Modulo p representations of reductive p-adic groups: functorial properties

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MODULO p REPRESENTATIONS OF REDUCTIVE p-ADIC GROUPS: FUNCTORIAL PROPERTIES

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ABSTRACT. Let F be a local field with residue characteristic p, let C be an algebraically closed field of characteristic p, and let $\mathbf{G}$ be a connected reductive F-group. In a previous paper, Florian Herzig and the authors classified irreducible admissible C-representations of $G = \mathbf{G}(F)$ in terms of supercuspidal representations of Levi subgroups of G. Here, for a parabolic subgroup P of G with Levi subgroup M and an irreducible admissible C-representation τ of M, we determine the lattice of subrepresentations of $\mathrm{Ind}_P^G \tau$ and we show that $\mathrm{Ind}_P^G \chi\tau$ is irreducible for a general unramified character χ of M. In the reverse direction, we compute the image by the two adjoints of Ind_P^G of an irreducible admissible representation π of G. On the way, we prove that the right adjoint of Ind_P^G respects admissibility, hence coincides with Emerton's ordinary part functor $\mathrm{Ord}_{\overline{P}}^G$ on admissible representations.

CONTENTS

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1. Introduction

1.1. **Classification results of** [AHHV17]**.** The present paper is a sequel to [AHHV17]. The overall setting is the same: p is a prime number, F a local field with finite residue field of characteristic p, $\mathbf{G}$ a connected reductive F-group and $G = \mathbf{G}(F)$ is seen as a topological locally pro-p group. We fix an algebraically closed field C of characteristic p and we study the smooth representations of G over C-vector spaces - we write $\mathrm{Mod}_C^\infty(G)$ for the category they form.

Let P be a parabolic subgroup of G with a Levi decomposition $P = MN$ and σ a supercuspidal C-representation of M, in the sense that it is irreducible, admissible, and does not appear as a subquotient of a representation of M obtained by parabolic induction from an irreducible, admissible C-representation of a proper Levi sugroup of M. Then there is a maximal parabolic subgroup $P(\sigma)$ of G containing P to which σ inflated to P extends; we write $e(\sigma)$ for that extension. For each parabolic subgroup Q of G with $P \subset Q \subset P(\sigma)$, we form

$$I_G(P,\sigma,Q) = \mathrm{Ind}_{P(\sigma)}^G(e(\sigma) \otimes \mathrm{St}_Q^{P(\sigma)})$$

where $\mathrm{St}_Q^{P(\sigma)} = \mathrm{Ind}_Q^{P(\sigma)} \mathbf{1}/\sum \mathrm{Ind}_{Q'}^{P(\sigma)} \mathbf{1}$, the sum being over parabolic subgroups Q' of G with $Q \subsetneq Q' \subset P(\sigma)$.

The classification result of [AHHV17] is that $I_G(P,\sigma,Q)$ is irreducible admissible, and that conversely any irreducible admissible C-representation of G has the form $I_G(P,\sigma,Q)$, where P is determined up to conjugation, and, once P is fixed, Q is determined and so is the isomorphism class of σ.

1.2. **Main results.** The classification raises natural questions: if G is a Levi subgroup of a parabolic subgroup R in a larger connected reductive group H, what is the structure of $\mathrm{Ind}_R^H \pi$ when π is a irreducible admissible C-representation of G?

We show that $\mathrm{Ind}_R^H \pi$ has finite length and multiplicity 1; we determine its irreducible constituents and the lattice of its subrepresentations: see section 3 for precise results and proofs. As an application, we answer a question of Jean-Francois Dat, in showing that $\mathrm{Ind}_R^H \chi\pi$ is irreducible when χ is a general unramified character of G.

If P_1 is a parabolic subgroup of G with Levi decomposition $P_1 = M_1N_1$, then $\mathrm{Ind}_{P_1}^G : \mathrm{Mod}_C^\infty(M_1) \to \mathrm{Mod}_C^\infty(G)$ has a left adjoint $L_{P_1}^G$, which is the usual Jacquet functor $(-)_{N_1}$ taking N_1-coinvariants, and also a right adjoint functor $R_{P_1}^G$ [Vig13]. It is natural to apply

$L^G_{P_1}$ and $R^G_{P_1}$ to π. They turn out to be irreducible or 0, in sharp contrast to the case of complex representations of G. To state precise results, we fix a minimal parabolic subgroup B of G and a Levi decomposition $B = ZU$ of B, and we consider only parabolic subgroups containing B and their Levi components containing Z. We simply say "let $P = MN$ be a standard parabolic subgroup of G" to mean that P contains B and M is the Levi component of P containing Z, N being the unipotent radical of P.

Theorem 1.1. *Let $P = MN$ and $P_1 = M_1N_1$ be standard parabolic subgroups of G, let σ be a supercuspidal C-representation of M and let Q be a parabolic subgroup of G with $P \subset Q \subset P(\sigma)$.*

(i) *$L^G_{P_1} I_G(P, \sigma, Q)$ is isomorphic to $I_{M_1}(P \cap M_1, \sigma, Q \cap M_1)$ if $P_1 \supset P$ and the group generated by $P_1 \cup Q$ contains $P(\sigma)$, and is 0 otherwise.*

(ii) *$R^G_{P_1} I_G(P, \sigma, Q)$ is isomorphic to $I_{M_1}(P \cap M_1, \sigma, Q \cap M_1)$ if $P_1 \supset Q$, and is 0 otherwise.*

See §6 and §7 for the proofs, with consequences already drawn in §6.1: in particular, we prove that an irreducible admissible C-representation π of G is supercuspidal exactly when $L^G_P \pi$ and $R^G_P \pi$ are 0 for any proper parabolic subgroup P of G.

As the construction of $I_G(P, \sigma, Q)$ involves parabolic induction, we are naturally led to investigate, as an intermediate step, the composite functors $L^G_{P_1} \operatorname{Ind}^G_P$ and $R^G_{P_1} \operatorname{Ind}^G_P$, for standard parabolic subgroups $P = MN$ and $P_1 = M_1N_1$ of G. In §5, we prove:

Theorem 1.2. *The functor $L^G_{P_1} \operatorname{Ind}^G_P : \operatorname{Mod}^\infty_C(M) \to \operatorname{Mod}^\infty_C(M_1)$ is isomorphic to the functor $\operatorname{Ind}^{M_1}_{P \cap M_1} L^M_{P_1 \cap M}$, and the functor $R^G_{P_1} \operatorname{Ind}^G_P : \operatorname{Mod}^\infty_C(M) \to \operatorname{Mod}^\infty_C(M_1)$ is isomorphic to the functor $\operatorname{Ind}^{M_1}_{P \cap M_1} R^M_{P_1 \cap M}$.*

We actually describe explicitly the functorial isomorphism for $L^G_{P_1} \operatorname{Ind}^G_P$ whereas the case of $R^G_{P_1} \operatorname{Ind}^G_P$ is obtained by adjunction properties. The fact that $R^G_{P_1}$ has no direct explicit description has consequence for the proof of Theorem 1.1 (ii). We first prove:

Theorem 1.3. *If π is an admissible C-representation of G, then $R^G_P \pi$ is an admissible C-representation of M.*

It follows that on admissible C-representations of G, R^G_P coincides with Emerton's ordinary part functor $\operatorname{Ord}^G_{\overline{P}}$ (as extended to the case of C-representations in [Vig13]). To prove Theorem 1.1 (ii) we in fact use $\operatorname{Ord}^G_{\overline{P}_1}$ in place of $R^G_{P_1}$. Note that, if the characteristic of F is 0 and π is an admissible C-representation of G, then $L^G_P \pi$ is admissible. But in contrast, when F has characteristic p, we produce in §4 an example, for $G = \mathrm{SL}(2, F)$, of an admissible C-representation π of G such that $L^G_B \pi$ is not admissible.

1.3. **Outline of the proof.** After the initial section §2 devoted to notation and preliminaries, our paper mainly follows the layout above. However admissibility questions are explored in §4, where Theorem 1.3 is established: as mentioned above, the result is used in the proof Theorem 1.1 (ii).

Without striving for the utmost generality, we have taken care not to use unnecessary assumptions. In particular, from section §4 on, we consider a general commutative ring R as coefficient ring, imposing conditions on R only when useful. The reason is that for arithmetic applications it is important to consider the case where R is artinian and p is nilpotent or invertible in R. Only when we use the classification do we assume $R = C$. Our results are valid for R noetherian and p nilpotent in R in sections §4 to §7. For example, when R is noetherian and p is nilpotent in R, Theorem 1.2 is valid (Theorem 5.5 and Corollary 5.6) and a version to Theorem 1.1 is obtained in Theorem 6.1 and Corollary 6.2. Likewise Theorem 1.3 is valid when R is noetherian and p is nilpotent in R (Theorem 4.11).

In a companion paper [AHV], the authors will investigate the effect of taking invariants under a pro-p Iwahori subgroup in the modules $I_G(P,\sigma,Q)$ of 1.1.

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2. Notation, useful facts and preliminaries

2.1. **The group G and its standard parabolic subgroups $P = MN$.** In all that follows, p is a prime number, F is a local field with finite residue field k of characteristic p; as usual, we write O_F for the ring of integers of F, P_F for its maximal ideal and val_F the absolute value of F normalised by $\mathrm{val}_F(F^*) = \mathbb{Z}$. We denote an algebraic group over F by a bold letter, like $\mathbf{H}$, and use the same ordinary letter for the group of F-points, $H = \mathbf{H}(F)$. We fix a connected reductive F-group $\mathbf{G}$. We fix a maximal F-split subtorus $\mathbf{T}$ and write $\mathbf{Z}$ for its $\mathbf{G}$-centralizer; we also fix a minimal parabolic subgroup $\mathbf{B}$ of $\mathbf{G}$ with Levi component $\mathbf{Z}$, so that $\mathbf{B} = \mathbf{ZU}$ where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$. Let $X^*(\mathbf{T})$ be the group of F-rational characters of $\mathbf{T}$ and Φ the subset of roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{G}$. Then $\mathbf{B}$ determines a subset Φ^+ of positive roots - the roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{U}$- and a subset of simple roots Δ. The $\mathbf{G}$-normalizer $\mathbf{N_G}$ of $\mathbf{T}$ acts on $X^*(\mathbf{T})$ and through that action, $\mathbf{N_G}/\mathbf{Z}$ identifies with the Weyl group of the root system Φ. Set $\mathcal{N} := \mathbf{N_G}(F)$ and note that $\mathbf{N_G}/\mathbf{Z} \simeq \mathcal{N}/Z$; we write $\mathbb{W}$ for $\mathcal{N}/Z$.

A standard parabolic subgroup of $\mathbf{G}$ is a parabolic F-subgroup containing $\mathbf{B}$. Such a parabolic subgroup $\mathbf{P}$ has a unique Levi subgroup $\mathbf{M}$ containing $\mathbf{Z}$, so that $\mathbf{P} = \mathbf{MN}$ where $\mathbf{N}$ is the unipotent radical of $\mathbf{P}$ - we also call $\mathbf{M}$ standard. By a common abuse of language to describe the preceding situation, we simply say "let $P = MN$ be a standard parabolic subgroup of G"; we sometimes write N_P for N and M_P for M. The parabolic subgroup of G opposite to P will be written $\overline{P}$ and its unipotent radical $\overline{N}$, so that $\overline{P} = M\overline{N}$, but beware that $\overline{P}$ is not standard ! We write $\mathbb{W}_M$ for the Weyl group $M \cap \mathcal{N}/Z$.

If $\mathbf{P} = \mathbf{MN}$ is a standard parabolic subgroup of $\mathbf{G}$, then $\mathbf{M} \cap \mathbf{B}$ is a minimal parabolic subgroup of $\mathbf{M}$. If Φ_M denotes the set of roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{M}$, with respect to $\mathbf{M} \cap \mathbf{B}$ we have $\Phi_M^+ = \Phi_M \cap \Phi^+$ and $\Delta_M = \Phi_M \cap \Delta$. We also write Δ_P for Δ_M as P and M determine each other, $P = MU$. Thus we obtain a bijection $P \mapsto \Delta_P$ from standard parabolic subgroups of G to subsets of Δ, with B corresponds to $\emptyset$ and G to Δ. If I is a subset of Δ, we sometimes denote by $P_I = M_I N_I$ the corresponding standard parabolic subgroup of G. If $I = \{\alpha\}$ is a singleton, we write $P_\alpha = M_\alpha N_\alpha$. We note a few useful properties. If P_1 is another standard parabolic subgroup of G, then $P \subset P_1$ if and only if $\Delta_P \subset \Delta_{P_1}$; we have $\Delta_{P \cap P_1} = \Delta_P \cap \Delta_{P_1}$ and the parabolic subgroup corresponding to $\Delta_P \cup \Delta_{P_1}$ is the subgroup $\langle P, P_1 \rangle$ of G generated by P and P_1. The standard parabolic subgroup of M associated to $\Delta_M \cap \Delta_{M_1}$ is $M \cap P_1 = (M \cap M_1)(M \cap N_1)$ [Car85, Proposition 2.8.9]. It is convenient to write G' for the subgroup of G generated by the unipotent radicals of the parabolic subgroups; it is also the normal subgroup of G generated by U, and we have $G = ZG'$.

For each $\alpha \in X^*(\mathbf{T})$, the homomorphism $x \mapsto \mathrm{val}_F(\alpha(x)) : T \to \mathbb{Z}$ extends uniquely to a homomorphism $Z \to \mathbb{Q}$ that we denote in the same way. This defines a homomorphism $Z \xrightarrow{v} X_*(T) \otimes \mathbb{Q}$ such that $\alpha(v(z)) = \mathrm{val}_F(\alpha(z))$ for $z \in Z, \alpha \in X^*(\mathbf{T})$.

An interesting situation occurs when $\Delta = I \sqcup J$ is the union of two orthogonal subsets I and J. In that case, $G' = M'_I M'_J$, M'_I and M'_J commute with each other, and their intersection is finite and central in G [AHHV17, II.7 Remark 5].

2.2. **Representations of G.** As apparent in the abstract and the introduction, our main interest lies in smooth C-representations of G, where C is an **algebraically closed field of characteristic** p, which we fix throughout. However many of our arguments do not necessitate so strong a hypothesis on coefficients, so we let R be a **fixed commutative ring**.

Occasionally we shall consider an $R[A]$-module V where A is a monoid. An element v of V is called **A-finite** if its translates under A generate a finitely generated submodule of V. If R is noetherian the A-finite elements in V generate a submodule of V, that we write V^{A-f}. When A is generated by an element t, we write V^{t-f} instead of V^{A-f}.

We speak indifferently of $R[H]$-modules and of R-representations of H for a locally profinite group H. An $R[H]$-module V is called **smooth** if every vector in V has an open stabilizer in H. The smooth R-representations of H and $R[H]$-linear maps form an abelian category $\mathrm{Mod}_R^\infty(H)$.

An R-representation V of a locally profinite group H is **admissible** if it is smooth and for any open compact subgroup J of H, the R-submodule V^J of J-fixed vectors is finitely generated. When R is noetherian, it is clear that it suffices to check this when J is small enough. When R is noetherian we write $\mathrm{Mod}_R^a(H)$ for the subcategory of $\mathrm{Mod}_R^\infty(H)$ made out of the admissible R-representations of H. We explore admissibility further in section 4.

If $P = MN$ is a standard parabolic subgroup of G, the parabolic induction functor $\mathrm{Ind}_P^G : \mathrm{Mod}_R^\infty(M) \to \mathrm{Mod}_R^\infty(G)$ sends $W \in \mathrm{Mod}_R^\infty(M)$ to the smooth $R[G]$-module $\mathrm{Ind}_P^G W$ made out of functions $f : G \to W$ satisfying $f(mngk) = mf(g)$ for $m \in M, n \in N, g \in G$ and k in some open subgroup K_f of G - the action of G is via right translation. The functor Ind_P^G has a left adjoint $L_P^G : \mathrm{Mod}_R^\infty(G) \to \mathrm{Mod}_R^\infty(M)$ which sends V in $\mathrm{Mod}_R^\infty(G)$ to the module of N-coinvariants V_N of V, which is naturally a smooth $R[M]$-module. The functor Ind_P^G has a right adjoint $R_P^G : \mathrm{Mod}_R^\infty(G) \to \mathrm{Mod}_R^\infty(M)$ [Vig13, Proposition 4.2].

When R is a **field**, a smooth R-representation of G is called **irreducible** if it is a simple $R[G]$-module. An R-representation of G is called **supercuspidal** it is irreducible, admissible, and does not appear as a subquotient of a representation of M obtained by parabolic induction from an irreducible, admissible representation of a proper Levi subgroup of M.

2.3. **On compact induction.** If X is a locally profinite space with a countable basis of open sets, and V is an R-module, we write $C_c^\infty(X, V)$ for the space of compactly supported locally constant functions $X \to V$. One verifies that the natural map $C_c^\infty(X, R) \otimes_R V \to C_c^\infty(X, V)$ is an isomorphism.

Lemma 2.1. *The R-module $C_c^\infty(X, R)$ is free. When X is compact, the submodule of constant functions is a direct factor of $C_c^\infty(X, R)$.*

Proof. The proof of [Ly15, Appendix A.1] when X is compact is easily adapted to $C_c^\infty(X, V)$ when X is not compact. □

Example 2.2. $C_c^\infty(X, R)^H$ is a direct factor of $C_c^\infty(X, R)$ when X is compact with a continuous action of a profinite group H with finitely many orbits (apply the lemma to the orbits which are open).

Let H be a locally profinite group and J a closed subgroup of H.

Lemma 2.3. *The quotient map $H \to J\backslash H$ has a continuous section.*

Proof. When H is profinite, this is [RZ10, Proposition 2.2.2]. In general, let K be a compact open subgroup of H. Cover H with disjoint double cosets JgK. It is enough to find, for any given g, a continuous section of the induced map $JgK \xrightarrow{\pi_g} J\backslash JgK$. The map $k \mapsto gk$ induces a continous bijective map $(K \cap g^{-1}Jg)\backslash K \xrightarrow{p} J\backslash JgK$. Because J is closed in H, both spaces are Hausdorff and $(K \cap g^{-1}Jg)\backslash K$ is compact since K is, so p is a homeomorphism. If σ is a continuous section of the quotient map $K \to (K \cap g^{-1}Jg)\backslash K$ then $x \mapsto g\sigma(p^{-1}(x))$ gives the desired section of π_g. □

Let σ be a continuous section of $H \to J\backslash H$, and let V be a smooth R-representation of J. Recall that $\text{c-Ind}_J^H V$ is the space of functions $f : H \to V$, left equivariant by J, of compact support in $J\backslash H$, and smooth for H acting by right translation. Immediately:

Lemma 2.4. *The map $f \mapsto f \circ \sigma : \text{c-Ind}_J^H V \to C_c^\infty(J\backslash H, V)$ is an R-module isomorphism.*

As a consequence we get a useful induction/restriction property: let W be a smooth R-representation of H.

Lemma 2.5. *The map $f \otimes w \mapsto (h \mapsto f(h) \otimes hw) : (\text{c-Ind}_J^H V) \otimes W \to \text{c-Ind}_J^H(V \otimes W)$ is an $R[H]$-isomorphism.*

Proof. The map is linear and H-equivariant. Lemma 2.4 implies that it is bijective. □

Remark 2.6. Arens' theorem says that if X is a homogeneous space for H and H/K is countable for a compact open subgroup K of H, then for $x \in X$ the orbit map $h \mapsto hx$ induces a homeomorphism $H/H_x \simeq X$. In particular, for two closed subgroups I, J of H such that $H = IJ$, we get a homeomorphism $I/(I \cap J) \simeq H/J$. Hence $(\text{c-Ind}_J^H V)|_I \simeq \text{c-Ind}_{I\cap J}^I V$ for any smooth R-representation V of J.

2.4. $I_G(P, \sigma, Q)$ **and minimality.** We recall from [AHHV17] the construction of $I_G(P, \sigma, Q)$, our main object of study.

Proposition 2.7. *Let $P = MN \subset Q$ be two standard parabolic subgroups of G and σ an R-representation of M. Then the following are equivalent:*

(i) *σ extends to a representation of Q where N acts trivially.*
(ii) *For each $\alpha \in \Delta_Q \setminus \Delta_P$, $Z \cap M'_\alpha$ acts trivially on σ.*

That comes from [AHHV17, II.7 Proposition] when $R = C$, but the result is valid for any commutative ring R [AHHV17, II.7 first remark 2]. Besides, the extension of σ to Q, when the conditions are fulfilled, is unique; we write it $e_Q(\sigma)$; it is trivial on N_Q and we view it equally as a representation of M_Q. The R-representation $e_Q(\sigma)$ of Q or M_Q is smooth, or admissible, or irreducible (when R is a field) if and only if σ is. Let $P_\sigma = M_\sigma N_\sigma$ be the standard parabolic subgroup of G with $\Delta_{P_\sigma} = \Delta_\sigma$ where

$$\Delta_\sigma = \{\alpha \in \Delta \setminus \Delta_P \mid Z \cap M'_\alpha \text{ acts trivially on } \sigma\}. \tag{1}$$

There is a largest parabolic subgroup $P(\sigma)$ containing P to which σ extends: $\Delta_{P(\sigma)} = \Delta_P \cup \Delta_\sigma$. Clearly when $P \subset Q \subset P(\sigma)$, the restriction to Q of $e_{P(\sigma)}(\sigma)$ is $e_Q(\sigma)$. If there is no risk of ambiguity, we write

$$e(\sigma) = e_{P(\sigma)}(\sigma).$$

Definition 2.8. An $R[G]$**-triple** is a triple (P,σ,Q) made out of a standard parabolic subgroup $P=MN$ of G, a smooth R-representation of M, and a parabolic subgroup Q of G with $P\subset Q\subset P(\sigma)$. To an $R[G]$-triple (P,σ,Q) is associated a smooth R-representation of G:

$$I_G(P,\sigma,Q)=\operatorname{Ind}_{P(\sigma)}^G(e(\sigma)\otimes \operatorname{St}_Q^{P(\sigma)})$$

where $\operatorname{St}_Q^{P(\sigma)}$ is the quotient of $\operatorname{Ind}_Q^{P(\sigma)}\mathbf{1}$, $\mathbf{1}$ denoting the trivial R-representation of Q, by the sum of its subrepresentations $\operatorname{Ind}_{Q'}^{P(\sigma)}\mathbf{1}$, the sum being over the set of parabolic subgroups Q' of G with $Q\subsetneq Q'\subset P(\sigma)$.

Note that $I_G(P,\sigma,Q)$ is naturally isomorphic to the quotient of $\operatorname{Ind}_Q^G(e_Q(\sigma))$ by the sum of its subrepresentations $\operatorname{Ind}_{Q'}^G(e_{Q'}(\sigma))$ for $Q\subsetneq Q'\subset P(\sigma)$ by Lemma 2.5.

We also remark that we have the identifications $\operatorname{Ind}_P^Q\sigma\simeq\operatorname{Ind}_{P/N_Q}^{Q/N_Q}\sigma$ and $\operatorname{St}_P^Q\simeq\operatorname{St}_{P/N_Q}^{Q/N_Q}$ where $P\subset Q$ are parabolic subgroups, N_Q the unipotent radical of Q and σ an representation of P with the trivial action of N_P (hence a representation of the Levi quotient of P). The subgroup P/N_Q of Q/N_Q is a parabolic subgroup.

It might happen that σ itself has the form $e_P(\sigma_1)$ for some standard parabolic subgroup $P_1=M_1N_1$ contained in P and some R-representation σ_1 of M_1. In that case, $P(\sigma_1)=P(\sigma)$ and $e(\sigma)=e(\sigma_1)$. We say that σ is e**-minimal** if $\sigma=e_P(\sigma_1)$ implies $P_1=P,\sigma_1=\sigma$.

Lemma 2.9. *Let $P=MN$ be a standard parabolic subgroup of G and let σ be an R-representation of M. There exists a unique standard parabolic subgroup $P_{\min,\sigma}=M_{\min,\sigma}N_{\min,\sigma}$ of G and a unique e-minimal representation of $\sigma_{\min}$ of $M_{\min,\sigma}$ with $\sigma=e_P(\sigma_{\min})$. Moreover $P(\sigma)=P(\sigma_{\min})$ and $e(\sigma)=e(\sigma_{\min})$.*

Proof. We have

$$\Delta_{P_{\min,\sigma}}=\{\alpha\in\Delta_P\mid\ Z\cap M'_\alpha \text{ does not act trivially on }\sigma\},\tag{2}$$

$\sigma_{\min}$ is the restriction of σ to $M_{\min,\sigma}$, and

$$\Delta_{\sigma_{\min}}=\{\alpha\in\Delta\mid\ Z\cap M'_\alpha\ \text{ acts trivially on }\ \sigma\}.\tag{3}$$

□

Lemma 2.10. *Let $P=MN$ be a standard parabolic subgroup of G and σ an e-minimal R-representation of M. Then Δ_P and Δ_σ are orthogonal.*

That comes from [AHHV17, II.7 Corollary 2]. That corollary of loc. cit. also shows that when R is a field and σ is supercuspidal, then σ is e-minimal. Lemma 2.10 shows that $\Delta_{P_{\min,\sigma}}$ and $\Delta_{\sigma_{\min}}$ are orthogonal.

Note that when Δ_P and Δ_σ are orthogonal of union $\Delta=\Delta_P\sqcup\Delta_\sigma$, then $G=P(\sigma)=MM'_\sigma$ and $e(\sigma)$ is the R-representation of G simply obtained by extending σ trivially on M'_σ.

Lemma 2.11. *Let (P,σ,Q) be an $R[G]$-triple. Then $(P_{\min,\sigma},\sigma_{\min},Q)$ is an $R[G]$-triple and $I_G(P,\sigma,Q)=I_G(P_{\min,\sigma},\sigma_{\min},Q)$.*

Proof. We already saw that $P(\sigma)=P(\sigma_{\min})$ and $e(\sigma)=e(\sigma_{\min})$. □

2.5. **Hecke algebras.** We fix a special parahoric subgroup $\mathcal{K}$ of G fixing a special vertex x_0 in the apartment $\mathcal{A}$ associated to T in the Bruhat-Tits building of the adjoint group of G. If V is an irreducible smooth C-representation of $\mathcal{K}$, we have the compactly induced representation $\text{c-Ind}_{\mathcal{K}}^G V$ of G, its endomorphism algebra $\mathcal{H}_G(\mathcal{K}, V)$ and the centre $\mathcal{Z}_G(\mathcal{K}, V)$ of $\mathcal{H}_G(\mathcal{K}, V)$. For a standard parabolic subgroup $P = MN$ of G, the group $M \cap \mathcal{K}$ is a special parahoric subgroup of M and $V_{N\cap\mathcal{K}}$ is an irreducible smooth C-representation of $M \cap \mathcal{K}$. For $W \in \operatorname{Mod}_C^\infty(M)$, there is an injective algebra homomorphism

$$\mathcal{S}_P^G : \mathcal{H}_G(\mathcal{K}, V) \to \mathcal{H}_M(M \cap \mathcal{K}, V_{N\cap\mathcal{K}})$$

for which the natural isomorphism $\operatorname{Hom}_G(\text{c-Ind}_{\mathcal{K}}^G V, \operatorname{Ind}_P^G W) \simeq \operatorname{Hom}_M(\text{c-Ind}_{M\cap\mathcal{K}}^M V_{N\cap\mathcal{K}}, W)$ is $\mathcal{S}_P^G$-equivariant [HV15], [HV12]. Moreover. $\mathcal{S}_P^G(\mathcal{Z}_G(\mathcal{K}, V)) \subset \mathcal{Z}_M(M \cap \mathcal{K}, V_{N\cap\mathcal{K}})$.

Let $Z(M)$ denote the maximal split central subtorus of M; it is equal to the group of F-points of the connected component in $\mathbf{T}$ of $\bigcap_{\alpha\in\Delta_M} \operatorname{Ker}\alpha$. Let $z \in Z(M)$. We say that z strictly contracts an open compact subgroup N_0 of N if the sequence $(z^k N_0 z^{-k})_{k\in\mathbb{N}}$ is strictly decreasing of intersection $\{1\}$. We say that z strictly contracts N if there exists an open compact subgroup $N_0 \subset N$ such that z strictly contracts N_0. Choose $z \in Z(M)$ which strictly contracts N. Let $\tau \in \mathcal{Z}_M(M \cap \mathcal{K}, V_{N\cap\mathcal{K}})$ be a non-zero element which supports on $(M\cap\mathcal{K})z(M\cap\mathcal{K})$. (Such an element is unique up to constant multiplication.) Then $\tau \in \operatorname{Im}\mathcal{S}_P^G$ and the algebra $\mathcal{H}_M(\mathcal{K} \cap M, V_{N\cap\mathcal{K}})$ (resp. $\mathcal{Z}_M(M \cap \mathcal{K}, V_{N\cap\mathcal{K}})$) is the localization of $\mathcal{H}_G(\mathcal{K}, V)$ (resp. $\mathcal{Z}_G(\mathcal{K}, V)$) at τ.

3. Lattice of subrepresentations of $\operatorname{Ind}_P^G \sigma$, σ irreducible admissible

3.1. **Result.** This section is a direct complement to [AHHV17]. Our coefficient ring is $R = C$. We are given a standard parabolic subgroup $P_1 = M_1 N_1$ of G and an irreducible admissible C-representation σ_1 of M_1. Our goal is to describe the lattice of subrepresentations of $\operatorname{Ind}_{P_1}^G \sigma_1$. We shall see that $\operatorname{Ind}_{P_1}^G \sigma_1$ has finite length and is multiplicity free, meaning that its irreducible constituents occur with multiplicity 1. We recall the main result of [AHHV17] :

Theorem 3.1 (Classification Theorem). *(A) Let $P = MN$ be a standard parabolic subgroup of G and σ a supercuspidal C-representation of M. Then $\operatorname{Ind}_P^G \sigma \in \operatorname{Mod}_C^\infty(G)$ has finite length and is multiplicity free of irreducible constituents the representations $I_G(P, \sigma, Q)$ for $P \subset Q \subset P(\sigma)$, and all $I_G(P, \sigma, Q)$ are admissible.*

(B) Let π be an irreducible admissible C-representation of G. Then, there is a $C[G]$- triple (P, σ, Q) with σ supercuspidal, such that π is isomorphic to $I_G(P, \sigma, Q)$ and π determines P, Q and the isomorphism class of σ.

By the classification theorem, there is a standard parabolic subgroup $P = MN$ of G and a supercuspidal C-representation σ of M such that σ_1 occurs in $\operatorname{Ind}_{P\cap M_1}^{M_1} \sigma$. More precisely, if $P(\sigma)$ is the largest standard parabolic subgroup of G to which σ extends, then by Proposition 2.7, $P(\sigma) \cap M_1$ is the largest standard parabolic subgroup of M_1 to which σ extends and

$$\sigma_1 \simeq I_{M_1}(P \cap M_1, \sigma, Q) \simeq \operatorname{Ind}_{P(\sigma)\cap M_1}^{M_1}(e_{P(\sigma)\cap M_1}(\sigma) \otimes \operatorname{St}_Q^{P(\sigma)\cap M_1})$$

for some parabolic subgroup Q of M_1 with $(P \cap M_1) \subset Q \subset (P(\sigma) \cap M_1)$. By transitivity of the parabolic induction,

$$\operatorname{Ind}_{P_1}^G \sigma_1 \simeq \operatorname{Ind}_{P(\sigma)}^G(e(\sigma) \otimes \operatorname{Ind}_{M(\sigma)\cap P_1}^{M(\sigma)} \operatorname{St}_Q^{P(\sigma)\cap M_1}),$$

and we need to analyse this representation. Our analysis is based on [Her11, §10]. We recall the structure of the lattice of subrepresentations of a finite length multiplicity free representation X. Let J be the set of its irreducible constituents. For $j \in J$, there is a unique subrepresentation X_j of X with cosocle j - it is the smallest subrepresentation of X with j as a quotient. Put the order relation $\leq$ on J, where $i \leq j$ if i is a constituent of X_j. Then the lattice of subrepresentations of X is isomorphic to the lattice of lower sets in $(J, \leq)$ - recall that such a lower set is a subset J' of J such that if $j_1 \in J, j_2 \in J'$ and $j_1 \leq j_2$ then $j_1 \in J'$. A subrepresentation of X is sent to the lower set made out of its irreducible constituents, and a lower set J' of J is sent to the sum of the subrepresentations X_j for $j \in J'$. We have $X_j = j$ if and only if j is minimal in $(J, \leq)$. If the cosocle of X is irreducible, then $(J, \leq)$ has the unique maximal element and $X_j = X$ if and only if j is maximal in $(J, \leq)$. The socle of X is the direct sum of the minimal $j \in (J, \leq)$ and the cosocle of X is the direct sum of the maximal $j \in (J, \leq)$.

In the sequel J will often be identified with $\mathcal{P}(I)$ for some subset I of Δ, both equipped with the order relation reverse to the inclusion. Thus we rather talk of upper sets in $\mathcal{P}(I)$ (for the inclusion). In that case the socle I of X and the cosocle $\emptyset$ of X are both irreducible.

Theorem 3.2. *With the above notations,* $\mathrm{Ind}_{P_1}^G \sigma_1$ *has finite length and is multiplicity free, of irreducible constituents the* $I_G(P, \sigma, Q')$ *where* Q' *is a parabolic subgroup of* G *satisfying* $P \subset Q' \subset P(\sigma)$ *and* $\Delta_{P_1} \cap \Delta_{Q'} = \Delta_Q$. *Sending* $I_G(P, \sigma, Q')$ *to* $\Delta_{Q'} \cap (\Delta \setminus \Delta_{P_1})$ *gives an isomorphism of the lattice of subrepresentations of* $\mathrm{Ind}_{P_1}^G \sigma_1$ *onto the lattice of upper sets in* $\mathcal{P}(\Delta_{P(\sigma)} \cap (\Delta \setminus \Delta_{P_1}))$.

The first assertion is a consequence of the classification theorem 3.1 since $\mathrm{Ind}_{P_1}^G \sigma_1$ is a subrepresentation of $\mathrm{Ind}_P^G \sigma$. For the rest of the proof, given in §3.2, we proceed along the classification, treating cases of increasing generality. As an immediate consequence of the theorem, we get an irreducibility criterion.

Corollary 3.3. *The representation* $\mathrm{Ind}_{P_1}^G \sigma_1$ *is irreducible if and only if* P_1 *contains* $P(\sigma)$.

Corollary 3.4. *The socle and the cosocle of* $\mathrm{Ind}_{P_1}^G \sigma_1$ *are both irreducible.*

This is very different from the complex case [LM16].

3.2. **Proof.** We proceed now to the proof of Theorem 3.2. The very first and basic case is when $P_1 = B$ and σ_1 is the trivial representation $\mathbf{1}$ of Z. The irreducible constituents of $\mathrm{Ind}_B^G \mathbf{1}$ are the St_Q^G for the different standard parabolic subgroups Q of G, each occuring with multiplicity 1.

Proposition 3.5. *Let* Q *be a standard parabolic subgroup of* G.

(i) *The submodule of* $\mathrm{Ind}_B^G \mathbf{1}$ *with cosocle* St_Q^G *is* $\mathrm{Ind}_Q^G \mathbf{1}$.
(ii) *Sending* St_Q^G *to* Δ_Q *gives an isomorphism of the lattice of subrepresentations of* $\mathrm{Ind}_B^G \mathbf{1}$ *onto the lattice of upper sets in* $\mathcal{P}(\Delta)$.

Proof. By the properties recalled before Theorem 3.2, (i) implies (ii). For (i) the proof is given in [Her11, §10] when G is split, using results of Grosse-Klönne [GK14]. The general case is due to T. Ly [Ly15, beginning of §9]. □

We have variants of Proposition 3.5. If Q is a standard parabolic subgroup of G, the subrepresentations of $\mathrm{Ind}_Q^G \mathbf{1}$ are the subrepresentations of $\mathrm{Ind}_B^G \mathbf{1}$ contained in $\mathrm{Ind}_Q^G \mathbf{1}$. So the lattice of subrepresentations of $\mathrm{Ind}_Q^G \mathbf{1}$ is isomorphic of the sublattice of upper sets in $\mathcal{P}(\Delta)$ consisting of subsets containing Δ_Q; intersecting with $\Delta \setminus \Delta_Q$ gives an isomorphism onto the lattice of upper sets in $\mathcal{P}(\Delta \setminus \Delta_Q)$. More generally,

Proposition 3.6. *Let P, Q be two standard parabolic subgroups of G with $Q \subset P$.*

(i) *The irreducible constituents of $\mathrm{Ind}_P^G \mathrm{St}_Q^P$ are the $\mathrm{St}_{Q'}^G$ where $Q' \cap P = Q$, and each occurs with multiplicity 1.*

(ii) *Sending $\mathrm{St}_{Q'}^G$ to $\Delta_{Q'} \cap (\Delta \setminus \Delta_P)$ gives an isomorphism of the lattice of subrepresentations of $\mathrm{Ind}_P^G \mathrm{St}_Q^P$ onto the lattice of upper sets in $\mathcal{P}(\Delta \setminus \Delta_P)$.*

Proof. For (i), note that $\mathrm{Ind}_P^G \mathrm{St}_Q^P$ is the quotient of $\mathrm{Ind}_Q^G \mathbf{1}$ by the sum of its subrepresentations $\mathrm{Ind}_{Q'}^G \mathbf{1}$ for Q' where $Q \subsetneq Q' \subset P$ and (i) is the content of [Ly15, Corollary 9.2]. The order $\mathrm{St}_{Q'}^G \leq \mathrm{St}_{Q''}^G$ on the irreducible constituents corresponds (as it does in $\mathrm{Ind}_B^G \mathbf{1}$) to $\Delta_{Q''} \subset \Delta_{Q'}$. Again (ii) follows for (i). □

Remark 3.7. Note that $\mathcal{P}(\Delta \setminus \Delta_P)$ does not depend on Q. The unique irreducible quotient of $\mathrm{Ind}_P^G \mathrm{St}_Q^P$ is St_Q^G, and its unique subrepresentation is $\mathrm{St}_{Q'}^G$ where $\Delta_{Q'} = \Delta_Q \cup (\Delta \setminus \Delta_P)$.

The next case where $P_1 = P, \sigma_1 = \sigma$ is a consequence of :

Proposition 3.8. *Let $P = MN$ be a standard parabolic subgroup of G and σ a supercuspidal C-representation of M. Then the map $X \mapsto \mathrm{Ind}_{P(\sigma)}^G (e(\sigma) \otimes X)$ gives an isomorphism of the lattice of subrepresentations of $\mathrm{Ind}_P^{P(\sigma)} \mathbf{1}$ onto the lattice of subrepresentations of $\mathrm{Ind}_P^G \sigma$.*

It has the immediate consequence:

Corollary 3.9. *Sending $I_G(P, \sigma, Q)$ to $\Delta_Q \setminus \Delta_P$ gives an isomorphism of the lattice of subrepresentations of $\mathrm{Ind}_P^G \sigma$ onto the lattice of upper sets in $\mathcal{P}(\Delta_{P(\sigma)} \setminus \Delta_P)$.*

The proposition 3.8 is proved in two steps, inducing first to $P(\sigma)$ and then to G. In the first step we may as well assume that $P(\sigma) = G$:

Lemma 3.10. *Let $P = MN$ be a standard parabolic subgroup of G and σ a supercuspidal C-representation of M such that $P(\sigma) = G$. Then the map $X \mapsto e(\sigma) \otimes X$ gives an isomorphism of the lattice of subrepresentations of $\mathrm{Ind}_P^G \mathbf{1}$ onto the lattice of subrepresentations of $e(\sigma) \otimes \mathrm{Ind}_P^G \mathbf{1} \simeq \mathrm{Ind}_P^G \sigma$.*

Proof. By the classification theorem 3.1, the map $X \mapsto e(\sigma) \otimes X$ gives a bijection between the irreducible constituents of $\mathrm{Ind}_P^G \mathbf{1}$ and those of $e(\sigma) \otimes \mathrm{Ind}_P^G \mathbf{1}$. It is therefore enough to show that, for a parabolic subgroup Q of G containing P, the subrepresentation of $e(\sigma) \otimes \mathrm{Ind}_P^G \mathbf{1}$ with cosocle $e(\sigma) \otimes \mathrm{St}_Q^G$ is $e(\sigma) \otimes \mathrm{Ind}_Q^G \mathbf{1}$. Certainly, $e(\sigma) \otimes \mathrm{St}_Q^G$ is a quotient of $e(\sigma) \otimes \mathrm{Ind}_Q^G \mathbf{1}$. Assume that $e(\sigma) \otimes \mathrm{St}_Q^G$ is a quotient of $e(\sigma) \otimes \mathrm{Ind}_{Q'}^G \mathbf{1}$ for some parabolic subgroup Q' of G containing P; we want to conclude that $Q' = Q$. Recall from §2.2 that σ being supercuspidal, Δ_P and Δ_σ are orthogonal . Also, $e(\sigma)$ is obtained by extending σ from M to $G = MM'_\sigma$ trivially on M'_σ. Upon restriction to M'_σ, therefore, $e(\sigma) \otimes \mathrm{Ind}_Q^G \mathbf{1}$ is a direct sum of copies of $\mathrm{Ind}_Q^G \mathbf{1}$ whereas $e(\sigma) \otimes \mathrm{St}_{Q'}^G$ is a direct sum of copies of $\mathrm{St}_{Q'}^G$. Thus there is a non-zero M'_σ-equivariant map $\mathrm{Ind}_Q^G \mathbf{1} \to \mathrm{St}_{Q'}^G$. Let $\mathbf{M}_\sigma^{\mathrm{is}}$ denote the isotropic part of the simply connected covering of the derived group $\mathbf{M}_\sigma$. Then M'_σ is the image of M_σ^{is} in M_σ [AHHV17, II.4 Proposition]; moreover, as a representation of M_σ^{is}, $\mathrm{Ind}_Q^G \mathbf{1}$ is simply $\mathrm{Ind}_{Q_\sigma^{\mathrm{is}}}^{M_\sigma^{\mathrm{is}}} \mathbf{1}$ where Q_σ^{is} is the parabolic subgroup of M_σ^{is} corresponding to $\Delta_Q \cap \Delta_\sigma$, whereas $\mathrm{St}_{Q'}^G$ is $\mathrm{St}_{Q_\sigma'^{\mathrm{is}}}^{M_\sigma^{\mathrm{is}}}$. It follows that $\mathrm{St}_{Q_\sigma'^{\mathrm{is}}}^{M_\sigma^{\mathrm{is}}}$ is a quotient of $\mathrm{Ind}_{Q_\sigma^{\mathrm{is}}}^{M_\sigma^{\mathrm{is}}} \mathbf{1}$, thus $\Delta_Q \cap \Delta_\sigma = \Delta_{Q'} \cap \Delta_\sigma$ which implies $\Delta_Q = \Delta_{Q'}$ and $Q = Q'$, since Δ_Q and $\Delta_{Q'}$ both contain Δ_P. $\square$

The second step in the proof of Proposition 3.8 is an immediate consequence of the following lemma, applied to $P(\sigma)$ instead of P.

Lemma 3.11. *Let $P = MN$ be a standard parabolic subgroup of G. Let W be a finite length smooth C-representation of M, and assume that for any irreducible subquotient Y of W, $\mathrm{Ind}_P^G Y$ is irreducible. The map $Y \mapsto \mathrm{Ind}_P^G Y$ from the lattice $\mathcal{L}_W$ of subrepresentations of W to the lattice $\mathcal{L}_{\mathrm{Ind}_P^G W}$ of subrepresentations of $\mathrm{Ind}_P^G W$ is an isomorphism.*

Proof. We recall from [Vig13, Theorem 5.3] that the functor Ind_P^G has a right adjoint R_P^G and that the natural map $\mathrm{Id} \to R_P^G \mathrm{Ind}_P^G$ is an isomorphism of functors. Let $\varphi : \mathcal{L}_W \to \mathcal{L}_{\mathrm{Ind}_P^G W}$ be the map $Y \mapsto \mathrm{Ind}_P^G Y$ and let $\psi : \mathcal{L}_{\mathrm{Ind}_P^G W} \to \mathcal{L}_W$ be the map $X \mapsto R_P^G X$. The composite $\psi \circ \varphi$ is a bijection. If ψ is injective, then ψ and φ are bijective, reciprocal to each other. To show that ψ is injective, we show first that $X \in \mathcal{L}_{\mathrm{Ind}_P^G W}$ and $R_P^G X \in \mathcal{L}_W$ have always the same length.

Step 1. An irreducible subquotient X of $\mathrm{Ind}_P^G W$ has the form $\mathrm{Ind}_P^G Y$ for an irreducible subquotient Y of W; in particular, $R_P^G X \simeq Y$ is irreducible. Thus, W and $\mathrm{Ind}_P^G W$ have the same length.

Step 2. Let X be a subquotient of $\mathrm{Ind}_P^G W$. Denote the length by $\lg(-)$. We prove that $\lg(R_P^G X) \leq \lg(X)$, by induction on $\lg(X)$. If $X \neq 0$, insert X in an exact sequence $0 \to X' \to X \to X'' \to 0$ with X'' irreducible; then the sequence $0 \to R_P^G X' \to R_P^G X \to R_P^G X''$ is exact and $R_P^G X''$ is irreducible. So $\lg(R_P^G X) \leq \lg(R_P^G X') + 1 \leq \lg(X') + 1 = \lg(X)$.

Step 3. Let $X \in \mathcal{L}_{\mathrm{Ind}_P^G W}$. We deduce from the steps 1 and 2 that $\lg(R_P^G X) = \lg(X)$. Indeed, the exact sequence $0 \to X \to \mathrm{Ind}_P^G W \to (\mathrm{Ind}_P^G W)/X \to 0$ gives an exact sequence $0 \to R_P^G X \to W \to R_P^G((\mathrm{Ind}_P^G W)/X)$. By Step 2, $\lg(R_P^G X) \leq \lg(X)$ and $\lg(R_P^G((\mathrm{Ind}_P^G W)/X)) \leq \lg((\mathrm{Ind}_P^G W)/X)$; by Step 1, $\lg(\mathrm{Ind}_P^G W) = \lg(W)$, so we get equalities instead of inequalities.

We can show now that ψ is injective. Let X, X' in $\mathcal{L}_{\mathrm{Ind}_P^G W}$ such that $R_P^G X = R_P^G X'$. Applying R_P^G to the exact sequence $0 \to X \cap X' \to X \oplus X' \to X + X' \to 0$ gives an exact sequence $0 \to R_P^G(X \cap X') \to R_P^G X \oplus R_P^G X' \to R_P^G(X + X')$ because R_P^G is compatible with direct sums. As R_P^G respects the length, the last map is surjective by length count. But then $R_P^G(X + X') = R_P^G(X) + R_P^G(X')$ inside W. Hence $R_P^G(X + X') = R_P^G X = R_P^G X'$. So $X = X' = X + X'$ by length preservation. $\square$

Remark 3.12. Note that $\lg(R_P^G X) = \lg(X)$ for a subquotient X of $\mathrm{Ind}_P^G W$. Indeed, insert X in an exact sequence $0 \to X' \to X'' \to X \to 0$ where X'' is a subrepresentation of $\mathrm{Ind}_P^G W$. The exact sequence $0 \to R_P^G X' \to R_P^G X'' \to R_P^G X$ and $\lg(R_P^G X') = \lg(X')$, $\lg(R_P^G X'') = \lg(X'')$ give $\lg(R_P^G X) \geq \lg(X)$; with Step 2, this inequality is an equality.

We are now finally in a position to prove Theorem 3.2. It follows from Proposition 3.8 that $X \mapsto \mathrm{Ind}_{P(\sigma)}^G(e(\sigma) \otimes X)$ gives an isomorphism of the lattice of subrepresentations of $\mathrm{Ind}_{P_1 \cap P(\sigma)}^{P(\sigma)} \mathrm{St}_Q^{M_1 \cap P(\sigma)}$ (a quotient of the $\mathrm{Ind}_P^{P(\sigma)} \mathbf{1}$) onto the lattice of subrepresentations of $\mathrm{Ind}_{P(\sigma)}^G(e(\sigma) \otimes \mathrm{Ind}_{P_1 \cap P(\sigma)}^{P(\sigma)} \mathrm{St}_Q^{M_1 \cap P(\sigma)})$ isomorphic to $\mathrm{Ind}_{P_1}^G \sigma_1$. The desired result then follows from Proposition 3.6 applied to $G = P(\sigma), P = P_1 \cap P(\sigma)$ describing the first lattice.

3.3. **Twists by unramified characters.** Recall the definition of unramified characters of G. If $X_F^*(\mathbf{G})$ is the group of algebraic F-characters of $\mathbf{G}$, we have a group homomorphism $H_G : G \to \mathrm{Hom}(X_F^*(\mathbf{G}), \mathbb{Z})$ defined by $H_G(g)(\chi) = \mathrm{val}_F(\chi(g))$ for $g \in G$ and $\chi \in X_F^*(\mathbf{G})$. The kernel ${}^0G$ of H_G is open and closed in G, and the image $H_G(G)$ has finite index in $\mathrm{Hom}(X_F^*(\mathbf{G}), \mathbb{Z})$. It is well known (see 2.12 in [HL17]) that ${}^0G$ is the subgroup of G generated by its compact subgroups. A smooth character $\chi : G \to C^*$ is **unramified** if it is trivial on ${}^0G$; the unramified characters of G form the group of C-points of the algebraic variety $\mathrm{Hom}_{\mathbb{Z}}(H_G(G), \mathbf{G}_m)$.

Let σ_1 be an irreducible admissible C-representation σ_1 of M_1 and we now examine the effect on $\mathrm{Ind}_{P_1}^G \sigma_1$ of twisting σ_1 by unramified characters of M_1. As announced in §1.2, we want to prove that for a general unramified character $\chi : M_1 \to C^*$, the representation $\mathrm{Ind}_{P_1}^G \chi\sigma_1$ is irreducible. For that we translate the irreducibility criterion $P(\chi|_M\sigma) \subset P_1$ given in Corollary 3.3 into more concrete terms. Note that $\chi|_M$ is an unramified character of M. By Proposition 2.7, $P(\chi|_M\sigma) \subset P_1$ means that for each $\alpha \in \Delta \setminus \Delta_{P_1}$, $\chi\sigma$ is non-trivial on $Z \cap M'_\alpha$. Because $\chi|_M\sigma$ is supercuspidal, when $\alpha \in \Delta$ is not orthogonal to Δ_P, $\chi\sigma$ is not trivial on $Z \cap M'_\alpha$. Let $\Delta_{nr}(\sigma)$ be the set of roots $\alpha \in \Delta \setminus \Delta_{P_1}$ orthogonal to Δ_P, such that there exists an unramified character $\chi_\alpha : M \to C^*$ such that $\chi_\alpha\sigma$ is trivial on $Z \cap M'_\alpha$; for $\alpha \in \Delta_{nr}(\sigma)$, choose such a χ_α.

Recall from [AHHV17, III.16 Proposition] that the quotient of $Z \cap M'_\alpha$ by its maximal compact subgroup is infinite cyclic; if we choose $a_\alpha \in Z \cap M'_\alpha$ generating the quotient, then $\chi\sigma$ is trivial on $Z \cap M'_\alpha$ is and only if $\chi(a_\alpha) = \chi_\alpha(a_\alpha)$. We conclude:

Proposition 3.13. *Let $\chi : M_1 \to C^*$ be an unramified C-character of M_1. Then $\mathrm{Ind}_{P_1}^G \chi\sigma_1$ is irreducible if and only if for all $\alpha \in \Delta_{nr}(\sigma)$ we have $\chi(a_\alpha) \neq \chi_\alpha(a_\alpha)$.*

The following corollary answers a question of J.-F. Dat.

Corollary 3.14. *The set of unramified C-characters χ of M_1 such that $\mathrm{Ind}_{P_1}^G \chi\sigma_1$ is reducible is a Zariski-closed proper subset of the space of unramified characters.*

Indeed by the proposition, the reducibility set is the union, possibly empty, of hypersurfaces with equation $\chi(a_\alpha) = \chi_\alpha(a_\alpha)$ for $\alpha \in \Delta_{nr}(\sigma)$.

4. Admissibility

4.1. **Generalities.** Let H be a locally profinite group and let R be a commutative ring. When R is noetherian, a subrepresentation of an admissible R-representation of H is admissible. If H is locally pro-p and p is invertible in R, then taking fixed points under a pro-p open subgroup of H is an exact functor [Vig96, I.4.6], so for noetherian R a quotient of an admissible R-representation of H is again admissible. This is not generally true, however when $p = 0$ in R, as the following example shows.

Example 4.1. Assume that $p = 0$ in R so that R is a $\mathbb{Z}/p\mathbb{Z}$-algebra. Let H be the additive group $(\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}}$, with the product of the discrete topologies on the factors; it is a pro-p group. The space $C^\infty(H, R)$ (§2.2) can be interpreted as the space of functions $H \to R$ which depend only on finitely many terms of a sequence $(u_n)_{n\in\mathbb{N}} \in H$. The group H acts by translation yielding a smooth R-representation of H; if J is an open subgroup of H, the J-invariant functions in $C^\infty(H, R)$ form the finitely generated free R-module of functions $J\backslash H \to R$. In particular, $V = C^\infty(H, R)$ is an admissible R-representation of H. However the quotient of V by its subrepresentation $V_0 = V^H$ of constant functions is not admissible. Indeed, a linear form $f \in \mathrm{Hom}_{\mathbb{Z}/p\mathbb{Z}}(H, R)$ contained in V satisfies $wf(v) - f(v) = f(w+v) - f(v) = f(w)$ for $v, w \in H$ so f produces an H-invariant vector in V/V_0. Such linear forms make an infinite rank free R-submodule of V and V/V_0 cannot be admissible. That example will be boosted below in §4.2.

Lemma 4.2. *Assume that R is noetherian. Let M be an R-module and t a nilpotent R-endomorphism of M. Then M is finitely generated if and only if* $\mathrm{Ker}\, t$ *is.*

Proof. If M is finitely generated so is its R-submodule $\mathrm{Ker}\, t$, because R is noetherian. Conversely assume that $\mathrm{Ker}\, t$ is a finitely generated R-module; we prove that M is finitely generated by induction over the smallest integer $r \geq 1$ such that $t^r = 0$. The case $r = 1$ is a tautology so we assume $r \geq 2$. By induction, the R-submodule $\mathrm{Ker}\, t^{r-1}$ is finitely generated. As t^{r-1} induces an injective map $M/\mathrm{Ker}\, t^{r-1} \to \mathrm{Ker}\, t$ of finitely generated image because R is noetherian, the R-module M is finitely generated. □

Lemma 4.3. *Assume that R is noetherian. Let H be a locally pro-p group and J an open pro-p subgroup of H. Let M be a smooth R-representation of H such that the multiplication p_M by p on M is nilpotent. Then the following are equivalent:*

(i) *M is admissible;*
(ii) *M^J is finitely generated over R;*
(iii) *$M^J \cap \mathrm{Ker}\, p_M$ is finitely generated over R/pR.*

Proof. Clearly (i) implies (ii) and the equivalence of (ii) and (iii) comes from Lemma 4.2 applied to $t = p_M$. Assume now (ii). To prove (i), it suffices to prove that for any open normal subgroup J' of J, the R-module $M^{J'}$ is finitely generated. By Lemma 4.2, it suffices to do it for $M^{J'} \cap \mathrm{Ker}\, p_M$, that is, we can assume $p = 0$ in R. Now $M^{J'} = \mathrm{Hom}_{J'}(R, M) \simeq \mathrm{Hom}_J(R[J/J'], M)$ as R-modules. The group algebra $\mathbb{F}_p[J/J']$ has a decreasing filtration by two sided ideals A_i for $0 \leq i \leq r$ with $A_0 = \mathbb{F}_p[J/J'], A_r = \{0\}$ and A_i/A_{i+1} of dimension 1 over $\mathbb{F}_p$ with trivial action of J/J'. By tensoring with R we get an analogous filtration with $B_i = R \otimes A_i$ for $R[J/J']$. By decreasing induction on i, we prove that $\mathrm{Hom}_J(B_i, M)$ is finitely generated over R. Indeed, the case $i = r$ is a tautology, the exact sequence

$$0 \to B_{i+1} \to B_i \to B_i/B_{i+1} \to 0$$

gives an exact sequence

$$0 \to \mathrm{Hom}_J(B_i/B_{i+1}, M) \to \mathrm{Hom}_J(B_i, M) \to \mathrm{Hom}_J(B_{i+1}, M)$$

and $\mathrm{Hom}_J(B_i/B_{i+1}, M) \simeq M^J$ is a finitely generated R-module by assumption. Since $\mathrm{Hom}_J(B_{i+1}, M)$ is finitely generated by induction, so is $\mathrm{Hom}_J(B_i, M)$ because R is noetherian. The case $i = 0$ gives what we want. □

4.2. **Examples.** Let us now take up the case of a reductive connected group $G = \mathbf{G}(F)$. Here the characteristic of F plays a role. When $\mathrm{char}(F) = 0$, G is an analytic p-adic group, in particular contains a uniform open pro-p subgroup, so that at least when R is a finite local $\mathbb{Z}_p$-algebra [Eme10] or a field of characteristic p [Hen09, 4.1 Theorem 1 and 2], a quotient of an admissible representation of G is still admissible. That does not survive when $\mathrm{char}(F) = p$, as the following example shows.

Example 4.4. An admissible representation of F^* with a non-admissible quotient, when $\mathrm{char}(F) = p > 0$ and $pR = 0$.

The group $1+P_F$ is a quotient of F^*. Choose a uniformizer t of F. For simplicity assume that $q=p$. Then it is known that the map $\prod_{(m,p)=1,m\geq 1}\mathbb{Z}_p \to 1+P_F$ sending (x_m) to $\prod_m(1+t^m)^{x_m}$ is a topological group isomorphism. The group H of Example 4.1 is a topological quotient of F^*. When $pR=0$ the admissible R-representation $C_c^\infty(H,R)$ of H with the non-admissible quotient $C_c^\infty(H,R)/C_c^\infty(H,R)^H$ inflates to an admissible R-representation V of F^* containing the trivial representation $V_0=V^{1+P_F}$ with a non-admissible quotient V/V_0.

That contrast also remains when we consider Jacquet functors. Let $P=MN$ be a standard parabolic subgroup of G. Assume that R is noetherian. The parabolic induction $\mathrm{Ind}_P^G : Mod_R^\infty(M) \to Mod_R^\infty(G)$ respects admissibility [Vig13, Corollary 4.7]. Its left adjoint L_P^G respects admissibility when R is a field of characteristic different from p [Vig96, II.3.4]. More generally,

Proposition 4.5. *Assume that R is noetherian and that p is invertible in V. Let $V\in \mathrm{Mod}_R^\infty(G)$ such that for any open compact subgroup J of G, the R-module V^J has finite length. Then for any open compact subgroup J_M of M, the R-module $V_N^{J_M}$ has finite length.*

Proof. Assume that p is invertible in V. We recall first the assertions (i) and (ii) of the last part of [Vig13]. Let $(K_r)_{r\geq 0}$ be a decreasing sequence of open pro-p subgroups of G with an Iwahori decomposition with respect to $P=MN$, with K_r normal in K_0, $\cap K_r=\{1\}$. We write $\kappa : V\to V_N$ for the natural map and $M_r=M\cap K_r, N_r=N\cap K_r, W_r=V^{K_rN_0}$. Let $z\in Z(M)$ strictly contracting N_0 (subsection 2.5). Then we have

For any finitely generated submodule X of $V_N^{M_r}$ there exists $a\in\mathbb{N}$ with $z^aX\subset\kappa(W_r)$.

We prove now the proposition. As K_rN_0 is a compact open subgroup of G, the R-module W_r has finite length, say ℓ. The R-modules $\kappa(W_r)$ and z^aX have finite length $\leq\ell$, hence X also. This is valid for all X hence $V_N^{M_r}$ has finite length $\leq\ell$. We have $z^aV_N^{M_r}\subset\kappa(W_r)\subset V_N^{M_r}$ for some $a\in\mathbb{N}$. The three R-modules have finite length hence $\kappa(W_r)=V_N^{M_r}$. As any open compact subgroup J_M of M contains M_r for r large enough, the proposition is proved. □

Remark 4.6. The proof is essentially due to Casselman [Cas], who gives it for complex coefficients. The proof shows that $V_N^{M_r}=\kappa(W_r)$ where $W_r\subset V^{N_0}$ for all $r\geq 0$. This implies $\kappa(V^{N_0})=V_N$ because V_N being smooth is equal to $\bigcup_{r\geq 0}V_N^{M_r}$.

When R is artinian, any finitely generated R-module has finite length, so the proposition implies:

Corollary 4.7. *L_P^G respects admissibility when R is artinian (in particular a field) and p is invertible in R.*

Remark 4.8. This corollary was already noted by Dat [Dat09]. The corollary is expected to be true for R noetherian when p is invertible in R. Using the theory of types, Dat proves it when G is a general linear group, a classical group with p odd, or a group of relative rank 1 over F.

Emerton has proved that L_P^G respects admissibility when R is a finite local $\mathbb{Z}_p$-algebra and $\mathrm{char}(F)=0$ [Eme10]. But again, his proof does not survive when $\mathrm{char}(F)=p>0$ and $pR=0$.

Example 4.9. An admissible representation of $\mathrm{SL}(2,F)$ with a non-admissible space of U-coinvariants, when $\mathrm{char}(F)=p>0$ and $pR=0$.

Assume $\mathrm{char}(F) = p > 0$ and $pR = 0$. Let $B = TU$ the upper triangular subgroup of $G = \mathrm{SL}(2, F)$ and identify T with F^* via $\mathrm{diag}(a, a^{-1}) \mapsto a$. Example 4.4 provides an admissible R-representation V of T containing the trivial representation V_0 (the elements of V fixed by the maximal pro-p subgroup of T), such that V/V_0 is not admissible. The representation $\mathrm{Ind}_B^G V$ of G contains $\mathrm{Ind}_B^G V_0$, which contains the trivial subrepresentation V_{00}. We claim that the quotient $W = (\mathrm{Ind}_B^G V)/V_{00}$ is admissible and that W_U is not admissible (as a representation of T).

For the second assertion, it suffices to prove that $W_U = V/V_0$. The Steinberg representation $\mathrm{St} = \mathrm{Ind}_B^G V_0/V_{00}$ of G is contained in W and W/St is isomorphic to $\mathrm{Ind}_B^G(V/V_0)$. We get an exact sequence

$$\mathrm{St}_U \to W_U \to (\mathrm{Ind}_B^G(V/V_0))_U \to 0.$$

It is known that $\mathrm{St}_U = 0$ (see the more general result in Corollary 6.10 below). Hence the module W_U is isomorphic to $(\mathrm{Ind}_B^G(V/V_0))_U \simeq V/V_0$ [Vig13, Theorem 5.3].

We now prove the admissibility of W. Let $\mathcal{U}$ be the pro-p Iwahori subgroup of G, consisting of integral matrices in $\mathrm{SL}(2, O_F)$ congruent modulo P_F to the strictly upper triangular subgroup of $\mathrm{SL}(2, k)$. We prove that $W^{\mathcal{U}} = \mathrm{St}^{\mathcal{U}}$, so W is admissible by Lemma 4.3, because St is admissible. Let $f \in \mathrm{Ind}_B^G V$ with a $\mathcal{U}$-invariant image in W, hence for $x \in \mathcal{U}$, there exists $v_x \in V_0$ with $f(gx) - f(g) = v_x$ for all $g \in G$. Put $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $f(sx) - f(x) = f(sx) - v_x - (f(x) - v_x) = f(s) - f(1)$. Put $v = f(s) - f(1) \in V$. If $x \in \overline{U}$, then $sxs^{-1} \in U$ and $f(sg) = f(sxs^{-1}sg) = f(sxg)$. If $x \in \mathcal{U} \cap U$ and $z \in \mathcal{U}$ we have $f(sz) = f(z) + v = f(xz) + v = f(sxz)$. An easy matrix calculation shows that $\mathcal{U}$ is generated by $\mathcal{U} \cap \overline{U}$ and $\mathcal{U} \cap U$, so the map $z \mapsto f(sz)$ from $\mathcal{U}$ to V is invariant under left multiplication by $\mathcal{U}$. We have $V_0 = V^{\mathcal{U} \cap T}$ and $\mathcal{U} \cap T$ is stable by conjugation by s. For $t \in \mathcal{U} \cap T$ and $z \in \mathcal{U}$ we have $f(sz) = f(stz) = sts^{-1} f(sz)$ and $f(z) = f(sz) - v = f(stz) - v = f(tz) = tf(z)$. Therefore, $f(sz)$ and $f(z)$ lie in V_0. But G is the union of $B\mathcal{U}$ and $Bs\mathcal{U}$, so $f(g) \in V_0$ for all $g \in G$, which means $f \in \mathrm{Ind}_B^G V_0$ and its image in W does belong to $\mathrm{St}^{\mathcal{U}}$.

4.3. **Admissibility and R_P^G.** We turn to the main result of this section (theorem 1.3 of the introduction) for a general connected reductive group G and a standard parabolic subgroup $P = MN$ of G.

Lemma 4.10. *Let V be a noetherian R-module, let t be an endomorphism of V, and view V as a $\mathbb{Z}[T]$-module with T acting through t. Then the map $f \mapsto f(1)$ yields an isomorphism e from $\mathrm{Hom}_{\mathbb{Z}[T]}(\mathbb{Z}[T, T^{-1}], M)$ onto the submodule $V^\infty = \cap_{n \geq 0} t^n V$ of infinitely t-divisible elements.*

Proof. A $\mathbb{Z}[T]$-morphism $f : \mathbb{Z}[T, T^{-1}] \to V$ is determined by the values $m_n = f(T^{-n})$ for $n \in \mathbb{N}$, which are only subject to the condition $tm_{n+1} = m_n$ for $n \in \mathbb{N}$. Certainly $f(1) = m_0$ is in V^∞. Let us prove that e is surjective. As V is noetherian, there is some $n \geq 0$ such that $\mathrm{Ker}\, t^{n+k} = \mathrm{Ker}\, t^n$ for $k \geq 0$. Let $m \in V^\infty$ and for $k \geq 0$ choose m_k such that $m = t^k m_k$. Then for $k \geq 0$, $m_{n+k} - tm_{n+k+1}$ belongs to $\mathrm{Ker}\, t^{n+k}$ so that $t^n m_{n+k} = t^{n+1} m_{n+k+1}$ Putting $\mu_k = t^n m_{n+k}$ we have $\mu_k = t\mu_{k+1}$ and $\mu_0 = m$. Therefore e is surjective. By [Bou12, §2, No 2, Proposition 2], the action of t on V^∞ being surjective is bijective because the R-module V^∞ is noetherian, so e is indeed bijective. □

Theorem 4.11. *Assume that R is noetherian and p is nilpotent in R. Then the functor $R_P^G : \mathrm{Mod}_R^\infty(G) \to \mathrm{Mod}_R^\infty(M)$ respects admissibility.*

Proof. Let π be an admissible R-representation of G and we prove $R_P^G(\pi)$ is admissible. By Lemma 4.3, we may replace π with $\mathrm{Ker}(p\colon \pi \to \pi)$, hence we assume that $p = 0$ in R.

Recall that we have fixed a special parahoric subgroup $\mathcal{K}$ in §2.5. Take a finite extension $\mathbb{F}$ of $\mathbb{F}_p$ such that all absolute irreducible representations of $\mathcal{K}$ in characteristic p are defined over $\mathbb{F}$. Then for any open pro-p subgroup J of $\mathcal{K} \cap M$, we have

$$\begin{aligned} R_P^G(\pi)^J \subset R_P^G(\mathbb{F} \otimes_{\mathbb{F}_p} \pi)^J &= \mathrm{Hom}_{\mathbb{F}[J]}(\mathbb{F}, R_P^G(\mathbb{F} \otimes_{\mathbb{F}_p} \pi)) \\ &= \mathrm{Hom}_{\mathbb{F}[\mathcal{K}\cap M]}(\mathrm{Ind}_J^{\mathcal{K}\cap M}(\mathbb{F}), R_P^G(\mathbb{F} \otimes_{\mathbb{F}_p} \pi)). \end{aligned}$$

Since we have a filtration on $\mathrm{Ind}_J^{\mathcal{K}\cap M}(\mathbb{F})$ whose successive quotients are absolute irreducible representations, it is sufficient to prove that the R-module

$$\mathrm{Hom}_{\mathbb{F}[\mathcal{K}\cap M]}(V, R_P^G(\mathbb{F} \otimes_{\mathbb{F}_p} \pi)).$$

is finitely generated for any irreducible $\mathbb{F}$-representation V of $\mathcal{K} \cap M$.

Put $\pi_1 = \mathbb{F} \otimes_{\mathbb{F}_p} \pi$. This is also admissible. Let V_0 be an irreducible $\mathbb{F}$-representation of $\mathcal{K}$ which is $\overline{P}$-regular [HV12, Definition 3.6] and $(V_0)_{N\cap\mathcal{K}} \simeq V$. This V_0 exists by the classification of absolute irreducible representations of $\mathcal{K}$ ([HV12, Theorem 3.7], [AHHV17, III.10 Lemma]). Then by [HV12, Theorem 1.2] we have

$$\mathrm{Ind}_P^G(\text{c-Ind}_{\mathcal{K}\cap M}^M(V)) \simeq \mathcal{H}_M(\mathcal{K} \cap M, V) \otimes_{\mathcal{H}_G(\mathcal{K},V_0)} \text{c-Ind}_{\mathcal{K}}^G(V_0).$$

Hence

$$\begin{aligned} \mathrm{Hom}_{\mathbb{F}[\mathcal{K}\cap M]}(V, R_P^G(\pi_1)) &= \mathrm{Hom}_{\mathbb{F}[M]}(\text{c-Ind}_{\mathcal{K}\cap M}^M(V), R_P^G(\pi_1)) \\ &= \mathrm{Hom}_{\mathbb{F}[G]}(\mathrm{Ind}_P^G(\text{c-Ind}_{\mathcal{K}\cap M}^M(V)), \pi_1) \\ &= \mathrm{Hom}_{\mathbb{F}[G]}(\mathcal{H}_M(\mathcal{K}\cap M, V) \otimes_{\mathcal{H}_G(\mathcal{K},V_0)} \text{c-Ind}_{\mathcal{K}}^G(V_0), \pi_1) \\ &= \mathrm{Hom}_{\mathcal{H}_G(\mathcal{K},V_0)}(\mathcal{H}_M(\mathcal{K}\cap M, V), \mathrm{Hom}_{\mathbb{F}[\mathcal{K}]}(V_0, \pi_1)). \end{aligned}$$

As $\mathcal{H}_M(\mathcal{K}\cap M, V)$ is a localization of $\mathcal{H}_G(\mathcal{K}, V_0)$ at some $\tau \in \mathcal{Z}_G(\mathcal{K}, V_0)$, the R-module

$$\mathrm{Hom}_{\mathcal{H}_G(\mathcal{K},V_0)}(\mathcal{H}_M(\mathcal{K}\cap M, V), \mathrm{Hom}_{\mathbb{F}[\mathcal{K}]}(V_0, \pi_1))$$

identifies with

$$\mathrm{Hom}_{\mathbb{F}[T]}(\mathbb{F}[T, T^{-1}], \mathrm{Hom}_{\mathbb{F}[\mathcal{K}]}(V_0, \pi_1))$$

with T acting on $\mathrm{Hom}_{\mathbb{F}[\mathcal{K}]}(V_0, \pi_1)$ through τ. Since the R-module $\mathrm{Hom}_{\mathbb{F}[\mathcal{K}]}(V_0, \pi_1)$ is finitely generated and R is noetherian, Lemma 4.10 show that $\mathrm{Hom}_{\mathbb{F}[T]}(\mathbb{F}[T, T^{-1}], \mathrm{Hom}_{\mathbb{F}[\mathcal{K}]}(V_0, \pi_1))$ is also a finitely generated R-module. □

Remark 4.12. Using [OV17, Proposition 4.6] instead of [HV12, Corollary 1.3], the argument works replacing $\mathcal{K}$ by a pro-p Iwahori subgroup. Note that the only irreducible representation of pro-p Iwahori subgroup in characteristic p is the trivial representation. So we may take $\mathbb{F} = \mathbb{F}_p$.

When R is noetherian, $\mathrm{Ind}_P^G : \mathrm{Mod}_R^\infty(M) \to \mathrm{Mod}_R^\infty(G)$ respects admissibility and induces a functor $\mathrm{Ind}_P^{G,a} : \mathrm{Mod}_R^a(M) \to \mathrm{Mod}_R^a(G)$ between the category of admissible representations. Emerton's **$\overline{P}$-ordinary part functor** $\mathrm{Ord}_{\overline{P}}^G$ is right adjoint to $\mathrm{Ind}_P^{G,a}$. For $V \in \mathrm{Mod}_R^\infty(G)$ admissible,

$$\mathrm{Ord}_{\overline{P}}^G V = (\mathrm{Hom}_{R[\overline{N}]}(C_c^\infty(\overline{N}, R), V))^{Z(M)-f}, \tag{4}$$

is the space of $Z(M)$-finite vectors of $\mathrm{Hom}_{R[\overline{N}]}(C_c^\infty(\overline{N}, R), V)$ with the natural action of M (the representation $\mathrm{Ord}_{\overline{P}}^G V$ of M is smooth) [Vig13, §8].

If R_P^G respects admissibility, the restriction of R_P^G to the category of admissible representations is necessarily right adjoint to $\mathrm{Ind}_P^{G,a}$, hence is isomorphic to $\mathrm{Ord}_{\overline{P}}^G$.

Corollary 4.13. *Assume R noetherian and p nilpotent in R. Then R_P^G is isomorphic to the $\overline{P}$-ordinary part functor $\mathrm{Ord}_{\overline{P}}^G$ on admissible R-representations of G.*

Corollary 4.14. *Assume that R is a field of characteristic p. Let V be an irreducible admissible R-representation of G which is a quotient of $\mathrm{Ind}_P^G W$ for some smooth R-representation W of M. Then V is a quotient of $\mathrm{Ind}_P^G W'$ for some irreducible admissible subquotient W' of W.*

The latter corollary was previously known only under the assumption that W admits a central character and R is algebraically closed [HV12, Proposition 7.10]. Its proof is as follows. By assumption, there is a non-zero M-equivariant map $f : W \to R_P^G V$. By the theorem $R_P^G V$ is admissible so $f(W)$ contains an irreducible admissible subrepresentation W' because $\operatorname{char} R = p$ [HV12, Lemma 7.9]. The inclusion of W' into $R_P^G V$ gives a non-zero G-equivariant map $\mathrm{Ind}_P^G W' \to V$, so that V is a quotient of $\mathrm{Ind}_P^G W'$.

Remark 4.15. When R is a field of characteristic $\neq p$ and R_P^G respects admissibility, then Corollary 4.14 remains true.

Proof. It suffices to modify the proof of Corollary 4.14 as follows. We reduce to a finitely generated R-representation W of M, by replacing W by the representation of M generated by the values of an element of $\mathrm{Ind}_P^G W$ with non-zero image in V. An admissible quotient of W is also finitely generated, thus is of finite length [Vig96, II.5.10], and in particular, contains an irreducible admissible subrepresentation W'. By the arguments in the proof of Corollary 4.14, V is a quotient of $\mathrm{Ind}_P^G W'$. ☐

Let $V \in \mathrm{Mod}_R^\infty(G)$. Obviously, $\mathrm{Ord}_{\overline{P}}^G(V)$ given by the formula (4)depends only on the restriction of V to $\overline{P}$, and $L_P^G V = V_N$ depends only on the restriction of V to P. We ask:

Question 4.16. *Does $R_P^G V$ depend only on the restriction of V to $\overline{P}$?*

To end this section we assume that R is noetherian and p is invertible in R and we compare L_P^G and Ord_P^G. In the same situation than in Proposition 4.5, we take up the same notations. For $V \in \mathrm{Mod}_R^a(G)$ we have the R-linear map

$$\varphi \mapsto \kappa(\varphi(\mathbf{1}_{N_0})) : \mathrm{Ord}_P^G(V) \xrightarrow{e_V} L_P^G(V) = V_N, \tag{5}$$

where $\mathbf{1}_{N_0}$ is the characteristic function of N_0. Replacing N_0 by a compact open subgroup $J_N \subset N$ multiplies e_V by the generalized index $[J_N : N_0]$ which is a power of p. Following the action of $m \in M$ which sends $\varphi \in \mathrm{Ord}_P^G(V)$ to $m \circ \varphi \circ m^{-1}$,

$$\kappa((m\varphi)(\mathbf{1}_{N_0})) = \kappa(m(\varphi(\mathbf{1}_{m^{-1}N_0m}))) = [m^{-1}N_0m : N_0]m(\kappa(\varphi(\mathbf{1}_{N_0}))),$$

we get that e_V is an $R[M]$-linear map $\mathrm{Ord}_P^G(V) \to \delta_P^{-1} L_P^G(V)$, and that $V \mapsto e_V$ defines on $\mathrm{Mod}_R^a(G)$ a morphism of functors $e : \mathrm{Ord}_P^G \to \delta_P^{-1} L_P^G$. Here $\delta_P(m) = [mN_0m^{-1} : N_0]$ for $m \in M$.

Proposition 4.17. *Assume R noetherian and p invertible in R. Let $V \in \mathrm{Mod}_R^\infty(G)$ such that for any open compact subgroup J of G, the R-module V^J has finite length. Then e_V is an isomorphism.*

Proof. 1) We recall the Hecke version of the Emerton's functor [Vig13, §7, §8] for $V \in \mathrm{Mod}_R^a(G)$. We fix an open compact subgroup N_0 of N as in [Eme10, §3.1.1]. The monoid $M^+ \subset M$ of $m \in M$ contracting N_0 acts on V^{N_0} by the Hecke action:

$$(m, v) \mapsto h_m(v) = \sum_{n \in N_0/mN_0m^{-1}} nmv : M^+ \times V^{N_0} \to V^{N_0}.$$

We write $I_{M^+}^M : \mathrm{Mod}_R(M^+) \to \mathrm{Mod}_R(M)$ for the induction, right adjoint of the restriction $\mathrm{Res}_{M^+}^M : \mathrm{Mod}_R(M) \to \mathrm{Mod}_R(M^+)$. Let $z \in Z(M)$ strictly contracting N_0 (subsection 2.5). *The map* $\Phi_V : \mathrm{Ord}_P^G(V) \to (I_{M^+}^M V^{N_0})^{z^{-1}-f}$ *defined by*

$$\Phi_V(\varphi)(m) = (m\varphi)(\mathbf{1}_{N_0}) \tag{6}$$

is an isomorphism in $\mathrm{Mod}_R^a(M)$ (loc. cit. Proposition 7.5 restricted to the smooth and $Z(M)$-finite part, and Theorem 8.1 which says that the right hand side is admissible, hence is smooth and $Z(M)$-finite). *For any $r \geq 0$, W_r is stable by h_z, the restriction from M to $z^{\mathbb{Z}}$ gives a $R[z^{\mathbb{Z}}]$-isomorphism*

$$((I_{M^+}^M V^{N_0})^{z^{-1}-f})^{M_r} \simeq (I_{z^{\mathbb{N}}}^{z^{\mathbb{Z}}}(V^{N_0 M_r}))^{z^{-1}-f} \tag{7}$$

(loc. cit. Remark 7.7 for z^{-1}-finite elements, Proposition 8.2), *the RHS of* (7) *is contained in $I_{z^{\mathbb{N}}}^{z^{\mathbb{Z}}}(W_r)$, and we have the isomorphism*

$$f \mapsto (f(z^{-n}))_{n \in \mathbb{N}} : I_{z^{\mathbb{N}}}^{z^{\mathbb{Z}}}(W_r) \to \{(x_n)_{n \geq 0},\ x_n \in h_z^\infty(W_r) = \cap_{n \in \mathbb{N}} h_z^n(W_r),\ h_z(x_{n+1}) = x_n\}$$

(loc. cit. Proposition 8.2, for the isomorphism Lemma 4.10).

2) The inclusion above is an equality $(I_{z^{\mathbb{N}}}^{z^{\mathbb{Z}}}(V^{N_0 M_r}))^{z^{-1}-f} = I_{z^{\mathbb{N}}}^{z^{\mathbb{Z}}}(W_r)$, because the map

$$f \to f(1) : I_{z^{\mathbb{N}}}^{z^{\mathbb{Z}}}(W_r) \to h_z^\infty(W_r) \tag{8}$$

is an isomorphism: on the finitely generated R-module $h_z^\infty(W_r)$, h_z is bijective as it is surjective (Lemma 4.10), hence any element $f \in I_{z^{\mathbb{N}}}^{z^{\mathbb{Z}}}(W_r)$ is z^{-1}-finite as $(z^{-n}f)(1) = f(z^{-n})$ for $n \in \mathbb{N}$ and a R-submodule of $h_z^\infty(W_r)$ is finitely generated.

Through the isomorphisms (6), (7), (8) the restriction of e_V to $(\mathrm{Ord}_P(V))^{M_r}$ translates into the restriction κ_r of κ to $h_z^\infty(W_r)$

$$h_z^\infty(W_r) \xrightarrow{\kappa_r} V_N^{M_r}.$$

3) The sequence $\mathrm{Ker}(h_z^n|_{W_r})$ is increasing hence stationary. Let n the smallest number such that $\mathrm{Ker}(h_z^n|_{W_r}) = \mathrm{Ker}(h_z^{n+1}|_{W_r})$. By [Cas, III.5.3 Lemma, beginning of the proof of III.5.4 Lemma],

$$\mathrm{Ker}(\kappa|_{W_r}) = \mathrm{Ker}(h_z^n|_{W_r}), \quad h_z^n(W_r) \cap \mathrm{Ker}(h_z^n|_{W_r}) = 0.$$

4) If the R-module W_r has finite length, $h_z^\infty(W_r) = h_z^n(W_r)$ and $W_r = h_z^n(W_r) \oplus \mathrm{Ker}(h_z^n|_{W_r})$. Indeed, the sequence $(h_z^m(W_r))_{m \in \mathbb{N}}$ is decreasing and $\lg(W_r) = \lg(\mathrm{Ker}(h_z^m|_{W_r})) + \lg(h_z^m(W_r))$. Therefore κ_r is injective of image $\kappa(W_r)$. As $\kappa(W_r) = V_N^{M_r}$ (proof of Proposition 4.5), κ_r is an isomorphism.

5) If the R-module W_r has finite length for any $r \geq 0$, then $\kappa(V^{N_0}) = V_N$ (Remark 4.6) and e_V is an isomorphism. □

Remark 4.18. The arguments in part 1) show that for $V \in \mathrm{Mod}_R^a(G)$, we have $\mathrm{Ord}_{\overline{P}}^G V = (\mathrm{Hom}_{R[\overline{N}]}(C_c^\infty(\overline{N}, R), V))^{z^{-1}-f}$ for any $z \in Z(M)$ strictly contracting $\overline{N}$ (subsection 2.5).

When R is artinian, any finitely generated R-module has finite length, so the proposition implies:

Corollary 4.19. *Assume R artinian (in particular a field) and p is invertible in R. On $\mathrm{Mod}_R^a(G)$, the functors $\mathrm{Ord}_{\overline{P}}^G$ and $\delta_P^{-1} L_P^G$ are isomorphic via e.*

Remark 4.20. We expect the corollary to be true for noetherian R with p invertible in R. We even expect that the functors $R_{\overline{P}}^G$ and $\delta_P^{-1} L_P^G$ are isomorphic on $\mathrm{Mod}_R^\infty(G)$ (second adjunction). That is proved by Dat for the same groups as in Remark 4.8, and for those groups $R_{\overline{P}}^G$ preserves admissibility.

4.4. Admissibility of $I_G(P, \sigma, Q)$.

Theorem 4.21. *Assume R noetherian. Let (P, σ, Q) be an $R[G]$-triple with σ admissible. If p is invertible or nilpotent in R, then $I_G(P, \sigma, Q)$ is admissible.*

It is already known that St_Q^G is admissible when R is noetherian (when G is split [GK14, Corollary B], in general [Ly15, Remark 5.10]).

Proof. Since parabolic induction preserves admissibility, we may assume $P(\sigma) = G$. If p is invertible in R, the result is easy because $I_G(P, \sigma, Q)$ is a quotient of $\mathrm{Ind}_P^G \sigma$: if σ is admissible so are $\mathrm{Ind}_P^G \sigma$ and all its subquotients. Therefore, it is enough to prove the theorem when p is nilpotent in R and $P(\sigma) = G$. Then $I_G(P, \sigma, Q) = e(\sigma) \otimes_R \mathrm{St}_Q^G$. Let $\mathcal{U}$ be a pro-p-Iwahori subgroup which has the Iwahori decomposition $\mathcal{U} = (\mathcal{U} \cap \overline{N})(\mathcal{U} \cap M)(\mathcal{U} \cap N)$. Using Lemma 4.3 that is a consequence of [AHV, Theorem 4.7] which shows that the natural linear map $e(\sigma)^{\mathcal{U}} \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}} \to (e(\sigma) \otimes_R \mathrm{St}_Q^G)^{\mathcal{U}}$ is an isomorphism, hence $(e(\sigma) \otimes_R \mathrm{St}_Q^G)^{\mathcal{U}}$ is a finitely generated R-module. $\square$

4.5. Ind_P^G **does not respect finitely generated representations.** We add a few remarks on finiteness: when R is the complex number field, the parabolic induction preserves the finitely generated representations [Ber84a, Variante 3.11]. However when $R = C$ (recall that C is an algebraically closed field of characteristic p), this does not hold as we see in the following.

Proposition 4.22. *Let $P = MN$ be a proper parabolic subgroup, V_0 an irreducible C-representation of $M \cap \mathcal{K}$. Set $\sigma = \text{c-}\mathrm{Ind}_{M\cap\mathcal{K}}^M V_0$. Then $\mathrm{Ind}_P^G \sigma$ is not finitely generated.*

Proof. Let V be an irreducible C-representation of $\mathcal{K}$ such that $V_{N\cap\mathcal{K}} \simeq V_0$ and V is $\overline{P}$-regular ([HV12, Theorem 3.7], [AHHV17, III.10 Lemma]). Let $I_V \colon \text{c-}\mathrm{Ind}_K^G V \to \mathrm{Ind}_P^G \sigma$ be the injective homomorphism defined in [HV12, Definition 2.1]. Then by [HV12, Theorem 1.2], I_V induces an isomorphism

$$\mathrm{Ind}_P^G \sigma \simeq \mathcal{H}_M(M \cap \mathcal{K}, V_0) \otimes_{\mathcal{H}_G(\mathcal{K}, V)} \text{c-}\mathrm{Ind}_{\mathcal{K}}^G V.$$

Set $X = \mathrm{Im}\, I_V$. As $\mathcal{H}_M(M \cap \mathcal{K}, V_0)$ is the localization of $\mathcal{H}_G(\mathcal{K}, V)$ at $\tau \in \mathcal{Z}_G(\mathcal{K}, V)$ (subsection 2.5), we have $\mathrm{Ind}_P^G \sigma = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \tau^{-n} X$.

Now assume that $\mathrm{Ind}_P^G \sigma$ is generated by finitely many vectors $f_1, \ldots, f_r \in \mathrm{Ind}_P^G \sigma$. Since $\mathrm{Ind}_P^G \sigma = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \tau^{-n} X$, there exists $n \in \mathbb{Z}_{\geq 0}$ such that $f_i \in \tau^{-n} X$ for all $i = 1, \ldots, r$. Since

$f_1, \dots, f_r$ generates $\mathrm{Ind}_P^G \sigma$, we have $\tau^{-n}X = \mathrm{Ind}_P^G \sigma$. Since τ is invertible on $\mathrm{Ind}_P^G \sigma$, we have $X = \mathrm{Ind}_P^G \sigma$. This contradicts the following lemma. ☐

Lemma 4.23. *Assume $R = C$. If $P \neq G$, then I_V is not surjective for any irreducible representation V of $\mathcal{K}$.*

Proof. Take $\tau \in \mathcal{Z}_G(\mathcal{K}, V)$ such that $\mathcal{H}_M(M \cap \mathcal{K}, V_{N\cap\mathcal{K}}) = \mathcal{H}_G(\mathcal{K}, V)[\tau^{-1}]$. Since the ring homomorphism $\mathcal{S}_P^G \colon \mathcal{H}_G(\mathcal{K}, V) \to \mathcal{H}_M(M \cap \mathcal{K}, V_{N\cap\mathcal{K}})$ is not surjective (this follows from the description of the image of $\mathcal{S}_B^G \colon \mathcal{H}_G(\mathcal{K}, V) \to \mathcal{H}_Z(Z\cap\mathcal{K}, V_{U\cap\mathcal{K}})$ [HV15]), τ is not invertible. Assume that I_V is surjective. Since τ is invertible on $\mathrm{Ind}_P^G(\text{c-}\mathrm{Ind}_{M\cap\mathcal{K}}^M V_{N\cap\mathcal{K}})$ and I_V is $\mathcal{H}_G(\mathcal{K}, V)$-equivariant, τ is invertible on $\text{c-}\mathrm{Ind}_K^G V$. Hence τ is a unit in $\mathrm{End}_G(\text{c-}\mathrm{Ind}_K^G V) = \mathcal{H}_G(\mathcal{K}, V)$. This is a contradiction. ☐

We also have the following.

Proposition 4.24. *If $P \neq G$ and $R = C$, then the functor R_P^G does not preserve infinite direct sums.*

Proof. For an infinite family of representations $\{\pi_n\}$ and a finitely generated representation σ of M, we have $\mathrm{Hom}_M(\sigma, \bigoplus_n R_P^G(\pi_n)) = \bigoplus_n \mathrm{Hom}(\sigma, R_P^G(\pi_n)) \simeq \bigoplus_n \mathrm{Hom}(\mathrm{Ind}_P^G \sigma, \pi_n)$. Hence it is sufficient to prove

$$\bigoplus_n \mathrm{Hom}_G(\mathrm{Ind}_P^G \sigma, \pi_n) \neq \mathrm{Hom}_G(\mathrm{Ind}_P^G \sigma, \bigoplus_n \pi_n)$$

for some $\{\pi_n\}$ and σ.

We take σ as in Proposition 4.22 and use the same notation as in the proof of Proposition 4.22. Set $\pi = \mathrm{Ind}_P^G \sigma$ and $X_n = \tau^{-n}X$. Then we have $\pi \neq X_n$ for all $n \in \mathbb{Z}_{\geq 0}$ and $\bigcup_n X_n = \pi$. The homomorphism $\mathrm{Ind}_P^G \sigma = \pi \to \bigoplus_n \pi/X_n$ induced by the projections $\pi \to \pi/X_n$ is not in $\bigoplus_n \mathrm{Hom}_G(\mathrm{Ind}_P^G \sigma, \pi/X_n)$. ☐

Remark 4.25. The functor R_P^G preserves infinite direct sums when $R_P^G = \delta_P L_{\overline{P}}^G$ (the second adjoint theorem) holds true. It is known when R is the complex number field [Ber], when R is an algebraically closed field of characteristic different from p [Vig96, II.3.8 (2)] and in many cases when p is invertible in R [Dat09, Théorème 1.5].

5. Composing Ind_P^G with adjoints of $\mathrm{Ind}_{P_1}^G$ when p is nilpotent

Let us keep a general reductive connected group G and a commutative ring R. Let $P = MN, P_1 = M_1N_1$ be two standard parabolic subgroups of G.

5.1. Results.

We start our investigations on the compositions of the functor Ind_P^G with $L_{P_1}^G$ and $R_{P_1}^G$ by some considerations on coinvariants.

Lemma 5.1. *Let H be a group and let V, W be $R[H]$-modules, and assume that H acts trivially on W. Then the R-modules $(V \otimes_R W)_H$ and $V_H \otimes_R W$ are isomorphic.*

Proof. We write as usual $V(H)$ for the R-submodule of V generated by the elements $hv - v$ for $h \in H, v \in V$. The exact sequence $0 \to V(H) \to V \to V_H \to 0$ of $R[H]$-modules gives by tensor product over R with W an exact sequence

$$V(H) \otimes_R W \to V \otimes_R W \to V_H \otimes_R W \to 0$$

of $R[H]$-modules. Because H acts trivially on W, $(V \otimes_R W)(H)$ is the image of $V(H) \otimes_R W$ in $V \otimes_R W$, hence the result. ☐

As a consequence of Lemma 5.1, if V is a $\mathbb{Z}[H]$-module and $W = R$ with the trivial action of H, the R-modules $(V \otimes_{\mathbb{Z}} R)_H$ and $V_H \otimes_{\mathbb{Z}} R$ are isomorphic.

Let us study now $C_c^\infty(H,R)_H = C_c^\infty(H,\mathbb{Z})_H \otimes_{\mathbb{Z}} R$. A right **Haar measure on H with values in** R is a non-zero element of $\mathrm{Hom}_R(C_c^\infty(H,R)_H, R)$.

Proposition 5.2. *Let H be a locally pro-p group having an infinite open pro-p subgroup J and W an R-module on which H acts trivially. The R-module of H-coinvariants $C_c^\infty(H,W)_H$ is isomorphic to $R[1/p] \otimes_R W$.*

Proof. Lemma 5.1 reduces us to the case $R = W = \mathbb{Z}$. We consider the right Haar measure on H with values in $\mathbb{Z}[1/p]$ sending the characteristic function $\mathbf{1}_J$ of J to 1. It induces a linear map $C_c^\infty(H,\mathbb{Z}) \to \mathbb{Z}[1/p]$. This map is surjective because J is infinite hence has open subgroups of index p^n for n going to infinity. Let f be in its kernel. We write f as a finite sum $\sum_i a_i h_i \mathbf{1}_{J'}$ where J' is a suitable open subgroup of J, $a_i \in \mathbb{Z}, h_i \in H$. Then $\sum_i a_i [J:J']^{-1} = 0$ in $\mathbb{Z}[1/p]$ hence $\sum_i a_i = 0$ and $f = \sum_i a_i (h_i \mathbf{1}_{J'} - \mathbf{1}_{J'})$ belongs to the kernel of the natural map $C_c^\infty(H,\mathbb{Z}) \to (C_c^\infty(H,\mathbb{Z}))_H$. We thus get an isomorphism $C_c^\infty(H,\mathbb{Z})_H \simeq \mathbb{Z}[1/p]$. Therefore $C_c^\infty(H,W)_H \simeq R[1/p] \otimes_R W$. $\square$

Corollary 5.3. $C_c^\infty(H,R)_H = \{0\}$ *if and only if p is nilpotent in R, and in general,* $C_c^\infty(H,W)_H = \{0\}$ *if and only if W is p-torsion.*

$\mathrm{Hom}_R(C_c^\infty(H,R)_H, R) = \{0\}$ *if and only if* $\mathrm{Hom}(\mathbb{Z}[1/p], R) = \{0\}$ *if and only if there is no Haar measure on H with values in R.*

Proof. $R[1/p] = \{0\}$ if and only if p is nilpotent in R by [Bou85, II.2 Corollary 2] and $R[1/p] \otimes_R W = \{0\}$ if and only if any element of W is killed by a power of p (W is called **p-torsion**). $\square$

The p-ordinary part of an R-module V is

$$V_{p-ord} = \bigcap_{k \geq 0} p^k V.$$

When R is a field, the three conditions: p nilpotent, $R_{p-ord} = \{0\}$, $\mathrm{Hom}(\mathbb{Z}[1/p], R) = \{0\}$, are equivalent to $\mathrm{char}(R) = p$. The equivalence of these three conditions is not true for a general commutative ring, contrary to what is claimed in [Vig96, I (2.3.1)], [Vig13, §5].

Lemma 5.4. *1) p is nilpotent in R if and only if $V_{p-ord} = \{0\}$ for all R-modules V.*

2) $R_{p-ord} = \{0\}$ implies $\mathrm{Hom}(\mathbb{Z}[1/p], R) = \{0\}$*. The converse is true if R is noetherian.*

Proof. 1) Let $n \in \mathbb{N}$ be the characteristic of R ($n\mathbb{Z}$ is the kernel of the canonical map $\mathbb{Z} \to R$). Then p is nilpotent in R if and only if $n = p^k$ for some $k \geq 1$. Clearly $p^k = 0$ in R implies $p^k V = 0$ for all R-modules V. Conversely, if p is not nilpotent there exists a prime ideal J of R not containing p. The fraction field of R/J is a field V of characteristic $\mathrm{char}(V) \neq p$.

2) For the last assertion see Lemma 4.10. $\square$

For $W \in \mathrm{Mod}_R^\infty(M)$, Frobenius reciprocity gives a natural map $L_P^G \mathrm{Ind}_P^G W \to W$ sending the image of $f \in \mathrm{Ind}_P^G W$ to $f(1)$; that yields a natural transformation $L_P^G \mathrm{Ind}_P^G \to \mathrm{Id}_{\mathrm{Mod}_R^\infty(M)}$. When p is nilpotent in R, that natural transformation is an isomorphism of functors [Vig13, Theorem 5.3] (this uses Proposition 5.2); by general nonsense it follows that the natural morphism $\mathrm{Id}_{\mathrm{Mod}_R^\infty(M)} \to R_P^G \mathrm{Ind}_P^G$ coming from the adjunction property is also an isomorphism of functors. We generalize these statements.

Theorem 5.5. *When p is nilpotent in R, the two functors $L^G_{P_1}\operatorname{Ind}^G_P$ and $\operatorname{Ind}^{M_1}_{P\cap M_1} L^M_{P_1\cap M}$ from $\operatorname{Mod}^\infty_R(M)$ to $\operatorname{Mod}^\infty_R(M_1)$ are isomorphic.*

Before proving the theorem, we deduce a corollary:

Corollary 5.6. *In the same situation, the two functors $R^G_{P_1}\operatorname{Ind}^G_P$ and $\operatorname{Ind}^{M_1}_{P\cap M_1} R^M_{P_1\cap M}$ from $\operatorname{Mod}^\infty_R(M)$ to $\operatorname{Mod}^\infty_R(M_1)$ are isomorphic.*

Proof. By Theorem 5.5 the functors $L^G_{P_1}\operatorname{Ind}^G_P$ and $\operatorname{Ind}^{M_1}_{P\cap M_1} L^M_{P_1\cap M}$ are isomorphic, so are their right adjoints $R^G_P\operatorname{Ind}^G_{P_1}$ and $\operatorname{Ind}^{M_1}_{P\cap M_1} R^M_{P_1\cap M}$. □

In fact, our results are more precise than Theorem 5.5 and Corollary 5.6. See Corollaries 5.8 and 5.9. Our proof of Theorem 5.5 is inspired by the proof of the "geometric lemma" in [BZ77]. But [BZ77] uses complex coefficients, also Haar measures on unipotent groups and normalized parabolic inductions which are not available p is nilpotent in R. In fact, our result is simpler than for complex coefficients. As will be apparent in the proof, the isomorphism comes from the natural maps $L^G_{P_1}\operatorname{Ind}^G_P W\to \operatorname{Ind}^{M_1}_{P\cap M_1} L^M_{P_1\cap M}W$ for $W\in\operatorname{Mod}^\infty_R(M)$ sending the class of $f\in\operatorname{Ind}^G_P W$ to the function $m_1\mapsto$ image of $f(m_1)$ in $W_{N_1\cap M}$. To control $L^G_{P_1}\operatorname{Ind}^G_P W$ we look at $\operatorname{Ind}^G_P W$ as a representation of P_1. The coset space $P\backslash G/P_1$ is finite and we choose a sequence $X_1,\dots,X_r$ of (P,P_1)-double cosets in G such that $G=X_1\sqcup\cdots\sqcup X_r, X_r=PP_1$ and $X_1\sqcup\cdots\sqcup X_i$ is open in G for $i=1,\dots,r$. We let I_i be the space of functions in $\operatorname{Ind}^G_P W$ with support included in $X_1\sqcup\cdots\sqcup X_i$, and put $I_0=\{0\}$. For $i=1,\dots,r$, restricting to X_i functions in I_i gives an isomorphism from I_i/I_{i-1} onto the space $J_i=\text{c-}\operatorname{Ind}^{X_i}_P W$ of functions $f: X_i\to W$ satisfying $f(mng)=mf(g)$ for $m\in M, n\in N, g\in X_i$, which are locally constant and of support compact in $P\backslash X_i$. That isomorphism is obviously compatible with the action of P_1 by right translations. For $i=1,\dots,r$, we have the exact sequence

$$0\to I_{i-1}\to I_i\to J_i\to 0$$

and by taking N_1-coinvariants, an exact sequence

$$(I_{i-1})_{N_1}\to (I_i)_{N_1}\to (J_i)_{N_1}\to 0.$$

Proposition 5.7. *Let $W\in\operatorname{Mod}^\infty_R(M)$.*

(i) *The R-linear map $\text{c-}\operatorname{Ind}^{PP_1}_P W\to \operatorname{Ind}^{M_1}_{P\cap M_1} W_{M\cap N_1}$ sending $f\in \text{c-}\operatorname{Ind}^{PP_1}_P W$ to the function $m_1\mapsto$ image of $f(m_1)$ in $W_{M\cap N_1}$, gives an isomorphism of $(\text{c-}\operatorname{Ind}^{PP_1}_P W)_{N_1}$ onto $\operatorname{Ind}^{M_1}_{P\cap M_1} W_{M\cap N_1}$ as representations of M_1.*

(ii) *Assume W is a p-torsion R-module. The space of N_1-coinvariants of $\text{c-}\operatorname{Ind}^{X_i}_P W$ is 0 for $i=1,\dots,r-1$.*

(iii) *Let $V\in\operatorname{Mod}^\infty_R(M_1)$ with $V_{p-ord}=0$. Then the space $\operatorname{Hom}_{M_1}((\text{c-}\operatorname{Ind}^{X_i}_P W)_{N_1},V)$ is 0 for $i=1,\dots,r-1$.*

The proof of Proposition 5.7 is given in §5.2. Composing the surjective map in Proposition 5.7 (i) with the restriction from $\operatorname{Ind}^G_P W$ to $\text{c-}\operatorname{Ind}^{PP_1}_P W$ we get a surjective functorial M_1-equivariant homomorphism

$$L^G_{P_1}\operatorname{Ind}^G_P W\to \operatorname{Ind}^{M_1}_{P\cap M_1} L^M_{P_1\cap M}W. \tag{9}$$

Corollary 5.8. *For any $W\in\operatorname{Mod}^\infty_R(M)$ which is p-torsion, (9) is an isomorphism:*

$$L^G_{P_1}\operatorname{Ind}^G_P W\simeq \operatorname{Ind}^{M_1}_{P\cap M_1} L^M_{P_1\cap M}W.$$

Proof. Proposition 5.7 (ii) shows by induction on i that $(I_i)_{N_1} = 0$ when $i \leq r-1$; when $i = r$ we have $J_r = \text{c-Ind}_P^{PP_1} W$ and with Proposition 5.7 (i), we get the isomorphism. $\square$

If p is nilpotent in R, every $W \in \mathrm{Mod}_R^\infty(M)$ is p-torsion (and conversely), and Theorem 5.5 follows from the corollary.

Let $V \in \mathrm{Mod}_R^\infty(M_1)$, and any $W \in \mathrm{Mod}_R^\infty(M)$, the surjective homomorphism (9) gives an injection

$$\mathrm{Hom}_{M_1}(\mathrm{Ind}_{P\cap M_1}^{M_1} L_{P_1\cap M}^{M} W, V) \to \mathrm{Hom}_{M_1}(L_{P_1}^{G} \mathrm{Ind}_P^G W, V). \tag{10}$$

Taking the right adjoints of the functors we get an injection

$$\mathrm{Hom}_{M_1}(W, \mathrm{Ind}_{P_1\cap M}^{M} R_{P\cap M_1}^{M_1} V) \to \mathrm{Hom}_{M_1}(W, R_P^G \mathrm{Ind}_{P_1}^G V) \tag{11}$$

which is functorial in W. Consequently, we have an M-equivariant injective homomorphism

$$\mathrm{Ind}_{P_1\cap M}^{M} R_{P\cap M_1}^{M_1} V \to R_P^G \mathrm{Ind}_{P_1}^G V \tag{12}$$

Corollary 5.9. *For any $V \in \mathrm{Mod}_R^\infty(M_1)$ with $V_{p-ord} = 0$, (12) is an isomorphism:*

$$\mathrm{Ind}_{P_1\cap M}^{M} R_{P\cap M_1}^{M_1} V \simeq R_P^G \mathrm{Ind}_{P_1}^G V.$$

Proof. Proposition 5.7 (i) and (iii) shows that (10) is a bijection for any $W \in \mathrm{Mod}_R^\infty(M)$. This means that (12) is an isomorphism. $\square$

Now assume that R is noetherian and V is admissible. If for any admissible $W \in \mathrm{Mod}_R^\infty(M)$, $L_{P_1\cap M}^{M} W$ is admissible, from (10) we get by right adjunction an injection

$$\mathrm{Hom}_{M_1}(W, \mathrm{Ind}_{P_1\cap M}^{M} \mathrm{Ord}_{\overline{P}\cap M_1}^{M_1} V) \to \mathrm{Hom}_{M_1}(W, \mathrm{Ord}_{\overline{P}}^{G} \mathrm{Ind}_{P_1}^G V) \tag{13}$$

which is functorial in admissible W. So, we have an M-equivariant injective homomorphism

$$\mathrm{Ind}_{P_1\cap M}^{M} \mathrm{Ord}_{\overline{P}\cap M_1}^{M_1} V \to \mathrm{Ord}_{\overline{P}}^{G} \mathrm{Ind}_{P_1}^G V. \tag{14}$$

As for Corollary 5.9, we deduce:

Corollary 5.10. *Assume that R is noetherian. Let $V \in \mathrm{Mod}_R^\infty(M_1)$ be admissible with $V_{p-ord} = 0$. If for any admissible $W \in \mathrm{Mod}_R^\infty(M)$, $L_{P_1\cap M}^{M} W$ is admissible, then (14) is an isomorphism:*

$$\mathrm{Ind}_{P_1\cap M}^{M} \mathrm{Ord}_{\overline{P}\cap M_1}^{M_1} V \simeq \mathrm{Ord}_{\overline{P}}^{G} \mathrm{Ind}_{P_1}^G V.$$

Remark 5.11. 1)If $P_1 \supset P$, $L_{P_1\cap M}^{M} W = W$ so the hypothesis on W is always satisfied.

2) If p is nilpotent in R then R_P^G respects admissibility and is isomorphic to $\mathrm{Ord}_{\overline{P}}^{G}$. Hence (12) gives an isomorphism

$$\mathrm{Ind}_{P_1\cap M}^{M} \mathrm{Ord}_{\overline{P}\cap M_1}^{M_1} V \simeq \mathrm{Ord}_{\overline{P}}^{G} \mathrm{Ind}_{P_1}^G V.$$

5.2. **Proofs.** To prove Proposition 5.7 (ii) and (iii), we control the action of N_1 on $\text{c-Ind}_P^{X_i} W$ for $i = 1, \dots, r-1$. Since B contains N_1 we may filter X_i by (P, B) double cosets, exactly as we did in §5.1. Reasoning exactly as in §5.1, it is enough to prove the following lemma.

Lemma 5.12. *Let $W \in \mathrm{Mod}_R^\infty(M)$ and $V \in \mathrm{Mod}_R^\infty(M_1)$. Let X be a (P, B) double coset not contained in PP_1.*

(i) *the space of N_1-coinvariants of $\text{c-Ind}_P^X W$ is 0 if W is p-torsion.*
(ii) $\mathrm{Hom}_R((\text{c-Ind}_P^X W)_{N_1}, V) = 0$ *if $V_{p-ord} = 0$.*

Proof. By the Bruhat decomposition $G = B\mathcal{N}B$, we may assume that $X = PnB$ for some $n \in \mathcal{N}$, and the assumption that X is not contained in PP_1 means the image w of n in $\mathbb{W} = \mathcal{N}/Z$ does not belong to $\mathbb{W}_M\mathbb{W}_{M_1}$. The map $u \mapsto Pnu : U \to P\backslash G$ is continuous and induces a bijection from $(n^{-1}Pn \cap U)\backslash U$ onto $P\backslash PnB$. By Arens's theorem that bijection is an homeomorphism. The group $n^{-1}Pn\cap U$ is Z-invariant and is equal to the product (in any order) of subgroups U_α for some reduced roots α. More precisely,

$$n^{-1}Pn \cap U = \prod_{\alpha \in \Phi^+_{red}, w(\alpha) \in \Phi_M \cup \Phi_N} U_\alpha,$$

where $\Phi_N = \Phi^+\backslash\Phi^+_M$ and Φ is the disjoint union $\Phi_M \sqcup \Phi_N \sqcup (-\Phi_N)$ (§2.1). We choose a reduced root β such that $w(\beta)$ belongs to $-\Phi_N$ (we check the existence of β in Lemma 5.13), and an ordering $\alpha_1, \dots, \alpha_r$ with $\alpha_r = \beta$ of the reduced roots $\alpha \in \Phi^+_{red}$ such that $w(\alpha) \in -\Phi_N$. Let U' denote the subset $U_{\alpha_1} \times \cdots \times U_{\alpha_{r-1}}$ of U. Then the product map $(n^{-1}Pn\cap U) \times U' \times U_\beta \to U$ is a bijection, indeed a homeomorphism, so we get a homeomorphism $U' \times U_\beta \to (n^{-1}Pn\cap U)\backslash U$, which moreover is U_β-equivariant for the right translation. All taken together we have an U_β-equivariant isomorphism of R-modules:

$$f \mapsto ((u', u_\beta) \mapsto f(nu'u_\beta)) : \text{c-Ind}_P^X W \to C_c^\infty(U' \times U_\beta, W).$$

Now $C_c^\infty(U' \times U_\beta, W)$ is $C_c^\infty(U', R) \otimes_R C_c^\infty(U_\beta, R) \otimes_R W$ where U_β acts only on the middle factor. By Proposition 5.2, $C_c^\infty(U_\beta, R)_{U_\beta}$ is isomorphic to $R[1/p]$. If W is p-torsion, $C_c^\infty(U_\beta, R)_{U_\beta} \otimes_R W = 0$ hence $(\text{c-Ind}_P^{PnB}(W))_{U_\beta} = 0$ and a fortiori $(\text{c-Ind}_P^{PnB}(W))_{N_1} = 0$ by transitivity of the coinvariants, since N_1 contains U_β. We get (i). Similarly, if $V_{p-ord} = 0$, $\mathrm{Hom}_R(C_c^\infty(U_\beta, R)_{U_\beta}, V) = 0$ hence we get (ii). □

Lemma 5.13. *Let $w \in \mathbb{W} \setminus \mathbb{W}_M\mathbb{W}_{M_1}$. Then there exists $\beta \in \Phi_{N_1}$ such that $w(\beta)$ belongs to $-\Phi_N$.*

We can take β reduced. If β is not reduced, replace it by $\beta/2$.

Proof. The property in Lemma 5.13 depends only on the double coset $\mathbb{W}_M w \mathbb{W}_{M_1}$ because Φ_N is stable by $\mathbb{W}_M$ and Φ_{N_1} is stable by $\mathbb{W}_{M_1}$. We suppose that w is the element of minimal length in $\mathbb{W}_M w\mathbb{W}_{M_1}$. This condition translates as:

(i) $w^{-1}(\Phi^-) \cap \Phi^+ \subset \Phi_{N_1}$,
(ii) $\Phi^- \cap w(\Phi^+) \subset -\Phi_N$.

Proceeding by contradiction we suppose $w(\Phi_{N_1}) \subset \Phi_M \cup \Phi_N$. This implies $w(\Phi_{N_1}) \cap \Phi^- \subset \Phi^-_M$ then (ii) implies $w(\Phi_{N_1}) \cap \Phi^- = \emptyset$ so $w(\Phi_{N_1}) \subset \Phi^+$. With (i) we get $\Phi^- \cap w(\Phi^+) \subset w(\Phi_{N_1}) \subset \Phi^+$. Then comparing with (ii), $w(\Phi^+) \subset \Phi^+$ which implies $w = 1$. This is absurd hence Lemma 5.13 is proved. □

This ends the proof of Proposition 5.7 (ii) and (iii). To prove Proposition 5.7 (i), we control $\text{c-Ind}_P^{PP_1} W$ as a representation of P_1. As the inclusion of P_1 in PP_1 induces an homeomorphism $(P \cap P_1)\backslash P_1 \to P\backslash PP_1$, we think of $\text{c-Ind}_P^{PP_1} W$ as the representation $\text{c-Ind}_{P\cap P_1}^{P_1} W$ of P_1. To identify $(\text{c-Ind}_{P\cap P_1}^{P_1} W)_{N_1}$ and $\text{Ind}_{P\cap M_1}^{M_1} W_{M\cap N_1}$ we proceed exactly as in [BZ77, 5.16 case IV_1]; indeed mutatis mutandis we are in that case: their $G = Q$ is our P_1, their $M = P$ is our $P \cap P_1$, their N is our M_1 and their V our N_1. Their reasoning applies to get the desired result: it is enough to realize that the equivalence relation between ℓ-sheaves on $(P \cap P_1)\backslash P_1$ and smooth representations of $P\cap P_1$ is valid for R as coefficients [BZ77, 5.10 to 5.14] and also that although N_1 is locally pro-p, forming N_1-coinvariants is still compatible with inductive limits [BZ77, 1.9 (9)]. This latter property is valid for any functor $\text{Mod}_R^\infty(G) \to \text{Mod}_R^\infty(M_1)$ having a right adjoint, because $\text{Mod}_R^\infty(G)$ is a Grothendieck category [Vig13, Proposition 2.9, lemma 3.2].

6. Applying adjoints of $\text{Ind}_{P_1}^G$ to $I_G(P, \sigma, Q)$

Let us keep a general reductive connected group G and a commutative ring R. Let $P_1 = M_1N_1$ be a standard parabolic subgroup of G and $(P = MN, \sigma, Q)$ an $R[G]$-triple (2.2).

6.1. **Results and applications.** We would like to compute $L_{P_1}^G I_G(P, \sigma, Q)$ when σ is p-torsion and $R_{P_1}^G I_G(P, \sigma, Q)$ when $\sigma_{p-ord} = 0$. Applying Corollaries 5.8 and 5.9 we may reduce to the case where $P(\sigma) = G$, so $I_G(P, \sigma, Q) = e(\sigma) \otimes \text{St}_Q^G$. But we have no direct construction of $R_{P_1}^G$. When R is noetherian and p is nilpotent in R, then for admissible $V \in \text{Mod}_R^\infty(G)$, $R_{P_1}^G V \simeq \text{Ord}_{\overline{P}_1}^G V$ (Corollary 4.13). Consequently, in the following Theorem 6.1, Part (ii) we may replace $\text{Ord}_{\overline{P}_1}^G$ by $R_{P_1}^G$ and $\text{Ord}_{M\cap\overline{P}_1}^M$ by $R_{M\cap P_1}^M$ when p is nilpotent in R.

Theorem 6.1. *Assume $P(\sigma) = G$. We have:*

(i) *Assume that σ is p-torsion. Then $L_{P_1}^G(e(\sigma) \otimes \text{St}_Q^G)$ is isomorphic to $e_{M_1}(L_{M\cap P_1}^M(\sigma)) \otimes \text{St}_{M_1\cap Q}^{M_1}$ if $\langle Q, P_1\rangle = G$, and is 0 otherwise.*

(ii) *Assume R noetherian, σ admissible, and $\sigma_{p-ord} = 0$. Then $\text{Ord}_{\overline{P}_1}^G(e(\sigma) \otimes \text{St}_Q^G)$ is isomorphic to $e_{M_1}(\text{Ord}_{M\cap\overline{P}_1}^M(\sigma)) \otimes \text{St}_{M_1\cap Q}^{M_1}$ if $\langle P, P_1\rangle \supset Q$, and is 0 otherwise.*

In part (i), the statement includes that $L_{M\cap P_1}^M(\sigma)$ extends to M_1 and similarly in part (ii) for $\text{Ord}_{M\cap\overline{P}_1}^M(\sigma)$. Before the proof of the theorem (§6.2, §7) we derive consequences.

Without any assumption on $P(\sigma)$, we get:

Corollary 6.2. (i) *Assume that σ is p-torsion. Then $L_{P_1}^G I_G(P, \sigma, Q)$ is isomorphic to*

$$\text{Ind}_{P(\sigma)\cap M_1}^{M_1}(e_{M_1\cap M(\sigma)}(L_{M\cap P_1}^M(\sigma)) \otimes \text{St}_{Q\cap M_1}^{M_1\cap M(\sigma)}) \tag{15}$$

when $\langle P_1 \cap P(\sigma), Q\rangle = P(\sigma)$, and is 0 otherwise.

(ii) *Assume R noetherian, σ admissible, and p nilpotent in R. Then $\text{Ord}_{\overline{P}_1}^G I_G(P, \sigma, Q)$ is isomorphic to*

$$\text{Ind}_{P(\sigma)\cap M_1}^{M_1}(e_{M_1\cap M(\sigma)}(\text{Ord}_{M\cap\overline{P}_1}^M(\sigma)) \otimes \text{St}_{Q\cap M_1}^{M_1\cap M(\sigma)}) \tag{16}$$

if $\langle P, P_1 \cap P(\sigma)\rangle \supset Q$, and is 0 otherwise.

In the corollary, $L^M_{M\cap P_1}(\sigma)$ might extend to a parabolic subgroup of M_1 bigger than $M_1 \cap P(\sigma)$. So we cannot write (15) as $I_{M_1}(P\cap M_1, L^M_{M\cap P_1}(\sigma), Q\cap M_1)$. A similar remark applies to (16).

Proof. (i) $L^G_{P_1} I_G(P,\sigma,Q) = L^G_{P_1} \operatorname{Ind}^G_{P(\sigma)}(e_{M(\sigma)}(\sigma)\otimes \operatorname{St}^{M(\sigma)}_{Q\cap M(\sigma)})$ is isomorphic to (Corollary 5.8) $\operatorname{Ind}^{M_1}_{P(\sigma)\cap M_1} L^{M(\sigma)}_{P_1\cap M(\sigma)} e_{M(\sigma)}(\sigma)\otimes \operatorname{St}^{M(\sigma)}_{Q\cap M(\sigma)}$. Applying Theorem 6.1, we get (i).

(ii) Similarly, $\operatorname{Ord}^G_{\overline{P}_1} I_G(P,\sigma,Q) \simeq \operatorname{Ind}^{M_1}_{P(\sigma)\cap M_1} \operatorname{Ord}^{M(\sigma)}_{M\cap\overline{P}_1}(e_{M(\sigma)}(\sigma)\otimes \operatorname{St}^{M(\sigma)}_{Q\cap M(\sigma)})$ by Remark 5.11 (2). Applying Theorem 6.1, we get (ii). □

Definition 6.3. A smooth R-representation V of G is called left cuspidal if $L^G_P V = 0$ for all proper parabolic subgroups P of G, and right cuspidal if $R^G_P V = 0$ for all proper parabolic subgroups P of G.

We may restrict to proper standard parabolic subgroups in this definition, since any parabolic subgroup of G is conjugate to a standard one.

Proposition 6.4. *Assume that R is a field of characteristic p. Then a supercuspidal representation is right-cuspidal.*

Proof. An irreducible admissible R-representation V of G such that $R^G_P V \neq 0$ is a quotient of $\operatorname{Ind}^G_P R^G_P V$ and by Corollary 4.14 is a quotient of $\operatorname{Ind}^G_P W$ for some irreducible admissible R-representation W of M because the characteristic of R is p (Corollary 4.14). If V is supercuspidal, then $P = G$, so V is right cuspidal. □

Corollary 6.5. *Assume that R is a field of characteristic p and (P,σ,Q) is an $R[G]$-triple with σ supercuspidal. Then $R^G_{P_1} I_G(P,\sigma,Q)$ is isomorphic to $I_{M_1}(P\cap M_1,\sigma,Q\cap M_1)$ if $P_1 \supset Q$, and is 0 otherwise.*

This corollary implies Theorem 1.1 (ii).

Proof. (i) Assume first $P(\sigma) = G$. As a supercuspidal representation is e-minimal, we may apply Theorem 6.1 Part (ii). Thus $R^G_{P_1} I_G(P,\sigma,Q) = 0$ unless $\langle P, P_1\rangle \supset Q$ in which case it is isomorphic to $e_{M_1}(R^M_{M\cap P_1}(\sigma))\otimes \operatorname{St}^{M_1}_{M_1\cap Q}$.

If P_1 does not contain P, then $P_1 \cap M$ is a proper parabolic subgroup of M and by Proposition 6.4, $R^M_{P_1\cap M}\sigma = 0$.

If $P_1 \supset P$, then $M\cap P_1 = M$ and $R^M_{P_1\cap M}\sigma = \sigma$. Moreover, $\langle P, P_1\rangle \supset Q$ if and only if $P_1 \supset Q$. This gives the result when $P(\sigma) = G$.

(ii) Without hypothesis on $P(\sigma)$, we proceed as in the proof of Corollary 6.2. □

We now turn to consequences where $R = C$.

We have the supersingular C-representations of G - we recall their definition. Recall the homomorphism $\mathcal{S}^G_P$ in §2.5. A homomorphism $\chi : \mathcal{Z}_G(\mathcal{K}, V) \to C$ is supersingular if it does not factor through $\mathcal{S}^G_P$ when $P \neq G$.

Definition 6.6. A C-representation π of G is called supersingular if it is irreducible admissible and for all irreducible smooth C-representations V of $\mathcal{K}$, the eigenvalues of $\mathcal{Z}_G(\mathcal{K}, V)$ in $\operatorname{Hom}_G(\text{c-Ind}^G_K V, \pi)$ are supersingular.

A C-representation π of G is supersingular if and only if it is supercuspidal [AHHV17, I.5 Theorem 5].

Proposition 6.7. *A supersingular C-representation of G is left-cuspidal.*

Proof. Let π be an admissible C-representation of G and $P = MN$ be a standard parabolic subgroup of G such that $L_P^G\pi \neq 0$. Putting $W = L_P^G\pi$, adjunction gives a G-equivariant map $\pi \to \operatorname{Ind}_P^G W$. Choose an irreducible smooth C-representation of the special parahoric subgroup $\mathcal{K}$ of G such that the space $\operatorname{Hom}_G(\text{c-Ind}_{\mathcal{K}}^G V, \pi)$ (isomorphic to $\operatorname{Hom}_{\mathcal{K}}(V, \pi)$ and finite dimensional) is not zero. The commutative algebra $\mathcal{Z}(\mathcal{K}, V)$ posseses an eigenvalue on this space; that eigenvalue is also an eigenvalue of $\mathcal{Z}(\mathcal{K}, V)$ on $\operatorname{Hom}_G(\text{c-Ind}_{\mathcal{K}}^G V, \operatorname{Ind}_P^G W)$ which necessarily factorizes through $\mathcal{S}_P^G$ (§2.5). If π is supersingular (in particular irreducible), $P = G$ hence π is left cuspidal. $\square$

The classification theorem 3.1, Propositions 6.4 and 6.7 imply:

Corollary 6.8. *Assume that (P, σ, Q) is a $C[G]$-triple with σ supercuspidal. In that situation $L_{P_1}^G I_G(P, \sigma, Q)$ is isomorphic to $I_{M_1}(P \cap M_1, \sigma, Q \cap M_1)$ if $P_1 \supset P$ and $\langle P_1, Q\rangle \supset P(\sigma)$, and is 0 otherwise.*

This corollary is Theorem 1.1 (i).

Proof. We proceed as for the proof of Corollary 6.5. With the same reasoning we get $L_{P_1\cap M}^M\sigma = 0$ if P_1 does not contain P and $L_{P_1\cap M}^M\sigma = \sigma$ if $P_1 \supset P$. Therefore, Theorem 6.1 Part (i) implies the result when $P(\sigma) = G$. Otherwise, we use Theorem 5.5 to reduce to the case $P(\sigma) = G$. $\square$

From Corollary 6.5 and 6.8 we deduce immediately:

Corollary 6.9. *An irreducible admissible C-representation of G is left and right cuspidal if and only if it is supercuspidal.*

Now it is easy to describe the left or right cuspidal irreducible admissible C-representations of G.

Corollary 6.10. *Let (P, σ, Q) be a $C[G]$-triple with σ supercuspidal. Then $I_G(P, \sigma, Q)$ is*

(i) *left cuspidal if and only if $Q = P$ and $P(\sigma) = G$, so $I_G(P, \sigma, Q) = e(\sigma) \otimes \operatorname{St}_P^G$;*
(ii) *right cuspidal if and only if $Q = P(\sigma) = G$, so $I_G(P, \sigma, Q) = e(\sigma)$.*

Proof. (i) By Theorem 1.1 Part (i), $I_G(P, \sigma, Q)$ is left cuspidal if and only if

$$\Delta_{P_1} \supset \Delta_P \text{ and } \Delta_{P_1} \cup \Delta_Q \supset \Delta_{P(\sigma)} \text{ implies } \Delta_{P_1} = \Delta.$$

This displayed property is equivalent to $\Delta_\sigma \setminus (\Delta_Q \cap \Delta_\sigma) = \Delta \setminus \Delta_P$, and this is equivalent to $Q = P$ and $P(\sigma) = G$.

(ii) By Theorem 1.1 Part (ii), $I_G(P, \sigma, Q)$ is right cuspidal if and only if $P_1 \supset Q$ implies $P_1 = G$. This latter property is equivalent to $Q = G$. But $Q \subset P(\sigma)$ hence $I_G(P, \sigma, Q)$ is right cuspidal if and only if $Q = P(\sigma) = G$. $\square$

Remark 6.11. We compare with the case where R is a field of characteristic $\neq p$. Then, L_P^G is exact, a subquotient of a left cuspidal smooth R-representation of G is also left cuspidal. For a representation π of G satisfying the second adjointness property $R_P^G\pi = \delta_P L_{\overline{P}}^G\pi$ for all parabolic subgroups P of G (see §4.3), then left cuspidal is equivalent to right cuspidal. For an irreducible smooth R-representation (hence admissible), supercuspidal implies obviously left and right cuspidal. The converse is true when R is an algebraically closed field of characteristic 0 or banal [Vig96, II.3.9]. When $G = \operatorname{GL}(2, \mathbb{Q}_p)$ and the characteristic ℓ of C divides $p+1$, the smooth C-representation $\mathrm{Ind}_B^G \mathbf{1}$ of G admits a left and right cuspidal irreducible subquotient [Vig89], which is not supercuspidal.

6.2. **The case of N_1-coinvariants.** We proceed to the proof of Theorem 6.1, Part (i). First we assume that Δ_M is orthogonal to $\Delta \setminus \Delta_M$. Recall that P_σ is the parabolic subgroup corresponding to Δ_σ and M_σ its Levi subgroup (subsection 2.4). Our assumption $P(\sigma) = G$ implies $\Delta_\sigma = \Delta \setminus \Delta_M$. The representation $e(\sigma)$ is obtained by extending σ from M to $G = MM'_\sigma$ trivially on M'_σ.

6.2.1. Assume $P_1 \supset P$, so that N_1 acts trivially on $e(\sigma)$ because $N_1 \subset M'_\sigma$. We start from the exact sequence defining St_Q^G and we tensor it by $e(\sigma)$

$$\bigoplus_{Q' \in \mathcal{Q}} e(\sigma) \otimes \mathrm{Ind}_{Q'}^G \mathbf{1} \to e(\sigma) \otimes \mathrm{Ind}_Q^G \mathbf{1} \to e(\sigma) \otimes \mathrm{St}_Q^G \to 0, \tag{17}$$

where $\mathcal{Q}$ is the set of parabolic subgroups of G containing strictly Q. Applying the right exact functor $L_{P_1}^G$ gives an exact sequence. As σ is p-torsion, Corollary 5.8 gives a natural isomorphism $L_{P_1}^G(e(\sigma) \otimes \mathrm{Ind}_Q^G \mathbf{1}) \simeq e_{M_1}(\sigma) \otimes \mathrm{Ind}_{M_1 \cap Q}^{M_1} \mathbf{1}$ and similarly for $Q' \in \mathcal{Q}$, so we get the exact sequence

$$\bigoplus_{Q' \in \mathcal{Q}} e_{M_1}(\sigma) \otimes \mathrm{Ind}_{M_1 \cap Q'}^{M_1} \mathbf{1} \to e_{M_1}(\sigma) \otimes \mathrm{Ind}_{M_1 \cap Q}^{M_1} \mathbf{1} \to L_{P_1}^G(e(\sigma) \otimes \mathrm{St}_Q^G) \to 0.$$

The map on the left is given by the natural inclusion for each summand. If for some $Q' \in \mathcal{Q}$ we have $M_1 \cap Q' = M \cap Q'$ then that map is surjective and $L_{P_1}^G(e(\sigma) \otimes \mathrm{St}_Q^G) = 0$. Otherwise $\langle Q, P_1 \rangle = G$ (see the lemma below) and from the exact sequence we have an isomorphism $L_{P_1}^G(e(\sigma) \otimes \mathrm{St}_Q^G) \simeq e_{M_1}(\sigma) \otimes \mathrm{St}_{M_1 \cap Q}^{M_1}$.

Lemma 6.12. *$\langle Q, P_1 \rangle = G$ if and only if $M_1 \cap Q' \neq M \cap Q'$ for all $Q' \in \mathcal{Q}$. In this case, the map $Q' \mapsto M_1 \cap Q'$ is a bijection from $\mathcal{Q}$ to the set of parabolic subgroups of M_1 containing strictly $Q \cap M_1$.*

Proof. The proof is immediate after translation in terms of subsets of Δ. □

6.2.2. Assume $\langle P, P_1 \rangle = G$. Then $P_1 \supset P_\sigma$, N_1 is contained in M' and acts trivially on St_Q^G because Δ_M and $\Delta \setminus \Delta_M$ are orthogonal. By Lemma 5.1 we find that $L_{P_1}^G(e(\sigma) \otimes \mathrm{St}_Q^G) \simeq L_{P_1}^G e(\sigma) \otimes \mathrm{St}_Q^G|_{M_1}$. Decomposing $P_1 = (P_1 \cap M)M'_\sigma = (M_1 \cap M)N_1 M'_\sigma$ and $M_1 = (M_1 \cap M)M'_\sigma$ we see that the $R[P_1]$-module $L_{P_1}^G e(\sigma)$ is $L_{M \cap P_1}^M \sigma = \sigma_{N_1}$ trivially extended to M'_σ. That is $L_{P_1}^G e(\sigma) = e_{M_1}(L_{M \cap P_1}^M \sigma)$. On the other hand, because $Q \supset M$ and $M_1 \supset M_\sigma$ we have $G = MM_\sigma = QM_1$ and the inclusion of M_1 in G induces an homeomorphism $(Q \cap M_1)\backslash M_1 \simeq Q \backslash G$. So, $(\mathrm{Ind}_Q^G \mathbf{1})|_{M_1}$ identifies with $\mathrm{Ind}_{M_1 \cap Q}^{M_1} \mathbf{1}$, this also applies to the $Q' \in \mathcal{Q}$ containing Q, thus $\mathrm{St}_Q^G|_{M_1} \simeq \mathrm{St}_{M_1 \cap Q}^{M_1}$. We get $L_{P_1}^G(e(\sigma) \otimes \mathrm{St}_Q^G) \simeq e_{M_1}(L_{M \cap P_1}^M \sigma) \otimes \mathrm{St}_{M_1 \cap Q}^{M_1}$ proving what we want when $P_1 \supset M_\sigma$, since $\Delta_Q \cup \Delta_{M_1} = \Delta$. Note that the assumption that σ is p-torsion was not used.

6.2.3. The case where P_1 is arbitrary can finally be obtained in two stages, using the transitivity property of the coinvariant functors: first apply $L_{P_2}^G$ where $P_2 = MP_1$ contains P then apply $L_{M_2 \cap P_1}^{M_2}$ where $\langle P \cap M_2, P_1 \cap M_2 \rangle = M_2$. Applying 6.2.1, $L_{P_2}^G(e(\sigma) \otimes \mathrm{St}_Q^G) = 0$ unless $\Delta_{P_2} \cup \Delta_Q = \Delta$ in which case $L_{P_2}^G(e(\sigma) \otimes \mathrm{St}_Q^G) \simeq e_{M_2}(\sigma) \otimes \mathrm{St}_{M_2 \cap Q}^{M_1}$. Applying 6.2.2, $L_{M_2 \cap P_1}^{M_2}(e_{M_2}(\sigma) \otimes \mathrm{St}_{M_2 \cap Q}^{M_2}) \simeq (e_{M_1}(L_{M \cap P_1}^M \sigma) \otimes \mathrm{St}_{M_1 \cap Q}^{M_1})$.

This ends the proof of Theorem 6.1 (i) when Δ_M is orthogonal to $\Delta \setminus \Delta_M$.

In general, we introduce $P_{\min} = M_{\min}N_{\min}$ and an e-minimal representation $\sigma_{\min}$ of $M_{\min}$ as in Lemma 2.9, such that $\sigma = e_P(\sigma_{\min})$. Then $\Delta_{M_{\min}} = \Delta_{\min}$ is orthogonal to $\Delta \setminus \Delta_{\min}$ (Lemma 2.10), and σ is p-torsion so is $\sigma_{\min}$ so we can apply Theorem 6.1 (i) to $\sigma_{\min}$. As $e(\sigma) = e(\sigma_{\min})$ we get:

$L^G_{P_1}(e(\sigma) \otimes \mathrm{St}^G_Q)$ is isomorphic to $e_{M_1}(L^{M_{\min}}_{M_{\min}\cap P_1}(\sigma_{\min})) \otimes \mathrm{St}^{M_1}_{M_1\cap Q}$ if $\langle Q, P_1\rangle = G$, and is 0 otherwise.

We prove now $e_{M_1}(L^{M_{\min}}_{M_{\min}\cap P_1}(\sigma_{\min})) = e_{M_1}(L^M_{M\cap P_1}(\sigma))$. Write $J = \Delta_M \setminus \Delta_{\min}$ and $\Delta_{M_1} = \Delta_1$. The orthogonal decomposition $\Delta_M \cap \Delta_1 = (\Delta_{\min} \cap \Delta_1) \perp (J \cap \Delta_1)$ implies $M \cap M_1 = (M_{\min} \cap M_1)(M_J \cap M_1)'$. But $(M_J \cap M_1)' \subset M'_J$ acts trivially on σ (§2.2), so we deduce that $\sigma_{M\cap N_1}$ extends $(\sigma_{\min})_{M_{\min}\cap N_1}$ and $e_{M_1}(L^{M_{\min}}_{M_{\min}\cap P_1}(\sigma_{\min})) = e_{M_1}(L^M_{M\cap P_1}(\sigma))$. This ends the proof of Theorem 6.1 (i).

7. Ordinary functor $\mathrm{Ord}^G_{P_1}$

Let us keep a general reductive connected group G and a commutative ring R. Let $P_1 = M_1N_1$ be a standard parabolic subgroup of G and $(P = MN, \sigma, Q)$ an $R[G]$-triple with $P(\sigma) = G$.

In this section §7, we prove Theorem 6.1, Part (ii) after establishing some general results in §7.1 and §7.2, with varying assumptions on R. As in §6 for the coinvariant functor L^G_P, first we assume that σ is e-minimal, so that Δ_M is orthogonal to $\Delta \setminus \Delta_M$; it suffices to consider two special cases $P_1 \supset P$ (§7.3) and $\langle P_1, P\rangle = G$ (§7.4) and the general case is obtained in two stages, introducing the parabolic subgroup $\langle P_1, P\rangle = MP_1$. When σ is no longer assumed to be e-minimal, we proceed as above, using $\sigma_{\min}$.

7.1. **Haar measure and t-finite elements.** Let H be a locally profinite group acting on a locally profinite topological space X and on itself by left translation. For $x \in X$, we denote by H_x the H-stabilizer of x. The group H acts on $C^\infty_c(X, R)$ by $(hf)(x) = f(h^{-1}x)$ for $h \in H, f \in C^\infty_c(X, R), x \in X$.

Proposition 7.1. *Assume that R is a field and that there is a non-zero $R[H]$-linear map $C^\infty_c(H, R) \to C^\infty_c(X, R)$. Then for some $x \in X$ there is an R-valued left Haar measure on H_x.*

Proof. We show that the proposition follows from Bernstein's localization principle [Ber84b, 1.4] which, we remark, is valid for an arbitrary field R.

Let $C^\infty_c(H, R) \xrightarrow{\varphi} C^\infty_c(X, R)$ be a non-zero linear map. We show that there exists $x \in X$ such that $\mathrm{Hom}_R(C^\infty_c(H \times \{x\}, R), R) \neq 0$. We view φ as providing an integration along the fibres of the projection map $H \times X \to X$, that is, a non-zero linear map $C^\infty_c(H \times X, R) \xrightarrow{\Phi} C^\infty_c(X, R)$ defined by

$$\Phi(f)(x) = \varphi(f_x)(x)$$

for $x \in X, f \in C^\infty_c(H \times X, R)$, where $f_x \in C^\infty_c(H, R)$ sends $h \in H$ to $f(h, x)$. The dual of Φ is a non-zero linear map

$$\mathrm{Hom}_R(C^\infty_c(X, R), R) \xrightarrow{{}^t\Phi} \mathrm{Hom}_R(C^\infty_c(H \times X, R), R)$$

of image the space of linear functionals on $C^\infty_c(H \times X, R)$ vanishing on the kernel of Φ.

But $C_c^\infty(X,R)$ is also an R-algebra for the multiplication $\psi_1\psi_2(x)=\psi_1(x)\psi_2(x)$ if $\psi_1,\psi_2\in C_c^\infty(X,R)$ and $x\in X$. Then, $C_c^\infty(H\times X,R)$ is naturally a $C_c^\infty(X,R)$-module: for $\psi\in C_c^\infty(X,R)$ and $f\in C_c^\infty(H\times X,R)$, then $\psi f\in C_c^\infty(H\times X,R)$ is the function $(h,x)\mapsto(\psi f)(h,x)=\psi(x)f(h,x)$. The map Φ is $C_c^\infty(X,R)$-linear: $(\psi f)_x=\psi(x)f_x$ and $\Phi(\psi f)(x)=\varphi((\psi f)_x)(x)=\psi(x)\varphi(f_x)(x)=\psi(x)\Phi(f)(x)$. The image of ${}^t\Phi$ is a $C_c^\infty(X,R)$-submodule: for $\psi\in C_c^\infty(X,R)$ and $L\in\mathrm{Hom}_R(C_c^\infty(H\times X,R),R)$ vanishing on $\operatorname{Ker}\Phi$, $(\psi L)(f)=L(\psi f)$.

By Bernstein's localization principle, $\mathrm{Im}({}^t\Phi)$ is the closure of the span of those functionals in $\mathrm{Im}({}^t\Phi)$ which are supported on $H\times\{x\}$ for some $x\in X$. Consequently, as $\mathrm{Im}({}^t\Phi)\neq 0$, there exists $x\in X$ and a non-zero $L\in\mathrm{Hom}_R(C_c^\infty(H\times X,R),R)$ vanishing on $\operatorname{Ker}\Phi$ which factors through the restriction map $C_c^\infty(H\times X,R)\xrightarrow{\text{res}}C_c^\infty(H\times\{x\},R)$. There is a non-zero element $\mu\in\mathrm{Hom}_R(C_c^\infty(H\times\{x\},R),R)$ such that $L=\mu\circ\text{res}$.

Now assume that φ is H-equivariant. We show that μ is H_x-invariant. Indeed, denote by χ the characteristic function of a small open neighborhood V of x. Let $f\in C_c^\infty(H,R)$. Take $f\otimes\chi$ in $C_c^\infty(H\times X,R)$. Then $\Phi(f\otimes\chi)=\varphi(f)\chi$ whereas $\Phi(hf\otimes\chi)=\varphi(hf)\chi=(h\varphi(f))\chi$ for $h\in H_x$. We can certainly take V small enough for $\varphi(f)$ and $h\varphi(f)$ to be constant on V; as $hx=x$, they are equal at x hence on all V. In particular $L(f\otimes\chi)=L(hf\otimes\chi)$ which implies that μ is H_x-invariant.

Now, for $x\in X$, applying Bernstein's localization principle to the natural map $H\to H_x\backslash H$, the existence of a non-zero H_x-invariant element of $\mathrm{Hom}_R(C_c^\infty(H\times\{x\},R),R)$ implies the existence of a R-valued left Haar measure on H_x.

□

There is a variant of Proposition 7.1 where R is replaced by an R-module V with zero p-ordinary part.

Corollary 7.2. *Assume that V is an R-module with $\bigcap_{k\geq 0}p^kV=\{0\}$ and that there is a non-zero $R[H]$-linear map $\varphi: C_c^\infty(H,R)\to C_c^\infty(X,V)$. Then for some $x\in X$ there is a $\mathbb{F}_p$-valued left Haar measure on H_x.*

Proof. As $\cap_{k\geq 0}p^kV=\{0\}$, there exists a largest integer k such that the image of φ is contained in p^kV but not in $p^{k+1}V$. The map φ induces a non-zero $(R/pR)[H]$-linear map $C_c^\infty(H,R/pR)\to C_c^\infty(X,p^kV/p^{k+1}V)$. By R/pR-linearity, it restricts to a non-zero $\mathbb{F}_p[H]$-linear map $\varphi_p: C_c^\infty(H,\mathbb{F}_p)\to C_c^\infty(X,p^kV/p^{k+1}V)$. The values of the functions in the image of φ_p is a non-zero $\mathbb{F}_p$-subspace V_p of $p^kV/p^{k+1}V$ and composing with a $\mathbb{F}_p$-linear form on V_p, we get a non-zero $\mathbb{F}_p[H]$-linear map $C_c^\infty(H,\mathbb{F}_p)\to C_c^\infty(X,\mathbb{F}_p)$. Applying Proposition 7.1 to $R=\mathbb{F}_p$, we get the desired result. □

In the special case $X=H$ acting on itself by left translation, all stabilizers H_x are trivial, and there are non-zero $R[H]$-endomorphisms of $C_c^\infty(H,R)$, for example those given by right translations by elements of H.

Consider the special situation, which appears later in the proof of the theorem, where there is an automorphism t of H and an open compact subgroup H^0 of H such that $t^k(H^0)\subset t^{k+1}(H^0)$ for $k\in\mathbb{Z}$, $H=\bigcup_{k\in\mathbb{Z}}t^k(H^0)$ and $\{0\}=\bigcap_{k\in\mathbb{Z}}t^k(H^0)$. Let moreover W be an R-module with a **trivial** action of H and an action of t via an automorphism. Then we have a natural action of t on $C_c^\infty(H,W)$ - that we identify with $C_c^\infty(H,R)\otimes W$ - and on $\mathrm{Hom}_{R[H]}(C_c^\infty(H,R),C_c^\infty(H,W))$ by

$$tf(h)=t(f(t^{-1}h)),\quad (t\varphi)(f)=t(\varphi(t^{-1}f)),$$

for $h\in H, f\in C_c^\infty(H,W),\varphi\in\mathrm{Hom}_{R[H]}(C_c^\infty(H,R),C_c^\infty(H,W))$.

We recall that, for a monoid A and an $R[A]$-module V, an element $v \in V$ is **A-finite** if the R-module generated by the A-translates of v is finitely generated.

We say that V is **A-locally finite** if every element of V is A-finite, If A is generated by an element t, we say **t-finite** instead of A-finite. When R is noetherian, the set V^{A-f} of A-finite vectors in V is a submodule of V.

If $w \in W$ is t-finite, then $f \mapsto f \otimes w$ in $\mathrm{Hom}_{R[H]}(C_c^\infty(H,R), C_c^\infty(H,W))$ is obviously t-finite. Conversely:

Proposition 7.3. *When R is noetherian, any t-finite element of*

$$\mathrm{Hom}_{R[H]}(C_c^\infty(H,R), C_c^\infty(H,W))$$

has the form $f \mapsto f \otimes w$ for some t-finite vector $w \in W$.

Proof. For $r \in \mathbb{Z}$ let $f_r \in C_c^\infty(H,R)$ be the characteristic function of $t^r(H^0)$ so that $t^k f_r = f_{k+r}$ for $k \in \mathbb{Z}$, hf_r is the characteristic function of $ht^r(H^0)$ for $h \in H$, and for $r' \geq r$, $f_{r'} = \sum_{h \in t^{r'}(H^0)/t^r(H^0)} hf_r$. Any $f \in C_c^\infty(H,R)$ is a linear combination of H-translates of f_r, $r \in \mathbb{Z}$.

Let $\varphi \in \mathrm{Hom}_{R[H]}(C_c^\infty(H,R), C_c^\infty(H,W))$. The support of $\varphi(f_0) \in C_c^\infty(H,W)$ is contained in $t^r(H^0)$ for some integer $r \geq 0$. For $r' \geq 0$, the H-equivariance of φ implies that $\varphi(f_{r'}) = \sum_{h \in t^{r'}(H^0)/H^0} h\varphi(f_0)$; in particular, $\varphi(f_r)$ has support contained in $t^r(H^0)$ and since $\varphi(f_r)$ is $t^r(H^0)$-invariant, it has the form $f_r \otimes w$ for some $w \in W$. For $r' \geq r$, we have similarly $\varphi(f_{r'}) = \sum_{h \in t^{r'}(H^0)/t^r H^0} h\varphi(f_r) = f_{r'} \otimes w$. For $k \geq 0$, we compute

$$(t^k\varphi)(f_{r'+k}) = t^k(\varphi(t^{-k} f_{r'+k})) = t^k(\varphi(f_{r'})) = t^k(f_{r'} \otimes w) = f_{r'+k} \otimes t^k w. \tag{18}$$

Assume now that φ is t-finite. Then there is an integer $n \geq 1$ such that the $t^k\varphi$, $0 \leq k \leq n-1$, generate the R-submodule V_φ generated by the $t^k\varphi$, $h \in \mathbb{N}$, and there is a relation

$$t^n\varphi = a_1 t^{n-1}\varphi + \cdots + a_{n-1}t\varphi + a_n\varphi, \tag{19}$$

with $a_1, \ldots, a_n \in R$. Applying (19) to f_{n+r} and using $(t^k\varphi)(f_{n+r}) = f_{n+r} \otimes t^k w$ for $0 \leq k \leq n$ by (18), we get

$$f_{n+r} \otimes t^n w = f_{n+r} \otimes (a_1 t^{n-1} w + \cdots + a_{n-1} tw + a_n w).$$

So that $t^n w = a_1 t^{n-1} w + \cdots + a_{n-1} tw + a_n w$ and w is t-finite.

We have already seen that $\varphi(f_{r'}) = f_{r'} \otimes w$ for $r' \geq r$. Let $k \geq 1$ and assume that $\varphi(f_{r'}) = f_{r'} \otimes w$ for $r' \geq k$. Noting that $(t^i\varphi)(f_{n+k-1}) = f_{n+k-1} \otimes t^i w$ for $0 \leq i \leq n-1$ because $n + k - 1 - i \geq k$, we apply (19) to f_{n+k-1} and we deduce

$$(t^n\varphi)(f_{n+k-1}) = f_{n+k-1} \otimes (a_1 t^{n-1} w + \cdots + a_{n-1} tw + a_n w) = f_{n+k-1} \otimes t^n w,$$

so that $t^n(\varphi(f_{k-1})) = t^n(f_{k-1} \otimes w)$ and finally $\varphi(f_{k-1}) = f_{k-1} \otimes w$. This proves the proposition by descending induction on k. $\square$

We suppose now that W is a **free** R-module with a **trivial** action of H and of t. Let V be an $R[H]$-module with a compatible action of t. As above, we have a natural action of t on $\mathrm{Hom}_{R[H]}(C_c^\infty(H,R), V)$ and on $\mathrm{Hom}_{R[H]}(C_c^\infty(H,R), V \otimes W)$.

Proposition 7.4. *When R is noetherian, the natural map $\mathrm{Hom}_{R[H]}(C_c^\infty(H,R), V) \otimes W \to \mathrm{Hom}_{R[H]}(C_c^\infty(H,R), V \otimes W)$ induces an isomorphism between the submodules of t-finite elements.*

Proof. The natural map sends $\varphi \otimes w$ to $f \mapsto \varphi(f) \otimes w$. It is an embedding because W is R-free. It sends a t-finite element to a t-finite element because t acts trivially on W. Let $\varphi \in \mathrm{Hom}_{R[H]}(C_c^\infty(H,R), V \otimes W)$ and let $(w_i)_{i\in I}$ be an R-basis of W. For $f \in C_c^\infty(H,R)$ we write uniquely $\varphi(f) = \sum_{i\in I} v_i(f) \otimes w_i$ for $v_i(f) \in V$. For each $i \in I$, the map v_i is $R[H]$-linear and for each f, $v_i(f)$ vanishes outside some finite subset $I(f)$ of I. But it is not clear if the map v_i vanishes outside a finite subset of I. Now assume that φ is t-finite. As in (19), there exists $n \geq 1$ and $a_1, \dots, a_n \in R$ such that for each $i \in I$,

$$t^n v_i(t^{-n} f) = a_1 t^{n-1} v_i(t^{-n+1} f) + \cdots + a_{n-1} t v_i(t^{-1} f) + a_n v_i(f). \tag{20}$$

Let $I_0 = I(f_0)$ be a finite subset of I such that $v_i(f_0) = 0$ for $i \in I \setminus I_0$. For $r \geq 0$, $v_i(f_r) = 0$ for $i \in I \setminus I_0$ because f_r is a sum of H-translates of f_0. Let $k \in \mathbb{Z}$ and assume that for $r \geq k$, $v_i(f_r) = 0$ for $i \in I \setminus I_0$. Apply (20) to $f = f_{n+k-1}$ for $i \in I \setminus I_0$. This gives $t^n v_i(f_{k-1}) = 0$ hence $v_i(f_{k-1}) = 0$. As any $f \in C_c^\infty(H,R)$ is a linear combination of H-translates of f_k, $k \in \mathbb{Z}$, we have $v_i(f) = 0$ for $i \in I \setminus I_0$ and $\varphi(f) = \sum_{i \in I_0} v_i(f) \otimes w_i$ does belong to $\mathrm{Hom}_{R[H]}(C_c^\infty(H,R), V) \otimes W$; each of the $v_i \in \mathrm{Hom}_{R[H]}(C_c^\infty(H,R), V)$ for $i \in I_0$ is t-finite (because φ is t-finite), and that proves the proposition. □

7.2. **Filtrations.** We analyze the sequence (17) defining St_Q^G, by filtering $\mathrm{Ind}_Q^G \mathbf{1}$ by subspaces of functions with support in a union of $(Q, \overline{B})$ double cosets. An important fact is that the $(Q, \overline{B})$-cosets outside $Q\overline{P}_1$ do not contribute.

For convenience of references to [AHHV17], we first consider (Q, B) double cosets - we shall switch to $(Q, \overline{B})$-cosets later. A (Q, B)-double coset has the form QnB for some $n \in \mathcal{N}$; if w is the image of n in the finite Weyl group $\mathbb{W} = \mathcal{N}/Z$ we write, as is customary, QwB instead of QnB. The coset $\mathbb{W}_Q w$ is uniquely determined by QwB and contains a single element of minimal length. We write ${}^Q\mathbb{W}$ for the set of $w \in \mathbb{W}$ with minimal length in $\mathbb{W}_Q w$; they are characterized by the condition $w^{-1}(\alpha) > 0$ for $\alpha \in \Delta_Q$ [Car85, 2.3.3]. We have the disjoint union

$$G = \bigsqcup_{w \in {}^Q\mathbb{W}} QwB.$$

By standard knowledge, for $w, w' \in {}^Q\mathbb{W}$, the closure of QwB contains $Qw'B$ if and only if $w \geq w'$ in the Bruhat order of W. As in [AHHV17, V.7], we let $A \subset {}^Q\mathbb{W}$ be a non-empty upper subset (if $a \leq w, a \in A, w \in {}^Q\mathbb{W}$, then $w \in A$) so that QAB is open in G, and we choose $w_A \in A$ minimal for the Bruhat order; letting $A' = A \setminus \{w_A\}$, $QA'B$ is open in G too. Let $\text{c-Ind}_Q^{QAB} \mathbf{1} \subset \mathrm{Ind}_Q^G \mathbf{1}$ be the subspace of functions with support in QAB,

$$\text{c-Ind}_Q^{QAB} \mathbf{1} \simeq C_c^\infty(Q\backslash QAB, R).$$

For a parabolic subgroup Q_1 of G containing Q, we have $\mathrm{Ind}_{Q_1}^G \mathbf{1} \subset \mathrm{Ind}_Q^G \mathbf{1}$ and we let

$$I_{Q_1}^{QAB} = \mathrm{Ind}_{Q_1}^G \mathbf{1} \cap \text{c-Ind}_Q^{QAB} \mathbf{1}.$$

It is the subspace of functions with support in the union of the cosets $Q_1 x$ contained in QAB. We have $I_{Q_1}^{QA'B} \subset I_{Q_1}^{QAB}$. We also use an abbreviation $I_{Q_1,A} = I_{Q_1}^{QAB}$.

Lemma 7.5. *For $Q_1 \supset Q$, the injective natural map $I_{Q_1}^{QAB}/I_{Q_1}^{QA'B} \to \text{c-Ind}_Q^{QAB} \mathbf{1}/\text{c-Ind}_Q^{QA'B} \mathbf{1}$ is an isomorphism if $w_A \in {}^{Q_1}\mathbb{W}$, and $I_{Q_1}^{QAB} = I_{Q_1}^{QA'B}$ otherwise.*

Proof. We write $w = w_A$. Assume first that $w \notin {}^{Q_1}\mathbb{W}$. Write $w = vw'$ with $v \in \mathbb{W}_{Q_1}\backslash\{1\}, w' \in {}^{Q_1}\mathbb{W}$. We have $w' < w$ and w is minimal in A hence $w' \notin A$. Let $\varphi \in I_{Q_1,A}$. If the support of φ meets QwB, it meets $w'B$ and this is impossible because $w' \notin A$. Thus $\varphi \in I_{Q_1,A'}$ and $I_{Q_1,A} = I_{Q_1,A'}$ as desired.

Assume now that $w \in {}^{Q_1}\mathbb{W}$ and let $\varphi \in I_{Q,A}$. As $w \in {}^{Q_1}\mathbb{W}$, the natural map $U \mapsto Q_1\backslash Q_1wB$ induces a homeomorphism $(w^{-1}Uw \cap U)\backslash U \xrightarrow{\simeq} Q_1\backslash Q_1wB$; as $w \in {}^{Q}\mathbb{W}$, the natural map $U \mapsto Q\backslash QwB$ induces also a homeomorphism $(w^{-1}Uw \cap U)\backslash U \xrightarrow{\simeq} Q\backslash QwB$ [AHHV17, V.7]. Consequently, there is a function ψ on Q_1wB left invariant under Q_1 and locally constant with compact support modulo Q_1 which has the same restriction as φ to QwB. Set $A_{1,\geq w} \subset {}^{Q_1}\mathbb{W}$ to be the upper subset of u with $u \geq w$. The set $Q_1A_{1,\geq w}B$ is open in G and Q_1wB is closed in $Q_1A_{1,\geq w}B$. There exists a function $\tilde{\psi}$ on $Q_1A_{1,\geq w}B$ left invariant under Q_1 and locally constant with compact support modulo Q_1 which is equal to ψ on Q_1wB. For $u \in A_{1,\geq w}$ the double coset Q_1uB is the union of double cosets $QtuB$ for $t \in \mathbb{W}_{Q_1}$ with $tu \in {}^{Q}\mathbb{W}$; as $tu \geq u \geq w$ we have $tu \in A$ hence $Q_1uB \subset QAB$ and naturally $Q_1A_{1,\geq w}B \subset QAB$. Now, we have $\tilde{\psi} \in I_{Q_1,A}$, $\tilde{\psi}$ and φ have the same restriction to QwB, hence the same image in $I_{Q,A}/I_{Q,A'}$, and the map of the lemma is surjective. □

Lemma 7.6. *If $\mathcal{P}$ is a set of parabolic subgroups of G containing Q, then*

$$\left(\sum_{Q_1\in\mathcal{P}} \text{c-Ind}_{Q_1}^{G} \mathbf{1}\right) \cap \text{c-Ind}_{Q}^{QAB} \mathbf{1} = \sum_{Q_1\in\mathcal{P}} \text{c-Ind}_{Q_1}^{QAB} \mathbf{1}.$$

Proof. The left hand side obviously contains the right hand side. The reverse inclusion is proved as in [AHHV17, V.16 Lemma 23] by descending induction on the order of A. The case where $A = {}^{Q}\mathbb{W}$ being a tautology, we assume the result for A and we prove it for $A' = A \setminus \{w_A\}$. As $(\sum_{Q_1\in\mathcal{P}} \text{Ind}_{Q_1}^{G} \mathbf{1}) \cap I_{Q,A'}$ is nothing else than $(\sum_{Q_1\in\mathcal{P}} I_{Q_1,A}) \cap I_{Q,A'}$, we pick $f_{Q_1} \in I_{Q_1,A}$ for $Q_1 \in \mathcal{P}$ and assume that $\sum_{Q_1\in\mathcal{P}} f_{Q_1} \in I_{Q,A'}$; we want to prove that $\sum_{Q_1\in\mathcal{P}} f_{Q_1} \in \sum_{Q_1\in\mathcal{P}} I_{Q_1,A'}$.

If $w_A \notin {}^{Q_1}\mathbb{W}$, $f_{Q_1} \in I_{Q_1,A'}$ by Lemma 7.5. We are done if $w_A \notin {}^{Q_1}\mathbb{W}$ for all $Q_1 \in \mathcal{P}$.

Otherwise, $Q_1 \in \mathcal{P}$ such that $w_A \in {}^{Q_1}\mathbb{W}$ is contained in the parabolic subgroup Q_2 associated to $\Delta_2 = \{\alpha \in \Delta, w_A^{-1}(\alpha) > 0\}$ and $w_A \in {}^{Q_2}\mathbb{W}$; we choose $f_{Q_2} \in I_{Q_2,A}$ such that $f_{Q_1} - f_{Q_2} \in I_{Q_1,A'}$, that is possible by Lemma 7.5. We write $\sum_{Q_1\in\mathcal{P}} f_{Q_1}$ as

$$\sum_{Q_1\in\mathcal{P}} f_{Q_1} = \sum_{Q_1\in\mathcal{P}, w_A\notin {}^{Q_1}\mathbb{W}} f_{Q_1} + \sum_{Q_1\in\mathcal{P}, w_A\in {}^{Q_1}\mathbb{W}} (f_{Q_1} - f_{Q_2}) + \sum_{Q_1\in\mathcal{P}, w_A\in {}^{Q_1}\mathbb{W}} f_{Q_2}.$$

The last term on the right belongs also to $I_{Q,A'}$ because the other terms do, and even to $I_{Q_2,A'}$. We have $I_{Q_2,A'} \subset I_{Q_1,A'}$, and the last term belongs to $I_{Q_1,A'}$ for any $Q_1 \in \mathcal{P}$ such that $w \in {}^{Q_1}\mathbb{W}$. This ends the proof of the lemma. □

To express Lemmas 7.5, 7.6 in terms of $(Q, \overline{B})$-double cosets we apply the remark that $QwBw_0 = Qww_0\overline{B}$ if w_0 is the longest element in $\mathbb{W}$, so translating by w_0^{-1} a function with support in QAB gives a function with support in $QAw_0\overline{B}$. For a parabolic subgroup $Q_1 \subset Q$,

$$I_{Q_1}^{QAw_0\overline{B}} = \text{Ind}_{Q_1}^{G} \mathbf{1} \cap \text{c-Ind}_{Q}^{QAw_0\overline{B}} \mathbf{1}$$

is the set of functions obtained in this way from $I_{Q_1}^{QAB}$. We have $w \leq w'$ if and only if $w'w_0 \geq ww_0$ for $w, w' \in \mathbb{W}$ [BB05, Proposition 2.5.4], ${}^{Q}\mathbb{W}w_0$ is the set of $w \in \mathbb{W}$ with maximal length in $\mathbb{W}_Q w$, Aw_0 is a non-empty lower subset of ${}^Q\mathbb{W}w_0$ and $w_A w_0$ is a maximal element of Aw_0 for the Bruhat order. We get:

Lemma 7.7. *For $Q_1 \supset Q$, the natural map*

$$I_{Q_1}^{QAw_0\overline{B}}/I_{Q_1}^{QA'w_0\overline{B}} \to \text{c-Ind}_Q^{QAw_0\overline{B}}\mathbf{1}/\,\text{c-Ind}_Q^{QA'w_0\overline{B}}\mathbf{1}$$

is an isomorphism if $w_A \in {}^{Q_1}\mathbb{W}$, and $I_{Q_1}^{QAw_0\overline{B}} = I_{Q_1}^{QA'w_0\overline{B}}$ otherwise.

Lemma 7.8. *If $\mathcal{P}$ is a set of parabolic subgroups of G containing Q, then*

$$\left(\sum_{Q_1\in\mathcal{P}} \text{c-Ind}_{Q_1}^G \mathbf{1}\right) \cap \text{c-Ind}_Q^{QAw_0\overline{B}}\mathbf{1} = \sum_{Q_1\in\mathcal{P}} \text{c-Ind}_{Q_1}^{QAw_0\overline{B}}\mathbf{1}.$$

Note that

$$\text{c-Ind}_Q^{QAw_0\overline{B}}\mathbf{1}/\,\text{c-Ind}_Q^{QA'w_0\overline{B}}\mathbf{1} \simeq \text{c-Ind}_Q^{Qw_Aw_0\overline{B}}\mathbf{1}$$

as representations of $\overline{B}$. The image of $\text{Ind}_Q^{QAw_0\overline{B}}\mathbf{1}$ in St_Q^G is denoted by $\text{St}_Q^{QAw_0\overline{B}}$.

Lemma 7.9. *The R-modules $\text{c-Ind}_Q^{QAw_0\overline{B}}\mathbf{1}$ and $\text{St}_Q^{QAw_0\overline{B}}$ are free.*

Proof. We denote $\text{St}_Q^G = \text{St}_Q^G(R)$ or $\text{St}_Q^{QAw_0\overline{B}} = \text{St}_Q^{QAw_0\overline{B}}(R)$ to indicate the coefficient ring R. The module $C_c^\infty(Q\backslash QAw_0\overline{B},\mathbb{Z})$ and $\text{St}_Q^G(\mathbb{Z})$ are free [Ly15] and a submodule of the free $\mathbb{Z}$-module $\text{St}_Q^G(\mathbb{Z})$ is free, hence $\text{St}_Q^{QAw_0\overline{B}}(\mathbb{Z})$ is also free. The exact sequence of free modules defining $\text{St}_Q^G(\mathbb{Z})$ or $\text{St}_Q^{QAw_0\overline{B}}(\mathbb{Z})$ remains exact when we tensor by R. As $C_c^\infty(Q\backslash QAw_0\overline{B},R) = C_c^\infty(Q\backslash QAw_0\overline{B},\mathbb{Z})\otimes_{\mathbb{Z}} R$, we have also $\text{St}_Q^G(\mathbb{Z})\otimes_{\mathbb{Z}} R = \text{St}_Q^G(R)$ and $\text{St}_Q^{QAw_0\overline{B}}(\mathbb{Z})\otimes_{\mathbb{Z}} R = \text{St}_Q^{QAw_0\overline{B}}(R)$. Thus, the lemma. □

Lemma 7.10. $\text{St}_Q^{QAw_0\overline{B}} = \text{St}_Q^{QA'w_0\overline{B}}$ *if $w_A \in {}^{Q_1}\mathbb{W}$ for some $Q_1 \in \mathcal{Q}$ (notation of (6.2.1)). Otherwise the map* $\text{c-Ind}_Q^{QAw_0\overline{B}}\mathbf{1} \to \text{St}_Q^{QAw_0\overline{B}}$ *induces an isomorphism*

$$\text{c-Ind}_Q^{QAw_0\overline{B}}\mathbf{1}/\,\text{c-Ind}_Q^{QA'w_0\overline{B}}\mathbf{1} \simeq \text{St}_Q^{QAw_0\overline{B}}/\text{St}_Q^{QA'w_0\overline{B}}.$$

Proof. Set $\overline{I}_{Q_1,A} = I_{Q_1}^{QAw_0\overline{B}}$. If $w_A \in {}^{Q_1}\mathbb{W}$ for some $Q_1 \in \mathcal{Q}$, then by Lemma 7.7, $\overline{I}_{Q,A} = \overline{I}_{Q_1,A}+\overline{I}_{Q,A'}$ and taking images in St_Q^G we get $\text{St}_Q^{QA'w_0\overline{B}} = \text{St}_Q^{QAw_0\overline{B}}$. Otherwise, $\overline{I}_{Q_1,A} = \overline{I}_{Q_1,A'}$ for all $Q_1 \in \mathcal{Q}$ by Lemma 7.7. The kernel of the map $\overline{I}_{Q,A} \to \text{St}_Q^{QAw_0\overline{B}}$ is $\sum_{Q_1\in\mathcal{Q}}\overline{I}_{Q_1,A}$ by Lemma 7.8 and similarly for A'. Hence the kernels of the maps $\overline{I}_{Q,A} \to \text{St}_Q^{QAw_0\overline{B}}$ and $\overline{I}_{Q,A'} \to \text{St}_Q^{QA'w_0\overline{B}}$ are the same, and we get the last assertion. □

Proposition 7.11. *Assume that P_1 and Q_1 contain Q but that P_1 does not contain Q_1. Then* $\text{Ind}_{Q_1}^G\mathbf{1} \cap \text{c-Ind}_Q^{Q\overline{P}_1}\mathbf{1} = 0$.

Proof. We prove that the assumptions of the proposition imply that $Q\overline{P}_1$ does not contain any coset $Q_1 x$. We note that $P_1 \supset Q$ implies

$$Q\overline{P}_1 = P_1\overline{P}_1 = N_1 M_1 \overline{N}_1. \tag{21}$$

The inclusion $P_1\overline{P}_1 \supset Q\overline{P}_1$ is obvious, and the inverse inclusion (and the second equality) follows from $N_1 \subset N_Q$ and $P_1\overline{P}_1 = N_1\overline{P}_1, Q\overline{P}_1 = N_Q\overline{P}_1$. If $Q\overline{P}_1$ contains a coset Q_1x, we can suppose that $x = \overline{p}_1$ with $\overline{p}_1 \in \overline{P}_1$. We have $N_1 \subset N_Q \subset Q_1$ and $Q_1\overline{p}_1 \subset P_1\overline{P}_1$ implies $Q_1 \subset P_1\overline{P}_1$, in particular $M_{Q_1} \subset P_1\overline{P}_1$. By that latter inclusion, for $y \in M_{Q_1}$ there exist unique $n_1 \in N_1, m_1 \in M_1, \overline{n}_1 \in \overline{N}_1$ with $y = n_1m_1\overline{n}_1$. For any central element z of M_{Q_1}, we have $zyz^{-1} = y$ and by uniqueness $zn_1z^{-1} = n_1,\ zm_1z^{-1} = m_1,\ z\overline{n}_1z^{-1} = \overline{n}_1$. But then, $n_1, m_1, \overline{n}_1 \in M_{Q_1}$ and we deduce $M_{Q_1} = (M_{Q_1} \cap N_1)(M_{Q_1} \cap M_1)(M_{Q_1} \cap \overline{N}_1)$; this contradicts the fact that $M_{Q_1} \cap P_1$ is a proper parabolic subgroup of M_{Q_1} when P_1 does not contain Q_1. □

Corollary 7.12. *For $P_1 \supset Q$, the exact sequence* (17) *induces an exact sequence of $\overline{P}_1$-modules*

$$0 \to \sum_{Q \subsetneq Q_1 \subset P_1} (\mathrm{Ind}_{Q_1}^G \mathbf{1} \cap \text{c-Ind}_Q^{Q\overline{P}_1} \mathbf{1}) \to \text{c-Ind}_Q^{Q\overline{P}_1} \mathbf{1} \to \mathrm{St}_Q^{Q\overline{P}_1} \to 0.$$

7.3. **Case $P_1 \supset P$.** Assume that σ is e-minimal, hence Δ_M is orthogonal to $\Delta \setminus \Delta_M$, and that $P_1 \supset P$ in this whole section §7.3. We start the proof of the theorem 6.1 (ii).

Proposition 7.13. *Assume $\sigma_{p-ord} = \{0\}$. When $w \in \mathbb{W} \setminus \mathbb{W}_Q\mathbb{W}_{M_1}$,*

$$\mathrm{Hom}_{\overline{N}_1}(C_c^\infty(\overline{N}_1, R), e(\sigma) \otimes \text{c-Ind}_Q^{Qw\overline{B}} \mathbf{1}) = 0$$

Note that $w \in \mathbb{W} \setminus \mathbb{W}_Q\mathbb{W}_{M_1}$ is equivalent to $Qw\overline{B} \not\subset Q\overline{P}_1$ and that $\overline{N}_1$ acts trivially on $e(\sigma)$ because $P_1 \supset P$ as in (6.2.1).

Proof. As $\sigma_{p-ord} = 0$, Corollary 7.2 applied to $H = \overline{N}_1,\ X = Q\backslash Qw\overline{B}, V$ the space of σ, implies

$$\mathrm{Hom}_{\overline{N}_1}(C_c^\infty(\overline{N}_1, R), e(\sigma) \otimes \text{c-Ind}_Q^{Qw\overline{B}} \mathbf{1}) = \mathrm{Hom}_{\overline{N}_1}(C_c^\infty(\overline{N}_1, R), e(\sigma) \otimes C_c^\infty(Q\backslash Qw\overline{B}, R) = 0,$$

if the $\overline{N}_1$-fixator of any coset Qx contained in $Qw\overline{B}$ is infinite (the infinite closed subgroups of $\overline{N}_1$ being locally pro-p-groups do not admit an $\mathbb{F}_p$-valued Haar measure). This latter property is equivalent to $Q \cap w\overline{N}_1w^{-1}$ infinite, because $\overline{N}_1$ is normalized by $\overline{P}_1 \supset \overline{U}$. Indeed, $Qw\overline{B} = Qw\overline{U}$ and $Qx = Qw\overline{u}$ with $\overline{u} \in \overline{U}$. For $\overline{n}_1 \in \overline{N}_1$, $Qw\overline{u}\overline{n}_1 = Qw\overline{u}$ if and only if $\overline{u}\overline{n}_1\overline{u}^{-1}$ fixes Qw if and only if $\overline{u}\overline{n}_1\overline{u}^{-1} \in w^{-1}Qw \cap \overline{N}_1$.

When $w \in \mathbb{W} \setminus \mathbb{W}_Q\mathbb{W}_{M_1}$, there exists $\beta \in -\Phi_{N_1} = \Phi_{\overline{N}_1}$ with $w(\beta) \in \Phi_{N_Q}$ by Lemma 5.13. The group $Q \cap w\overline{N}_1w^{-1}$ is infinite because it contains $U_{w(\beta)}$. We get the proposition. □

Corollary 7.14. *When $\sigma_{p-ord} = \{0\}$, we have*

$$\mathrm{Hom}_{\overline{N}_1}(C_c^\infty(\overline{N}_1, R), e(\sigma) \otimes \mathrm{Ind}_Q^G \mathbf{1}) = \mathrm{Hom}_{\overline{N}_1}(C_c^\infty(\overline{N}_1, R), e(\sigma) \otimes \text{c-Ind}_Q^{Q\overline{P}_1} \mathbf{1}),$$
$$\mathrm{Hom}_{\overline{N}_1}(C_c^\infty(\overline{N}_1, R), e(\sigma) \otimes \mathrm{St}_Q^G) = \mathrm{Hom}_{\overline{N}_1}(C_c^\infty(\overline{N}_1, R), e(\sigma) \otimes \mathrm{St}_Q^{Q\overline{P}_1}).$$

Proof. $Q\overline{P}_1$ is open in G (a union of Q-translates of $N_1\overline{P}_1$) and there is a sequence of double cosets $Qw_i\overline{B},\ w_i \in \mathbb{W},\ i = 1, \ldots, r$, disjoint form each other and not contained in $Q\overline{P}_1$ such that

$$X_i = Q\overline{P}_1 \sqcup \left(\bigsqcup_{j \le i} Qw_j\overline{B} \right)$$

is open in G and $G = X_r$. We reason by descending induction on $i \leq r$. Consider the exact sequence of free R-modules (Lemma 7.9)

$$0 \to \text{c-Ind}_Q^{X_{i-1}} \mathbf{1} \to \text{c-Ind}_Q^{X_i} \mathbf{1} \to \text{c-Ind}_Q^{Qw_i\overline{B}} \mathbf{1} \to 0.$$

Tensoring by $e(\sigma)$ keeps an exact sequence, and applying $\mathrm{Hom}_{\overline{N}_1}(C_c^\infty(\overline{N}_1, R), -)$ we obtain an isomorphism (Proposition 7.13 and the latter functor is left exact)

$$\mathrm{Hom}_{\overline{N}_1}(C_c^\infty(\overline{N}_1, R), e(\sigma) \otimes \text{c-Ind}_Q^{X_{i-1}} \mathbf{1}) \xrightarrow{\simeq} \mathrm{Hom}_{\overline{N}_1}(C_c^\infty(\overline{N}_1, R), e(\sigma) \otimes \text{c-Ind}_Q^{X_i} \mathbf{1}).$$

Composing these isomorphisms we get the first equality of the corollary. For the second equality, we suppose that each w_i has maximal length in the coset $\mathbb{W}_Q w_i$ and is maximal in $\{w_1, \ldots, w_i\}$ for the Bruhat order. This is possible because $Q\overline{P}_1 = \bigcup_{w \in \mathbb{W}_Q \mathbb{W}_{M_1}} Qw\overline{P}_1$ and $\mathbb{W}_Q \mathbb{W}_{M_1}$ is a lower set for the Bruhat order hence there are no $w, w' \in \mathbb{W}$ of maximal length in their cosets $\mathbb{W}_Q w, \mathbb{W}_Q w'$ with $w \geq w'$ and $Qw \subset Q\overline{P}_1$ but $Qw' \not\subset Q\overline{P}_1$. Now, we have the exact sequence of free R-modules (Lemma 7.9),

$$0 \to \mathrm{St}_Q^{X_{i-1}} \to \mathrm{St}_Q^{X_i} \to Y_i \to 0$$

where Y_i is either 0 or $\text{c-Ind}_Q^{Qw_i\overline{B}} \mathbf{1}$ by lemma 7.10. Then proceeding as above for the first equality, we get the second equality of the corollary. □

Proposition 7.15. *Assume R noetherian, σ admissible, $\sigma_{p-ord} = 0$ and $P_1 \supset Q$. Then $\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma) \otimes \mathrm{Ind}_Q^G \mathbf{1})$ and $\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma) \otimes \mathrm{St}_Q^G)$ are naturally isomorphic to $e_{M_1}(\sigma) \otimes \mathrm{Ind}_{Q \cap M_1}^{M_1} \mathbf{1}$ and $e_{M_1}(\sigma) \otimes \mathrm{St}_{Q \cap M_1}^{M_1}$.*

Proof. Noting that $Q\overline{P}_1 = P_1 \overline{N}_1$ because $P_1 \supset Q$ and $N_1 \subset N_Q$, the $\overline{P}_1$-module $\text{c-Ind}_Q^{Q\overline{P}_1} \mathbf{1}$ identifies with

$$\text{c-Ind}_{Q \cap M_1}^{M_1} \mathbf{1} \otimes C_c^\infty(\overline{N}_1, R)$$

where $\overline{N}_1$ acts by right translation on $C_c^\infty(\overline{N}_1, R)$ and trivially on $\text{c-Ind}_{Q \cap M_1}^{M_1} \mathbf{1}$, whereas M_1 acts by conjugation on $\overline{N}_1$ on the second factor and right translation on the first. If $\sigma_{p-ord} = 0$, it suffices to recall Corollary 7.14 to identify $\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma) \otimes \mathrm{Ind}_Q^G \mathbf{1}) = \mathrm{Ord}_{\overline{P}_1}^G(e(\sigma) \otimes \text{c-Ind}_Q^{Q\overline{P}_1} \mathbf{1})$ with the subspace of $Z(M_1)$-finite vectors in

$$\mathrm{Hom}_{R[\overline{N}_1]}(C_c^\infty(\overline{N}_1, R), e(\sigma) \otimes \mathrm{Ind}_{Q \cap M_1}^{M_1} \mathbf{1} \otimes C_c^\infty(\overline{N}_1, R)). \tag{22}$$

By Remark 4.18 we may even take only t-finite vectors where $t = z^{-1}$ and $z \in Z(M)$ contracts strictly N (subsection 2.5). Put $W = e_{M_1}(\sigma) \otimes \mathrm{Ind}_{M_1 \cap Q}^{M_1} \mathbf{1}$ and then $W \otimes \mathrm{Id}$ for the subspace of (22) made of the maps $\varphi \mapsto f \otimes \varphi$ for $f \in W$. If R is noetherian, $W \otimes \mathrm{Id}$ is $Z(M_1)$-locally finite because W is an admissible R-representation of M_1 (a vector $w \in W$ is fixed by an open compact subgroup J of M_1 and W^J is a finitely generated R-module, invariant by $Z(M_1)$). Hence $\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma) \otimes \text{c-Ind}_Q^G \mathbf{1})$ contains $W \otimes \mathrm{Id}$. Applying Proposition 7.3 with $H = \overline{N}_1$ and some suitable $t \in Z(M_1)$ we find that $W \otimes \mathrm{Id}$ is the space of t-finite vectors in (22). This provides an isomorphism

$$\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma) \otimes \mathrm{Ind}_Q^G \mathbf{1}) \simeq e_{M_1}(\sigma) \otimes \mathrm{Ind}_{Q \cap M_1}^{M_1} \mathbf{1}.$$

Similarly, for $Q \subset Q_1 \subset P_1$, $\text{c-Ind}_{Q_1}^{Q_1\overline{P}_1} \mathbf{1} \simeq \mathrm{Ind}_{Q_1 \cap M_1}^{M_1} \mathbf{1} \otimes C_c^\infty(\overline{N}_1, R)$, as $R[\overline{P}_1]$-modules.

The exact sequence in Corollary 7.12 is made of free R-modules (Lemma 7.9) hence remains exact under tensorisation by $e(\sigma)$, we get a $R[\overline{P}_1]$-isomorphism

$$e_{M_1}(\sigma)\otimes \mathrm{St}_Q^{Q\overline{P}_1}\simeq e_{M_1}(\sigma)\otimes \mathrm{St}_{Q\cap M_1}^{M_1}\otimes C_c^\infty(\overline{N}_1,R)$$

As R is noetherian and $\sigma_{p-ord}=0$, $\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma)\otimes \mathrm{St}_Q^G)=\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma)\otimes \mathrm{St}_Q^{Q\overline{P}_1})$ identifies (Corollary 7.14) with the subspace of $Z(M_1)$-finite vectors in

$$\mathrm{Hom}_{R[\overline{N}_1]}(C_c^\infty(\overline{N}_1,R),e_{M_1}(\sigma)\otimes \mathrm{St}_{Q\cap M_1}^{M_1}\otimes C_c^\infty(\overline{N}_1,R)),$$

which is made out of the maps $\varphi\mapsto f\otimes\varphi$ for $f\in \mathrm{St}_{Q\cap M_1}^{M_1}$ by the same reasoning as above, thus providing an isomorphism

$$\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma)\otimes \mathrm{St}_Q^G)\simeq e_{M_1}(\sigma)\otimes \mathrm{St}_{Q\cap M_1}^{M_1}.$$

This ends the proof of the proposition. □

Proposition 7.16. *When $P_1\not\supset Q$ and $\sigma_{p-ord}=\{0\}$, then*

$$\mathrm{Hom}_{\overline{N}_1}(C_c^\infty(\overline{N}_1,R),e(\sigma)\otimes \mathrm{Ind}_Q^G\mathbf{1})=\mathrm{Hom}_{\overline{N}_1}(C_c^\infty(\overline{N}_1,R),e(\sigma)\otimes \mathrm{St}_Q^G)=0.$$

Proof. As allowed by Corollary 7.14, we work with

$$\mathrm{Hom}_{\overline{N}_1}(C_c^\infty(\overline{N}_1,R),e(\sigma)\otimes \text{c-Ind}_Q^{Q\overline{P}_1}\mathbf{1}),\quad \mathrm{Hom}_{\overline{N}_1}(C_c^\infty(\overline{N}_1,R),e(\sigma)\otimes \mathrm{St}_Q^{Q\overline{P}_1}).$$

We filter $Q\overline{P}_1$ by double cosets $Qw\overline{B}$, $w\in\mathbb{W}_{M_1}$, as above. We simply need the following lemma. □

Lemma 7.17. *When $P_1\not\supset Q$, $w\in\mathbb{W}_{M_1}$ and $\sigma_{p-ord}=\{0\}$, then*

$$\mathrm{Hom}_{R[\overline{N}_1]}(C_c^\infty(\overline{N}_1,R),e(\sigma)\otimes \text{c-Ind}_Q^{Qw\overline{B}}\mathbf{1})=0.$$

Proof. As in Proposition 7.13, assuming $\sigma_{p-ord}=0$ that follows from Corollary 7.2 applied to $H=\overline{N}_1$ and $X=Q\backslash Qw\overline{B}, V=e(\sigma)$ if $Q\cap w\overline{N}_1w^{-1}$ is not trivial. When $w\in\mathbb{W}_{M_1}$, we have $\overline{N}_1=w\overline{N}_1w^{-1}$ and the hypothesis that P_1 does not contains Q implies that there is $\alpha\in\Delta_Q$ not contained in Δ_{P_1}. The group $Q\cap w\overline{N}_1w^{-1}=Q\cap\overline{N}_1$ is not trivial because it contains $U_{-\alpha}$. We get the lemma. □

Corollary 7.18. *Assume R noetherian, σ admissible, $\sigma_{p-ord}=\{0\}$, and $P_1\not\supset Q$. Then* $\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma)\otimes \mathrm{Ind}_Q^G\mathbf{1})=\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma)\otimes \mathrm{St}_Q^G)=0$.

7.4. **Case $\langle P,P_1\rangle=G$.** Assume that σ is e-minimal and that $\langle P,P_1\rangle=G$.

Proposition 7.19. *Assume R noetherian, σ admissible. For X_Q^G equal to $\mathrm{Ind}_Q^G\mathbf{1}$ or St_Q^G, we have*

$$\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma)\otimes X_Q^G)\simeq e_{M_1}(\mathrm{Ord}_{M\cap\overline{P}_1}^M(\sigma))\otimes X_{M_1\cap Q}^{M_1}.$$

Proof. We have $P_1\supset P_\sigma$, or equivalently $M_1\supset M_\sigma$ and $N_1\subset N_\sigma$. As $N_1\subset M'$, N_1 acts trivially on $\mathrm{Ind}_Q^G\mathbf{1}$ (hence on its quotient St_Q^G) because $G=M'M_\sigma$ acts on $\mathrm{Ind}_Q^G\mathbf{1}$ trivially on M' (Δ_M and Δ_σ are orthogonal of union Δ). As $M_1\supset M_\sigma$, $Z(M_1)$ commutes with M_σ and acts trivially on St_Q^G. We can apply Proposition 7.4 to $H=\overline{N}_1, V=e(\sigma), W=X_Q^G$ and $t\in Z(M_1)$ strictly contracting N_1 (subsection 2.5), to get isomorphisms

$$\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma)\otimes X_Q^G)\simeq \mathrm{Ord}_{\overline{P}_1}^G(e(\sigma))\otimes X_Q^G,$$

as representations of M_1. As $M_1 \supset M_\sigma$, the restriction to M_1 of X_Q^G is $X_{Q\cap M_1}^{M_1}$. To prove the desired result, we need to identify $\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma))$ and $e_{M_1}(\mathrm{Ord}_{M\cap\overline{P}_1}^M(\sigma))$. Put $Y = \mathrm{Hom}_{R[\overline{N}_1]}(C_c^\infty(\overline{N}_1, R), V)$. Then $\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma)) = Y^{Z(M_1)-f}$ and $\mathrm{Ord}_{M\cap\overline{P}_1}^M(\sigma) = Y^{Z(M_1\cap M)-f}$. As $Z(M_1 \cap M) \supset Z(M_1)$, a $Z(M_1 \cap M)$-finite vector is also $Z(M_1)$-finite. On the other hand, $Z(M_1 \cap M) \cap M'_\sigma$ acts trivially on $\overline{N}_1$ and V hence on Y. The maximal compact subgroup $Z(M_1 \cap M)^0$ of $Z(M_1 \cap M)$ acts smoothly on Y, hence all vectors in Y are $Z(M_1 \cap M)^0$-finite.

Lemma 7.20. *$Z(M_1)Z(M_1 \cap M)^0(Z(M_1 \cap M) \cap M'_\sigma)$ has finite index in $Z(M_1 \cap M)$.*

Granted that lemma, the inclusion $Y^{Z(M_1)-f} \subset Y^{Z(M_1\cap M)-f}$ which is obviously $M_1 \cap M$-equivariant is an isomorphism. As $Y^{Z(M_1)-f}$ is a representation of M_1 it is $e_{M_1}(Y^{Z(M_1\cap M)-f})$, which is what we want to prove.

We have $Z(M_1\cap M)^0 = Z(M_1\cap M)\cap T^0$. It suffices to prove that the image of $Z(M_1)(Z(M_1\cap M) \cap M'_\sigma)$ in $X_*(\mathbf{T})$ via the map $v : Z \to X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ defined in §2.1, has finite index in the image of $Z(M_1 \cap M)$. The orthogonal of $Z(M_1 \cap M)$ in $X^*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is contained in the orthogonal of $Z(M_1)(Z(M_1 \cap M) \cap M'_\sigma)$. It suffices to show the inverse inclusion. The orthogonal of $Z(M_1)$ in $X^*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by Δ_{M_1}. The image by v of $Z(M_1\cap M)\cap M'_\sigma$ in $X_*(\mathbf{T})$ containing the coroots of Δ_σ, its orthogonal is contained in Δ_M. We see that the orthogonal for $Z(M_1)(Z(M_1 \cap M)\cap M'_\sigma)$ in $X^*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is contained in $\Delta_{M_1} \cap \Delta_M$. As $\Delta_{M_1\cap M} = \Delta_{M_1} \cap \Delta_M$ is the orthogonal of $Z(M_1 \cap M)$ in $X^*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$, the lemma is proved. □

This ends the proof of Proposition 7.19.

7.5. **General case.** 1) First we assume that σ is e-minimal. We prove Theorem 6.1 (ii) in stages, introducing the standard parabolic subgroup $P_2 = \langle P_1, P\rangle$ and taking successively $\mathrm{Ord}_{\overline{P}_2}^G$ and $\mathrm{Ord}_{M_2\cap\overline{P}_1}^{M_2}$ using the transitivity of $\mathrm{Ord}_{\overline{P}_1}^G$. For X_Q^G equal to $\mathrm{Ind}_Q^G \mathbf{1}$ or St_Q^G, we have

$$\begin{aligned}
\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma) \otimes X_Q^G) &= \mathrm{Ord}_{M_2\cap\overline{P}_1}^{M_2}(\mathrm{Ord}_{\overline{P}_2}^G(e(\sigma) \otimes X_Q^G)) \\
&= \begin{cases} \mathrm{Ord}_{M_2\cap\overline{P}_1}^{M_2}(e_{M_2}(\sigma) \otimes X_{Q\cap M_2}^{M_2}) & \text{if } P_2 \supset Q \\ 0 & \text{if } P_2 \not\supset Q \end{cases} \\
&= \begin{cases} e_{M_1}(\mathrm{Ord}_{M\cap\overline{P}_1}^M \sigma) \otimes X_{Q\cap M_1}^{M_1} & \text{if } P_2 \supset Q \\ 0 & \text{if } P_2 \not\supset Q. \end{cases}
\end{aligned}$$

The second equality follows from Proposition 7.15 for the first case and Corollary 7.18 for the second case, and the third one from Proposition 7.19. This ends the proof of Theorem 6.1, Part (ii) when Δ_M is orthogonal to $\Delta \setminus \Delta_M$.

2) General case. As at the end of §6.2, we introduce $P_{\min} = M_{\min}N_{\min}$ and an e-minimal representation $\sigma_{\min}$ of $M_{\min}$. The case 1) gives

$$\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma_{\min}) \otimes X_Q^G) = \begin{cases} e_{M_1}(\mathrm{Ord}_{M_{\min}\cap\overline{P}_1}^{M_{\min}} \sigma_{\min}) \otimes X_{Q\cap M_1}^{M_1} & \text{if} \langle P_1, P_{\min}\rangle \supset Q, \\ 0 & \text{if} \langle P_1, P_{\min}\rangle \not\supset Q. \end{cases} \tag{23}$$

We have $e(\sigma) = e(\sigma_{\min})$. So we can suppress *min* on the left hand side. We show that we can also suppress *min* on the right hand side.

If $\langle P_1, P\rangle \not\supset Q$ then $\langle P_1, P_{\min}\rangle \not\supset Q$ as $P_{\min} \subset P$, hence $\mathrm{Ord}_{\overline{P}_1}^G(e(\sigma) \otimes X_Q^G) = 0$.

If $\langle P_1, P\rangle \supset Q$ but $\langle P_1, P_{\min}\rangle \not\supset Q$, then $\mathrm{Ord}^G_{\overline{P}_1}(e(\sigma)\otimes X^G_Q) = 0$ and we now prove $\mathrm{Ord}^M_{M\cap\overline{P}_1}\sigma = 0$. Our hypothesis implies that there exists a root $\alpha\in\Delta_P$ which does not belong to $\Delta_1\cup\Delta_{min}$. The root subgroup $U_{-\alpha}$ is contained in $M\cap\overline{N}_1$ and acts trivially on σ. Reasoning as in the proof of Proposition 7.13, $\mathrm{Hom}_{M\cap\overline{N}_1}(C^\infty_c(M\cap\overline{N}_1, R),\sigma) = 0$ hence $\mathrm{Ord}^M_{M\cap\overline{P}_1}\sigma = 0$.

If $\langle P_1, P_{\min}\rangle \supset Q$ then $J\subset\Delta_1 = \Delta_{P_1}$ where $J = \Delta_M\setminus\Delta_{\min}$. The extensions to M_1 of

$$\mathrm{Ord}^M_{M\cap\overline{P}_1}\sigma = (\mathrm{Hom}_{R[M\cap\overline{N}_1]}(C^\infty_c(M\cap\overline{N}_1, R),\sigma))^{Z(M\cap M_1)-f}$$

(see (4)) and of $\mathrm{Ord}^{M_{\min}}_{M_{\min}\cap\overline{P}_1}\sigma_{\min}$ are equal as we show now:

The group $M\cap\overline{N}_1$ is generated by the root subgroups U_α for α in Φ^-_M not in Φ_1. Noting that $\Phi_M\setminus\Phi_{\min} = \Phi_J$ is disjoint from $\Phi_{\min}$ and contained in $\Phi_1 = \Phi_{M_1}$, a root α in Φ^-_M not in Φ_1 belongs to $\Phi_{\min}$; hence $M\cap\overline{N}_1 = M_{\min}\cap\overline{N}_1$.

The group $Z(M\cap M_1)$ is contained in $Z(M_{\min}\cap M_1)$. Moreover $T\cap M'_J$ acts trivially on σ and on $M\cap\overline{N}_1$ and, reasoning as in 7.20, $Z(M\cap M_1)(Z(M_{\min}\cap M_1)\cap M'_J)$ has finite index in $Z(M_{\min}\cap M_1)$. Consequently taking $Z(M_{\min}\cap M_1)$-finite vectors or $Z(M\cap M_1)$-finite vectors in $\mathrm{Hom}_{R[M\cap\overline{N}_1]}(C^\infty_c(M\cap\overline{N}_1, R),\sigma)$ gives the same answer. This finishes the proof of Theorem 6.1 (ii) .

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