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Abstract

We consider an abstract second order evolution equation of the form

$$u'' + g(t, u') + \nabla F(u) = 0$$

set for $t \in (0, T)$ on a Hilbert space V such that $F \in C^1(V)$ with velocities in a larger Hilbert space H identified with its dual. The term $g(t, u')$ represents a dissipation mechanism and the restoring force $-\nabla F(u)$ dominates in a certain sense the dissipative term which is strictly superlinear at infinity with respect to the velocity. Under relevant conditions we establish that the energy $E(t) := \frac{1}{2}|u'(t)| + F(u(t))$ of any strong solution u is bounded for any $t \in (0, T)$ by a constant multiple of a negative power of t, thus independently of the initial state $(u(0), u'(0)) \in V \times H$. Applications are given to the general solution of some nonlinear wave, plate and Kirchhoff equations in a bounded domain.

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1 Introduction

It is well known that a certain number of dynamical systems $S(t)$ defined on a Banach space X have the property of universal boundedness for all $t > 0$, in the sense that

$$\forall t > 0, \quad S(t)X \text{ is a bounded subset of } X.$$

As a simple example we can consider the first order scalar ODE

$$u' + \delta|u|^\rho u = 0$$

with δ, ρ positive. Indeed for any initial state $u(0) = u_0$ we have

$$|u(t)| \leq \frac{C}{t^{1/\rho}}$$

Where $C = (\rho\delta)^{-1/\rho}$ is independent of u_0. This property extends classically to some classes of nonlinear parabolic PDEs, for instance the semilinear parabolic equation

$$u' - \Delta u + \delta|u|^\rho u = 0$$

with either Dirichlet or Neumann homogeneous boundary conditions, the result follows at once from the maximum principle. For a more elaborate quasilinear case, cf. Jacques Simon [10].

It is natural to ask whether second order ODEs with superlinear dampings such as

$$u'' + \omega^2 u + \delta|u'|^\rho u' = 0$$

have the same property, however the equation

$$u'' + u + |u'|u' = 0$$

has an explicit solution defined for all t on the line, more precisely $u(t) = \frac{1}{4}t^2 - \frac{1}{2}$ is a solution for $t \leq 0$ which extends uniquely for $t \geq 0$ and has an unbounded range. In such a case, due to the autonomous character of the equation, the entire range of the unbounded solution is contained in $S(t)\mathbb{R}^2$ for all $t > 0$ and universal boundedness fails. The next step is to consider the scalar second order ODE

$$u'' + |u'|^\alpha u' + |u|^\beta u = 0, \tag{1.1}$$

For this equation, Philippe Souplet [11] gave a definitive negative answer when $\alpha \geq \beta \geq 0$. On the other hand if $0 < \alpha < \beta$, it was shown very recently in [2] that the universal boundedness holds and moreover the method of [2] yields the optimal estimate

$$\forall t > 0, \quad E(t) \leq C \max\{t^{-\frac{2}{\alpha}},\ t^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}}\} \tag{1.2}$$

where

$$E(t) = \frac{1}{2}u'(t)^2 + \frac{1}{\beta+2}|u(t)|^{\beta+2} \tag{1.3}$$

and C does not depend on the initial data. After this result, which can be easily extended in the finite dimensional vector framework, it was reasonable to ask what happens for wave equations of the type

$$u'' - \Delta u + |u|^\beta u + |u'|^\alpha u' = 0, \tag{1.4}$$

with either Dirichlet or Neumann boundary conditions. Since in the absence of the nonlinear term $|u|^\beta u$, the universal boundedness does not take place as shown in [4], and there is no such maximum principle as in the parabolic case, the issue seems to be non-trivial and one would rather expect the result to hold for quasilinear equations of the form

$$u'' - \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{\beta}{2}} \Delta u + |u'|^\alpha u' = 0, \tag{1.5}$$

with the problem that the interpretation of "solutions" is not at all clear in that case. Then one would think of limiting the investigation to equations like

$$u'' - \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{\beta}{2}} \Delta u + \left(\int_\Omega |u'|^2 dx\right)^{\frac{\alpha}{2}} u' = 0, \tag{1.6}$$

for which we have a nice interpretation for solutions emanating from a pair of finite linear combination of eigenfunctions of the Laplacian. It turns out that a natural slight change of the method, inspired by a technique devised in [7] involving a power of the total energy, gives the result in all cases which enter a natural functional framework containing as particular cases the three situations. The main object of the present paper is to report on this rather unexpected general result. The plan of the paper is as follows. In Section 2 we give a general functional setting for the problem. The main result concerning general second order evolutions is stated and proved in Section 3. Section 4 is devoted to examples in the PDE framework and Section 5 to some counterexamples showing the limitation of the results. Finally in Section 6, we state and prove a universal decay property relying on our main result for the solutions of semilinear wave equations.

2 Functional setting

Let H be a real Hilbert space with inner product denoted by $(,.)$ and norm denoted by $|.|$. Let $V \subset H$ be another Hilbert space continuously imbedded into H. The norm and inner product on V will be denoted by $||.||$ and $((,.))$. We assume that V is dense in H, so that

$$V \subset H \subset V'$$

Let $F \in C^1(V)$ and

$$\mathcal{A} = \nabla F \in C(V, V')$$

In the sequel we shall be concerned with local solutions around $t = 0$ of the evolution equation

$$u'' + \mathcal{A}u(t) + g(t, u') = 0 \tag{2.1}$$

where g satisfies the following properties.

Let $J = (0, T)$ with $T > 0$. We assume that $g : J \times V \to V'$ is such that

$$\forall v := v(t) \in L^\infty(J, V), \quad g(t, v(t)) \in L^1(J, V')$$

Definition 2.1. *A function $u : J \to V$ is called a strong solution of (2.1) if*

$$u \in W^{1,\infty}(J, V) \cap W^{2,1}(J, V') \tag{2.2}$$

and

$$u'' + \mathcal{A}u(t) + g(t, u') = 0 \quad \textit{in } L^1(J, V') \tag{2.3}$$

Remark 2.2. *It is clear that*

$$W^{1,\infty}(J, V) \cap W^{2,1}(J, V') \subset C([0, T], V) \cap C^1([0, T], H).$$

In particular the values $u(t) \in V$ and $u'(t) \in H$ are well defined for all $t \in [0, T]$.

Remark 2.3. *For any strong solution u of (2.1) the fiunction*

$$E(t) := \frac{1}{2}|u'(t)| + F(u(t)$$

satisfies $E \in W^{1,\infty}(J, \mathbb{R})$ with

$$E'(t) = -\langle g(t, u'(t)), u'(t)\rangle_{V',V} \quad \textit{a.e. on } J. \tag{2.4}$$

For the main result of the next section, we shall assume that F and g satisfy the following additional conditions. There exist two Banach spaces X, Y such that

$$V \subset Y \subset X \subset H$$

with continous and dense imbeddings for which we have for some positive constants $\alpha, \beta, \delta_1, \delta_2, \delta_3, C_1, C_2, C_3, C_4$ the inequalities

$$\forall t \in J, \forall v \in V, \quad \langle g(t, v), v\rangle_{V',V} \geq \delta_1 ||v||_X^{\alpha+2} - C_1 \tag{2.5}$$

$$\forall t \in J, \forall v \in V, \quad ||g(t, v)||_{Y'} \leq C_2(1 + ||v||_X^{\alpha+1}) \tag{2.6}$$

$$\forall u \in V, \quad F(u) \geq \delta_2 ||u||_Y^{\beta+2} - C_3 \tag{2.7}$$

$$\forall u \in V, \quad \langle \mathcal{A}u, u\rangle \geq \delta_3 F(u) - C_4 \tag{2.8}$$

3 Universal bound when $0 < \alpha < \beta$.

Theorem 3.1. *Let F, g be as in section 2 and assume that conditions* (2.5)*,* (2.6)*,* (2.7) *and* (2.8)*are satisfied with $0 < \alpha < \beta$. Then there is a constant $C > 0$ such that for any strong solution u of* (2.1) *on $J = (0, T)$ we have*

$$\forall t \in (0,T), \quad E(t) \leq Ct^{-\lambda} \tag{3.1}$$

with

$$\lambda = \max\{\frac{2}{\alpha}, \frac{(\alpha+1)(\beta+2)}{\beta-\alpha}\}$$

Proof. First of all we observe that by replacing F by $F+C_3+1$ we may assume $E(t) \geq 1$ for all t and $C_4 = 0$. We introduce the modified energy

$$\Phi(t) := E(t) + \varepsilon E(t)^{\gamma}(u, u') \tag{3.2}$$

where γ and $\varepsilon > 0$ will be chosen later. First, in order for Φ to be a small perturbation of the energy E we require that for some constant K

$$|E(t)^{\gamma}(u, u')| \leq KE(t) \tag{3.3}$$

But by definition of the energy we have $|u'| \leq 2E^{1/2}$ and (2.7) implies $|u| \leq K_1 E^{\frac{1}{\beta+2}}$. So the conclusion will follow if $\gamma + \frac{1}{\beta+2} \leq 1/2$, equivalent to

$$\gamma \leq \frac{\beta}{2(\beta+2)}$$

Assuming this condition, for all ε small enough we shall have throughout the interval J

$$\frac{1}{2}E \leq \Phi \leq 2E$$

We compute

$$\Phi' = E'(1 + \gamma\varepsilon E(t)^{\gamma-1}(u, u')) + \varepsilon E(t)^{\gamma}(|u'|^2 + \langle u'', u\rangle)$$

and by (3.3) and (2.4) we deduce, by distinguishing the 2 cases for the sign of $\langle g(u'), u'\rangle$ that for $\varepsilon < \frac{1}{2\gamma K}$

$$\forall t \in (0,T), \quad \Phi' \leq -\frac{\delta_1}{2}||u'||_X^{\alpha+2} + \frac{3}{2}C_1 + \varepsilon E(t)^{\gamma}(|u'|^2 + \langle u'', u\rangle) \tag{3.4}$$

Let us for the moment forget about the term $\varepsilon E(t)^{\gamma}|u'|^2$ and concentrate on the more interesting product

$$\langle u'', u\rangle = -\langle \mathcal{A}u, u\rangle - \langle g(t, u'), u\rangle \leq -\delta_3 F(u) + C_4 + ||g(t, u')||_{Y'}||u||_Y$$

By using the identity $F(u) = E - \frac{1}{2}|u'|^2$ we now obtain (reducing to $C_4 = 0$)

$$\Phi' \leq -\varepsilon\delta_3 E^{1+\gamma} - \frac{\delta_1}{2}||u'||_X^{\alpha+2} + \varepsilon(1+\delta_3/2)E(t)^\gamma|u'|^2 + \varepsilon E^\gamma||g(t,u')||_{Y'}||u||_Y + \frac{3}{2}C_1 \quad (3.5)$$

We now choose

$$\gamma = \min\left\{\frac{\alpha}{2}, \frac{\beta-\alpha}{(\alpha+1)(\beta+2)}\right\} := \gamma_0 \quad (3.6)$$

and we claim that in all cases $\gamma_0 \leq \frac{\beta}{2(\beta+2)}$ and consequently (3.3) is satisfied . Indeed this is clearly true if $\alpha \leq \frac{\beta}{\beta+2}$. On the other hand if $\alpha \geq \frac{\beta}{\beta+2}$ an easy calculation shows that $\frac{\beta-\alpha}{(\alpha+1)(\beta+2)} = \gamma_0$ and we just need to check that $\beta \geq \frac{2(\beta-\alpha)}{\alpha+1}$, an immediate consequence of $\alpha \geq \frac{\beta}{\beta+2}$. We now need to show that the third and fourth term in (3.5) are dominated by the sum of the two first terms. For the third term we write

$$E(t)^\gamma|u'|^2 \leq \mu E^{1+\gamma} + C(\mu)|u'|^{2(1+\gamma)}$$

with $2+2\gamma \leq 2+\alpha$ and by fixing μ small enough, we obtain under a smallness condition on ε

$$\Phi' \leq -\varepsilon\frac{\delta_3}{2}E^{1+\gamma} - \frac{\delta_1}{4}||u'||_X^{\alpha+2} + \varepsilon E^\gamma||g(t,u')||_{Y'}||u||_Y + C_5$$

Finally we have

$$E^\gamma||g(t,u')||_{Y'}||u||_Y \leq C_6 E^{\gamma+\frac{1}{\beta+2}}(1+||u'||_X^{\alpha+1})$$

The term $E^{\gamma+\frac{1}{\beta+2}}$ is easily absorbed since $\gamma+\frac{1}{\beta+2} < 1+\gamma$. Moreover we have by Young's inequality

$$E^{\gamma+\frac{1}{\beta+2}}||u'||_X^{\alpha+1} \leq \nu E^{(\gamma+\frac{1}{\beta+2})(\alpha+2)} + C(\nu)||u'||_X^{\alpha+2}$$

We claim that

$$(\gamma+\frac{1}{\beta+2})(\alpha+2) \leq 1+\gamma$$

indeed this inequality reduces to

$$\gamma(\alpha+1) \leq 1 - \frac{\alpha+2}{\beta+2} = \frac{\beta-\alpha}{\beta+2}$$

Finally by choosing ε sufficiently small we find

$$\Phi' \leq -\varepsilon\frac{\delta_3}{4}E^{1+\gamma} + C_6 \leq -\rho\Phi^{1+\gamma} + C_6$$

for some $\rho > 0$ and the conclusion follows in one step from the standard comparison principle since the function

$$\Psi(t) = \left(\frac{1}{\gamma\rho t}\right)^{\frac{1}{\gamma}} + \left(\frac{C_6}{\rho}\right)^{\frac{1}{1+\gamma}}$$

satisfies the inequality

$$\Psi'(t) + \rho\Psi^{1+\gamma} \geq C_6$$

Hence $\Psi \geq \Phi$ since $\Psi \geq \Phi$ for $t \to 0^+$. The inequality $\Phi \leq \Psi$ is referred to in [12], III, Lemma 5.1 as Ghidaglia' inequality. $\square$

Remark 3.2. *In the finite dimensional case at least, it should be possible to generalize this result to some singular singular equations and systems such as those studied in [1, 3].*

4 Examples of application

Throughout this section the partial derivatives with respect to time of a function $u(t, x)$ will be denoted as u' instead of u_t to keep the notation of the abstract section. It also gives more compact formulas in the calculations. The spatial L^p norm of u will often be written as $||u||_p$, with the exception of the L^2 norm which will be denoted as $|u|$. These conventions should bring no specific difficulty for the reader.

4.1 Semilinear wave equations

Let Ω be a bounded domain of $\mathbb{R}^N$. Given the positive constants α, β, c, b and the real constants λ, μ, we consider the semilinear equation

$$u'' - \Delta u + b|u|^\beta u - \lambda u + c|u'|^\alpha u' - \mu u' = h(t, x), \quad (t, x) \in (0, T) \times \Omega \tag{4.1}$$

with either Dirichlet homogeneous boundary conditions

$$u(t, x) = 0 \quad (t, x) \in (0, T) \times \partial\Omega \tag{4.2}$$

or Neumann homogeneous boundary conditions

$$\frac{\partial u(t, x)}{\partial n} = 0 \quad (t, x) \in (0, T) \times \partial\Omega \tag{4.3}$$

Assuming $h \in L^1(0, T; L^2(\Omega))$, the initial-value problem corresponding to (4.1) is classically well-posed for initial data

$$(u(0), u'(0)) = (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$$

in the case of BC (4.2) and for initial data

$$(u(0), u'(0)) = (u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$$

in the case of BC (4.3). In both cases, under the condition

$$(N - 2)\alpha \leq 2, \tag{4.4}$$

the solutions lie in the regularity class

$$u \in C([0,T],V) \cap C^1([0,T],L^2(\Omega)) \cap W^{2,1}([0,T],V')$$

with $V = H_0^1(\Omega)$ in the case of BC (4.2) ; $V = H^1(\Omega)$ in the case of BC (4.3). Moreover the additional regularity conditions $(u_0,u_1) \in H^2(\Omega) \times H^1(\Omega))$ in the case of BC (4.3) (resp. $(u_0,u_1) \in H^2(\Omega) \times H_0^1(\Omega))$ in the case of BC (4.2) and $h \in W^{1,1}(0,T;L^2(\Omega))$ imply that the solution is strong in the sense of definition 2.1.

Corollary 4.1. *Let $0 < \alpha < \beta$ and assume $h \in L^\infty(0,T;L^2(\Omega))$, $\quad (N-2)\alpha \le 2$. Then there is a constant $C > 0$ such for any (weak) solution u of* (4.1) *satisfying either* (4.2) *or* (4.3)*we have*

$$\forall t \in (0,T), \qquad \int_\Omega \left[u'^2(t,x) + |\nabla u|^2(t,x) + |u|^{\beta+2}\right] dx \le Ct^{-\lambda}$$

with

$$\lambda = \max\left\{\frac{2}{\alpha}, \frac{(\alpha+1)(\beta+2)}{\beta-\alpha}\right\}$$

Proof. We apply Theorem 3.1 with

$$H = L^2(\Omega), \quad X = L^{\alpha+2}(\Omega) \quad Y = L^{\beta+2}(\Omega)$$

The result is immediate for strong solutions and follows by density in the case of weak solutions (cf. e.g. [8], Proposition II.2.2.1 and Theorem II.3.2.1 for the construction of weak and strong solutions.) □

The following consequence of Corollary 4.1 is straightforward.

Corollary 4.2. *Under the hypotheses of the previous corollary, there are constants $C_1, C_2 > 0$ such for any (weak) solution u of* (4.1) *satisfying either* (4.2) *or* (4.3)*we have for all $t \in (0,T)$*

$$||u(t,.)||_{L^{\beta+2}(\Omega)} \le C_1 t^{-\mu} \quad \textit{with } \mu = \max\left\{\frac{2}{\alpha(\beta+2)}, \frac{(\alpha+1)}{\beta-\alpha}\right\}$$

$$||u(t,.)||_{H^1(\Omega)} + ||u'(t,.)||_{L^2(\Omega)} \le C_2 t^{-\nu} \quad \textit{with } \nu = \max\left\{\frac{1}{\alpha}, \frac{(\alpha+1)(\beta+2)}{2(\beta-\alpha)}\right\}$$

Remark 4.3. *There is no difference here between the Dirichlet and Neumann case since the coercivity for large energies is driven by the super-quadratic power $\beta+2$. The situation will be different for large time estimates when $h = 0$ (cf. Section 6).*

Remark 4.4. *It is difficult to understand why we get different bounds for the $L^{\beta+2}$ norm and the H^1 norm of u. One possible explanation would be an imperfect transfer of dissipation on higher order modes. We do not know if this corresponds to a real phenomenon.*

4.2 Semilinear plate equations

Let Ω be a bounded domain of $\mathbb{R}^N$. Given the positive constants α, β, c, b and the real constants λ, μ, we consider the semilinear equation

$$u'' + \Delta^2 u + b|u|^\beta u - \lambda u + c|u'|^\alpha u' - \mu u' = h(t,x), \quad (t,x) \in (0,T) \times \Omega \tag{4.5}$$

with either hinged boundary conditions

$$u(t,x) = \Delta u(t,x) = 0 \quad (t,x) \in (0,T) \times \partial\Omega \tag{4.6}$$

or clamped boundary conditions

$$u(t,x) = \frac{\partial u(t,x)}{\partial n} = 0 \quad (t,x) \in (0,T) \times \partial\Omega \tag{4.7}$$

Assuming $h \in L^1(0,T; L^2(\Omega))$, the initial-value problem corresponding to (4.1) is classically well-posed for initial data

$$(u(0), u'(0)) = (u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$$

in the case of BC (4.7) and for initial data

$$(u(0), u'(0)) = (u_0, u_1) \in H^2 \cap H_0^1(\Omega) \times L^2(\Omega)$$

in the case of BC (4.6) in both cases the solutions lie in the regularity class

$$u \in C([0,T], H^2(\Omega)) \cap C^1([0,T], L^2(\Omega))$$

under the condition

$$(N-4)\alpha \leq 4$$

and assuming $h \in W^{1,1}(0,T; L^2(\Omega))$, the additional regularity condition $(u_0, u_1) \in H^4(\Omega) \times H_0^2(\Omega))$ in the case of BC (4.7)(resp. $(\Delta u_0, u_1) \in (H^2 \cap H_0^1(\Omega)))^2$ in the case of BC (4.6) imply that the solution is strong in the sense of definition 2.1.

Corollary 4.5. *Let $0 < \alpha < \beta$ and assume $h \in L^\infty(0,T; L^2(\Omega))$, $\quad (N-4)\alpha \leq 4$. Then there is a constant $C > 0$ such for any (weak) solution u of* (4.5) *satisfying either* (4.2) *or* (4.3)*we have*

$$\forall t \in (0,T), \qquad \int_\Omega \left[u'^2(t,x) + |\Delta u|^2(t,x)\right] dx \leq C t^{-\lambda}$$

with

$$\lambda = \max\{\frac{2}{\alpha}, \frac{(\alpha+1)(\beta+2)}{\beta - \alpha}\}$$

Proof. We apply Theorem 3.1 with

$$H = L^2(\Omega), \quad X = L^{\alpha+2}(\Omega) \quad Y = L^{\beta+2}(\Omega)$$

and $V = H^2 \cap H_0^1(\Omega)$ in the case of BC (4.6) ; $V = H_0^2(\Omega)$ in the case of BC (4.7). The result is immediate for strong solutions and follows by density in the case of weak solutions. □

Corollary 4.6. *Under the hypotheses of the previous corollary, there are constants* $C_1, C_2 > 0$ *such for any (weak) solution* u *of* (4.1) *satisfying either* (4.2) *or* (4.3)*we have for all* $t \in (0, T)$

$$||u(t,.)||_{L^{\beta+2}(\Omega)} \leq C_1 t^{-\mu} \quad \textit{with } \mu = \max\left\{\frac{2}{\alpha(\beta+2)}, \frac{(\alpha+1)}{\beta-\alpha}\right\}$$

$$||u(t,.)||_{H^2(\Omega)} + ||u'(t,.)||_{L^2(\Omega)} \leq C_1 t^{-\nu} \quad \textit{with } \nu = \max\left\{\frac{1}{\alpha}, \frac{(\alpha+1)(\beta+2)}{2(\beta-\alpha)}\right\}$$

Remark 4.7. *As previously we do not understand why we get different bounds for the* $L^{\beta+2}$ *norm and the* H^2 *norm of* u.

4.3 A Kirchhoff equation with averaged damping

Let Ω be a bounded domain of $\mathbb{R}^N$. Given the positive constants α, β, c and the real constants λ, μ, we consider the equation

$$u'' - \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{\beta}{2}} \Delta u + c\left(\int_\Omega |u'|^2 dx\right)^{\frac{\alpha}{2}} u' - \lambda u - \mu u' = 0 \quad (t,x) \in (0,T) \times \Omega \quad (4.8)$$

with either Dirichlet homogeneous boundary conditions (4.2)or Neumann homogeneous boundary conditions (4.3) the initial-value problem corresponding to (4.8) is not presently known to be well-posed in any classical regularity class except for analytic initial data satisfying compatibility conditions. However if

$$(u(0), u'(0)) = (u_0, u_1)$$

is a pair of finite linear combinations of eigen-functions of the Laplacian, then the equation is equivalent to a finite dimensional system of ODEs and then the problem is well-posed with a solution which is trivially strong in the sense of definition 2.1. For such solutions our main result is applicable, and we get a universal bound in the energy space. We skip the details which are fairly obvious, the situation is in fact identical to the case of scalar equation.

Corollary 4.8. *Let $0 < \alpha < \beta$. Then there is a constant $C > 0$ such for any solution u of* (4.1) *satisfying either* (4.2) *or* (4.3) *and having as initial conditions a pair of finite linear combinations of eigen-functions of the Laplacian, we have*

$$\forall t \in (0,T), \quad \int_\Omega u'^2(t,x)dx + \left[\int_\Omega |\nabla u|^2(t,x)dx\right]^{1+\frac{\beta}{2}} \leq Ct^{-\lambda}$$

with

$$\lambda = \max\{\frac{2}{\alpha}, \frac{(\alpha+1)(\beta+2)}{\beta-\alpha}\}$$

Corollary 4.9. *Under the hypotheses of the previous corollary, there are constants $C_1, C_2 > 0$ such for any (weak) solution u of* (4.1) *satisfying either* (4.2) *or* (4.3)*we have for all $t \in (0,T)$*

$$||u(t,.)||_{H^1(\Omega)} \leq C_1 t^{-\mu} \quad \textit{with } \mu = \max\left\{\frac{2}{\alpha(\beta+2)}, \frac{(\alpha+1)}{\beta-\alpha}\right\}$$

$$||u'(t,.)||_{L^2(\Omega)} \leq C_1 t^{-\nu} \quad \textit{with } \nu = \max\left\{\frac{1}{\alpha}, \frac{(\alpha+1)(\beta+2)}{2(\beta-\alpha)}\right\}$$

Remark 4.10. *Of course a similar result for t small would be applicable to strong solutions of* (1.5)*, but an existence theory of both strong and weak solutions is missing here as in the conservative case, in particular no existence result is known with a life time bounded away from* 0 *for large initial energies.*

5 Negative results in the PDE case

The general theorem 3.1 is optimal since it is already the case in the particular case of scalar ODEs as recalled in the inroduction. However it is not useless to summarize the impact of counterexamples in the framework of PDEs .

5.1 The semilinear wave equation with Neumann boundary conditions

More specifically let us consider the model equation

$$u'' - \Delta u + |u|^\beta u + |u'|^\alpha u' = 0$$

with Neumann boundary conditions. In this case, since solutions homogeneous in space coincide with the solutions of the ODE, we know from Souplet's Theorem 1 in [11] that for $\beta \leq \alpha$, the universal bound does not exist.

5.2 The semilinear wave equation with Dirichlet boundary conditions

In that case, we know from Theorem 1 by Ana Carpio [4] that for the equation

$$u'' - \Delta u + |u'|^{\alpha} u' = 0$$

the universal bound does not exist. It would be interesting to construct a counterexample for the model equation (1.4) with $0 < \beta \leq \alpha$, but this does not seem to follow from any result presently established.

5.3 Kirchhoff's equation.

For the equation (1.6), any solution for which both initial data are multiples of the same eigenfunction of the Laplacian is the product of the eigenfunction with solutions of a scalar ODE of the form (1.1), therefore Theorem 1 of [11] implies that for $\beta \leq \alpha$, the universal bound does not exist.

6 Universal decay bounds when $h = 0$.

Here we state, in order to be complete, the consequences of uniform boundedness for $t > 0$ on the asymptotic form of trajectories for t large when $h = 0$. We limit the statements to the case of the wave equation, the other cases are left to the reader. Here the results will be different for Dirichlet and Neumann. More precisely we have

Proposition 6.1. *Let $0 < \alpha < \beta$ and $(N-2)\beta \leq 2$. Then for any pair of constants $b, c > 0$ and $\lambda < \lambda_1(\Omega)$ there is a constant $D > 0$ such that for any weak solution u of*

$$u'' - \Delta u + b|u|^{\beta} u - \lambda u + c|u'|^{\alpha} u' = 0, \quad (t, x) \in \mathbb{R}^+ \times \Omega$$

with boundary conditions (4.2) *we have*

$$\forall t \geq 1, \quad \int_{\Omega} \left[u'^2(t,x) + |\nabla u|^2(t,x) \right] dx \leq D t^{-\frac{2}{\alpha}} \tag{6.1}$$

Proof. We set

$$E(t) := \frac{1}{2} \int_{\Omega} \left[u'^2(t,x) + |\nabla u|^2(t,x) - \lambda u^2(t,x) \right] dx + \frac{b}{\beta+2} \int_{\Omega} |u|^{\beta+2}(t,x) dx$$

$$=: \frac{1}{2}|u'|^2 + F(u(t))$$

and after observing that

$$E' = -c||u'||^{\alpha+2}_{L^{\alpha+2}(\Omega)} := -c||u'||^{\alpha+2}_{\alpha+2}$$

we introduce the modified energy

$$\Phi(t) := E(t) + \varepsilon E(t)^{\gamma}(u, u') \tag{6.2}$$

where γ and $\varepsilon > 0$ will be chosen later. First we know that $0 \leq E(t) \leq M$ for $t \geq 1/2$ and since E dominates the inner product (u, u') , for all ε small enough we shall have

$$\forall t \geq 1/2, \quad \frac{1}{2}E \leq \Phi \leq 2E$$

Moreover since the energy is non-increasing, either it vanishes on a halfline $[T, \infty)$ or we may assume $E > 0$ for $t \geq 1/2$. Now

$$\Phi' = E'(1 + \gamma\varepsilon E(t)^{\gamma-1}(u, u')) + \varepsilon E(t)^{\gamma}(|u'|^2 + \langle u'', u\rangle)$$

and for ε small enough this yields

$$\forall t \in (0, T), \Phi' \leq -\frac{c}{2}||u'||_{\alpha+2}^{\alpha+2} + \varepsilon E(t)^{\gamma}(|u'|^2 + \langle u'', u\rangle) \tag{6.3}$$

Let us for the moment forget about the term $\varepsilon E(t)^{\gamma}|u'|^2$ which will appear a second time later and will be estimated at the end. We compute

$$\langle u'', u\rangle = \langle \Delta u + \lambda u - b|u|^{\beta}u, u\rangle - \langle c|u'|^{\alpha}u', u\rangle \leq -2F(u) + \delta||u||_{\alpha+2}^{\alpha+2} + C(\delta)||u'||_{\alpha+2}^{\alpha+2}$$
$$\leq -2F(u) + \delta K M^{\alpha} F(u) + C(\delta)||u'||_{\alpha+2}^{\alpha+2} \leq -F(u) + C||u'||_{\alpha+2}^{\alpha+2}$$

by a suitable choice of δ. By using the identity $F(u) = E - \frac{1}{2}|u'|^2$ we now obtain

$$\Phi' \leq -\varepsilon E^{1+\gamma} - \frac{c}{2}||u'||_{\alpha+2}^{\alpha+2} + 2\varepsilon E(t)^{\gamma}|u'|^2 + C'\varepsilon||u'||_{\alpha+2}^{\alpha+2}$$

so that for ε small enough we find

$$\Phi' \leq -\varepsilon E^{1+\gamma} - \frac{c}{4}||u'||_{\alpha+2}^{\alpha+2} + 2\varepsilon E(t)^{\gamma}|u'|^2$$

We now choose $\gamma = \frac{\alpha}{2}$. Then we can see that $\frac{\alpha+2}{\alpha}\gamma = 1 + \gamma$ and

$$2\varepsilon E(t)^{\gamma}|u'|^2 \leq \frac{1}{2}\varepsilon E^{1+\gamma} + N\epsilon||u'||_{\alpha+2}^{\alpha+2}$$

Finally by choosing ε sufficiently small we find

$$\forall t \geq 1/2, \quad \Phi' \leq -\frac{\varepsilon}{2}E^{1+\gamma} \leq -\frac{\varepsilon}{2^{2+\gamma}}\Phi^{1+\gamma}$$

and the conclusion follows in one step $\square$

Remark 6.2. *Even in one dimension, we do not know whether this estimate is optimal. This is connected to a classical problem for individual trajectories of the simpler equation*

$$u'' - \Delta u + |u'|^{\alpha}u' = 0$$

cf. e.g. [9] , Problem 4.1.

Proposition 6.3. *Let $0 < \alpha < \beta$ and $(N-2)\beta \leq 2$. Then for any pair of constants $b, c > 0$ there are positive constants D_1, D_2 such that for any weak solution u of*

$$u'' - \Delta u + b|u|^\beta u + c|u'|^\alpha u' = 0, \quad (t,x) \in \mathbb{R}^+ \times \Omega$$

with boundary conditions (4.3) *we have*

$$\forall t \geq 1, \quad ||u(t,.)||_{L^{\beta+2}(\Omega)} \leq D_1 t^{-\mu} \quad \textit{with } \mu = \min\left\{\frac{2}{\alpha(\beta+2)}, \frac{(\alpha+1)}{\beta-\alpha}\right\} \tag{6.4}$$

$$\forall t \geq 1, \quad ||u'(t,.)||_{L^2(\Omega)} + ||\nabla u(t,.)||_{L^2(\Omega,\mathbb{R}^N)} \leq D_2 t^{-\nu} \tag{6.5}$$

with

$$\nu = \min\left\{\frac{1}{\alpha}, \frac{(\alpha+1)(\beta+2)}{2(\beta-\alpha)}\right\} \tag{6.6}$$

Proof. The proof is similar to the previous one with the same energy (but with $\lambda = 0$) and the more complicated choice $\gamma = \max\{\frac{\alpha}{2}, \frac{\beta-\alpha}{(\alpha+1)(\beta+2)}\}$. Actually with respect to the previous proof there are two non-trivial additional steps. First we need that $|u||u'|E^{\gamma-1}$ be bounded. Since $|u||u'| \leq CE^{(\frac{1}{2}+\frac{1}{\beta+2})}$ it is sufficient to have $\frac{1}{2} + \frac{1}{\beta+2} \geq 1-\gamma$ or equivalently $\gamma \geq \frac{\beta}{2(\beta+2)}$ If $\alpha \geq \frac{\beta}{(\beta+2)}$ we can choose $\gamma = \frac{\alpha}{2}$ as in the Dirichlet case. Otherwise we need

$$\gamma \geq \frac{\beta}{2(\beta+2)}$$

Now there is another problematic term coming from the product $E^\gamma\langle u'', u\rangle$, we need to control $E^\gamma ||u'||_{\alpha+2}^{\alpha+1}||u||_{\alpha+2}$ by $\delta E^{1+\gamma} + C(\delta)||u'||_{\alpha+2}^{\alpha+2}$. This reduces to the condition

$$(\alpha+2)(\gamma + \frac{\beta}{\beta+2}) \geq 1+\gamma$$

Finally we need

$$\gamma(\alpha+1) \geq \frac{\beta-\alpha}{\beta+2}$$

It is then not difficult to check that if $\alpha \geq \frac{\beta}{(\beta+2)}$, the choice $\gamma = \frac{\alpha}{2}$ fulfills this second condition. On the other hand if $\alpha < \frac{\beta}{(\beta+2)}$, the choice $\gamma = \frac{\beta-\alpha}{(\alpha+1)(\beta+2)}$ satisfies the second condition, but we also have $\frac{\beta-\alpha}{(\alpha+1)(\beta+2)} \geq \frac{\beta}{2(\beta+2)}$ so that the first condition is also satisfied. The remaining details are left to the reader □

Remark 6.4. *Here the consideration of solutions constant in space shows that our estimates are optimal, except maybe for the estimate on $||\nabla u(t,.)||_{L^2(\Omega,\mathbb{R}^N)}$.*

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