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Abstract

The scalar 2-D Helmholtz' equation (i.e., "membrane waves") can be used to model surface-wave propagation in a laterally smooth, lossless half space. Building on this known result, we develop an algorithm to localize earthquake sources based on surface-wave data, via numerical time-reversal on a membrane, where monochromatic waves propagate with the phase velocity of Rayleigh or Love waves at the same frequency. By conducting monochromatic membrane-wave time-reversal simulations at various frequencies and combining the results, broadband time-reversed surface waves can be modeled. Importantly, membrane-wave modeling is computationally much less expensive than three-dimensional surface-wave modeling. We first explain rigorously the relationship between surface waves and membrane waves. Our mathematical treatment is slightly different from those found in the literature, in that it does not invoke variational principles. We next implement our time-reversal algorithm via spectral elements as well as simple ray tracing. Both implementations account for the effects of lateral variations in phase velocity. We validate the two resulting tools by means of several numerical experiments. This includes synthetic tests, as well as the localization of a virtual source based on a data set of real ambient-noise cross correlations, and the localization of the epicenter of a real earthquake from real, raw data. In this study, applications are limited to Northern Italy and the Alpine arc, where we have access to recent, high resolution phase velocity maps, ambient-noise cross correlations and data from a recent, relatively large earthquake. The accuracy of epicenter location despite non-uniformity in station coverage encourages further applications of our method, in particular to the task of mapping large-earthquake rupture in space and time.

1 Introduction

Estimates of seismic slip as a function of position and time for a given earthquake are obtained today in different ways, depending on the magnitude and depth of the earthquake, and on the instrumental coverage. Several different types of seismic and geodetic observations are employed. Dense networks of strong-motion accelerometers are currently deployed in seismic regions worldwide; they are designed to record the high-frequency oscillations generated by a nearby event, but they have little sensitivity to the lower frequencies, and cannot be used to constrain the properties of far earthquakes. At the opposite end of the frequency spectrum, data from GPS networks and satellite geodesy are used as constraints of the final slip associated with an earthquake; they provide good resolution of the surface expression of the rupture, but have little or no sensitivity to fault geometry at depth [e.g. *Mai et al.*, 2016, and references therein]. Wherever the coverage provided by nearby instruments is insufficient, local and/or global broadband seismic networks at teleseismic distances are used to image slip. As a general rule, fault geometry is particularly hard to constrain on the basis of seismic data alone, and is determined based on geodetic data or, wherever possible, field geology observations.

Once a data set for a given event has been compiled, seismic oscillations and geodetic offsets are translated to slip on the fault via (1) least-squares inversions, (2) seismic time reversal, or (3) the back-projection method.

(1) Least-squares inversions are based on the representation theorem [e.g. *Aki and Richards*, 2002], i.e. the mathematical expression of the physical law relating the geometry of an arbitrary rupture to the resulting deformation at any point of a given medium. Because the spatiotemporal evolution of seismic ruptures is generally very complex, it is not surprising that their solutions tend to be very non-unique, as shown in detail by *Mai et al.* [2016].

(2) The physics of acoustic or seismic time reversal can be heuristically summarized as follows: a signal is emitted by a source and recorded by multiple receivers; if receivers are then turned into sources, each emitting its own recorded signal (with the corresponding delay) flipped with respect to time, the resulting wave field will "focus" at the original source location [e.g. *Fink*, 1999]. This means that by recording real data from an unknown source and then conducting the time-reversal exercise numerically, the location of the source could be determined, provided, of course, that the error associated with modeling of propagation is small, that is to say, that the complexity of the medium of propagation is accounted for within a good approximation. From the standpoint of seismology, this amounts to a kinematic, extended-source inversion, with the additional possibility of monitoring the backward propagation of time-reversed waves before focusing at the source. In seismology, applications of time reversal [e.g. *Larmat et al.*, 2006, 2008] are hindered by the high computational costs of accurate wave-propagation modeling, unless only very long periods are considered.

(3) The back-projection method as described, e.g., by *Ishii et al.* [2005] and currently employed by many authors in seismology, is usually thought of as a simplification of wavefield reverse-time migration, a tool for imaging structure in reflection seismology. This is in many ways similar to time reversal, but involves some further, fundamental simplifications. Namely, the term back-projection refers to studies where the effects of time-reversed wave propagation are not modeled, but approximately corrected for by stacking the signals recorded by an array of nearby receivers. One of the practical consequences of this is that the physical nature of the computed, time-reversed wave field that focuses at the source remains undefined. Its interpretation in terms of rupture mechanics is complicated by the fact, e.g., that it is not known whether it more closely approximates a slip or a rate of slip [*Fukahata et al.*, 2014].

We provide in this study the building blocks of a new algorithm for constraining extended-source geometry and time evolution. The algorithm is based on the time-reversal concept, and thus overcomes the limitations of P-wave back-projection, but it is designed so as to reduce significantly the computational costs of full-waveform time reversal. One of the key aspects of our method is that surface waves, instead of P waves, are time-reversed and backward-propagated. This is preferable for several reasons: (i) Surface waves are dispersive, i.e. they "spread out" along the time axis as they travel across the surface of the earth: time reversal turns this process around, enhancing the focusing of backward-propagating waves onto the source. (ii) The problem of surface-wave propagation modeling, although inherently three-dimensional, can be reduced to two dimensions by separating the signal into narrow frequency bands [e.g. *Tanimoto*, 1990; *Tromp and Dahlen*, 1993; *Peter et al.*, 2007, 2009], to be back-projected separately, and subsequently "stacked" together: this reduces the computational costs drastically. (iii) Our knowledge of the three-dimensional structure of the Earth's deep interior, essential to backward-propagate numerically the time-reversed signal, is limited; but surface-wave propagation is confined to the upper mantle, which is relatively well known; recent, robust global phase-velocity maps of Rayleigh- and Love-wave velocities are available in the frequency band relevant to this project, at the global and, where possible, regional scales [e.g. *Ekström*, 2011; *Kaestle et al.*, 2017]. In seismology/acoustics jargon, point (iii) is equivalent to saying that very accurate surface-wave "Green's functions" are available and will be used to backward-propagate time-reversed surface-wave data. This further enhances focusing of the time-reversed wave field, and thus the robustness and resolution of mapped seismic slip.

We expect our method to be effective over a broad range of epicentral distances. At distances of 30° or more from the epicenter, surface waves carry more energy than body waves, and they can be easily identified and isolated on seismograms. At shorter epicentral distances, where they are obscured by the body-wave coda, surface waves can still emerge in a time-reversal exercise as a result of focusing: this is confirmed by our results, discussed in sec. 6.3.

Today, broadband "full-waveform" information is not routinely utilized by researchers interested in mapping the seismic source. Tentative implementations of fault imaging via seismic-waveform time reversal such as those by *Larmat et al.* [2006] and *Larmat et al.* [2008] were successful from a theoretical standpoint, but seem too computationally heavy for systematic practical application. Most seismologists only back-project seismogram peaks associated with the arrival of P waves [e.g. *Ishii et al.*, 2005] so as to avoid costly simulations of broad-band seismic-wave propagation in a heterogeneous, three-dimensional medium (the Earth), whose heterogeneity is only approximately known. The only published experiment in surface-wave back-projection that we are aware of is that of *Roten et al.* [2012]. While the basic idea of *Roten et al.* [2012] is similar to some of the concepts presented here, their approach is essentially a form of back-projection, with the inherent approximations.

We provide in sec. 2 a description of surface-wave propagation in terms of "potentials" [e.g. *Udías*, 1999; *Aki and Richards*, 2002]. *Aki and Richards* [2002] state (box 7.5) that "potentials are of no direct interest, and are awkward to use...". We maintain that, as shown e.g. by *Tanimoto* [1990] or *Peter et al.* [2007], there is interest in using potentials, particularly for surface waves. For instance, if only the phase, and not the amplitude of the data is studied, many useful applications (e.g., imaging, backward-propagation) become possible by using the potentials and the associated simple, two-dimensional (2-D) equations, without having to solve the more cumbersome radial equations, or the general three-dimensional equations. This is strictly true within a high-frequency approximation, but applications to real data have often shown that, in practice, this approximation works remarkably well. Our theoretical formulation in sec. 2 is different from that of *Aki and Richards* [2002] in that we use potentials, and from those of *Tanimoto* [1990] and *Tromp and Dahlen* [1993] in that we do not invoke variational principles.

The main implication of sec. 2 is that the scalar 2-D Helmholtz' equation can be used to model surface-wave propagation in a laterally smooth, lossless half space, confirming earlier results by *Tanimoto* [1990] and *Tromp and Dahlen* [1993]. In secs. 3 and 4 we accordingly derive the theory of time reversal in a 2-D "acoustic" medium (i.e., a medium whose deformations are described by the 2-D Helmholtz' equation). Finally (sec. 6), theoretical results are validated by direct application to synthetic and real surface-wave data. The applications presented here are limited to two-dimensions; in future work, we shall explore the resolving power of our method in the vertical direction, combining the results of multiple, Love- and Rayleigh-wave 2-D time-reversal simulations conducted at different frequencies.

2 Surface waves and the two-dimensional Helmholtz equation

The scalar 2-D Helmholtz' equation can be used to model surface-wave propagation in a laterally smooth, lossless half space. We shall give a simplified proof of this fundamental result by briefly summarizing some parts of earlier studies by *Tanimoto* [1990] and *Tromp and Dahlen* [1993]. Let us start by writing the displacement equation for an elastic, isotropic medium in the frequency (ω) domain [*Udías*, 1999, eq. (2.60)],

$$\frac{\partial}{\partial x_j}\left[\lambda\delta_{ij}\frac{\partial u_k}{\partial x_k}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]=-\rho\omega^2u_j, \tag{1}$$

where x_1, x_2, x_3 are Cartesian coordinates, with the x_3-axis perpendicular to Earth's surface (which we assume to be flat) and oriented downward; δ_{ij} is Kronecker's delta, ρ denotes density and λ, μ Lamé's parameters. Repeated indices are implicitly summed over. Following *Tanimoto* [1990], we assume the Earth to be smooth laterally (horizontal derivatives of ρ, λ, μ, etc. are negligible) but not vertically (x_3-derivatives of the same parameters are not negligible); eq. (1) then takes a slightly different form for i=3 with respect to i=1,2; namely

$$(\lambda+\mu)\frac{\partial}{\partial x_{1,2}}\left(\frac{\partial u_k}{\partial x_k}\right)+\mu\nabla^2u_{1,2}+\frac{\partial\mu}{\partial x_3}\left(\frac{\partial u_{1,2}}{\partial x_3}+\frac{\partial u_3}{\partial x_{1,2}}\right)=-\rho\omega^2u_{1,2}, \tag{2}$$

and

$$(\lambda+\mu)\frac{\partial}{\partial x_3}\left(\frac{\partial u_k}{\partial x_k}\right)+\mu\nabla^2u_3+2\ \frac{\partial\mu}{\partial x_3}\ \frac{\partial u_3}{\partial x_3}+\frac{\partial\lambda}{\partial x_3}\ \frac{\partial u_k}{\partial x_k}=-\rho\omega^2u_3. \tag{3}$$

The displacement equations (1) or (2) and (3) are accompanied by the requirement that no tractions exist on the outer surface of the Earth ("free surface" boundary conditions); for an isotropic elastic medium, the stress tensor

$$\sigma_{ij} = \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} + 2\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \tag{4}$$

and the zero-traction requirement at the outer (horizontal) surface is equivalent to requiring that σ_{13}=σ_{23}=σ_{33}=0 when x_3=0. Displacements and stresses are also usually required to be continuous across all discontinuities.

We next introduce a Rayleigh-wave displacement *Ansatz* in the frequency domain,

$$\mathbf{u}_R = U(x_3, \omega)\mathbf{x}_3 \phi_R(x_1, x_2, \omega) + V(x_3, \omega)\nabla_1 \phi_R(x_1, x_2, \omega), \tag{5}$$

where the unit-vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are parallel to the Cartesian axes, and $\nabla_1 = \mathbf{x}_1 \frac{\partial}{\partial x_1} + \mathbf{x}_2 \frac{\partial}{\partial x_2}$. The functions $U(x_3, \omega)$ and $V(x_3, \omega)$ control the dependence of surface-wave amplitude on depth; they do not need to be known explicitly at this stage. The function ϕ_R can be thought of as a "Rayleigh-wave potential". For Love waves,

$$\mathbf{u}_L = W(x_3, \omega)(-\mathbf{x}_3 \times \nabla_1)\phi_L(x_1, x_2, \omega), \tag{6}$$

with ϕ_L the "Love-wave potential", and $W(x_3, \omega)$ playing the same role as U and V above. It can be seen by inspection of expressions (5) and (6) that they indeed describe Rayleigh- and Love-wave motion, respectively. The functions U, V, W need not be specified at this point, but, if only surface-wave solutions are of interest, it must be required that

$$\lim_{x_3 \longrightarrow \infty} U(x_3, \omega) = 0\,; \ \lim_{x_3 \longrightarrow \infty} V(x_3, \omega) = 0\,; \ \lim_{x_3 \longrightarrow \infty} W(x_3, \omega) = 0. \tag{7}$$

We next use our surface-wave *Ansätze* (5) and (6), together with the mentioned boundary conditions, to simplify and solve the displacement equations (2) and (3).

2.1 Love waves

We first substitute $\mathbf{u}$ in eqs. (2), (3) with the expression (6) for $\mathbf{u}_L$. It is useful to notice that the $\mathbf{x}_3$-component of $\mathbf{u}_L$ is 0, and that $\mathbf{u}_L$ is divergence-free; as a result, eq. (3) is always verifed by $\mathbf{u}_L$ as given by (6), whatever the functions $W(x_3)$ and $\phi_L(x_1, x_2)$. After some algebra, the remaining equations are reduced to

$$\left(\mu \frac{\partial^2 W}{\partial x_3^2} + \frac{\partial \mu}{\partial x_3} \frac{\partial W}{\partial x_3} + \rho \omega^2 W \right) \frac{\partial \phi_L}{\partial x_2} + \mu W \left(\frac{\partial^3 \phi_L}{\partial x_1^2 \partial x_2} + \frac{\partial^3 \phi_L}{\partial x_2^3} \right) = 0, \tag{8}$$

$$\left(\mu \frac{\partial^2 W}{\partial x_3^2} + \frac{\partial \mu}{\partial x_3} \frac{\partial W}{\partial x_3} + \rho \omega^2 W \right) \frac{\partial \phi_L}{\partial x_1} + \mu W \left(\frac{\partial^3 \phi_L}{\partial x_2^2 \partial x_1} + \frac{\partial^3 \phi_L}{\partial x_1^3} \right) = 0. \tag{9}$$

Remember that ϕ_L is only a function of x_1, x_2, while $\mu = \mu(x_3)$, $\rho = \rho(x_3)$ and $W = W(x_3)$. If we divide eq. (8) by $\mu W \frac{\partial \phi_L}{\partial x_2}$ we find

$$\frac{\mu \frac{\partial^2 W}{\partial x_3^2} + \frac{\partial \mu}{\partial x_3} \frac{\partial W}{\partial x_3} + \rho \omega^2 W}{\mu W} = - \frac{\frac{\partial^3 \phi_L}{\partial x_1^2 \partial x_2} + \frac{\partial^3 \phi_L}{\partial x_2^3}}{\frac{\partial \phi_L}{\partial x_2}}, \tag{10}$$

which can be solved by separation of variables [e.g. *Tromp and Dahlen*, 1993, sec. 3] since the right-hand side depends only on x_1, x_2, and the left-hand side only on x_3. This means that we can introduce a constant k_L such that

$$\mu\frac{\partial^2 W}{\partial x_3^2}+\frac{\partial\mu}{\partial x_3}\frac{\partial W}{\partial x_3}+\rho\omega^2 W=\mu k_L^2 W \tag{11}$$

and

$$\frac{\partial^3\phi_L}{\partial x_1^2\partial x_2}+\frac{\partial^3\phi_L}{\partial x_2^3}=-k_L^2\frac{\partial\phi_L}{\partial x_2}. \tag{12}$$

It might be noticed that the "radial" eq. (11) is equivalent to equation (46) of *Takeuchi and Saito* [1972], or eq. (7.24) of *Aki and Richards* [2002], even though those treatments are limited to plane waves (which affects ϕ_L but not W).

Applying the same procedure to eq. (9) additionally gives

$$\frac{\partial^3\phi_L}{\partial x_1\partial x_2^2}+\frac{\partial^3\phi_L}{\partial x_1^3}=-k_L^2\frac{\partial\phi_L}{\partial x_1}, \tag{13}$$

and a sufficient condition for ϕ_L to solve both (12) and (13) is the Helmholtz' equation

$$\frac{\partial^2\phi_L}{\partial x_1^2}+\frac{\partial^2\phi_L}{\partial x_2^2}=-k_L^2\phi_L. \tag{14}$$

The boundary conditions can also be simplified when applied to our Love-wave *Ansatz*: it follows from (4) and (6) that, for Love waves, σ_{33}=0, and $\sigma_{13}=-\sigma_{23}=\mu\frac{\partial W}{\partial x_3}\frac{\partial\phi_L}{\partial x_1}$. The zero-traction boundary condition at the outer surface thus reduces to

$$\frac{\partial W}{\partial x_3}=0 \text{ at } x_3=0. \tag{15}$$

2.1.1 Love-wave radial equation

Several different approaches to the (semi-analytical or numerical) solution of the "radial" equation (11) are reviewed in sections 7.1 and 7.2 of *Aki and Richards* [2002], starting with a simple one-layer-over-half-space model and then generalizing to the cases of an arbitrary number of layers, and of continuous velocity and density profiles. We need not repeat here the detailed treatment of *Aki and Richards* [2002], but it is useful to point out some of its essential implications.

Equation (11) is a second-order ordinary differential equation, whose general solution thus contains two arbitrary constants. Two boundary conditions must be taken into account: eqs. (7) and (15). These two equations allow in principle to determine both arbitrary constants.

If the Earth is modeled as a set of one or more uniform, horizontal layers overlying a half space, then within each layer i we have $\frac{\partial\mu}{\partial x_3}=0$, and eq. (11) is simplified to the Helmholtz' equation

$$\frac{\partial^2 W}{\partial x_3^2}+(\rho_i\omega^2-\mu_i k_L^2)W=0, \tag{16}$$

where ρ_i and μ_i denote the (constant) values of density and rigidity within layer i, respectively. Each layer adds one second-order equation, and therefore two arbitrary constants to the problem, but also one interface with the associated two continuity conditions (on W and

$\frac{\partial W}{\partial x_3}$): again, all arbitrary constants can be determined.

The parameters ω and k_L, however, have not been specified, and, as a consequence, one cannot simply identify a unique solution for W to be substituted into the *Ansatz* (6). According to *Aki and Richards* [2002], this problem is solved in general as follows: (i) a numerical value ω_0 is assigned to ω; (ii) a numerical, "trial" value is likewise assigned to k_L; (iii) the selected numerical values ω_0 and k_L are substituted into eq. (11) which can then be integrated numerically, or via the "propagator matrix" method [e.g. *Aki and Richards*, 2002, sec. 7.2.2], starting with W=0 at large depth x_3; (iv) it is verified whether condition (15) is met at x_3=0; (v) if this condition is not met, eq. (11) is integrated again, with the same ω_0 but a different trial value for k_L; (vi) if, instead, the condition (15) is met, the whole process is repeated for a new value ω_0, until the frequency range of interest is entirely covered.

It is found that a discrete set of one or more (depending on ω_0) values of k_L for which the free-surface boundary condition is met can be determined [e.g. *Aki and Richards*, 2002, figures 7.2, 7.3]. These values are dubbed "eigenvalues" in analogy with free-oscillation theory, and each corresponds to a different solution, or "mode," for W. If more than one eigenvalue exist at a given frequency, the mode corresponding to the largest k_L eigenvalue is referred to as "fundamental mode," followed by "higher modes" ("overtones").

2.1.2 Helmholtz' equation for the Love-wave potential ϕ_L

The parameters k_L and ω in eq. (14) must be substituted with one of the eigenvalues of k_L, and with the corresponding value ω_0, respectively, before this equation is solved for ϕ_L. Substitution of $\phi_L(x_1, x_2, \omega_0)$ and of the corresponding $W(x_3, \omega_0)$ into expression (6) yields a monochromatic Love-wave solution. The process can be iterated at each frequency ω_0 for which the eigenvalues k_L and eigenfunctions W have been determined as described in sec. 2.1.1.

Notice that, for a monochromatic wave of frequency ω_0, eq. (14) coincides with the 2-D wave equation with wavespeed $\frac{\omega_0}{k_L}$. The curve $k_L = k_L(\omega_0)$ is thus the "dispersion curve" describing how surface-wave phase velocity depends on frequency.

It is easy to show that a monochromatic *plane* wave ϕ_L would solve eq. (14), and in fact most seismology textbooks replace ϕ_L (and ϕ_R) with plane-wave formulae in the surface-wave *Ansätze* [e.g. *Aki and Richards*, 2002]. In view of the applications to be discussed here, however, circular (cylindrical) surface waves are more relevant. This case can be described, in our formulation, starting with the known solution G_{2D} to the Green's problem associated with equation (14), obtained, e.g., in Appendix E of *Boschi and Weemstra* [2015]; $G_{2D}(x_1, x_2, \omega)$ is clearly not a monochromatic wave, but the response of the medium to a monochromatic point source can be obtained, according to eq. (E34) of *Boschi and Weemstra* [2015], by time-domain convolution or frequency-domain multiplication of $G_{2D}(x_1, x_2, \omega)$ with a sinusoidal signal $\delta(\omega - \omega_0)$.

2.2 Rayleigh waves

In analogy with section 2.1, we next substitute $\mathbf{u}$ in eqs. (2), (3) with the expression (5) for $\mathbf{u}_R$. This results, after some algebra, in the system of equations

$$\left[\mu\frac{\partial^2 V}{\partial x_3^2}+\frac{\partial \mu}{\partial x_3}\left(U+\frac{\partial V}{\partial x_3}\right)+(\lambda+\mu)\frac{\partial U}{\partial x_3}+\rho\omega^2 V\right]\frac{\partial \phi_R}{\partial x_1}+(\lambda+2\mu)V\left(\frac{\partial^3\phi_R}{\partial x_1^3}+\frac{\partial^3\phi_R}{\partial x_2^2\partial x_1}\right)=0, \tag{17}$$

$$\left[\mu\frac{\partial^2 V}{\partial x_3^2}+\frac{\partial \mu}{\partial x_3}\left(U+\frac{\partial V}{\partial x_3}\right)+(\lambda+\mu)\frac{\partial U}{\partial x_3}+\rho\omega^2 V\right]\frac{\partial \phi_R}{\partial x_2}+(\lambda+2\mu)V\left(\frac{\partial^3\phi_R}{\partial x_2^3}+\frac{\partial^3\phi_R}{\partial x_1^2\partial x_2}\right)=0, \tag{18}$$

$$\left[(\lambda+2\mu)\frac{\partial^2 U}{\partial x_3^2}+2\frac{\partial \mu}{\partial x_3}\frac{\partial U}{\partial x_3}+\frac{\partial \lambda}{\partial x_3}\frac{\partial U}{\partial x_3}+\rho\omega^2 U\right]\phi_R+\left[(\lambda+\mu)\frac{\partial V}{\partial x_3}+\frac{\partial \lambda}{\partial x_3}V+\mu U\right]\left(\frac{\partial^2\phi_R}{\partial x_1^2}+\frac{\partial^2\phi_R}{\partial x_2^2}\right)=0, \tag{19}$$

which, again, can be solved by the method of separation of variables. After dividing it by $(\lambda+\mu)\frac{\partial V}{\partial x_3}+\frac{\partial \lambda}{\partial x_3}V+\mu U$, eq. (19) can be separated into

$$(\lambda+2\mu)\frac{\partial^2 U}{\partial x_3^2}+2\frac{\partial \mu}{\partial x_3}\frac{\partial U}{\partial x_3}+\frac{\partial \lambda}{\partial x_3}\frac{\partial U}{\partial x_3}+\rho\omega^2 U=k_R^2\left[(\lambda+\mu)\frac{\partial V}{\partial x_3}+\frac{\partial \lambda}{\partial x_3}V+\mu U\right] \tag{20}$$

and the Helmholtz' equation

$$\frac{\partial^2\phi_R}{\partial x_1^2}+\frac{\partial^2\phi_R}{\partial x_2^2}=-k_R^2\phi_R, \tag{21}$$

where k_R is, at this point, an arbitrary constant. If one then substitues eq. (21) into (17) and (18), it becomes apparent that a sufficient condition for both of them to be solved is given by

$$(\lambda+\mu)\frac{\partial U}{\partial x_3}+\mu\frac{\partial^2 V}{\partial x_3^2}+\frac{\partial \mu}{\partial x_3}\left(U+\frac{\partial V}{\partial x_3}\right)+\rho\omega^2 V-k_R^2(\lambda+2\mu)V=0. \tag{22}$$

The "radial" eqs. (20) and (22) form a linear system of second-order ordinary differential equations that can be solved to determine U and V. Since two equations and two unknown functions are now involved, the solution is more cumbersome, but qualitatively similar to the Love-wave case of sec. 2.1.1. Again, as shown by *Takeuchi and Saito* [1972] and *Aki and Richards* [2002] for the plane-wave case, a set of Rayleigh-wave "modes" can be found by numerical integration: each mode is defined by a frequency ω_0 and a value of k_R for which (20) and (22) are solved, and the boundary conditions met. The definitions of fundamental mode and overtone given in sec. 2.1.1 naturally holds also for Rayleigh waves.

The discussion of sec. 2.1.2 on the Love-wave potential ϕ_L also applies to the Rayleigh-wave potential ϕ_R, which is controlled by the Helmholtz' equation (21); in analogy with sec. 2.1.2, k_R can be interpreted as the ratio between the frequency and phase velocity of the corresponding Rayleigh-wave mode.

3 Reciprocity theorem in two dimensions

Consider the non-homogeneous 2-D Helmholtz' equation

$$\nabla_1^2 p(x_1,x_2,\omega)+\frac{\omega^2}{c^2}p(x_1,x_2,\omega)=-\mathrm{i}\omega q(x_1,x_2,\omega), \tag{23}$$

where p could represent the displacement of a stretched membrane (whose density and tension determine the parameter c), and the forcing term $-\mathrm{i}\omega q$ a pressure exerted on the membrane per unit of surface density [e.g., *Kinsler et al.*, 1999, secs. 4.2 and 4.8]. Here and in the following we denote by $f(\omega)$ the Fourier transform of a generic function $f(t)$, and by i the imaginary unit. The following mathematical treatment makes it convenient to denote forcing as $-\mathrm{i}\omega q(x_1, x_2, \omega)$.

Let us define a vector $\mathbf{v} = -\frac{1}{\mathrm{i}\omega}\nabla_1 p$, such that

$$\nabla_1 p + \mathrm{i}\omega\mathbf{v} = \mathbf{0}. \tag{24}$$

Substituting eq. (24) into (23), we then find

$$\nabla_1 \cdot \mathbf{v} + \frac{\mathrm{i}\omega}{c^2} p - q = 0. \tag{25}$$

The following treatment follows closely that of *Boschi and Weemstra* [2015], who summarized earlier results by, e.g., *Wapenaar and Fokkema* [2006] and *Snieder* [2007], limited to three-dimensional space. Let us consider a surface S bounded by the closed curve ∂S. (∂S is just an arbitrary closed curve within a 2-D medium, and generally does not represent a physical boundary.) Let $q_A(x_1, x_2, \omega)$, $p_A(x_1, x_2, \omega)$ and $\mathbf{v}_A(x_1, x_2, \omega)$ denote a possible combination of the fields q, p and $\mathbf{v}$ co-existing at (x_1, x_2) in S and ∂S. A different forcing q_B would give rise, through eqs. (24) and (25), to a different "state" B, defined by $p_B(x_1, x_2, \omega)$ and $\mathbf{v}_B(x_1, x_2, \omega)$.

A useful relationship between the states A and B, known as "reciprocity theorem", is obtained by combining eqs. (24) and (25) as follows,

$$\int_S \mathrm{d}^2\mathbf{x}\, [(24)_A \cdot \mathbf{v}_B^* + (24)_B^* \cdot \mathbf{v}_A + (25)_A\, p_B^* + (25)_B^*\, p_A] = 0, \tag{26}$$

where $\mathbf{x}{=}(x_1, x_2)$, $\mathrm{d}^2\mathbf{x} = \mathrm{d}x_1 \mathrm{d}x_2$, and * denotes complex conjugation. $(24)_A$ is short for the expression one obtains after substituting $p = p_A(\mathbf{x}, \omega)$ and $\mathbf{v} = \mathbf{v}_A(\mathbf{x}, \omega)$ into the left-hand side of eq. (24), etc. Namely,

$$(24)_A \cdot \mathbf{v}_B^* = \nabla_1 p_A \cdot \mathbf{v}_B^* + \mathrm{i}\omega\mathbf{v}_A \cdot \mathbf{v}_B^* \tag{27}$$

$$(24)_B^* \cdot \mathbf{v}_A = \nabla_1 p_B^* \cdot \mathbf{v}_A - \mathrm{i}\omega\mathbf{v}_B^* \cdot \mathbf{v}_A \tag{28}$$

$$(25)_A\, p_B^* = \nabla_1 \cdot \mathbf{v}_A\, p_B^* + \frac{\mathrm{i}\omega}{c^2} p_A p_B^* - q_A p_B^* \tag{29}$$

$$(25)_B^*\, p_A = \nabla_1 \cdot \mathbf{v}_B^*\, p_A - \frac{\mathrm{i}\omega}{c^2} p_B^* p_A - q_B^* p_A. \tag{30}$$

After substituting expressions (27) through (30) into eq. (26), the latter simplifies to

$$\int_S \mathrm{d}^2\mathbf{x}\, (\nabla_1 p_A \cdot \mathbf{v}_B^* + \nabla_1 p_B^* \cdot \mathbf{v}_A + \nabla_1 \cdot \mathbf{v}_A\, p_B^* + \nabla_1 \cdot \mathbf{v}_B^*\, p_A) = \int_S \mathrm{d}^2\mathbf{x}\, (q_A p_B^* + q_B^* p_A)\,. \tag{31}$$

The integrand at the left-hand side of (31) can be further simplified via the relationship $\nabla_1 \cdot (p_A \mathbf{v}_B^*) = \nabla_1 p_A \cdot \mathbf{v}_B^* + \nabla_1 \cdot \mathbf{v}_B^* p_A$ (which naturally holds also if A and B are swapped). We next apply the 2-D version of the divergence theorem to the resulting expression, and eq.

(31) collapses to

$$\int_{\partial S} \mathrm{d}\mathbf{x}\,(p_A \mathbf{v}_B^* + p_B^* \mathbf{v}_A)\cdot\mathbf{n} = \int_S \mathrm{d}^2\mathbf{x}\,(q_A p_B^* + q_B^* p_A)\,, \tag{32}$$

where $\mathbf{n}$ is a unit vector everywhere perpendicular to ∂S. For instance, "Green's identity" (4.22) of *Baker and Copson* [1950], or the "reciprocity theorem of the correlation type", eq. (5) of *Wapenaar and Fokkema* [2006], are three-dimensional versions of eq. (32).

3.1 Application of the reciprocity theorem to impulsive point sources: exact equations

Let us next consider the states A and B resulting from the impulsive forcing terms $q_A = \delta(\mathbf{x}-\mathbf{x}_A)$ and $q_B = \delta(\mathbf{x}-\mathbf{x}_B)$, respectively, with $\mathbf{x}_A$, $\mathbf{x}_B$ two arbitrary locations on S. It follows that $p_A = \mathscr{G}_{2D}(\mathbf{x},\mathbf{x}_A,\omega)$ and $p_B = \mathscr{G}_{2D}(\mathbf{x},\mathbf{x}_B,\omega)$, with $\mathscr{G}_{2D}$ the Green's function corresponding to a 2-D membrane excited by a nonzero right-hand side in eq. (23), and eq. (24) then implies that $\mathbf{v}_{A,B} = -\frac{1}{\mathrm{i}\omega}\nabla_1\mathscr{G}_{2D}(\mathbf{x},\mathbf{x}_{A,B},\omega)$.

$\mathscr{G}_{2D}$ is the solution of the non-homogeneous eq. (23) with $q = \delta(\mathbf{x}-\mathbf{x}_{A,B})$. Based on eq. (E34) of *Boschi and Weemstra* [2015],

$$\begin{aligned}\mathscr{G}_{2D}(\mathbf{x},\mathbf{x}_{A,B},\omega) &= \int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{x}'\; G_{2D}(\mathbf{x},\mathbf{x}',\omega)(-\mathrm{i}\omega)\delta(\mathbf{x}'-\mathbf{x}_{A,B})\\ &= -\mathrm{i}\omega G_{2D}(\mathbf{x},\mathbf{x}_{A,B},\omega),\end{aligned} \tag{33}$$

where $G_{2D}(\mathbf{x},\mathbf{x}',\omega)$ is the Green's function associated with a nonzero *initial velocity* at $\mathbf{x}'$, derived explicitly e.g. by *Boschi and Weemstra* [2015]. To translate the time-domain formula of *Boschi and Weemstra* [2015] into frequency domain, it is useful to notice that eq. (E34) of *Boschi and Weemstra* [2015] involves the time-domain convolution of G_{2D} with the non-homogeneous term (forcing term) of the wave equation, and to remember that a convolution in the time-domain maps to a product in the frequency domain.

Replacing $p_{A,B}$ and $\mathbf{v}_{A,B}$ in eq. (32) with their expressions in terms of G_{2D}, and $q_{A,B}$ with a Dirac delta,

$$\begin{aligned}&G_{2D}^*(\mathbf{x}_A,\mathbf{x}_B,\omega) - G_{2D}(\mathbf{x}_B,\mathbf{x}_A,\omega)\\ &= \int_{\partial S} \mathrm{d}\mathbf{x}'\left[G_{2D}^*(\mathbf{x}',\mathbf{x}_B,\omega)\nabla_1 G_{2D}(\mathbf{x}',\mathbf{x}_A,\omega) - G_{2D}(\mathbf{x}',\mathbf{x}_A,\omega)\nabla_1 G_{2D}^*(\mathbf{x}',\mathbf{x}_B,\omega)\right]\cdot\mathbf{n}.\end{aligned} \tag{34}$$

Eq. (34) can be thought of as the 2-D version of eq. (19) in *Wapenaar and Fokkema* [2006] or eq. (96) in *Boschi and Weemstra* [2015].

The above treatment holds if $\mathbf{x}_A$ and $\mathbf{x}_B$ are not within S; in that case, $q_{A,B}$ are zero within S. The integral at the right-hand side of eq. (32) is therefore zero, and so is, as a result, the left-hand side of (34). It follows that the integral at the right-hand side of (34) is zero if $\mathbf{x}_A$ and $\mathbf{x}_B$ are not within S [*Baker and Copson*, 1950, sec. 6.2].

3.2 Application of the reciprocity theorem to impulsive point sources: far-field/high-frequency approximation

Equation (34) can be simplified by the "far-field" approximation, which requires that the locations $\mathbf{x}_A$ and $\mathbf{x}_B$ be separated from one another and from δS by at least a few wavelengths. We additionally require that $\mathbf{x}-\mathbf{x}_A \approx \mathbf{x}-\mathbf{x}_B$ for any point $\mathbf{x}$ on ∂S (i.e., $\mathbf{x}_A$ and $\mathbf{x}_B$ are both very far from ∂S). The Green's function G_{2D} can be replaced by its far-field approximation, which reads

$$G_{2D}(\mathbf{x},\mathbf{y},\omega) \approx \frac{1}{4\mathrm{i}\pi c^{3/2}} \frac{e^{-\mathrm{i}\left(\frac{\omega|\mathbf{x}-\mathbf{y}|}{c}-\frac{\pi}{4}\right)}}{\sqrt{\omega|\mathbf{x}-\mathbf{y}|}} \tag{35}$$

[e.g., *Boschi and Weemstra*, 2015, eq. (E17)]. We next take advantage of this approximation to find a simple expression for $\nabla_1 G_{2D}$. Let us consider for example $\nabla_1 G_{2D}(\mathbf{x},\mathbf{x}_A,\omega)$ and call $r=|\mathbf{x}-\mathbf{x}_A|$. Then,

$$\begin{aligned}
\nabla_1 G_{2D}(\mathbf{x},\mathbf{x}_A,\omega) &\approx \frac{1}{4\mathrm{i}\pi c^{3/2}}\left(\mathbf{x}_1\frac{\partial}{\partial x_1}+\mathbf{x}_2\frac{\partial}{\partial x_2}\right)\left[\frac{e^{-\mathrm{i}\left(\frac{\omega r}{c}-\frac{\pi}{4}\right)}}{\sqrt{\omega r}}\right]\\
&= \frac{1}{4\mathrm{i}\pi c^{3/2}}\left(\mathbf{x}_1\frac{\partial r}{\partial x_1}+\mathbf{x}_2\frac{\partial r}{\partial x_2}\right)\frac{\partial}{\partial r}\left[\frac{e^{-\mathrm{i}\left(\frac{\omega r}{c}-\frac{\pi}{4}\right)}}{\sqrt{\omega r}}\right]\\
&= \frac{1}{4\mathrm{i}\pi c^{3/2}}\left[\frac{\mathrm{i}\omega}{c}+\frac{1}{2r}\right]\frac{e^{-\mathrm{i}\left(\frac{\omega r}{c}-\frac{\pi}{4}\right)}}{\sqrt{\omega r}}\nabla_1 r\\
&= G_{2D}(\mathbf{x},\mathbf{x}_A,\omega)\left[\frac{\mathrm{i}\omega}{c}+\frac{1}{2r}\right]\nabla_1 r.
\end{aligned} \tag{36}$$

In the far-field approximation, r is large and r^{-1} is much larger than r^{-2}: the second term inside square brackets in eq. (36) can be neglected. If one takes the origin, e.g., at $\mathbf{x_A}$, the condition $\mathbf{x}-\mathbf{x}_A \approx \mathbf{x}-\mathbf{x}_B$ implies that both $\mathbf{x}-\mathbf{x}_A$ and $\mathbf{x}-\mathbf{x}_B$ can be replaced by $\mathbf{x}$, and $\nabla_1 r \approx \frac{\mathbf{x}}{|\mathbf{x}|}$. We are left with

$$\nabla_1 G_{2D}(\mathbf{x},\mathbf{x}_A,\omega) \approx \frac{\mathrm{i}\omega}{c} G_{2D}(\mathbf{x},\mathbf{x}_A,\omega)\frac{\mathbf{x}}{|\mathbf{x}|}, \tag{37}$$

which we can finally substitute into eq. (34), to find

$$\begin{aligned}
&G^*_{2D}(\mathbf{x}_A,\mathbf{x}_B,\omega) - G_{2D}(\mathbf{x}_B,\mathbf{x}_A,\omega)\\
&\approx \frac{\mathrm{i}\omega}{c}\int_{\partial S} \mathrm{d}\mathbf{x}' \ \left[G_{2D}(\mathbf{x}',\mathbf{x}_A,\omega)G^*_{2D}(\mathbf{x}',\mathbf{x}_B,\omega)+G^*_{2D}(\mathbf{x}',\mathbf{x}_B,\omega)G_{2D}(\mathbf{x}',\mathbf{x}_A,\omega)\right]\frac{\mathbf{x}'}{|\mathbf{x}'|}\cdot\mathbf{n}.
\end{aligned} \tag{38}$$

Remember that the closed curve ∂S does not correspond to a physical boundary. We choose it to be circular (we shall see in the following that this assumption does not affect the relevant physical interpretations of our results), so that $\frac{\mathbf{x}}{|\mathbf{x}|}=\mathbf{n}$ on ∂S. Eq. (38) collapses to

$$G^*_{2D}(\mathbf{x}_A,\mathbf{x}_B,\omega) - G_{2D}(\mathbf{x}_B,\mathbf{x}_A,\omega) \approx \frac{2\mathrm{i}\omega}{c}\int_{\partial S}\mathrm{d}\mathbf{x}'\ G_{2D}(\mathbf{x}',\mathbf{x}_A,\omega)G^*_{2D}(\mathbf{x}',\mathbf{x}_B,\omega), \tag{39}$$

which is the 2-D counterpart of eq. (102) in *Boschi and Weemstra* [2015]. (It is also consistent with eq. (65) of the same study, valid for a 2-D medium, derived via the stationary-phase approximation and setting source density to 1.)

4 Implications for surface waves: diffuse-field interferometry, time reversal

We know from sec. 2 that the Rayleigh- and Love-wave potentials ϕ_R, ϕ_L, just like the "membrane-wave" field p, obey the Helmholtz' equation (23). It follows that eqs. (34) and (39) continue to be valid if p is replaced by potentials ϕ_R or ϕ_L, and if c is the Rayleigh- or Love-wave phase velocity at that frequency. We also know that the vertical displacement associated with a Rayleigh wave is proportional to ϕ_R and thus obeys (23) exactly at the frequency ω [e.g. *Boschi and Weemstra*, 2015]; slightly more complicated relations exist between Love-wave displacement (and the horizontal component of Rayleigh-wave displacement) and the Love-wave (Rayleigh-wave) potential, which are given e.g. by *Kaestle et al.* [2016]. In summary, the results of sec. 3 can be applied to the propagation of seismic surface waves, which will be the focus of the remainder of this study.

Eqs. (34) and its approximate version (39) describe the physics underlying both ambient-noise interferometry and acoustic/seismic time reversal. Analogies between these two techniques were first discussed by *Derode et al.* [2003].

4.1 Analogy with diffuse-field interferometry

In the context of diffuse-field interferometry, the far-field eq. (39) is invoked more often than its exact counterpart (34). The points $\mathbf{x}_A$, $\mathbf{x}_B$ in eq. (39) are taken to represent the locations of two receivers, while the points on δS are thought of as point sources. The right-hand side of (39) is the cross-correlation of the signal recorded at receiver $\mathbf{x}_A$ with that recorded at receiver $\mathbf{x}_B$, averaged (integrated, "stacked"...) over all sources. It is usually assumed that sources are approximately distributed along a closed curve surrounding the receivers in the far field. (If that is the case, it has also been shown that the cross-correlation of signals generated by different sources that act simultaneously will tend to cancel out; see *Boschi and Weemstra* [2015] for a more detailed discussion.) Eq. (39) then implies that the receiver-receiver cross-correlation at its right-hand side coincides approximately with the imaginary part of the frequency-domain Green's function G_{2D} at the left-hand side. Since G_{2D} is real in the time-domain, and nonzero only at positive times, $G_{2D}(\mathbf{x}_A, \mathbf{x}_B, t)$ is determined without ambiguity by the imaginary part of its Fourier transform $G_{2D}(\mathbf{x}_A, \mathbf{x}_B, \omega)$ [*Boschi and Weemstra*, 2015]. It follows that the surface-wave Green's function between two locations can be reconstructed from the cross-correlation of a diffuse surface-wave field recorded at those locations.

Recall now that, at the beginning of sec. 3.1, the assumption has been made that sources be impulsive. In practice, this amounts to selecting $q(\mathbf{x}, \omega) = \delta(\mathbf{x} - \mathbf{x}_{A,B})$, i.e., in the time domain, $q(\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{x}_{A,B})\delta(t)$. Let us next consider the case of arbitrary, unspecified time-dependence $h(t)$ of the source signal, i.e. $q(\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{x}_{A,B})h(t)$. Eq. (33) here was obtained from eq. (E34) of *Boschi and Weemstra* [2015], replacing the generic source signal there with the right-hand side of the non-homogeneous Helmholtz' eq. (23). The solution p corresponding to an arbitrary source signal $h(t)$ is therefore obtained by updating the right-hand side of eq. (23), which in the frequency domain now reads $-\mathrm{i}\omega\delta(\mathbf{x} - \mathbf{x}_{A,B})h(\omega)$. After substituting this into eq. (E34) of *Boschi and Weemstra* [2015], we find

$$p(\mathbf{x}, \mathbf{x}_{A,B}, \omega) = -\mathrm{i}\omega G_{2D}(\mathbf{x}, \mathbf{x}_{A,B}, \omega)h(\omega), \tag{40}$$

which replaces our eq. (33).

Substituting (40) and $q(\mathbf{x}, \omega) = \delta(\mathbf{x} - \mathbf{x}_{A,B})h(\omega)$ into eqs. (24) and (32), we find that introducing the time-dependence $h(t)$ of the source boils down to multiplying both sides of eq. (34), and therefore (39), by the squared Fourier spectrum $|h(\omega)|^2$.

In ambient-noise interferometry, this means that if all noise sources had the same spectrum then the cross-correlation of recorded ambient signal would also exhibit that spectrum (squared): consequently, we would not be reconstructing the Green's function but rather its time-domain convolution with the source-related term $|h(\omega)|^2$. Indeed, it is well known that the spectrum of seismic ambient-noise cross correlation is dominated by peaks that correspond to the spectrum of oceanic microseisms [*Longuet-Higgins*, 1950; *Stehly et al.*, 2006]. In many derivations of ambient-noise theory, the signals emitted by different noise sources are simply assumed to be random and uncorrelated, which results in the $|h(\omega)|^2$ factor canceling out [e.g. *Campillo and Roux*, 2014].

4.2 Surface-wave time reversal

If eq. (34) is to be used as an illustration of time-reversal acoustics [e.g., *Fink*, 1999], $\mathbf{x}_B$ should be thought of as the location of a source; $G_{2D}(\mathbf{x}', \mathbf{x}_B, \omega)$ is the Fourier-transform of the signal generated at $\mathbf{x}_B$ and recorded by a far-away receiver at $\mathbf{x}'$; its complex-conjugate $G^*_{2D}(\mathbf{x}', \mathbf{x}_B, \omega)$ is the Fourier transform of the same signal, reversed in time. Imagine that the time-reversed signal be then emitted from $\mathbf{x}'$ and recorded at another point $\mathbf{x}_A$: this amounts to convolving (in the frequency domain, multiplying) the time-reversed signal with the Green's function $G_{2D}(\mathbf{x}_A, \mathbf{x}', \omega)$. Eq. (39) then shows that by repeating time reversal and propagation ("backward in time") for all points $\mathbf{x}'$ on ∂S and summing all the resulting traces at $\mathbf{x}_B$, the imaginary part of the Green's function between $\mathbf{x}_B$ and $\mathbf{x}_A$ is obtained.

If ∂S is in the near field of $\mathbf{x}_A$, $\mathbf{x}_B$, the approximate eq. (39) should be replaced by (34), which is the 2-D version of eq. (3) in *Fink* [2006]. In practice, this means that to reconstruct the Green's function between $\mathbf{x}_A$ and $\mathbf{x}_B$ one needs to (i) time-reverse (in the frequency domain, take the complex-conjugate of) the signal $G_{2D}(\mathbf{x}', \mathbf{x}_B, \omega)$ emitted by $\mathbf{x}_B$ and recorded at $\mathbf{x}'$; (ii) take the spatial derivative of the time-reversed signal in the $\mathbf{n}$ direction at $\mathbf{x}'$, i.e. $\mathbf{n} \cdot \nabla_1 G^*_{2D}(\mathbf{x}', \mathbf{x}_B, \omega)$; (iii) convolve the time-reversed signal $G^*_{2D}(\mathbf{x}', \mathbf{x}_B, \omega)$ with the *dipole* response (see appendix) $\mathbf{n} \cdot \nabla_1 G_{2D}(\mathbf{x}_A, \mathbf{x}', \omega)$ between $\mathbf{x}_A$ and $\mathbf{x}'$; (iv) convolve its spatial-derivative with the impulse response between the same two points; (v) sum the two signals obtained at (iii) and (iv). In other words, rather than simply backward-propagating the signal recorded at receivers on ∂S, as in the far-field case, we must backward-propagate the sum of a dipole and a monopole source, to which the initial signal itself and its spatial derivative are "fed", respectively.

The backward-propagated signal so obtained coincides, approximately (if eq. (39) is implemented) or exactly (eq. (34)), with the difference

$$G^*_{2D}(\mathbf{x}_A, \mathbf{x}_B, \omega) - G_{2D}(\mathbf{x}_B, \mathbf{x}_A, \omega) = -2\mathrm{i}\,\Im\left[G_{2D}(\mathbf{x}_A, \mathbf{x}_B, \omega)\right]. \tag{41}$$

To understand the physical meaning of this expression, let us take its inverse Fourier transform ($\mathscr{F}^{-1}$). It follows from eqs. (B6) and (B9) of *Boschi and Weemstra* [2015] that

$$\mathscr{F}^{-1}\left\{-2\mathrm{i}\Im\left[G_{2D}(\mathbf{x}_A, \mathbf{x}_B, \omega)\right]\right\} = G_{2D}(\mathbf{x}_A, \mathbf{x}_B, t) - G_{2D}(\mathbf{x}_A, \mathbf{x}_B, -t), \tag{42}$$

similar, e.g., to eq. (6) of *Fink* [2006]. Consider an arbitrary observation point $\mathbf{x}_A$ within ∂S, and recall that G_{2D} is nonzero only at positive time. As t grows from -∞ to 0, only the second term at the right-hand side of (42) is nonzero, which means that $\mathbf{x}_A$ records a time-reversed Green's function. In space, a time-reversed impulsive circular wave converging towards the original source location $\mathbf{x}_B$ is observed. As $t \longrightarrow 0$, the value of $\mathbf{x}_A$ for which the field is maximum approaches $\mathbf{x}_B$, where the backward-propagating circular wave eventually "focuses." As t grows from 0 to ∞, only the first term at the right-hand side of (42) is nonzero, and $\mathbf{x}_A$ records a regular Green's function with inverted sign. That is to say, another circular wave is emitted from $\mathbf{x}_B$ after focusing.

In the words of *Fink* [2006], "if we were able to create a film of the propagation of the acoustic field during" propagation of the signal from the original source at $\mathbf{x}_B$ to receivers on ∂S, "the final result could be interpreted as a projection of this film in the reverse order, immediately followed by a reprojection in the initial order." *Fink* [2006] notes that acoustic time reversal, as described here, does not involve the "time reversal of the source," and in "an ideal time-reversed experiment, the initial active source (that injects some energy into the system) must be replaced by a sink (the time reversal of a source)," i.e., "a device that absorbs all arriving energy without reflecting it."

Result (41) is limited to impulsive signals. If the signal emitted at $\mathbf{x}_B$ is an arbitrary function of time, $h(t)$, the signal recorded at each receiver location $\mathbf{x}'$ is the convolution $G_{2D}(\mathbf{x}', \mathbf{x}_B, \omega)h(\omega)$. Accordingly, let us replace $G^*_{2D}(\mathbf{x}', \mathbf{x}_B, \omega)$ at the right-hand side of eq. (34) with the convolution $G^*_{2D}(\mathbf{x}', \mathbf{x}_B, \omega)h^*(\omega)$. Since the function h does not depend on any other variable but t (or ω in the frequency domain), it can be pulled out of the $\mathbf{x}'$-integral; it then follows from eq. (34) itself that

$$\begin{aligned} &h^*(\omega)\left[G^*_{2D}(\mathbf{x}_A, \mathbf{x}_B, \omega) - G_{2D}(\mathbf{x}_B, \mathbf{x}_A, \omega)\right] \\ &= \int_{\partial S} \mathrm{d}\mathbf{x}' \left\{ \left[h(\omega)G_{2D}(\mathbf{x}', \mathbf{x}_B, \omega)\right]^* \nabla_1 G_{2D}(\mathbf{x}', \mathbf{x}_A, \omega) - G_{2D}(\mathbf{x}', \mathbf{x}_A, \omega)\nabla_1 \left[h(\omega)G_{2D}(\mathbf{x}', \mathbf{x}_B, \omega)\right]^* \right\} \cdot \mathbf{n}. \end{aligned} \tag{43}$$

If one denotes $s(\mathbf{x}', \mathbf{x}_B, \omega) = h(\omega)G_{2D}(\mathbf{x}', \mathbf{x}_B, \omega)$, eq. (43) takes the more compact form

$$\begin{aligned} &h^*(\omega)\left[G^*_{2D}(\mathbf{x}_A, \mathbf{x}_B, \omega) - G_{2D}(\mathbf{x}_B, \mathbf{x}_A, \omega)\right] \\ &= \int_{\partial S} \mathrm{d}\mathbf{x}' \left[s^*(\mathbf{x}', \mathbf{x}_B, \omega)\nabla_1 G_{2D}(\mathbf{x}', \mathbf{x}_A, \omega) - G_{2D}(\mathbf{x}', \mathbf{x}_A, \omega)\nabla_1 s^*(\mathbf{x}', \mathbf{x}_B, \omega)\right] \cdot \mathbf{n}. \end{aligned} \tag{44}$$

Alternatively, the far-field approximation (37) can be applied to (43), which, in analogy with sec. 3.1, collapses to

$$h^*(\omega)\left[G^*_{2D}(\mathbf{x}_A, \mathbf{x}_B, \omega) - G_{2D}(\mathbf{x}_B, \mathbf{x}_A, \omega)\right] \approx \frac{2\mathrm{i}\omega}{c} \int_{\partial S} \mathrm{d}\mathbf{x}' \left[s^*(\mathbf{x}', \mathbf{x}_B, \omega)G_{2D}(\mathbf{x}', \mathbf{x}_A, \omega)\right]. \tag{45}$$

Eqs. (44) and (45) stipulate that the same results obtained above for impulsive signals also apply to arbitrary signals $h(t)$, except that in this case the backward-propagating Green's function is convoluted with the time-reversed signal, $h^*(\omega)$ or $h(-t)$, itself. Importantly, the backward-propagating wave field focuses, again, on the source location $\mathbf{x}_B$.

5 Implementation

The so-called membrane-wave approach is based on the horizontal/radial decoupling of the equation of motion illustrated in sec. 2, where it is shown that the membrane eq. (23) holds for both the Love- and Rayleigh-wave potentials ϕ_L and ϕ_R. In sec. 3, some properties of the solution of (23), that naturally apply to both ϕ_R and ϕ_L, are derived. Their most important implication in the context of our study is explained in sec. 4: the theory of acoustic time reversal as developed by, e.g., *Fink* [2006] holds on a flat membrane, and, as a result, the time-reversed potentials ϕ_L, ϕ_R can be otbained from eqs. (44) or (45), i.e., by time-reversing and backward propagating the potentials associated with the recorded waveforms. Surface-wave time reversal then consists of (i) extracting ϕ_L and ϕ_R from the data, for a broad and dense set of surface-wave fundamental and higher modes; (ii) determining radial eigenfunctions (U and V, or W) for each mode; (iii) backward propagating ϕ_L and ϕ_R for each mode; (iv) combining potentials with radial eigenfunctions at all available frequencies, via eqs. (5) and (6), to find the time-reversed displacements $\mathbf{u}_R$ and $\mathbf{u}_L$.

An important limitation of this procedure, as discussed in some detail in sec. 4.2, is that the time-reversed wave field necessarily includes an impulse propagating away from the reconstructed source location *after* focusing. This is not a problem for point sources (or of less-than-wavelength spatial extent, as in this study), but the time-reversed wave field at each point of a *finite-extent* source will include a non-physical contribution that cannot easily be subtracted from it, and that pollutes images of seismic slip. It should be noted that back-projection methods suffer from the same problem, although this is rarely (if ever) discussed. This issue will have to be addressed in future work. One possible strategy would be to subdivide the source-imaging process into two steps. First, time reversal could be interrupted before focusing occurs: this way, the surface-wave field in the immediate vicinity of the source could be reconstructed. In a second step, the reconstructed near-field displacement could be treated as data in a classic linear inverse problem, based on the representation theorem [e.g. *Ide*, 2007]: the unknown being slip on the fault. The accuracy of near-field displacement as reconstructed by time reversal would significantly reduce nonuniqueness.

Only monochromatic, fundamental-mode Rayleigh-wave propagation is implemented here. At each frequency of interest ω, propagation of the corresponding sinusoidal Rayleigh wave is modeled in the time domain. It is apparent from eq. (5) that, at frequency ω, ϕ_R is directly proportional to the vertical component of displacement, narrow-band-pass-filtered around ω; i.e., before time reversal, $\phi_R(\omega)$ can be obtained by the vertical component of the displacement by simply multiplying it by $1/U(\omega)$. The implementation of time reversal is exactly the same for Love waves (except that membrane-wave propagation of the Love-wave potential must naturally be modeled in a Love-wave phase velocity map); the Love-wave potential ϕ_L, however, needs to be extracted from the *transverse* component of cross correlations, which will require some more subtle data processing to be addressed in future work.

Accordingly, we do not yet reconstruct time-reversed displacements from time-reversed potentials. This requires that the eigenfunctions U, V and W be computed for a selected reference model. Because the crust/lithosphere depth range (i.e., the depth range of interest to surface-wave propagation) is characterized by large lateral heterogeneity, it is likely that a 3-D reference model will need to be employed, through the implementation of "local" radial eigenfunctions [e.g. *Boschi and Ekström*, 2002]. Studying the focusing of the Rayleigh-wave vertical component at various frequencies is, however, sufficient to verify the feasibility of our approach, which is the main goal of the present study. In the following, we model the propagation of time-reversed surface-wave potentials via two different approaches: ray theory and the spectral-element method.

In the ray-theory case, the value of G_{2D} for any given source and receiver position is determined approximately by tracing the ray between source and receiver, computing the propagation time along such ray path, and shifting by such time the signal prescribed at a source. Rays are traced by means of the algorithm described by *Fang et al.* [2015]. Geometrical spreading is accounted for approximately by simply multiplying the signal by the inverse squared root of the source-receiver distance, according, e.g., to eq. (E17) of *Boschi and Weemstra* [2015].

In the spectral-element case, following *Tape et al.* [2007], SPECFEM2D [*Komatitsch and Tromp*, 1999] is used to simulate the propagation, on a stretched, flat membrane, of a displacement perpendicular to the unperturbed membrane surface. Displacement is generated by prescribing a point force/acoustic pressure (rather than an initial displacement as in the ray-theory case), which implies, importantly, that our comparison between ray-theory and SPECFEM2D results is only qualitative. Additionally, to model wave propagation via SPECFEM2D, we need to project our spherical-Earth phase-velocity map onto a flat surface. This is done via a transverse Mercator projection centered at 12°E, 46°N. Errors are introduced near the corners of the region of interest, that will (slightly) alter modeled waveforms and might reduce the quality of focusing: the flat-membrane approach is adequate to the feasibility study presented here, but curved membranes will have to be implemented for future applications.

6 Validation

We test both ray-theory and spectral-element methods on synthetic membrane-wave data, on ambient-noise vertical-component cross correlations (which are theoretically equivalent to recordings of Rayleigh-wave impulse responses) and on vertical-component recordings of a 5.6-magnitude earthquake. To make sure that we can rely on robust, high-resolution Rayleigh-wave phase-velocity maps and a dense station coverage, we select Northern Italy, including most of the Alpine mountain range, as our study region. This area is characterized by complex tectonics, and at this scale surface waves are difficult to identify as they are hidden in the body-wave coda: if we can validate our theory in such an unfavorable situation, we can then expect that it will hold also at teleseismic scales. Furthermore, by limiting the experiments presented here to a relatively small region, we reduce the associated computational costs.

Earthquake data were downloaded from EIDA (http://www.orfeus-eu.org/data/eida/) and from all permanent broadband stations that recorded the earthquake within the region of interest; this includes INGV [*INGV Seismological Data Centre*, 1997], SED [*Swiss Seismological Service (SED) at ETH Zurich*, 1983], OGS [*Istituto Nazionale di Oceanografia e di Geofisica Sperimentale*, 2002], MedNet [*MedNet project partner institutions*, 1988] and the University of Genova data archive. Continuous ambient data for the year 2010 were downloaded from all available permanent broadband stations that were active during that time,

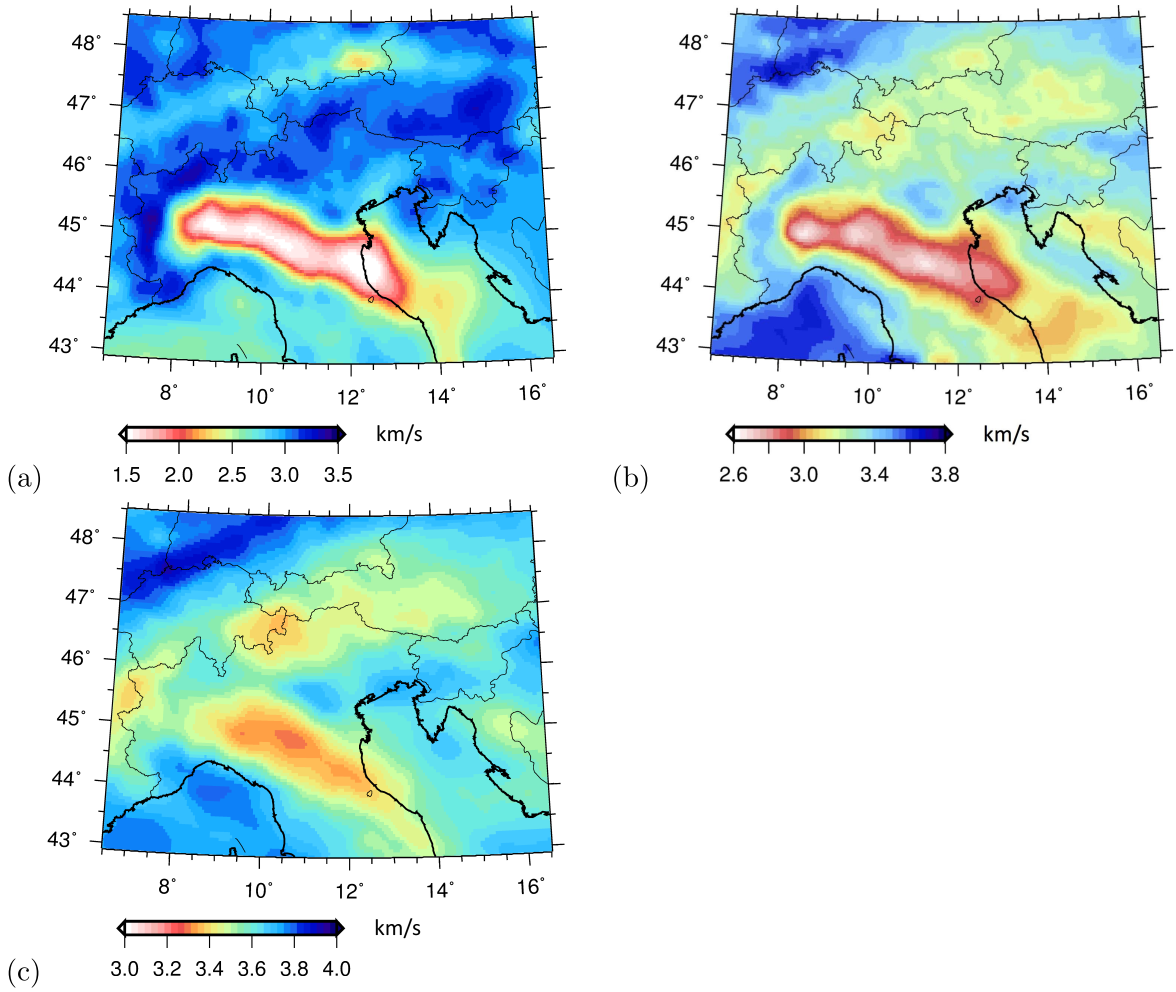

Figure 1: Rayleigh-wave phase-velocity maps of *Kaestle et al.* [2017], at periods of (a) 6 s, (b) 16 s and (c) 25 s.

via the INGV data center. Time-domain cross correlations were computed as described by *Molinari et al.* [2018].

As a general rule, computational costs are much reduced with respect to typical 3-D wave-propagation modeling applications in seismology. A time-reversal simulation, such as the ones shown in the following, involves one single run of SPECFEM2D with multiple sources (one per station), which requires about two hours on a single CPU. Ray-theory based simulations are much cheaper: a time-reversal simulation can be completed in less than two minutes on similar hardware.

6.1 Synthetic tests

Theoretical traces associated with a selected point-source location and a realistic station distribution in the region of interest are obtained via ray theory and SPECFEM2D. The source signal $h(t)$ is a Ricker wavelet as implemented in SPECFEM2D, Butterworth-filtered between 6 and 26 s. Membrane waves are propagated through the 16s Rayleigh-wave phase-velocity map of *Kaestle et al.* [2017], shown here in Fig. 1b. While only one particular surface-wave mode is implemented for this synthetic test, it is understood that the exact same procedure can be applied in the calculation of other Rayleigh- and Love-wave fundamental modes and overtones. No random noise is added to the synthetic signal.

Because the station distribution is nonuniform, the curve ∂S and, as a consequence, the vector $\mathbf{n}$ in eq. (34) are not uniquely defined. We avoid this difficulty by replacing eq. (34) with its far-field approximation (39), which can be implemented without specifying $\mathbf{n}$. Preliminary experiments show that, despite the small size of the study region, the location of the backward propagating wave field's focus is not visibly affected by this approximation. We plan to find ways to implement (34) exactly in future work, but we believe that the simplified approach employed here is adequate to the scope of this article. We accordingly time-reverse the traces, and propagate them backward in time, essentially implementing the right-hand side of eq. (45). Again, waves are propagated through the map of Fig. 1b.

We obtain a pair of animations, one based on ray theory and the other on SPECFEM2D. Samples of both are shown in Fig. 2. Fig. 3 shows the prescribed and reconstructed signal at the known source location, again for both methods. While the backward propagating wave fields differ because of the mentioned physical differences in the implementation (excitation by initial displacement vs. point force, curved membrane vs. Mercator projection), it appears from Fig. 2c,d that the maximum amplitude with respect to time and position in both simulations corresponds to the known, initial source location. This confirms the validity of surface-wave time reversal as a tool to localize/image a seismic source, despite the severe nonuniformity in receiver distribution. The maximum is less pronounced in the spectral-element simulation, resulting in normalized amplitudes throughout the simulation to be larger than ray-theory-based amplitudes. After focusing, in the absence of an "acoustic sink" (sec. 4.2), a non-physical wave field propagates away from the source. There are no major differences in the quality of focusing achieved by ray-theory vs. SPECFEM2D backward-propagation.

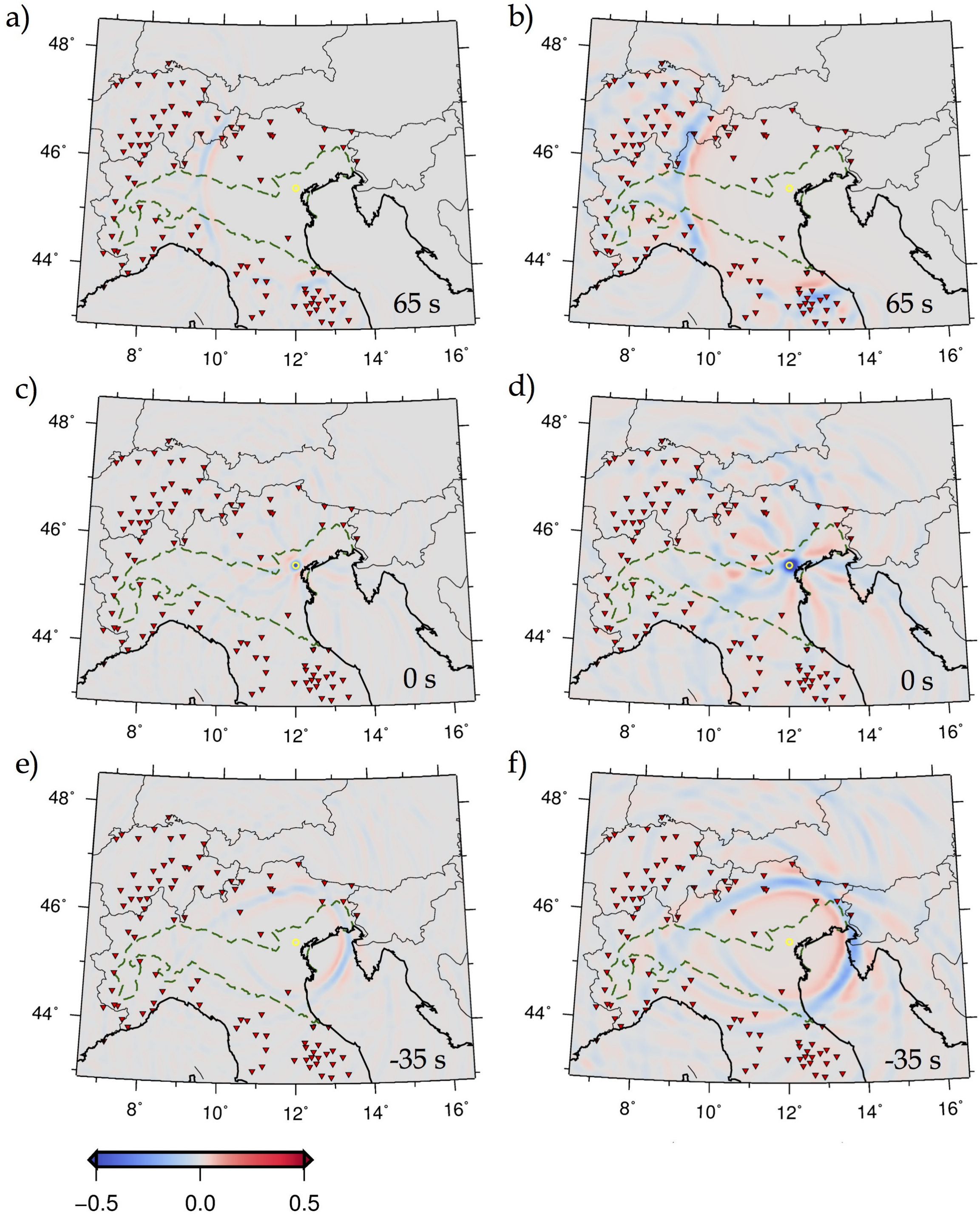

Figure 2: Snapshots of the ray-theory (left) and SPECFEM2D (right) synthetic-data time-reversal simulations (sec. 6.1). Station locations are denoted by red triangles, the source location by a yellow circle. We define t=0 as the time when the source experiences the maximum displacement according to the Ricker wavelet in the *forward* simulations; backward propagation starts at the time corresponding to the last data sample employed in our exercise, and time increments in time-reversal simulations are considered to be negative. For each of the two time-reversal simulations, amplitudes are normalized to the maximum value obtained in the simulation, corresponding to source location at t=0. Snapshots a and b are taken at time t=65 s; c and d at t=0 s, e and f at t=-35 s. As explained in sec. 6, ray-theory and SPECFEM2D wave fields can be compared only qualitatively. Snapshots c and d show that the time-reversed wave field focuses at the "epicenter" location.

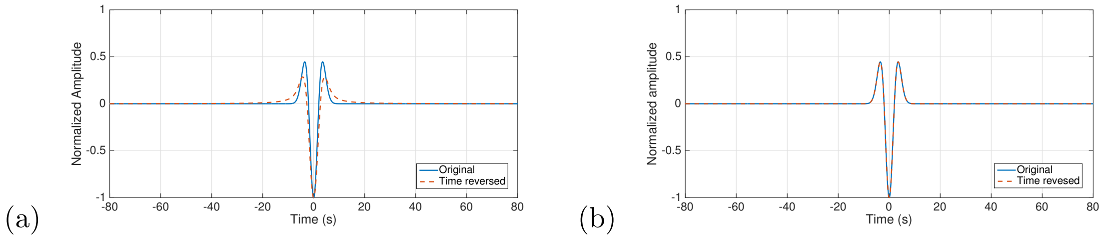

Figure 3: Synthetic test of sec. 6.1: Normalized time-reversed and backward-propagated displacement (dashed red curves) computed at the known location of the source, via (a) SPECFEM2D and (b) ray theory. In both cases, the known source time function is shown (blue curve) for comparison. In panel a, the difference between forcing term and reconstructed signal is explained by the fact that, in SPECFEM2D, displacement is initiated by prescribing a point force, rather than a displacement as in the ray-theory case.

6.2 Ambient-noise cross correlations

Cross-correlations of ambient signal form a perfectly suited data set to validate a source-localization method: each cross-correlation is an approximation for the corresponding receiver-receiver Green's function, and the location of both receivers is naturally well known. We select station LSD.GU (Fig. 5) as our virtual "test" source, and time-reverse and backward propagate ambient-noise based Green's functions associated with it. (Noise cross correlations will be described in a separate study [*Molinari et al.*, 2018].) This amounts to implementing the right-hand side of eq. (39) via our two algorithms. We first Butterworth-filter vertical-component cross correlations around 16 s (low and high corner frequencies corresponding to periods of 26 and 6 s, respectively), and, as in sec. 6.1, propagate time-reversed signal through the Rayleigh-wave phase-velocity map of Fig. 1b. The results of this exercise are shown in Fig. 4 and Fig. 5. Again, despite the poor azimuthal station coverage in this example, the time-reversed wave field focuses quite precisely on the virtual source, in both the ray-theory and SPECFEM2D implementations. In both cases, the maxima of the time-reversed wave field at the known source location is correctly achieved at t=0. Similar to sec. 6.1, non-physical signal naturally emerges after focusing.

If station coverage were uniform and the noise-based Green's functions perfectly reconstructed, the time-reversed signal at LSD.GU (Fig. 5) should closely approximate an impulse, which is not the case. We have seen, however, from the results of sec. 6.1 and in particular Fig. 3, that the source time function can be reconstructed well even when the station coverage is poor. We infer that artifacts in the trace of Fig. 5 result from inaccuracies in the reconstructed Green's function. This is not surprising because, while the phase of Green's functions is reconstructed well by ambient-noise cross correlation, their amplitude probably is not [e.g. *Ekström et al.*, 2009].

We next iterate the ray-theory procedure for the 4-to-10 s and 20-to-30 s period bands, and show the results in Figs. 6 and 7. Membrane-wave propagation is modeled using phase-velocity maps at 6 and 25 s periods (Fig. 1a, c), respectively, again from *Kaestle et al.* [2017]. The quality of focusing is comparable to the intermediate-period case (Fig. 4), and can be considered high, in view of the nonuniformity of station distribution. This result confirms that our algorithm can be applied to a variety of surface-wave modes, and, as far as reliable phase-velocity and Green's function estimates are available, is fairly independent of period,

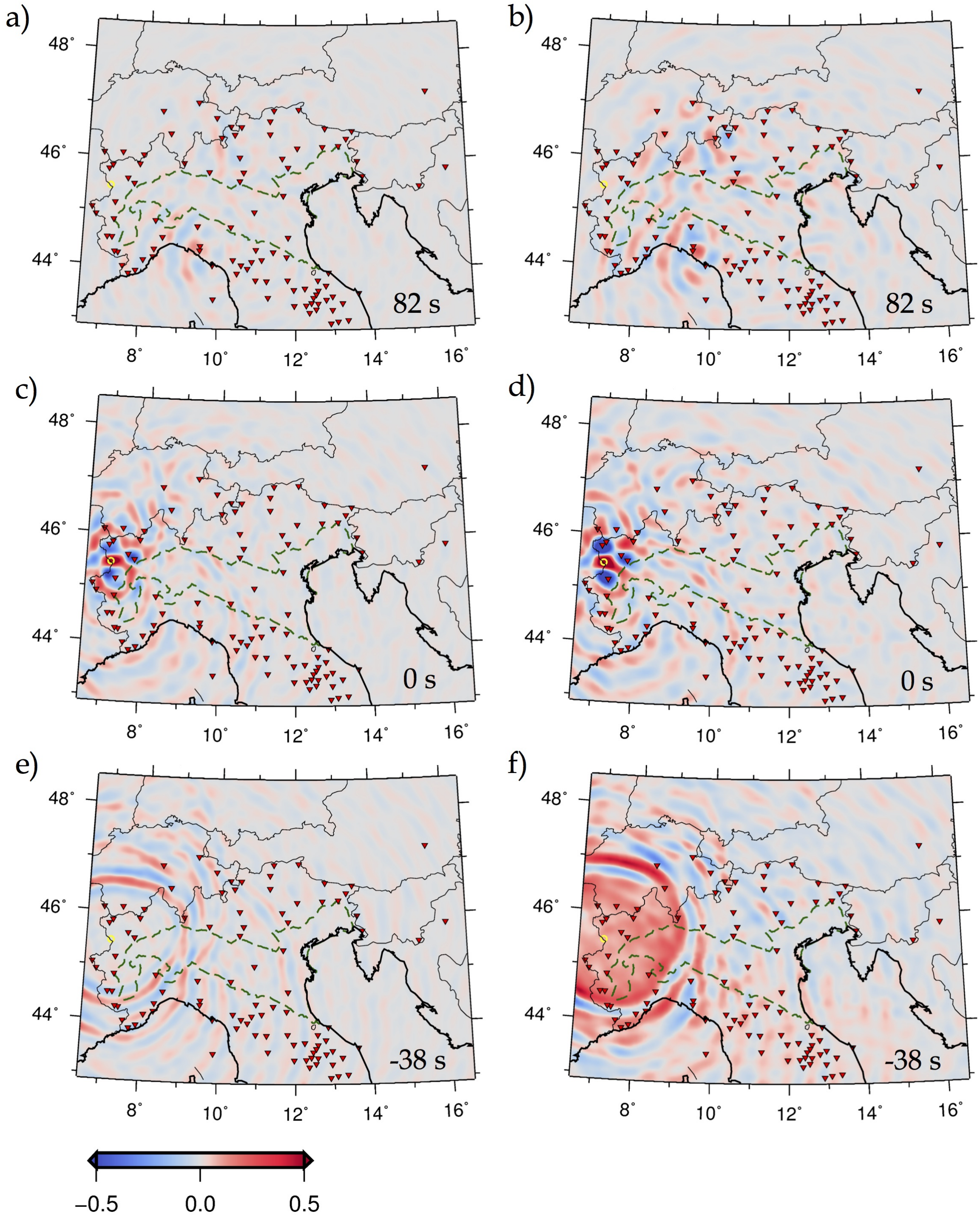

Figure 4: Snapshots of the ray-theory (left) and SPECFEM2D (right) time-reversal simulations of real noise cross-correlations described in sec. 6.2, in the 6-to-26 s period band. This is similar to Fig. 2, but synthetic traces are replaced by cross-correlations of ambient data recorded at station LSD.GU (yellow circle) and all other stations whose locations are denoted by red triangles. Ambient-noise cross-correlations approximate the Green's function for each station pair, and, in this exercise, station LSD.GU can accordingly be thought of as a "virtual source." Snapshots a and b are taken at time t=82 s; c and d at t=0 s, e and f at t=-38. Snapshots c and d show that the time-reversed wave field indeed focuses at the location of station LSD.GU.

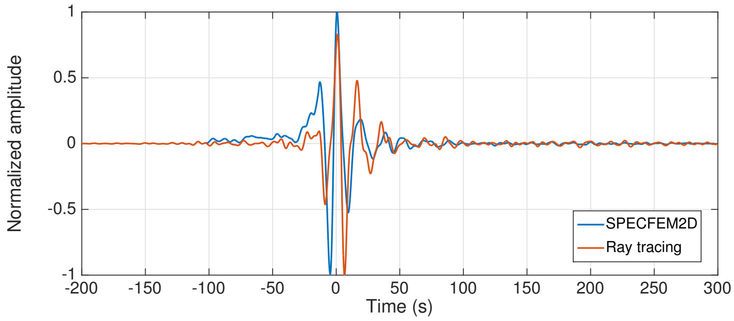

Figure 5: Time-reversed and backward-propagated empirical, ambient-noise based Green's functions (sec. 6.2), computed at the location of the virtual source, i.e. station LSD.GU, via SPECFEM2D (blue curve) and ray theory (red).

and of the width of the passband.

6.3 Recordings of the Emilia earthquake of May 29, 2012

We apply our ray-theory- and SPECFEM2D-based algorithms to vertical-component recordings of the magnitude Mw=5.6 (Ml=5.8) Emilia earthquake of May 29, 2012, 7:00:03 AM. These data, discussed in detail by *Molinari et al.* [2015], are shown here in Fig. 8. Traces are filtered around 16 s, the same way as in sec. 6.2, before time-reversal and backward propagation; propagation is modeled according to the 16s Rayleigh-wave phase velocity map of Fig. 1b. Results are summarized in Figs. 9 and 10. Early time-steps (e.g., Fig. 9a,b) are characterized by the emergence of time-reversed late arrivals, that we believe to be associated with reverberations, e.g. at the sharp boundaries between Po plain and surrounding mountain ranges. This signal does not focus sharply anywhere on our membrane, and can accordingly be neglected in this context. Direct-arrival surface waves, on the other hand, do focus at the known epicenter location in both our implementations (Fig. 9c,d). Similar to secs. 6.1 and 6.2, non-physical signal again emerges after focusing (Fig. 9e,f).

Fig. 10 shows that the maximum amplitude of the reconstructed vertical displacement at the epicenter occurs at t=22s according to spectral-element time reversal; this delay with respect to the reported earthquake origin time is comparable with the considered surface-wave period, and, in order of magnitude, with typical discrepancies between body- and surface-wave-based estimates of rupture times. The ray-theory simulation results in multiple maxima between 0 and 50 s. All this presumably reflects the complexity of surface-wave generation at the source, as well as errors introduced by the mentioned, non-physical propagation of the time-reversed wave field after focusing.

We repeat ray-theory time reversal in the 4-to-10 s and 20-to-30 s passbands, and show the results in Figs. 11 and 12. Focusing of the time-reversed wave field is less sharp both in space and time (although, interestingly, in late snapshots of the time-reversal simulation (Fig. 11e,f), wavefronts are nicely centered on the earthquake epicenter). We ascribe the loss in source localization accuracy to the significant reduction in the width of the passbands, with respect to the previously discussed, 6-to-26 s simulation: we had anticipated in sec. 1 that focusing of the time-reversed wave field is enhanced by combining as many time-reversed surface-wave modes as possible. In our future work, we plan to more rigorously take advantage of this effect, multiplication surface-wave potential and their horizontal gradients by the radial

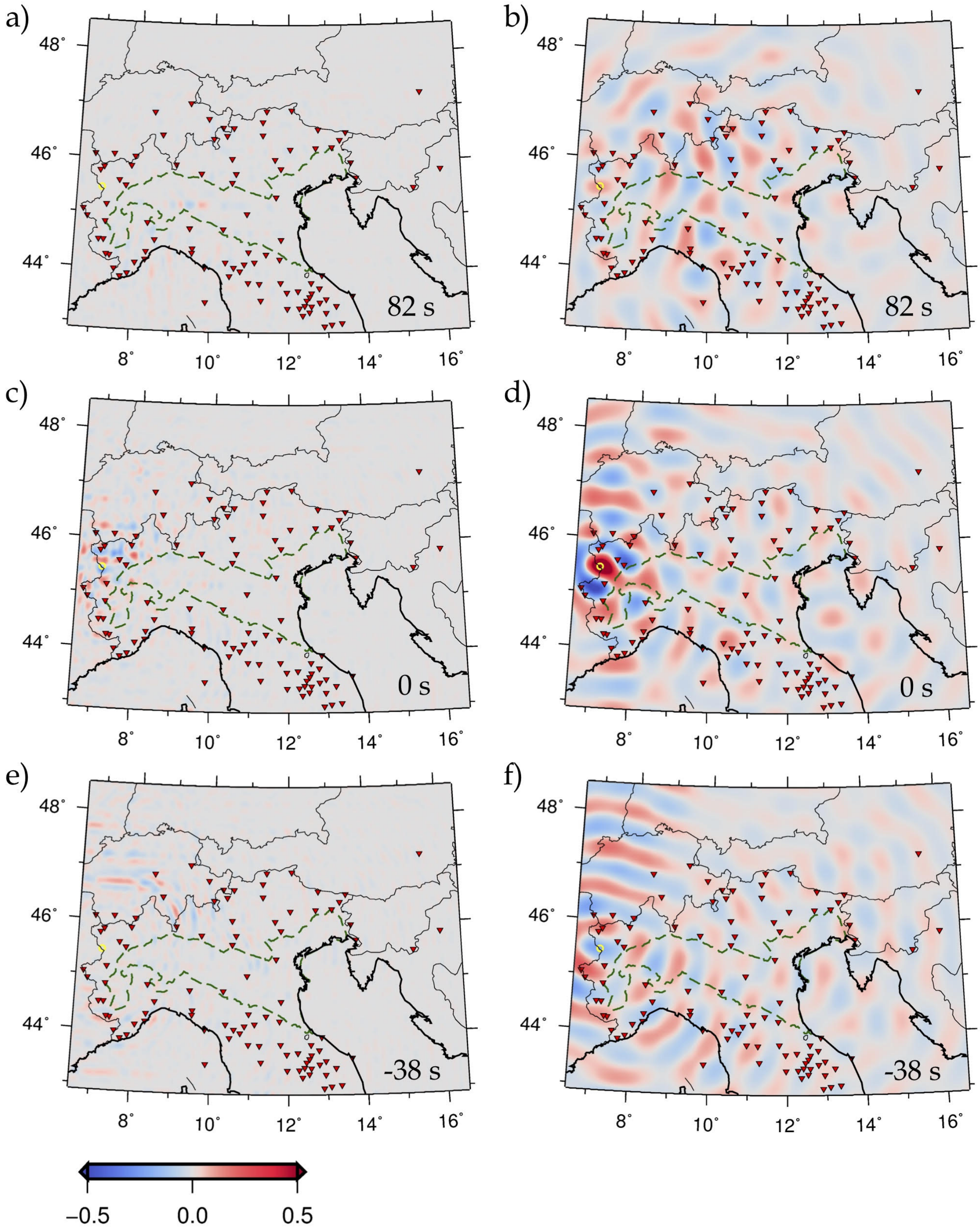

Figure 6: Snapshots of time-reversal simulations of real noise cross-correlations, in the 4-to-10 s (left) and 20-to-30 s (right) period bands. As in Fig. 4, cross-correlated data were recorded at station LSD.GU (yellow circle) and all other stations whose locations are denoted by red triangles. Snapshots were selected at the same times as in Fig. 4. Snapshots c and d show that, also in these period bands, the time-reversed wave field focuses at the location of station LSD.GU.

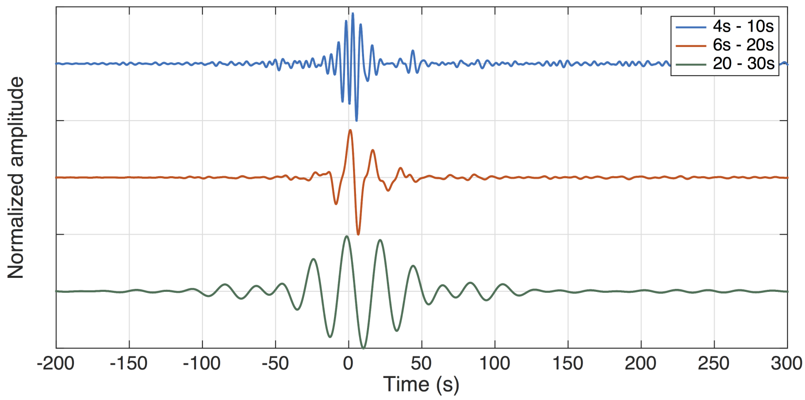

Figure 7: Same as Fig. 5, but traces obtained (via ray theory only) in the period bands 4-to-10 s (blue curve), 6-to-26 s (red) and 10-to-20 s (green) are shown.

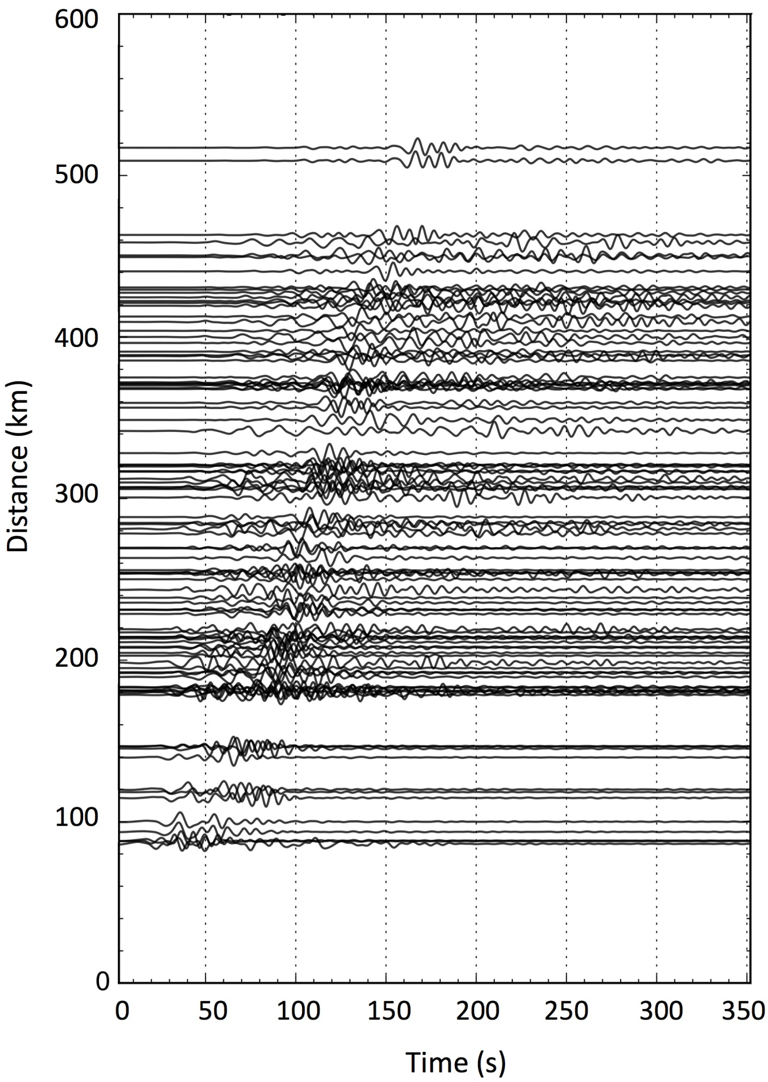

Figure 8: Normalized vertical-component recordings of the Mw=5.6 (Ml=5.8) Emilia earthquake of May 29, 2012 [e.g. *Molinari et al.*, 2015], that we time-reverse and backward-propagate as discussed in sec. 6.3. The vertical axis corresponds to epicentral distance, and each trace is plotted about its associated epicentral distance. All traces are Butterworth-filtered around 16 s as described in sec. 6.3.

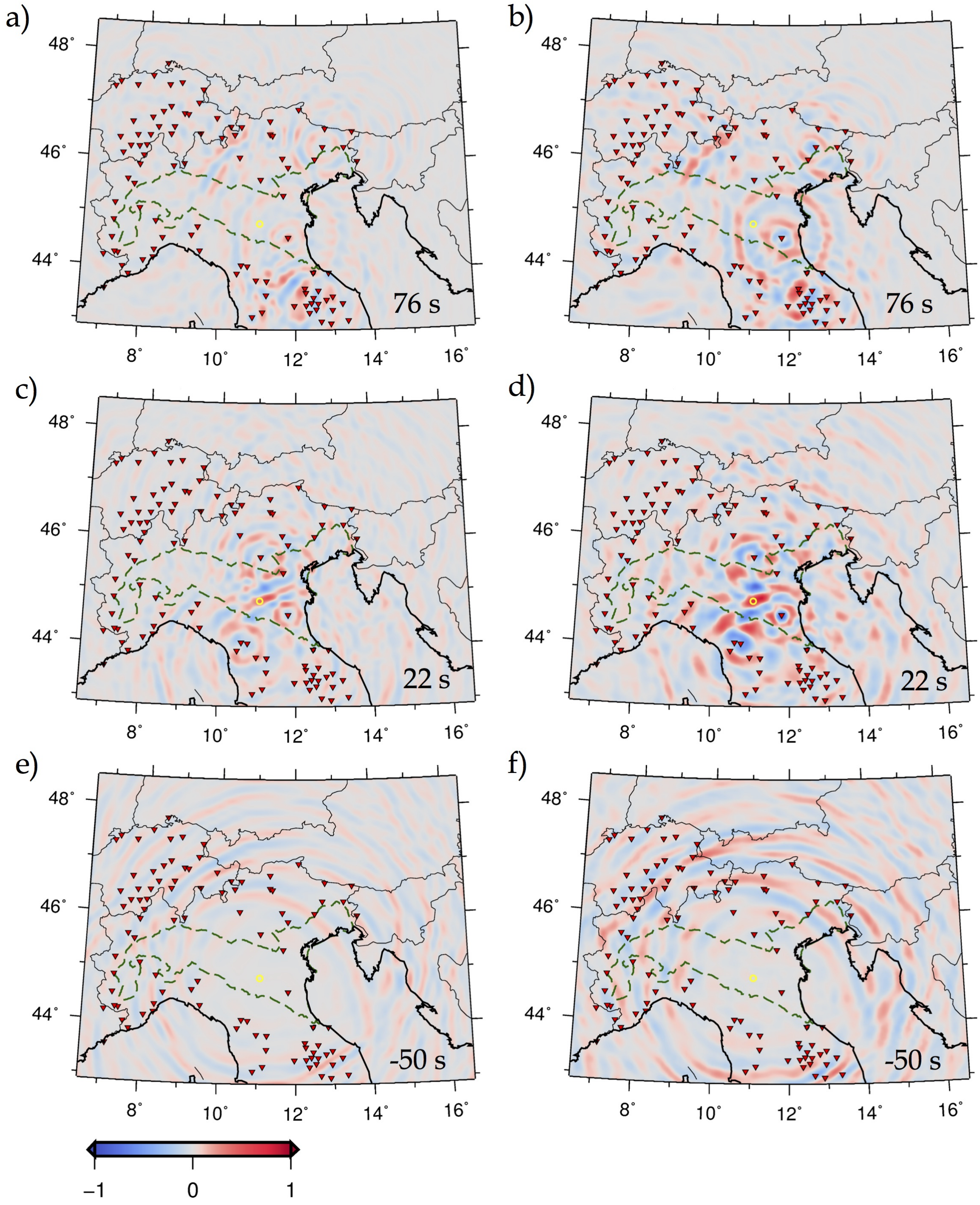

Figure 9: Snapshots of the ray-theory (left) and SPECFEM2D (right) time-reversal simulations of real earthquake data described in sec. 6.3. Again, the locations of stations utilized in the time-reversal simulation are denoted by red triangles, while the earthquake epicenter is marked by a yellow circle. We define t=0 as the earthquake origin time as reported by the *Centro Nazionale Terremoti* at INGV. Snapshots a and b are taken at time t=76 s; c and d at t=22 s, e and f at t=-50. Snapshots c and d show that the time-reversed wave field focuses at the epicenter of the earthquake.

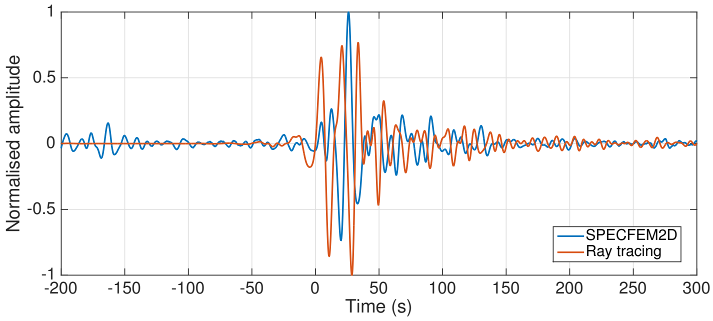

Figure 10: Time-reversed signal at the epicenter of the Emilia earthquake as reconstructed by SPECFEM2D (blue curve) and ray theory (red) time reversal. Again, we define t=0 as the earthquake origin time; t should be intepreted as in Fig. 9, i.e. negative t corresponds to time *after* focusing in a time-reversal simulation.

eigenfunctions $U(\omega)$, $V(\omega)$, $W(\omega)$ according to eqs. (5) and (6), before integrating over the entire surface-wave frequency range.

Importantly, our analysis of time-reversed earthquake data shows that even at relatively short epicentral distances, where they are obscured by the body-wave coda, surface waves can still emerge in a time-reversal exercise. Focusing of the backward-propagated signal at the source can be thought of as a form of constructive interference. For time-reversed waves emitted at various station locations to interfere constructively, their backward propagation has to be modeled correctly. In our approach, time-reversed seismograms are filtered around one surface-wave frequency, and backward-propagated via the known Green's function (i.e. phase-velocity map) for that frequency. In other words, only the propagation of time-reversed signal associated with surface waves at that frequency is modeled correctly, and it is only this signal that will contribute to "constructive interference" and to focusing of the time-reversed wave field. Accordingly, circular wave fronts that can be associated with body-wave signal, and that do not focus at the epicenter (or elsewhere) are visible in Fig. 9a, b. We infer that surface-wave time reversal can indeed function as a source-imaging method also at relatively small epicentral distances, independently of how clearly surface waves can be identified visually on seismograms.

7 Conclusions

By taking advantage of the theory of surface-wave "potentials," we have reduced the problem of surface-wave propagation to two dimensions ("membrane waves"). We have shown that 3-D wave fields can then be reconstructed from monochromatic 2-D ones, once radial surface-wave eigenfunctions (sec. 2) are known; in this study, however, we only studied the propagation of surface-wave potentials in 2-D. We implemented a surface-wave time-reversal algorithm that can rely on either spectral-element or ray-theory models of wave propagation. In both cases, the theory is validated by application to real seismometer arrays in Central Europe. First, a synthetic test is implemented by computing approximately monochromatic membrane-wave seismograms at all receiver positions, from an arbitrary selected source location in Northern Italy. In a second experiment, synthetic traces are replaced by approximate Green's functions, obtained by cross-correlating the real ambient signal recorded at one station of the array with

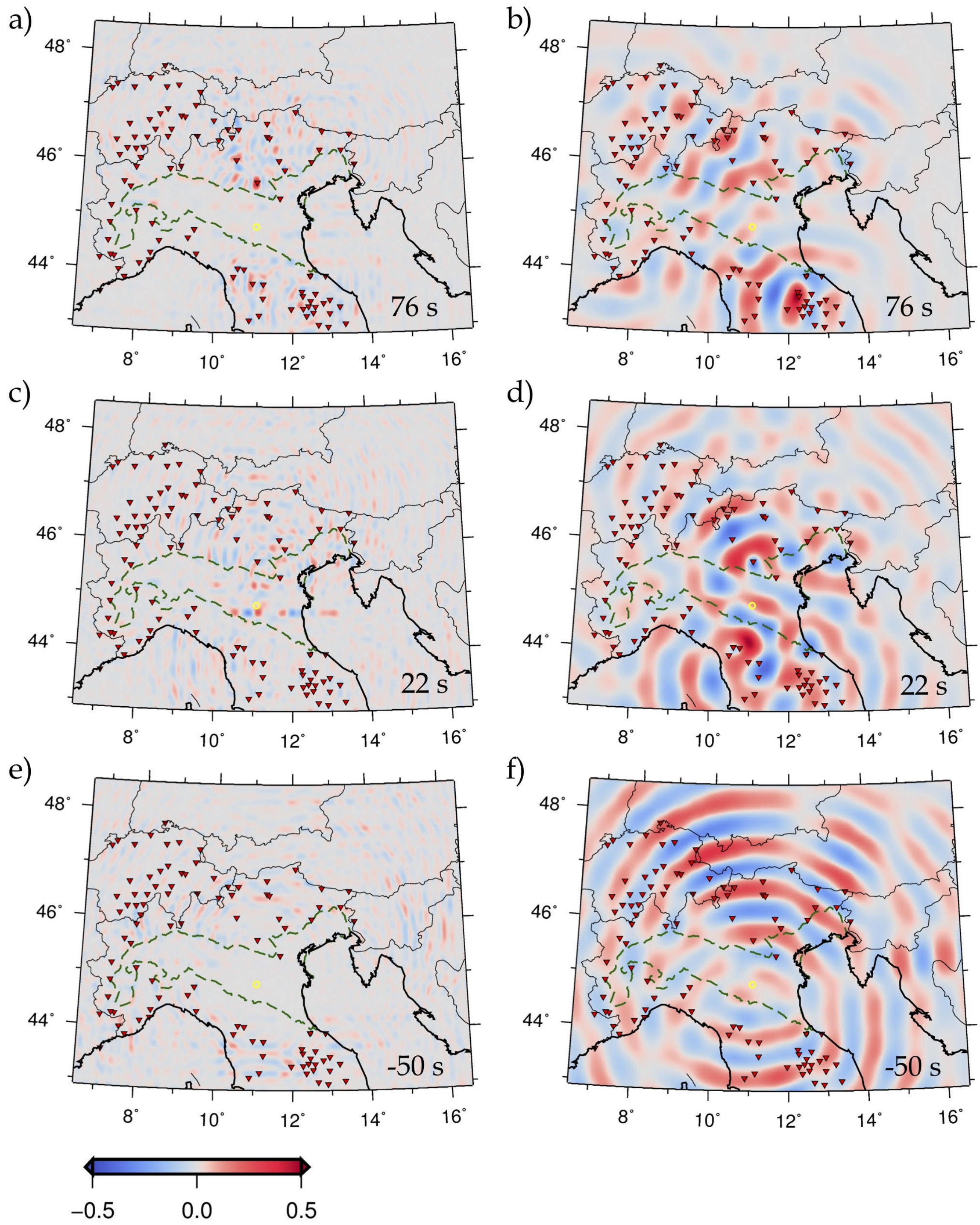

Figure 11: Snapshots of ray-theory time-reversal simulations of real earthquake data, in the 4-to-8 s (left) and 20-to-30 s (right) period bands. Snapshots were selected at the same times as in Fig. 9. All symbols are defined as in Fig. 9.

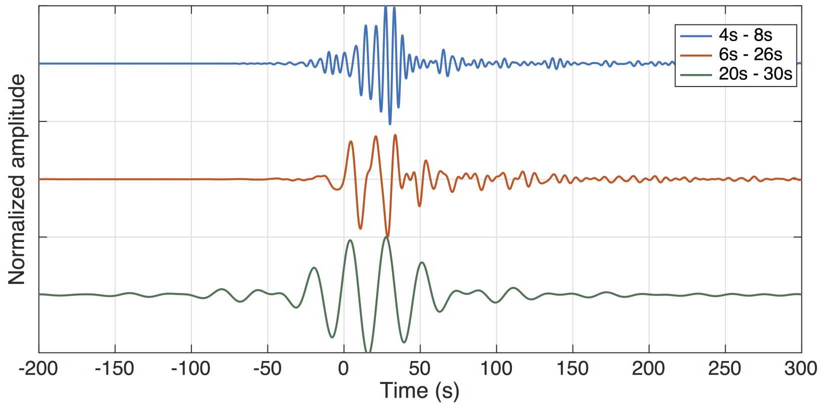

Figure 12: The ray-tracing based trace of Fig. 10 (red curve, 6-to-26s period band) is compared to analogous traces obtained for the 4-to-8s (blue) and 20-to-30s (green) bands. Each trace is normalized to its maximum.

that recorded at all other stations. Finally, waveforms from a magnitude-5.6 event in the Po plain are used. In all three cases, time-reversal and backward propagation of the data result in focusing of the signal at the location and time of the source, despite the severe nonuniformity of data coverage, inaccuracies in ambient-noise-based Green's function reconstruction, and difficulties in disentangling surface-wave signal from the body-wave coda. Importantly, our experiment described in sec. 6.3 suggests that time reversal and backward propagation using the surface-wave Green's function result in focusing of surface waves at the epicenter even at distances less than teleseismic, where surface waves carry less energy than body waves and their coda. These results encourage further applications of our method, in particular to the task of mapping, in space and time, rupture processes associated with large earthquakes.

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Appendix: dipole sources

The term "dipole source" refers here to the superposition of two impulsive point sources *of opposite sign*, located at two different points separated by a *very small* distance d. In this study, the concept of dipole emerges from the physical interpretation (sec. 4.2) of equation (34), relating the time-reversed, backward propagating wave field to the signals initally recorded by a receiver array. In the general context of wave physics, dipole sources are used, e.g., to formulate a modern, "corrected" version of Huygens' principle [*Baker and Copson*, 1950; *Miller*, 1991].

The mathematical expression for a dipole source can be obtained by first writing the forcing term q defined in sec. 3 as the sum of two equal source distributions $f(\mathbf{x},\omega)$ shifted in space by the vector $\mathbf{d}$ (of magnitude d) and switched in sign one with respect to the other, i.e.

$$q(\mathbf{x},\omega) = f\left(\mathbf{x}+\mathbf{d}\right) - f(\mathbf{x}). \tag{46}$$

A first-order Taylor expansion around the point $\mathbf{x}_S$ then gives

$$f(\mathbf{x}+\mathbf{d}) \approx f(\mathbf{x}) + \mathbf{d}\cdot\nabla_1 f(\mathbf{x},\mathbf{x}_S). \tag{47}$$

Substituting expression (47) into (46), we find

$$q(\mathbf{x},\omega) \approx \mathbf{d}\cdot\nabla_1 f(\mathbf{x},\mathbf{x}_S). \tag{48}$$

Finally, the sought expression is found by replacing f with a Dirac $\delta(\mathbf{x}-\mathbf{x}_S)$; since q accordingly becomes infinitely large at $\mathbf{x}_S$ and zero elsewhere, the magnitude of $\mathbf{d}$ ceases to have meaning and $\mathbf{d}$ can be replaced by the corresponding unit vector $\hat{\mathbf{d}}$, so that

$$q(\mathbf{x},\omega) = \hat{\mathbf{d}}\cdot\nabla_1\delta(\mathbf{x}-\mathbf{x}_S) \tag{49}$$

[e.g., *Wapenaar and Berkhout*, 1989, sec. I.3.1].

Let us next find a simple expression for the response of a medium to dipole forcing. Recall that we have introduced the Green's function $\mathscr{G}_{2D}(\mathbf{x},\mathbf{x}_S,\omega)$ in sec. 3 as the solution of eq. (23) with $q(\mathbf{x},\omega)=\delta(\mathbf{x}-\mathbf{x}_S)$, i.e.

$$\nabla_1^2\mathscr{G}_{2D}(\mathbf{x},\mathbf{x}_S,\omega) + \frac{\omega^2}{c^2}\mathscr{G}_{2D}(\mathbf{x},\mathbf{x}_S,\omega) = -\mathrm{i}\omega\delta(\mathbf{x}-\mathbf{x}_S). \tag{50}$$

Applying the operator $\hat{\mathbf{d}}\cdot\nabla_1$ to both sides of eq. (50) yields

$$\nabla_1^2\left[\hat{\mathbf{d}}\cdot\nabla_1\mathscr{G}_{2D}(\mathbf{x},\mathbf{x}_S,\omega)\right] + \frac{\omega^2}{c^2}\,\hat{\mathbf{d}}\cdot\nabla_1\mathscr{G}_{2D}(\mathbf{x},\mathbf{x}_S,\omega) = -\mathrm{i}\omega\,\hat{\mathbf{d}}\cdot\nabla_1\delta(\mathbf{x}-\mathbf{x}_S). \tag{51}$$

We infer from eq. (51) that the solution of (23) with $q(\mathbf{x},\omega)=\hat{\mathbf{d}}\cdot\nabla_1\delta(\mathbf{x}-\mathbf{x}_S)$ is simply $\hat{\mathbf{d}}\cdot\nabla_1\mathscr{G}_{2D}(\mathbf{x},\mathbf{x}_S,\omega)$.

Alternatively, *Boschi and Weemstra* [2015] (eqs. (E1)-(E3)) define the Green's function G_{2D} in the time domain as the solution of

$$\nabla^2 G_{2D} - \frac{1}{c^2}\frac{\partial^2 G_{2D}}{\partial t^2} = 0 \tag{52}$$

with initial conditions

$$G_{2D}(\mathbf{x},\mathbf{x}_S,0) = 0, \tag{53}$$

$$\frac{\partial G_{2D}}{\partial t}(\mathbf{x},\mathbf{x}_S,0) = \delta(\mathbf{x}-\mathbf{x}_S). \tag{54}$$

Applying, again, $\hat{\mathbf{d}}\cdot\nabla_1$ to both sides of eqs. (52)-(54), we find that $\hat{\mathbf{d}}\cdot\nabla_1 G_{2D}(\mathbf{x},\mathbf{x}_S,t)$ is the field resulting from a *dipole initial velocity* at $\mathbf{x}_S$. This, or rather its Fourier transform $\hat{\mathbf{d}}\cdot\nabla_1 G_{2D}(\mathbf{x},\mathbf{x}_S,\omega)$, is what we call "dipole response" throughout this study.