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A SIGNATURE INVARIANT FOR KNOTTED KLEIN GRAPHS

CATHERINE GILLE AND LOUIS-HADRIEN ROBERT

ABSTRACT. We define some signature invariants for a class of knotted trivalent graphs using branched covers. We relate them to classical signatures of knots and links. Finally, we explain how to compute these invariants through the example of Kinoshita's knotted theta graph.

1. INTRODUCTION

The notion of knotted graph generalizes the notion of link. It has direct applications in stereo-chemistry [Fla00, Sim87]. On the one hand, the classification of knotted graphs can be seen as an extension of the classification problem for knots. On the other hand, given a knotted graph, one can look at all its sub-links. Kinoshita [Kin58, Kin72] gave an example (see Figure 1) of a non-trivial knotted theta graph such that all the three sub-knots are trivial (see [JKL⁺16] for more example of *Brunnian* theta graphs). Hence it is necessary to develop specific invariants for knotted graphs.

We restrict ourselves to a certain class of trivalent graphs in \mathbb{S}^3 with an edge-coloring called 3-Hamiltonian Klein graphs. The aim of this paper is to define some signature-like invariants for knotted such graphs.

In [GL78], Gordon and Litherland explain how to compute signatures of knots from a non-orientable spanning surface F in \mathbb{B}^4 . It is the signature of the double branched cover of \mathbb{B}^4 along F corrected by the normal Euler number of F (see also [KT76]). In this paper, we adopt this 4-dimensional point of view.

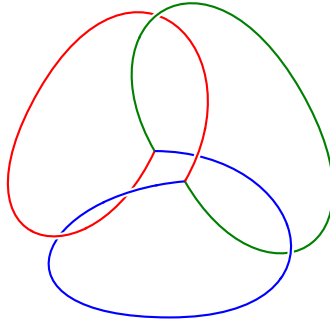


FIGURE 1. Kinoshita's knotted theta graph.

A Klein graph Γ is a trivalent graph endowed with a 3-coloring of its edges. For any knotted Klein graph in \mathbb{S}^3 , one can consider its *Klein cover*, it is a branched

$\mathbb{Z}_2 \times \mathbb{Z}_2$ -cover with Γ as branched locus. Given a spanning foam¹ F for Γ , one can construct W_F the Klein cover of \mathbb{B}^4 along F .

We define invariants of Γ using the signature of W_F . For doing so, we define normal Euler numbers of foams. It turns out that if Γ is 3-Hamiltonian², there are two ways to define the normal Euler numbers yielding different invariants (see Theorem 3.11). The computation for the Kinoshita knotted graph shows that this graph is non-trivial (and even chiral).

Moreover, we investigate the relations between our invariants and the signatures of the different knots and links related to the knotted graph Γ . The identities we find (see Proposition 3.14) can be thought of as consistency constraints between these signatures. The concept of foam enables to interpret these constraints geometrically.

For defining our invariants, we need the notion of normal Euler numbers of immersed surfaces with boundaries. For this, we use linking numbers of rationally null-homologous curves in arbitrary 3-manifolds (see [Les15] for a gentle introduction on this notion). The invariance of the signatures follows from the G-signature theorem [AS68] in dimension 4 (see [Gor86] for an elementary approach). Finally, in order to compute our invariants on an example, we use a result of Przytycki and Yasuhara [PY04] which calculates the modification by surgery of the linking matrix of a link in a rational homology spheres.

1.1. Structure of the paper. In Section 2, we introduce the notion of Klein graph, Klein foam and Klein cover. In Section 3.2 we define the invariants. For this we recall the notion of normal Euler numbers in Section 3.1. The rest of Section 3 is dedicated to the proof of invariance: Section 3.3 contains two technical lemmas about normal Euler numbers. Section 3.4 contains the proof of the Theorem 3.11. Finally in Section 4, we compute our signature invariants on Kinoshita's knotted graph.

1.2. Acknowledgment. The authors thank Christian Blanchet, Christine Lescop and Pierre Vogel for enlightening conversations and Brendan Owens for his historical remarks on a previous version of this paper. L.-H. R thanks the Université Paris Diderot for its hospitality. L.-H. R. was supported by the NCCR SwissMAP, funded by the Swiss National Science Foundation.

2. KLEIN COVERS

2.1. Graphs and foams. Throughout the paper D_4 denotes the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (we use the multiplication convention for the group law) and D_4^* denote the set $D_4 \setminus \{1\}$. The elements of D_4^* are denoted by a , b and c and are represented in pictures by red, blue and green respectively.

Definition 2.1. An abstract *Klein graph* is a finite trivalent multi-graph Γ with an edge-coloring by D_4^* (as usual in graph theory we require that the colors of two adjacent edges are different). It is *3-Hamiltonian* if for any two elements i and j

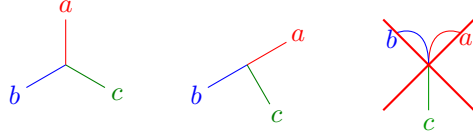
¹Foams are natural cobordisms when working with graphs: they are surfaces with some singularities (see Section 2.1). Here we do not suppose any kind of orientability conditions on F .

²If the graph Γ is *not* 3-Hamiltonian, we can still define one signature invariant, which turns out to depends only on the sub-links of Γ .

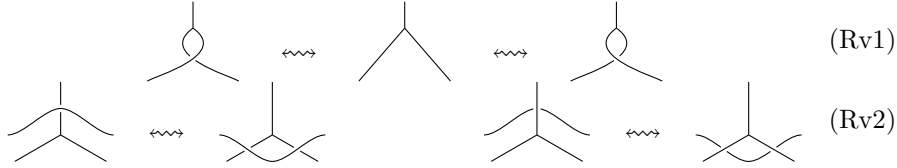
of D_4^* , the sub-graph Γ_{ij} consisting of edges colored by i or j is connected. From Section 3 on, all Klein graphs are supposed to be 3-Hamiltonian.

A *knotted Klein graph* is a Klein graph Γ together with a smooth embedding of Γ in a manifold of dimension 3. If the manifold is not given it is assumed to be \mathbb{S}^3 .

Remark 2.2. (1) We should explain what is meant by *smooth embedding* of an abstract Klein graph. We require that each edge is smoothly embedded and that for every vertex, none of the three tangent vector is positively co-linear with any of the two others.



- (2) We consider embedded Klein graphs up to ambient isotopy. In a diagrammatic approach, this means that a graph is considered up to the classical Reidemeister moves and the following additional ones (see [Lew13, Proposition 1.6]):



- (3) One may wonder which trivalent graphs can be endowed with a structure of Klein graphs. It is known to be the case for planar graphs with no bridge (this follows from the 4-color theorem) and for bipartite graphs (this follows from König's theorem). However, the 3-Hamiltonian condition is more complicated to ensure. Four examples are given in Figure 2.

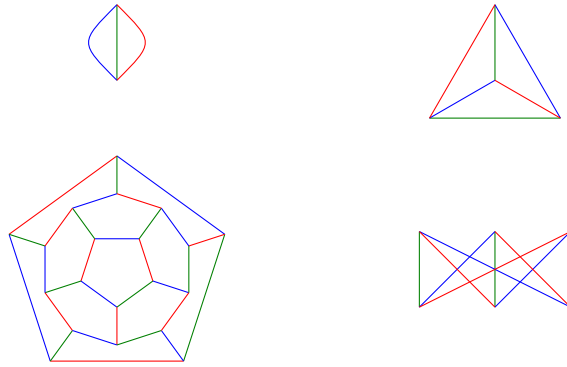


FIGURE 2. Examples of 3-Hamiltonian Klein graphs.

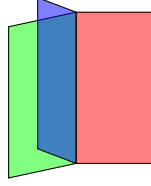
- (4) The 3-Hamiltonian condition is preserved by performing connected sum along a vertex. This operation is described in Figure 3.

Definition 2.3. A *closed embedded Klein foam* E is the realization of a finite CW-complex in a manifold of dimension 4 (if the manifold is not given, it is meant



FIGURE 3. Connected sum along a vertex.

to be \mathbb{S}^4) and some data attached to it. For every point of the CW complex, we require that there exists a neighborhood which is either diffeomorphic to a disk or to following picture (the singular one-dimensional cell is called a *binding*):



The data attached to E is a coloring of its facets by D_4^* , such that the three facets adjacent to a common binding have different colors.

The intersection F of an embedded Klein foam E with a sub-manifold of dimension 4 with boundary is a *Klein foam with boundary* if:

- the intersection of E with ∂W is a knotted Klein graph Γ in ∂W ,
- there exists a tubular neighborhood U of ∂W such that $(U, E \cap U)$ is diffeomorphic to $(\partial W \times]0, 1[, \Gamma \times]0, 1[)$.

In this case, Γ is the *boundary* of F and we write $\Gamma = \partial F$. We say as well that F is a *spanning foam* for Γ .

Proposition 2.4. [Proof in Appendix A] Let Γ be a knotted Klein graph in \mathbb{S}^3 . There exists a spanning foam for Γ in \mathbb{B}^4 .

2.2. Klein covers.

Definition 2.5. Let M be a closed oriented manifold. If D_4 acts on M by positive diffeomorphisms and if for every g in D_4^* , the set M^g of fix points of g is a sub-manifold of co-dimension 2, we say that M is a *Klein manifold*. We set $M^{\cup D_4} := \bigcup_{g \in D_4^*} M^g$.

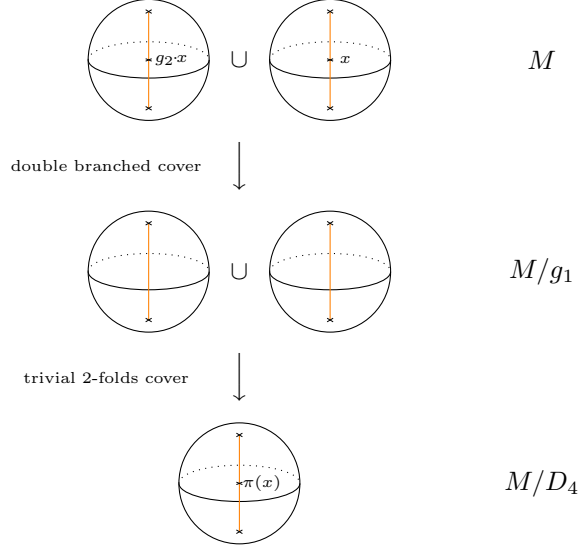
Proposition 2.6. Let M be a Klein manifold of dimension 3 (resp. of dimension 4), then M/D_4 is a closed oriented manifold of the same dimension and $M^{\cup D_4}$ is mapped on a Klein graph (resp. on a Klein foam) by $\pi : M \rightarrow M/D_4$.

Proof. Dimension 3 and 4 are analogous. We only treat dimension 3. Let g_1 be an element of D_4^* and x in M be a fixed point for g_1 . The diffeomorphism induced by g_1 being an involution, the action of g_1 on $T_x M$ is diagonalizable and we can find a basis of $T_x M$ such that the matrices of the linear map induced by g_1 is equal to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let us call g_2 and g_3 the two remaining elements of D_4 . We have $g_2 \cdot x = g_2 g_1 \cdot x = g_3 \cdot x$. Hence, if x is fixed by g_2 if and only if it is fixed by g_3 .

Suppose that x is not fixed by g_2 . Then the restriction of $\pi : M \rightarrow M/D_4$ to a neighborhood of $\{x, g_2 \cdot x\}$ is isomorphic to a double branched cover followed by a trivial 2-folds cover.

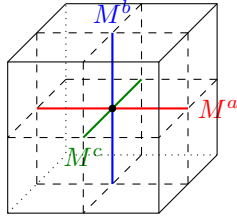


This implies that $\pi(x)$ has a neighborhood homeomorphic to a ball.

Suppose now that x is fixed by all the elements of D_4^* . We look at the action of D_4 over $T_x M$. This can be seen as a map from $\phi : D_4 \rightarrow GL_3(\mathbb{R})$. The matrices $\phi(g_1), \phi(g_2)$ and $\phi(g_3)$ are simultaneously diagonalizable and we can find a basis of $T_x M$ such that these matrices are equal to:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, in a chart, the action of g_1, g_2 and g_3 is given by these matrices and the fixed point loci looks like

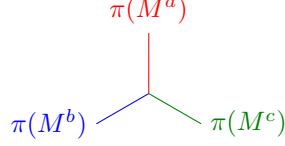


in a neighborhood of x . This implies that a neighborhood of $\pi(x)$ in M/D_4 is given by gluing the two cubes



along the three faces containing the distinguished vertex. This gives a 3-dimensional ball. This proves that M/D_4 is a manifold and that in a neighborhood of $\pi(x)$ the

image of $\pi(M^{\cup D_4})$ is given in a chart by:



Since M is compact, there are finitely many points which are fixed by the whole group D_4 . Altogether, this implies that M/D_4 is a manifold and that the image of $M^{\cup D_4}$ is a Klein graph. For the 4-dimensional statements, all local discussions have to be multiplied by an interval. \square

Definition 2.7. Let M be a closed manifold of dimension 3 (resp. of dimension 4) and Γ (resp. F) be an embedded Klein graph (resp. a Klein foam). Suppose that N is a Klein manifold of the same dimension such that $N/D_4 \simeq M$ and that $\pi(N^{\cup D_4})$ is identified with Γ (resp. F). Then we say that N is a *Klein cover of M along Γ (resp. along F)*.

- Proposition 2.8.** (1) *For any embedded Klein graph Γ in \mathbb{S}^3 , there exists a unique (up to diffeomorphism) Klein cover of \mathbb{S}^3 along Γ . It is denoted by M_Γ .*
- (2) *For any Klein foam F embedded in \mathbb{S}^4 , there exists a unique (up to diffeomorphism) Klein cover of \mathbb{S}^4 along F . It is denoted by V_F .*
- (3) *For any Klein foam F properly embedded in \mathbb{B}^4 , there exists a unique (up to diffeomorphism) Klein cover of \mathbb{B}^4 along F . It is denoted by W_F . (The first point actually implies that $\partial W_F \simeq M_{\partial F}$).*

Proof. We only prove the first statement, the two others are analogous. Fix an arbitrary orientation on $\Gamma = (V, E)$. The first homology group of $M := \mathbb{S}^3 \setminus \Gamma$ is generated by elements $[\gamma_e]_e \in E$, where γ_e is a small loop wrapping positively around the edge e . More precisely

$$H_1(M, \mathbb{Z}) \simeq \left(\bigoplus_{z \in E} \mathbb{Z} \right) / \langle \pm[\gamma_{e_1(v)}] + \pm[\gamma_{e_2(v)}] + \pm[\gamma_{e_3(v)}] = 0 \mid \text{for all } v \text{ in } V \rangle$$

where $e_1(v), e_2(v)$ and $e_3(v)$ are the three edges adjacent to v and the signs ambiguity is given by the way the orientations are toward v . The coloring of Γ gives a morphism

$$\phi : H_1(M) \rightarrow D_4$$

which sends $[\gamma_e]$ to the color of e in Γ . Note here that the ambiguity in the orientation of Γ is not a problem since all non-trivial elements of D_4 have order 2. This is indeed a morphism, since in D_4 we have $g_1 g_2 g_3 = 1$. We can promote ϕ to a morphism:

$$\phi : \pi_1(M) \rightarrow D_4.$$

We consider the covering M' associated with this morphism. We can complete M' in order to make it a Klein cover of \mathbb{S}^3 . For the edges and the vertices we use the local models described in the proof of Proposition 2.6.

Uniqueness comes from the uniqueness of the covering M' and from the fact that the local models for the singular points and the edges are the only possible ones

(see proof of 2.6). The proof in dimension 4 is analogous. Note that in this case, the local models are given by the ones we described times an interval.

□

3. A SIGNATURE INVARIANT

3.1. Normal Euler numbers. The aim of this part is to recall the definition of normal Euler numbers of surfaces with boundaries (Definition 3.5) and to give an additivity property that they satisfy. They have been studied by Gilmer [Gil93] and require linking numbers of rationally null-homologous knots in arbitrary 3-manifolds. We refer to the lecture notes of Lescop [Les15, Section 1.5] for details about such linking numbers. We start with the normal Euler number of a closed surface:

Definition 3.1. Let Σ be a smooth, not necessarily orientable, closed surface immersed in a smooth oriented manifold of dimension 4 and let s be a section of the normal bundle of Σ transverse to the null section s_0 . Then the intersection (computed with local orientations) of s_0 with s is called the *normal Euler number* of Σ and is denoted by $e(\Sigma)$. As the definition suggests the integer $e(\Sigma)$ does not depends on s .

Remark 3.2. If we choose a local system of orientations $O(\Sigma)$ of Σ , we can define the Euler class $e(N_\Sigma)$ of the normal bundle N_Σ . This is an element of $H^2(\Sigma, O(\Sigma))$. We consider $[\Sigma] \in H_2(\Sigma, O(\Sigma))$, the fundamental class of Σ in this local system of orientations. The normal Euler number of Σ is then equal to $e(N_\Sigma)([\Sigma])$.

If the 4-manifold and the surface has boundary, we can extend this definition relatively to the boundary.

Definition 3.3 (Relative normal Euler number). Let W be an oriented 4-manifold with boundary and Σ be a smooth, properly immersed surface with boundary in W . Let $L = l_1 \cup \dots \cup l_k$ be the boundary of Σ . Let us choose $\tilde{l}_1, \dots, \tilde{l}_k$ some parallels of l_1, \dots, l_k in ∂W . The normal Euler number of Σ relatively to $\tilde{l}_1, \dots, \tilde{l}_k$ is the intersection number of Σ with a section s of the normal bundle of Σ (transverse to the null section s_0), such that $\partial s = \tilde{l}_1 \cup \dots \cup \tilde{l}_k$. We denote it by $e(\Sigma; \tilde{l}_1, \dots, \tilde{l}_k)$.

Proposition 3.4 ([Gil93, p. 311]). Let W be an oriented 4-manifold with boundary and Σ be a smooth, properly immersed surface with boundary. Let $L = l_1 \cup \dots \cup l_k$ be the boundary of Σ . Suppose that each l_i is rationally null-homologous. Let us choose $\tilde{l}_1^1, \dots, \tilde{l}_k^1$ and $\tilde{l}_1^2, \dots, \tilde{l}_k^2$ two sets of parallels of l_1, \dots, l_k . For each i pick an orientation of l_i and orient \tilde{l}_i^1 and \tilde{l}_i^2 accordingly. Then for every i in $[1, k]$, $\text{lk}(l_i, \tilde{l}_i^1) - \text{lk}(l_i, \tilde{l}_i^2)$ is an integer, and we have:

$$e(\Sigma; \tilde{l}_1^2, \dots, \tilde{l}_k^2) - e(\Sigma; \tilde{l}_1^1, \dots, \tilde{l}_k^1) = \sum_{i=1}^k (\text{lk}(l_i, \tilde{l}_i^2) - \text{lk}(l_i, \tilde{l}_i^1)).$$

Proof. Let T_i be a tubular neighborhood of l_i in ∂W . An homotopy from \tilde{l}_i^1 to \tilde{l}_i^2 can be thought of a section s_{h_i} of the normal bundle of $l_i \times [-\epsilon, \epsilon]$ in $T_i \times [-\epsilon, \epsilon]$. The intersection of $s_{h_i}(l_i \times [-\epsilon, \epsilon])$ with $l_i \times [-\epsilon, \epsilon]$ is equal to $\text{lk}(l_i, \tilde{l}_i^1) - \text{lk}(l_i, \tilde{l}_i^2)$. Let

s_1 be a section of the normal bundle of Σ used to compute $e(\Sigma; \tilde{l}_1^1, \dots, \tilde{l}_k^1)$. Gluing the s_{h_i} 's to s_1 , one obtains a section s_2 which bounds $\tilde{l}_1^1 \cup \dots \cup \tilde{l}_k^1$. Hence:

$$\Sigma \cap s_2(\Sigma) - \Sigma \cap s_1(\Sigma) = \sum_{i=1}^k (\text{lk}(l_i, \tilde{l}_i^2) - \text{lk}(l_i, \tilde{l}_i^1)).$$

□

This yields the following definition:

Definition 3.5 (Normal Euler numbers, boundary case). Let W be an oriented 4-manifold with boundary and Σ be a smooth surface with boundary properly immersed. Suppose that every component l_1, \dots, l_k of the boundary of Σ is rationally null-homologous, and let s be a section of the normal bundle of Σ (transverse to s_0) and denote l'_1, \dots, l'_k the parallels of l_1, \dots, l_k induced by s . We define $e(\Sigma) = \Sigma \cap s(\Sigma) - \sum_{i=1}^k \text{lk}(l_i, l'_i)$.

Remark 3.6. (1) Suppose that every component of Σ has a non-empty boundary. Then it is possible to find a nowhere vanishing section s of the normal bundle of Σ . Let us denote l'_1, \dots, l'_k the parallels of l_1, \dots, l_k induced by s . Then we have:

$$e(\Sigma) = - \sum_{i=1}^k \text{lk}(l_i, l'_i).$$

(2) If ∂W is a rational homology sphere, then the conditions on the l_i 's are automatically satisfied.

Proposition 3.7. *Let W be an oriented 4-manifold with boundary and Σ be a smooth surface with boundary properly immersed in W . Suppose that each connected component of $\partial \Sigma$ is rationally null-homologous.*

- *The normal Euler number $e(\Sigma, -W)$ of Σ in the manifold W endowed with the opposite orientation is equal to $-e(\Sigma, W)$.*
- *Let W' be an oriented 4-manifold with boundary and Σ' be a smooth surface with boundary immersed in W' such that $\partial \Sigma' \subseteq \partial W'$. Suppose that ϕ is an orientation-reversing diffeomorphism from ∂W to $\partial W'$ which maps $\partial \Sigma$ on $\partial \Sigma'$. Then $e(\Sigma \cup_{\phi} \Sigma', W \cup_{\phi} W') = e(\Sigma, W) + e(\Sigma', W')$. Note that $\Sigma \cup_{\phi} \Sigma'$ is a closed surface in a closed oriented manifold.*

Proof. The first assertion directly follows from the definition. For the second one, we choose a set of parallels $\tilde{l}_1, \dots, \tilde{l}_k$ for $\partial \Sigma$ and a section s of the normal bundle of Σ bounding them. We choose a section s' of the normal bundle of Σ' bounding $\phi(\tilde{l}_1), \dots, \phi(\tilde{l}_k)$. We can glue s and s' along ϕ , this gives a section s'' of the normal

bundle of $\Sigma' \cup_\phi \Sigma$. Hence we have:

$$\begin{aligned}
e(\Sigma' \cup_\phi \Sigma) &= (\Sigma' \cup_\phi \Sigma) \cap s''(\Sigma' \cup_\phi \Sigma) \\
&= \Sigma \cap s(\Sigma) + \Sigma' \cap s'(\Sigma') \\
&= \Sigma \cap s(\Sigma) - \sum_{i=1}^k \text{lk}_{\partial W}(l_i, \tilde{l}_i) + \sum_{i=1}^k \text{lk}_{\partial W}(l_i, \tilde{l}_i) + \Sigma' \cap s(\Sigma') \\
&= \Sigma \cap s(\Sigma) - \sum_{i=1}^k \text{lk}_{\partial W}(l_i, \tilde{l}_i) - \sum_{i=1}^k \text{lk}_{\partial W'}(\phi(l_i), \phi(\tilde{l}_i)) + \Sigma' \cap s(\Sigma') \\
&= e(\Sigma, W) + e(\Sigma', W').
\end{aligned}$$

□

3.2. The invariants. We start by introducing some notations which will be used throughout the rest of the paper.

Notation 3.8. Let F be a Klein foam with boundary properly embedded in \mathbb{B}^4 . Recall that we have a Klein cover $\pi : W_F \rightarrow W_F/D_4 \simeq \mathbb{B}^4$. For every element i in D_4^* , we denote by:

- \widehat{F}_i the fixed-points surface of the diffeomorphism i in W_F ,
- W_F/i the manifold $W_F/\langle i \rangle$ (where $\langle i \rangle$ is the subgroup of order two generated by i),
- \widetilde{F}_i^i the image of \widehat{F}_i in W_F/i ,
- \widetilde{F}_{jk}^i the image of $\widehat{F}_j \cup \widehat{F}_k$ in W_F/i where j and k denote the two other elements of D_4^* ,
- F_{jk} the image of $\widehat{F}_j \cup \widehat{F}_k$ in \mathbb{B}^4 .

These notations are summarized below in Figure 4.

Note that \widetilde{F}_i^i and \widetilde{F}_{jk}^i are properly embedded surfaces in W_F/i and that F_{jk} is a properly embedded surface in \mathbb{B}^4 : in fact this is the union of the facets of F colored by j and k . Moreover the Klein cover π is the composition of two double branched covers $W_F \rightarrow W_F/i \rightarrow \mathbb{B}^4$. The last one can be seen as the double branched cover of \mathbb{B}^4 along F_{jk} and the first one as a double branched cover of W_F/i along \widetilde{F}_i^i . Denoting by a , b and c the three elements of D_4^* , the following diagram explains

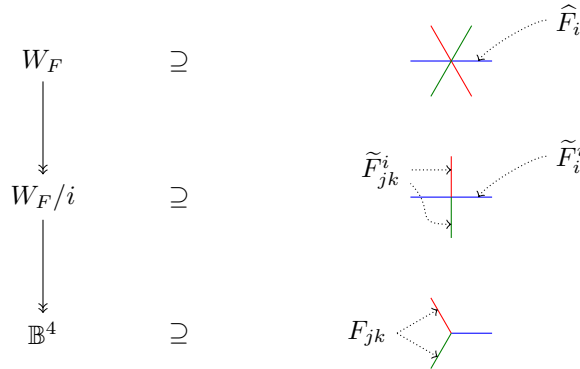
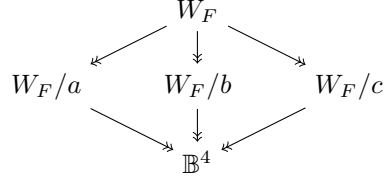


FIGURE 4. Decomposition of a Klein cover in two double branched covers

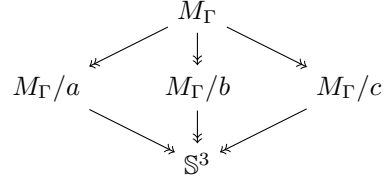
the three decompositions of the Klein cover of \mathbb{B}^4 on F in double branched covers:



Moreover we will use the same system of notations:

- for a Klein foam without boundary E embedded in \mathbb{S}^4 ,
- for a Klein graph Γ embedded in \mathbb{S}^3 .

In this last case, the Klein cover M_Γ has the following decomposition in double branched covers:



Remark 3.9. (1) The sub-graphs Γ_{ab} , Γ_{bc} and Γ_{ca} are links in \mathbb{S}^3 . For $\{i, j, k\} = \{a, b, c\}$, $\tilde{\Gamma}_{jk}^i$ and $\hat{\Gamma}_i^i$ are links in M_Γ/i and $\hat{\Gamma}_a$, $\hat{\Gamma}_b$ and $\hat{\Gamma}_c$ are links in M_Γ .
(2) If Γ is 3-Hamiltonian, Γ_{ab} , Γ_{bc} and Γ_{ca} are knots. In this case M_Γ/a , M_Γ/b and M_Γ/c are rational homology spheres whose first homology group has no 2-torsion (see for instance [Rol90, p. 213]).

Finally, we define normal Euler numbers for Klein foams as follows.

Definition 3.10. Let F be a Klein foam with boundary in $(\mathbb{B}^4, \mathbb{S}^3)$. The *weak normal Euler number* $e(F)$ of F is given by the following formula:

$$e(F) = e(F_{ab}) + e(F_{bc}) + e(F_{ca}).$$

If ∂F is 3-Hamiltonian, we define the *strong normal Euler number* $\tilde{e}(F)$ of F by:

$$\tilde{e}(F) = e(\tilde{F}_a^a) + e(\tilde{F}_b^b) + e(\tilde{F}_c^c).$$

We are now able to state the main theorem and to define our invariants. Recall that for a knotted Klein graph Γ in \mathbb{S}^3 there exists a spanning foam for Γ in \mathbb{B}^4 (Proposition A.1).

Theorem 3.11. Let Γ be a knotted 3-Hamiltonian Klein graph in \mathbb{S}^3 . Let F be a spanning foam for Γ in \mathbb{B}^4 . We denote the signature of the 4-manifold W_F by $\sigma(W_F)$.

- The integer $\sigma(\Gamma) := \sigma(W_F) + \frac{1}{2}e(F)$,
- the rational $\tilde{\sigma}(\Gamma) := \sigma(W_F) + \frac{1}{2}\tilde{e}(F)$,
- the rational $\delta(\Gamma) := \frac{1}{2}\tilde{e}(F) - \frac{1}{2}e(F)$

- and the rationals:

$$\begin{aligned}\delta_{ab}(\Gamma) &:= \frac{1}{4}e(\tilde{F}_a^a) + \frac{1}{4}e(\tilde{F}_b^b) - \frac{1}{2}e(F_{ab}), \\ \delta_{bc}(\Gamma) &:= \frac{1}{4}e(\tilde{F}_b^b) + \frac{1}{4}e(\tilde{F}_a^a) - \frac{1}{2}e(F_{bc}), \\ \delta_{ca}(\Gamma) &:= \frac{1}{4}e(\tilde{F}_c^c) + \frac{1}{4}e(\tilde{F}_a^a) - \frac{1}{2}e(F_{ca})\end{aligned}$$

only depend on the knotted Klein graph Γ . These quantities are called signature invariants of Γ .

Of course, we have the following relations between these invariants:

$$\delta(\Gamma) = \tilde{\sigma}(\Gamma) - \sigma(\Gamma) \quad \text{and} \quad \delta(\Gamma) = \delta_{ab}(\Gamma) + \delta_{bc}(\Gamma) + \delta_{ca}(\Gamma).$$

Before relating our invariants with signatures of knots and links, let us recall the following definition due to Gilmer [Gil92, Gil93]:

Definition 3.12. Let M be a rational homology sphere with a first homology group of odd order and L a (unoriented) link in M . We can find an oriented four dimensional manifold W and a surface F such that the pair (W, F) bounds a r copies of (M, L) for a positive integer r . Then the *signature* $\xi(L)$ of L is defined by the following formula:

$$\xi(L) = \frac{1}{r} \left(\sigma(W_F) - 2\sigma(W) + \frac{1}{2}e(F) \right)$$

where W_F denotes the double branched cover of W along F .

Remark 3.13. This definition is given in [Gil92] for an oriented link and in a more general setting where ξ depends on a choice of a branched covering of M along L . With our assumptions, such a choice is unique and ξ does not depend on the orientation of L (see [Gil93, p. 295]). If M is \mathbb{S}^3 this definition agrees with the signature of an unoriented link given by Murasugi [Mur70].

Proposition 3.14. Let Γ be a knotted 3-Hamiltonian Klein graph in \mathbb{S}^3 . The following relations holds:

$$\begin{aligned}\sigma(\Gamma) &= \sigma(\Gamma_{ab}) + \sigma(\Gamma_{bc}) + \sigma(\Gamma_{ca}), \\ \tilde{\sigma}(\Gamma) &= \xi(\tilde{\Gamma}_a^a) + \xi(\tilde{\Gamma}_b^b) + \xi(\tilde{\Gamma}_c^c), \\ \delta_{ij}(\Gamma) &= \frac{1}{2}\xi(\tilde{\Gamma}_i^i) + \frac{1}{2}\xi(\tilde{\Gamma}_j^j) - \sigma(\Gamma_{ij}) \quad \text{for all } i \neq j \text{ in } D_4^*\end{aligned}$$

Note that $\tilde{\Gamma}_a^a$, $\tilde{\Gamma}_b^b$ and $\tilde{\Gamma}_c^c$ are links in rational homology spheres. In these formulas $\sigma(K)$ denotes the signature of a knot K in \mathbb{S}^3 .

The proofs of Theorem 3.11 and Proposition 3.14 are postponed to Section 3.4.

Remark 3.15. (1) In some cases, this proposition enables to compute signature of the knots $\tilde{\Gamma}_i^i$ by taking advantage of some symmetries. See Section 4 for an example.

(2) The invariants σ , $\tilde{\sigma}$ and δ_{ij} are additive with respect to connected sum along a vertex (see Remark 2.2(4)). For σ , it follows from the additivity of

the signature of knots with respect to connected sum. For the invariants δ_{ij} , it follows from the additivity of the normal Euler numbers of surfaces. From this, one easily deduces the additivity of $\tilde{\sigma}$.

The proofs of the results of this section require some properties on normal Euler numbers and coverings which are described in the next section.

3.3. Normal Euler numbers and double branched covers. In this part, we explain how normal Euler numbers behave when considering double branched coverings. We consider three different situations. The first two appear in Lemma 3.16 and are classical. The third one is quite specific to our situation. It is given in Lemma 3.18.

Lemma 3.16. (1) *Let Σ_1 be a surface embedded in a 4-manifold W and \widetilde{W} be the double branched cover of W along Σ_1 . We define $\widetilde{\Sigma}_1$ to be the pre-image of Σ_1 in \widetilde{W} . We have:*

$$e(\Sigma_1) = 2e(\widetilde{\Sigma}_1).$$

(2) *Suppose that another surface Σ_2 intersects Σ_1 transversely. We denote by $\widetilde{\Sigma}_2$ the pre-image of Σ_2 in \widetilde{W} . We have:*

$$e(\Sigma_2) = \frac{1}{2}e(\widetilde{\Sigma}_2).$$

Proof. (1) We denote by $\pi : \widetilde{W} \rightarrow W$ the canonical projection. We choose a section \tilde{s} of the normal bundle of $\widetilde{\Sigma}_1$ transverse to the trivial section \tilde{s}_0 . The section \tilde{s} induces via π a section s of the normal bundle of Σ_1 . This section s is not transverse to the normal section s_0 of Σ_1 . However there are only finitely many intersection points. In a neighborhood of every intersection point, the double branched cover can be written in an appropriate chart as:

$$\begin{aligned} \pi : \mathbb{B}^4 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| \leq 1, |z_2| \leq 1\} &\rightarrow \mathbb{B}^4 \\ (z_1, z_2) &\mapsto (z_1, z_2^2) \end{aligned}$$

with $\widetilde{\Sigma}_1 = \{(z_1, 0) \mid |z_1| \leq 1\}$ and $\tilde{s}(\widetilde{\Sigma}_1) = \{(z_1, z_1) \mid |z_1| \leq 1\}$. Locally the intersection number of $\widetilde{\Sigma}_1$ with \tilde{s} is ± 1 , the sign depending on the orientation of \mathbb{B}^4 . With this setting, $\Sigma_1 = \{(z_1, 0) \mid |z_1| \leq 1\}$ and $s = \{(z_1, z_1^2) \mid |z_1| \leq 1\}$. The local intersection number of Σ_1 with s is ± 2 , with the same sign. We obtain the result by summing over all intersection points.

(2) We consider s a section of the normal bundle of Σ_2 transverse to the null section. We may assume that $\Sigma_2 \cap s(\Sigma_2)$ and Σ_1 are disjoint. The section s induces a section \tilde{s} of the normal bundle of $\widetilde{\Sigma}_2$. The section \tilde{s} is transverse to the null section. Moreover the map π induces a 2 to 1 correspondence between $\widetilde{\Sigma}_2 \cap \tilde{s}(\widetilde{\Sigma}_2)$ and $\Sigma_2 \cap s(\Sigma_2)$ which respects the intersection signs. \square

Definition 3.17. Let Σ and Σ' be two surfaces embedded in a 4-manifold W . We say that Σ and Σ' intersect *quasi-transversely* if for all x in $\Sigma \cap \Sigma'$, there exists a

³When working in a chart, we identify a section of the normal bundle of a surface with its image in the ambient 4-manifold by an appropriate exponential map. In particular the null section is identified with the surface itself.

neighborhood U of x in W such that:

$$(U, \Sigma \cap U, \Sigma' \cap U, x) \simeq (\mathbb{R}^4, \mathbb{R}^2 \times \{(0, 0)\}, \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\}, \{(0, 0, 0, 0)\}).$$

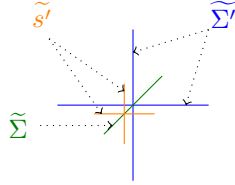
Lemma 3.18. *Let Σ be a surface embedded in an oriented 4-manifold W . Let \widetilde{W} a double branched cover of W along Σ and $\widetilde{\Sigma}$ the pre-image of Σ in \widetilde{W} . Suppose that a surface Σ' in W intersects Σ quasi-transversely. Let $\widetilde{\Sigma}'$ be the pre-image of Σ' in \widetilde{W} . We have:*

$$e(\Sigma') = \frac{1}{2}e(\widetilde{\Sigma}')$$

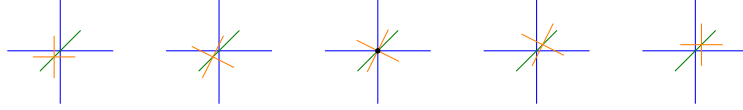
Note that the surface $\widetilde{\Sigma}'$ is not embedded.

Proof. As a double branched cover, the manifold \widetilde{W} is naturally endowed with an involution. We denote it by τ . The intersection of Σ and Σ' is a finite collection \mathcal{C} of circles.

We construct a τ -invariant section \widetilde{s}' of the normal bundle of $\widetilde{\Sigma}'$ in a tubular neighborhood of \mathcal{C} . In order to do this, we fix a base point x_C on each circle C and define the section locally. This can be extended along the circle.

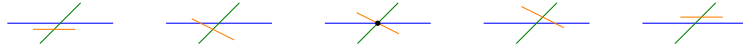


When reaching the base point x_C , we may face a gluing problem. In that case we complete the section as described in the picture:



This yields a pair of transverse intersection points of the \widetilde{s}' with the null section of the normal bundle.

The image s' of this (incomplete) section gives an (incomplete) transverse section of the normal bundle of Σ' transverse to the trivial section. The image of the neighborhoods of the intersection points is given by:



Hence, to each pair of intersection points in \widetilde{W}' corresponds a single intersection point of the same sign. We extend the section s' into a complete section of the normal bundle of Σ' . Its pre-image extend \widetilde{s}' into a complete section of the normal bundle of \widetilde{W}' . What happens far from the circles is given by part (2) of Lemma 3.16. \square

3.4. Proof of the main theorem. The principle of the proof is inspired by [GL78, GL79] which use the 4-dimensional point of view of Kauffman and Taylor [KT76] on signature of knots. It uses the G -signature theorem by Atiyah and Singer [AS68].

Following [Gor86] let us recall the definition of signature in the context of 4-dimensional manifolds. Let W be a compact oriented 4-manifold (with or without

boundary). The intersection form of W induces an hermitian form φ on $H_2(W, \mathbb{C})$. Take a finite group G which acts on W by orientation-preserving diffeomorphisms. Then for every g in G , g_* preserves the form φ on $H_2(W, \mathbb{C})$ and we may choose a G -invariant φ -orthogonal decomposition $H_2(W, \mathbb{C}) = H^+ \oplus H^- \oplus H^0$ such that φ is positive definite on H^+ , negative definite on H^- and zero on H^0 . The *signature of g* is defined by:

$$\sigma(W, g) := \sigma(H_2(W), g_*) = \text{tr}(g_{*|H^+}) - \text{tr}(g_{*|H^-}),$$

where tr denotes the trace of linear endomorphisms.

Remark 3.19. The signature $\sigma(W)$ of a manifold W is the signature of the identity of W .

The following formula connects the sum of the signature of the elements of G and the signature of the quotient manifold W/G and is known in the literature as a standard transfer argument.

Proposition 3.20. *Let W be a compact oriented 4-manifold endowed with an action of a finite group G by orientation-preserving diffeomorphisms. Then we have:*

$$\sum_{g \in G} \sigma(W, g) = |G| \sigma(W/G)$$

Sketch of the proof. Let π be the projection $\pi : W \rightarrow W/G$. Then there exists cellular decompositions of W and W/G compatible with the action of G such that π is cellular. Moreover, at the level of chains one can define a map which assigns to each 2-chain of W/G the sum of its pre-images by π_* . This induces a transfer map $t : H_2(W/G) \rightarrow H_2(W)^G \hookrightarrow H_2(W)$ respecting intersections forms in the sense that $\varphi_W(tx, ty) = |G| \varphi_{W/G}(x, y)$. Moreover we have $\pi_* \circ t = |G| \text{id}_{H_2(W/G)}$ and $t \circ \pi_* = \sum_{g \in G} g_*$ on $H_2(W)$. It follows that:

$$\begin{aligned} \sum_{g \in G} \sigma(W, g) &= \sigma(H_2(W), t \circ \pi_*) \\ &= \sigma(H_2(W)^G, t \circ \pi_*) \\ &= |G| \sigma(H_2(W)^G, \text{id}) \\ &= |G| \sigma(W/G). \end{aligned}$$

□

The G -signature theorem for 4-manifolds [Gor86, Theorem 2] takes an especially simple form for involutions:

Theorem 3.21. *Let W be an oriented closed 4-manifold and τ be an orientation-preserving smooth involution. Then the set of fixed points of τ consists of a surface F_τ and some isolated points and we have:*

$$\sigma(W, \tau) = e(F_\tau).$$

Remark 3.22. If τ is not an involution (but still has finite order), the statement is a bit more complicated and there are some contribution coming from the isolated fixed points.

Proof of the Theorem 3.11. We first prove the invariance of σ , and $\tilde{\sigma}$. This implies the invariance of δ . The invariance of the δ_{ij} 's is postponed to the proof of Proposition 3.14.

Let us consider two spanning foams (\mathbb{B}_1^4, F_1) and (\mathbb{B}_2^4, F_2) for (\mathbb{S}^3, Γ) (the indices of the \mathbb{B}^4 are simply labels with no geometrical meaning). Then we can consider the gluing $(\mathbb{S}^4, E) = (\mathbb{B}_1^4, F_1) \cup_{(\mathbb{S}^3, \Gamma)} (-\mathbb{B}_2^4, F_2)$. Lifting to Klein covers we get the gluing $V_E = W_{F_1} \cup_{M_\Gamma} (-W_{F_2})$. Novikov additivity [Kir89, Theorem 5.3] gives:

$$(1) \quad \sigma(V_E) = \sigma(W_{F_1}) - \sigma(W_{F_2}).$$

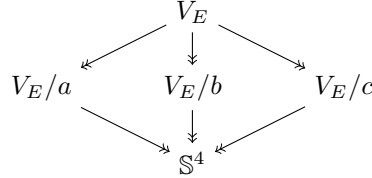
The action of the Klein group D_4 on V_E defines a collection of signatures which satisfies, thanks to Proposition 3.20, the relation:

$$\sigma(V_E) + \sum_{g \in D_4^*} \sigma(V_E, g) = 4\sigma(\mathbb{S}^4) = 0.$$

Together with Theorem 3.21 this gives:

$$(2) \quad -\sigma(V_E) = \sigma(V_E, a) + \sigma(V_E, b) + \sigma(V_E, c) = e(\widehat{E}_a) + e(\widehat{E}_b) + e(\widehat{E}_c).$$

In the diagram



the three top arrows are double branched covers along \tilde{E}_a^a , \tilde{E}_b^b and \tilde{E}_c^c respectively. So applying first part of Lemma 3.16, we get:

$$-\sigma(V_E) = \frac{1}{2}e(\tilde{E}_a^a) + \frac{1}{2}e(\tilde{E}_b^b) + \frac{1}{2}e(\tilde{E}_c^c).$$

Note that in the gluing $V_E/a = (W_{F_1}/a) \cup_{M_\Gamma/a} (-W_{F_2}/a)$ the 3-manifold M_Γ/a is a rational homology sphere (see Remark 3.15) so that thanks to Proposition 3.7 and Definition 3.10 we obtain:

$$\begin{aligned} -\sigma(V_E) &= \frac{1}{2} \left(e((\widetilde{F_1})_a^a) - e((\widetilde{F_2})_a^a) + e((\widetilde{F_1})_b^b) - e((\widetilde{F_2})_b^b) + e((\widetilde{F_1})_c^c) - e((\widetilde{F_2})_c^c) \right) \\ &= \frac{1}{2}\tilde{e}(F_1) - \frac{1}{2}\tilde{e}(F_2). \end{aligned}$$

From this and Relation (1) we deduce:

$$\sigma(W_{F_1}) + \frac{1}{2}\tilde{e}(F_1) = \sigma(W_{F_2}) + \frac{1}{2}\tilde{e}(F_2).$$

This proves the invariance of $\tilde{\sigma}$.

On the other hand, starting again from relation (2) we get:

$$\begin{aligned}
-\sigma(V_E) &= \frac{1}{2} \left(e(\widehat{E}_a) + e(\widehat{E}_b) \right) + \frac{1}{2} \left(e(\widehat{E}_b) + e(\widehat{E}_c) \right) + \frac{1}{2} \left(e(\widehat{E}_c) + e(\widehat{E}_a) \right) \\
&= \frac{1}{2} e(\widehat{E}_a \cup \widehat{E}_b) + \frac{1}{2} e(\widehat{E}_b \cup \widehat{E}_c) + \frac{1}{2} e(\widehat{E}_c \cup \widehat{E}_a) \\
&= e(\widetilde{E}_{a,b}^c) + e(\widetilde{E}_{b,c}^a) + e(\widetilde{E}_{c,a}^b) \\
&= \frac{1}{2} (e(E_{ab}) + e(E_{bc}) + e(E_{ca})) \\
&= \frac{1}{2} e(F_1) - \frac{1}{2} e(F_2).
\end{aligned}$$

The third and fourth equalities follows from Lemmas 3.18 and 3.16, while the last one holds because of Proposition 3.7. Again from this and relation (1) we deduce:

$$\sigma(W_{F_1}) + \frac{1}{2} e(F_1) = \sigma(W_{F_2}) + \frac{1}{2} e(F_1)$$

This proves the invariance of σ . □

Before proving Proposition 3.14, we need the following lemma.

Lemma 3.23. *Let F be foam with boundary in \mathbb{B}^4 , then we have:*

$$\sigma(W_F) = \sigma(W_F/a) + \sigma(W_F/b) + \sigma(W_F/c).$$

Proof. Using Proposition 3.20 for the manifold W_F endowed with the action of the Klein group D_4 we get:

$$\sigma(W_F) + \sigma(W_F, a) + \sigma(W_F, b) + \sigma(W_F, c) = 4\sigma(\mathbb{B}^4) = 0.$$

Now the same proposition applied to the action of the involution a on W_F gives:

$$\sigma(W_F) + \sigma(W_F, a) = 2\sigma(W_F/a)$$

and analogous formulas for involutions b and c . The result follows easily. □

Proof of Proposition 3.14. Let F be an arbitrary spanning foam for Γ in \mathbb{B}^4 . The surface F_{ab} is a spanning surface for the knot Γ_{ab} . Thanks to [GL78, Theorem 2 and Corollary 5], we have:

$$(3) \quad \sigma(\Gamma_{ab}) = \sigma(W_F/c) + \frac{1}{2} e(F_{ab}).$$

We obviously have analogous formulas for $\sigma(\Gamma_{bc})$ and $\sigma(\Gamma_{ca})$. This gives:

$$\begin{aligned}
\sigma(\Gamma_{ab}) + \sigma(\Gamma_{bc}) + \sigma(\Gamma_{ca}) &= \sigma(W_F/c) + \sigma(W_F/a) + \sigma(W_F/b) \\
&\quad + \frac{1}{2} (e(F_{ab}) + e(F_{bc}) + e(F_{ca})) \\
&= \sigma(W_F) + \frac{1}{2} e(F) \\
&= \sigma(\Gamma),
\end{aligned}$$

where the second equality comes from Lemma 3.23. This is exactly the first part of Proposition 3.14.

The second part is in some sense a generalization. Here $\widetilde{\Gamma}_a^a$ is a link in M_Γ/a and \widetilde{F}_a^a is a spanning surface for this link (living in W_F/a). Recall that M_Γ/a is a

rational homology whose first homology group has odd order. Hence Definition 3.12 gives:

$$(4) \quad \xi(\tilde{\Gamma}_a^a) = \sigma(W_F) - 2\sigma(W_F/a) + \frac{1}{2}e(\tilde{F}_a^a).$$

Doing the sum with the analogous formulas for $\xi(\tilde{\Gamma}_b^b)$ and $\xi(\tilde{\Gamma}_c^c)$ we get:

$$\begin{aligned} \xi(\tilde{\Gamma}_a^a) + \xi(\tilde{\Gamma}_b^b) + \xi(\tilde{\Gamma}_c^c) &= 3\sigma(W_F) - 2(\sigma(W_F/a) + \sigma(W_F/b) + \sigma(W_F/c)) \\ &\quad + \frac{1}{2}(e(\tilde{F}_a^a) + e(\tilde{F}_b^b) + e(\tilde{F}_c^c)) \\ &= \sigma(W_F) + \frac{1}{2}\tilde{e}(F) \\ &= \tilde{\sigma}(\Gamma), \end{aligned}$$

where the second equality follows from Lemma 3.23.

It remains to show that

$$\delta_{ij}(\Gamma) = \frac{1}{2}\xi(\tilde{\Gamma}_i^i) + \frac{1}{2}\xi(\tilde{\Gamma}_j^j) - \sigma(\Gamma_{ij}) \quad \text{for all } i \neq j \text{ in } D_4^*.$$

Note that this formula implies the invariance of the δ_{ij} 's and therefore completes the proof of Theorem 3.11.

By symmetry it is enough to consider $i = a$ and $j = b$. We have:

$$\begin{aligned} &\frac{1}{2}\xi(\tilde{\Gamma}_a^a) + \frac{1}{2}\xi(\tilde{\Gamma}_b^b) - \sigma(\Gamma_{ab}) \\ &= \frac{1}{2}\left(\sigma(W_F) - 2\sigma(W_F/a) + \frac{1}{2}e(\tilde{F}_a^a) + \sigma(W_F) - 2\sigma(W_F/b) + \frac{1}{2}e(\tilde{F}_b^b)\right) \\ &\quad - \left(\sigma(W_F/c) + \frac{1}{2}e(F_{ab})\right) \\ &= \sigma(W_F) - (\sigma(W_F/a) + \sigma(W_F/b) + \sigma(W_F/c)) + \frac{1}{4}e(\tilde{F}_a^a) + \frac{1}{4}e(\tilde{F}_b^b) - \frac{1}{2}e(F_{ab}) \\ &= \delta_{ab}(\Gamma). \end{aligned}$$

The last equality follows from Lemma 3.23. \square

4. AN EXAMPLE

In this section we compute our signature invariants on Γ , the Kinoshita knotted graph. We describe a spanning foam F for Γ by a movie given in Figure 5. In between the successive frames of the movie, one has a canonical foamy cobordism. The spanning foam F of Γ is obtained by composing all these cobordisms together and finally glue this foam with a trivial half-theta foam. This gives a foam whose boundary is Γ .

Since all sub-links of Γ are trivial, we have $\sigma(\Gamma) = 0$. In order to determine the other signature invariants it is enough to compute $e(F_{ab})$, $e(F_{bc})$, $e(F_{ac})$, $e(\tilde{F}_a^a)$, $e(\tilde{F}_b^b)$ and $e(\tilde{F}_c^c)$.

As explained in Section 3.1, normal Euler numbers of surfaces with boundary can be computed via some linking numbers. We first inspect the surfaces F_{ij} .

The standard theta diagram can be seen as a framed graph equipped with three specific parallels (see the first frame in Figure 5). Following these parallels during the various steps of the movie gives for each pair $\{i, j\}$ a *non-vanishing* section of the normal bundle of the surface (with boundary) F_{ij} . The boxes with integers or

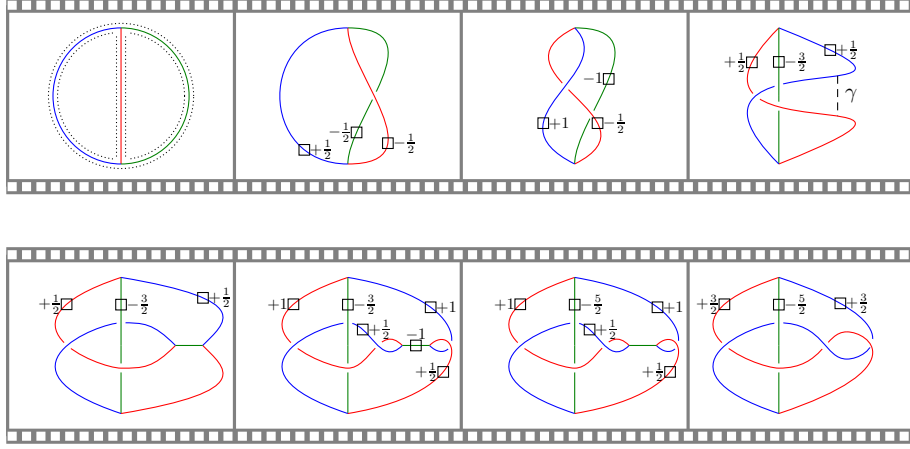


FIGURE 5. Movie describing a spanning foam for the Kinoshita knotted graph

half-integers drawn in the movie encode twists or half-twists between an edge and its parallel.

The normal Euler number $e(F_{ij})$ is equal to $-l_{ij}$, where l_{ij} is the linking number in \mathbb{S}^3 of the knot $\Gamma_{ij} = \partial F_{ij}$ with its parallel (see Definition 3.5 and Remark 3.6 (3)): that is the sum of all values of the boxes in the last frame plus the number of crossings counted algebraically.

Note that between the frames 6 and 7, a box on a green (c) edge travels the other green edge (because we need to unzip the first edge). This does not impact the validity of the computation.

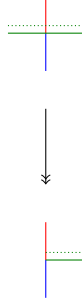
We have:

- $e(F_{ab}) = -(1 + \frac{3}{2} + \frac{3}{2}) = -4$,
- $e(F_{bc}) = e(F_{ac}) = -(1 + \frac{3}{2} - \frac{5}{2}) = 0$.

We now explain how to compute $e(\tilde{F}_i^i)$ for i in $\{a, b, c\}$. As we will see, it is possible to deduce $e(\tilde{F}_a^a)$ and $e(\tilde{F}_b^b)$ from $e(\tilde{F}_c^c)$ by using symmetries. For computing $e(\tilde{F}_c^c)$, we consider a section of the normal bundle of \tilde{F}_c^c (in W_F/c) transverse to the trivial section. Such a section can be read on the movie. Indeed, consider each step of the movie as a framed graph where every edge colored by c comes with a parallel attached to its adjacent edges⁴. The pre-image of these parallels in the double cover along the knot consisting of edges colored by a and b is a genuine

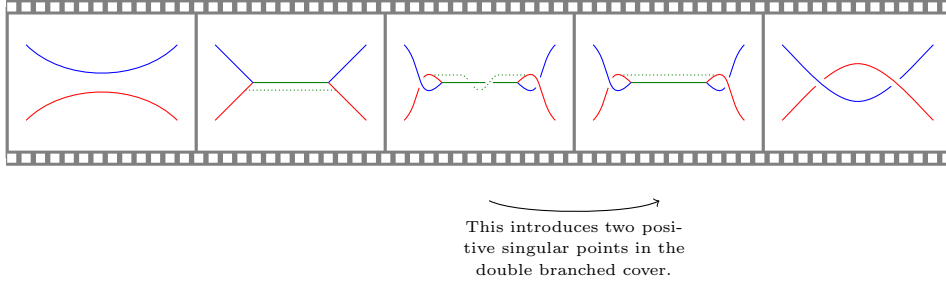
⁴Here the apparent choice of side has no effect.

parallel of the pre-image of the edge, as shown in the following picture:



Following these parallels during the movie gives an appropriate section of \tilde{F}_c^c in W_F/c . The surface \tilde{F}_c^c has two connected components: a knotted sphere S coming from the small horizontal green edge on the frames 5, 6 and 7 and another component Σ whose boundary is the knot $\tilde{\Gamma}_c^c$.

Let us first deal with the knotted sphere S . We isolate the interesting part of the movie:



Hence we have $e(S) = +2$.

In order to compute $e(\Sigma)$, we consider the section of Σ given by the pre-image of the parallels. This section does not intersect Σ , hence $e(\Sigma)$ is equal to $-\ell_c$ where ℓ_c is equal to the linking number of the knot $\tilde{\Gamma}_c^c$ with the pre-image in $\partial W_F/c$ of its preferred parallel of the edge of Γ colored by c .

For computing ℓ_c , we use the following theorem:

Theorem 4.1 ([PY04, Theorem 1.1], restricted to our case). *Let K_1 and K_2 be two knots in \mathbb{S}^3 , $J = (J_1, \dots, J_l)$ a link in \mathbb{S}^3 disjoint from K_1 and K_2 and r_1, \dots, r_l some rational numbers. Let M be the manifold obtained by Dehn surgery along J with coefficients r_1, \dots, r_l . Suppose that M is a rational homology sphere, then*

$$\begin{aligned} \text{lk}_M(K_1, K_2) - \text{lk}_{\mathbb{S}^3}(K_1, K_2) \\ = -(\text{lk}_{\mathbb{S}^3}(K_1, J_1), \dots, \text{lk}_{\mathbb{S}^3}(K_1, J_l)) G^{-1} (\text{lk}_{\mathbb{S}^3}(K_2, J_1), \dots, \text{lk}_{\mathbb{S}^3}(K_2, J_l))^t, \end{aligned}$$

where $G = (g_{ij})_{\substack{1 \leq i \leq l \\ 1 \leq j \leq l}}$ is the $l \times l$ matrix defined by:

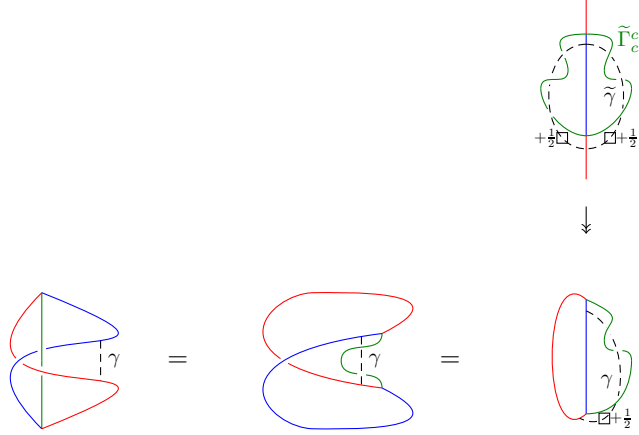
$$g_{ij} = \begin{cases} \text{lk}_{\mathbb{S}^3}(J_i, J_j) & \text{if } i \neq j, \\ r_i & \text{if } i = j. \end{cases}$$

Indeed, Montesinos' trick [Mon75, Section 2] tells us that if one changes a knot by the local moves:



the double branched cover of the new knot is obtained by a surgery along $\tilde{\gamma}$ with coefficient $-\frac{1}{2}$ for a positive clasp move, $+\frac{1}{2}$ for a negative clasp move, where $\tilde{\gamma}$ is the pre-image of γ in the double branched cover. These coefficients are given in the canonical⁵ basis longitude/meridian of $\tilde{\gamma}$.

Since the graph (denoted by Γ') in the 4th frame is still a trivial theta graph, all double branched cover are diffeomorphic to \mathbb{S}^3 . The following picture describes the branched double cover of \mathbb{S}^3 along Γ'_{ab} :



The knot $\tilde{\gamma}$ is a framed trivial knot with framing $+1$. In Montesinos' trick, the surgery coefficient is $-\frac{1}{2}$ for the basis given by the framing. For the standard basis it is therefore $\frac{-1+2}{2} = \frac{1}{2}$. Moreover, we have $\text{lk}_{\mathbb{S}^3}(\tilde{\Gamma}'_c, \tilde{\gamma}) = \pm 3$ (the sign depends on which orientation we take for $\tilde{\gamma}$). Thanks to Theorem 4.1, we have: $\ell_c = -(\pm 3)^2(\frac{1}{2})^{-1} = -18$ and finally $e(\tilde{F}_c^c) = 2 + 18 = 20$.

The remaining normal Euler numbers are $e(\tilde{F}_a^a)$ and $e(\tilde{F}_b^b)$.

First of all, the three bi-colored knots of Γ are trivial so that $\sigma(\Gamma_{ab}) = \sigma(\Gamma_{bc}) = \sigma(\Gamma_{ca}) = 0$. Moreover because of the symmetry of Γ , the knots $\tilde{\Gamma}_a^a$, $\tilde{\Gamma}_b^b$ and $\tilde{\Gamma}_c^c$ are the same. Denote by s the signature of this knot. By symmetry of the movie relatively to the colors a and b we have $e(\tilde{F}_a^a) = e(\tilde{F}_b^b)$. Denote by e this value. We have:

$$\begin{cases} \delta_{ab}(\Gamma) = \delta_{bc}(\Gamma) = \delta_{ca}(\Gamma) = \frac{1}{2}s + \frac{1}{2}s - 0 = s \\ \delta_{ab}(\Gamma) = \frac{1}{4}e(\tilde{F}_a^a) + \frac{1}{4}e(\tilde{F}_b^b) - \frac{1}{2}e(F_{ab}) = \frac{1}{2}e + 2, \\ \delta_{bc}(\Gamma) = \frac{1}{4}e(\tilde{F}_b^b) + \frac{1}{4}e(\tilde{F}_c^c) - \frac{1}{2}e(F_{bc}) = \frac{1}{4}e + 5. \end{cases}$$

⁵The longitude is required to be the pre-image of the dotted arc.

It implies that $e = 12$ and $s = 8$. In conclusion we have:

$$\begin{cases} \delta_{ab}(\Gamma) = \delta_{bc}(\Gamma) = \delta_{ac}(\Gamma) = 8, \\ \sigma(\Gamma) = 0, \\ \tilde{\sigma}(\Gamma) = 24. \end{cases}$$

In particular the Kinoshita graph is not trivial.

Remark 4.2. As an intermediate result, our computations give the signature of the knot $\tilde{\Gamma}_a^a$ in \mathbb{S}^3 (which is 8). Note that we did not need to determine this knot which happens to be the mirror image of 10_{124} (see for example [CMB16]).

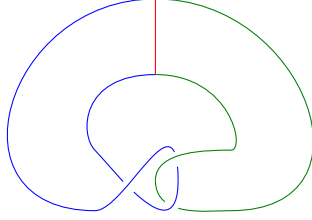


FIGURE 6. The trefoil theta graph.

The same kind of computations can be done for other knotted graphs. For the trefoil theta, we find:

$$\begin{cases} \delta_{ab}(\Gamma) = \delta_{ca}(\Gamma) = \frac{4}{3} \\ \delta_{bc}(\Gamma) = 4 \\ \sigma(\Gamma) = -2, \\ \tilde{\sigma}(\Gamma) = \frac{14}{3}. \end{cases}$$

As for the Kinoshita graph, the symmetries give an intermediate result: the signature of $\sigma(\tilde{\Gamma}_a^a)$ which is a knot in the lens space $L(3, 1)$ is equal to $\frac{2}{3}$.

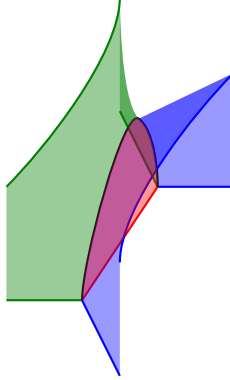
APPENDIX A. SPANNING FOAMS

The aim of this section is to prove the following statement:

Proposition A.1. *Let Γ be a knotted Klein graph in \mathbb{S}^3 . There exists a spanning foam for Γ in \mathbb{B}^4 .*

We need to introduce some elementary foamy cobordisms. A *clasp* is a cobordism which performs a bicolor crossing change. It is described by the movie at the bottom of page 19. An *unzip* is simply the second part of a clasp. It removes two vertices

of the graph. Locally an *unzip* looks like the picture drawn below.



Lemma A.2. *Let Γ be a knotted Klein graph in \mathbb{S}^3 . There exists a cobordism F in $\mathbb{S}^3 \times I$ from Γ to a Klein graph with no vertices (i. e. a link).*

Proof. Let D be a diagram for Γ , we consider all edges⁶ of D colored by a , with a sequence of Reidemeister move, we can shrink these edges until they are not involved in any crossing. Then we unzip these edges. Before performing the unzip, we may have to twist one end of these edges with the Rv1 Reidemeister move. \square

Lemma A.3. *Let Γ be a knotted Klein graph in \mathbb{S}^3 with no vertices. This can be seen as a link L colored by D_4^* , this means that $L = L_a \cup L_b \cup L_c$. There exists a (foamy) cobordism from L to a link $L' = L'_a \cup L'_b \cup L'_c$, where the component L'_a , L'_b and L'_c are in three disjoint balls.*

Proof. In order to unlink the component L_a , L_b and L_c of L , it is enough to perform crossing changes on bi-colored crossings. These crossing changes can be achieved by clasps. \square

Proof of Proposition A.1. Thanks to Lemmas A.2 and A.3 we can construct a cobordism F in $\mathbb{S}^3 \times I$ from Γ to a link $L' = L'_a \cup L'_b \cup L'_c$, where the component L'_a , L'_b and L'_c are in disjoint balls. We now pick Seifert surfaces for L'_a , L'_b and L'_c , push them in \mathbb{B}^4 and concatenate them with F . This gives a spanning foam for Γ . \square

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⁶Here, we mean *real* edges, i. e. not circles.

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UNIV PARIS DIDEROT, INSTITUT DE MATHÉMATIQUES DE JUSSIEU-PARIS RIVE GAUCHE, CNRS, SORBONNE UNIVERSITÉ. CAMPUS DES GRANDS MOULINS, BÂTIMENT SOPHIE-GERMAIN, CASE 7012, 75205 PARIS CEDEX 13, FRANCE

E-mail address: catherine.gille@imj-prg.fr

UNIVERSITÉ DE GENÈVE, 2-4 RUE DU LIÈVRE, CASE POSTALE 64, 1211 GENÈVE 4, SWITZERLAND

E-mail address: louis-hadrien.robert@unige.ch