm} itself cannot be decomposed in a similar way, as some authors have wrongly claimed; that is, it is not a simple sum of a function of T and a function of f , as is obvious from equation (7)₂.

B Appendix : on the relative importance of Greenwood and Johnson (1965)’s and Magee (1966)’s mechanisms and their experimental separation

In some cases Magee (1966)’s mechanism is important and may even completely dominate over that of Greenwood and Johnson (1965). The most obvious case is that of shape memory alloys, in which the difference of specific volume between the phases is very small so that Greenwood and Johnson (1965)’s mechanism is virtually absent. As a result, the thermomechanical behaviour of such materials during phase transformations is governed by the pseudo-elastic transformation strain; the origin of the “memory effects” observed lies in the reversible nature of this strain.

Another example is that of “TRIP steels”. Although the acronym TRIP stands for “TRansformation Induced Plasticity”, the situation it refers to is rather different from that considered here. TRIP steels are characterized by the presence of a significant amount of untransformed austenite at room temperature. When such steels are subjected to an increasing stress at this temperature, this “retained” austenite is transformed into α -phase with a significant accompanying deformation, and this generates an important increase of ductility. Since the stress applied has a decisive impact upon the *kinetics of the transformation*,¹² it is highly probable that it also has a large influence upon the *orientation of the transformation strain* (Magee (1966)’s mechanism). This large impact of the stress applied upon the transformation marks a significant difference with the case considered here of transformations essentially governed by the decrease of the temperature, rather than by application of a stress.

In the case considered here of transformations occurring during the cooling period of thermomechanical processes, the situation is less clear. The classical metallurgical view is that Greenwood and Johnson (1965)’s mechanism should dominate for diffusion-controlled transformations, and that of Magee (1966) for martensitic ones. To go beyond these qualitative ideas, one may try to experimentally distinguish between the two mechanisms by using the different physical natures - plastic for Greenwood and Johnson (1965)’s mechanism, pseudo-elastic for that of Magee (1966) - of the microscopic strains assumed to be responsible for transformation plasticity. One possibility is to subject a stressed specimen to repeated thermal cycles. The transformation plastic strain must be perfectly cumulative with the number of cycles if it arises from Greenwood and Johnson (1965)’s mechanism, because nothing prevents the accumulation of standard plastic strains; but

¹² This is a consequence of the highly unstable character of retained austenite at room temperature: a small increase of “driving force” is sufficient to trigger the transformation.

it must “saturate” at some stage if it is due to Magee (1966)’s mechanism, because a maximum value of the overall strain is obtained when the transformation strains of the various transforming regions are all collinear. The experiment has been performed several times on different metals and alloys (Greenwood and Johnson, 1965; Desalos, 1981; Taleb and Petit, 2006) with the conclusion, for the materials considered, of perfect cumulation of the transformation plastic strain with the number of cycles, and therefore of domination of Greenwood and Johnson (1965)’s mechanism.

Another possibility is to exploit memory effects expected to arise, for Magee (1966)’s mechanism, from the reversibility of the transformation strain. This can be done by applying some stress on the material during cooling, then eliminating this stress and reheating. If Greenwood and Johnson (1965)’s mechanism prevails, the transformation plastic strain obtained at the end of cooling must not be recovered during re-heating since standard plasticity is irreversible; but if Magee (1966)’s mechanism dominates, this strain must be at least partially recoverable. The experiment was performed at least once, by Gigou (1985), with the result that the recoverable part of the transformation plastic strain did not amount to more than 10% of this strain; the ensuing conclusion was again that Greenwood and Johnson (1965)’s mechanism was dominant in the alloy considered.

The general conclusion is that for the materials and types of thermomechanical histories considered in the present paper, although Magee (1966)’s mechanism cannot be ruled out *a priori* without a dedicated experimental study, Greenwood and Johnson (1965)’s mechanism often dominates.