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SINGULAR HOCHSCHILD COHOMOLOGY VIA THE SINGULARITY CATEGORY

BERNHARD KELLER

Abstract. We show that the singular Hochschild cohomology (=Tate–Hochschild cohomology) of an algebra \( A \) is isomorphic, as a graded algebra, to the Hochschild cohomology of the differential graded enhancement of the singularity category of \( A \). The existence of such an isomorphism is suggested by recent work of Zhengfang Wang.

1. Introduction

Let \( k \) be a commutative ring. We write \( \otimes \) for \( \otimes_k \). Let \( A \) be a right noetherian (non commutative) \( k \)-algebra projective over \( k \). The stable derived category or singularity category of \( A \) is defined as the Verdier quotient

\[
\text{Sg}(A) = \mathcal{D}^b(\text{mod } A)/\text{per}(A)
\]

of the bounded derived category of finitely generated (right) \( A \)-modules by the perfect derived category \( \text{per}(A) \), i.e. the full subcategory of complexes quasi-isomorphic to bounded complexes of finitely generated projective modules. It was introduced by Buchweitz in an unpublished manuscript [4] in 1986 and rediscovered, in its scheme-theoretic variant, by Orlov in 2003 [24]. Notice that it vanishes when \( A \) is of finite global dimension and thus measures the degree to which \( A \) is ‘singular’, a view confirmed by the results of [24].

Let us suppose that the enveloping algebra \( A^e = A \otimes A^{op} \) is also right noetherian. In analogy with Hochschild cohomology, in view of Buchweitz’ theory, it is natural to define the Tate–Hochschild cohomology or singular Hochschild cohomology of \( A \) to be the graded algebra with components

\[
HH^n_{\text{sg}}(A, A) = \text{Hom}_{\text{Sg}(A^e)}(A, \Sigma^n A), \ n \in \mathbb{Z},
\]

where \( \Sigma \) denotes the suspension (=shift) functor. It was studied for example in [10, 2, 23] and more recently in [31, 32, 30, 33, 29, 5]. Wang showed in [31] that, like Hochschild cohomology [11], singular Hochschild cohomology carries a structure of Gerstenhaber algebra. Now recall that the Gerstenhaber algebra structure on Hochschild cohomology is a small part of much richer higher structure on the Hochschild cochain complex \( C(A, A) \) itself, namely the structure of a \( B_\infty \)-algebra in the sense of Getzler–Jones [12, 5.2] given by the brace operations [1, 16]. In [29], Wang improves on [31] by defining a singular Hochschild cochain complex \( C_{\text{sg}}(A, A) \) and endowing it with a \( B_\infty \)-structure which in particular yields the Gerstenhaber algebra structure on \( HH^*_{\text{sg}}(A, A) \).

Using [17] Lowen–Van den Bergh showed in [21, Theorem 4.4.1] that the Hochschild cohomology of \( A \) is isomorphic to the Hochschild cohomology of the canonical differential graded (=dg) enhancement of the (bounded or unbounded) derived category of \( A \) and that the isomorphism lifts to the \( B_\infty \)-level (cf. Corollary 7.6 of [26] for a related statement). Together with the complete structural analogy between Hochschild and singular
Hochschild cohomology described above, this suggests the question whether the singular Hochschild cohomology of $A$ is isomorphic to the Hochschild cohomology of the canonical dg enhancement $S_{dg}(A)$ of the singularity category $Sg(A)$ (note that such an enhancement exists by the construction of $Sg(A)$ as a Verdier quotient [19, 6]). Chen–Li–Wang show in [5] that this does hold at the level of Gerstenhaber algebras when $A$ is the radical square zero algebra associated with a finite quiver without sources or sinks. Our main result is the following.

**Theorem 1.1.** There is a canonical isomorphism of graded algebras between the singular Hochschild cohomology of $A$ and the Hochschild cohomology of the dg singularity category $S_{dg}(A)$.

**Conjecture 1.2.** The isomorphism of the theorem lifts to an isomorphism

$$C_{sg}(A, A) \cong C(S_{dg}(A), S_{dg}(A))$$

in the homotopy category of $B_\infty$-algebras.

Notice that the $B_\infty$-structure on Hochschild cohomology of dg categories is preserved (up to quasi-isomorphism) under Morita equivalences, cf. [17].

Let us mention an application of Theorem 1.1 obtained in joint work with Zheng Hua. Suppose that $k$ is algebraically closed of characteristic 0 and let $P$ the power series algebra $k[[x_1, \ldots, x_n]]$.

**Theorem 1.3** ([15]). Suppose that $Q \in P$ has an isolated singularity at the origin and $A = P/(Q)$. Then $A$ is determined up to isomorphism by its dimension and the dg singularity category $S_{dg}(A)$.

In [8, Theorem 8.1], Efimov proves a related but different reconstruction theorem: He shows that if $Q$ is a polynomial, it is determined, up to a formal change of variables, by the differential $Z/2$-graded endomorphism algebra $E$ of the residue field in the differential $Z/2$-graded singularity category together with a fixed isomorphism between $H^* B$ and the exterior algebra $\Lambda(k^n)$.

In section 2, we generalize Theorem 1.1 to the non noetherian setting and prove the generalized statement. We comment on a possible lift of this proof to the $B_\infty$-level in section 3. We prove Theorem 1.3 in section 4.

### 2. Generalization and proof

#### 2.1. Generalization to the non noetherian case.

We assume that $A$ is an arbitrary $k$-algebra projective as a $k$-module. Its singularity category $Sg(A)$ is defined as the Verdier quotient $\mathcal{H}^{-b}(\text{proj} A)/\mathcal{H}^{b}(\text{proj} A)$ of the homotopy category of right bounded complexes of finitely generated projective $A$-modules by its full subcategory of bounded complexes of finitely generated projective $A$-modules. Notice that when $A$ is right noetherian, this is equivalent to the definition given in the introduction.

The (partially) completed singularity category $\hat{Sg}(A)$ is defined as the Verdier quotient of the bounded derived category $D^b(\text{Mod} A)$ of all right $A$-modules by its full subcategory consisting of all complexes quasi-isomorphic to bounded complexes of arbitrary projective modules.

**Lemma 2.2.** The canonical functor $Sg(A) \to \hat{Sg}(A)$ is fully faithful.

**Proof.** Let $M$ be a right bounded complex of finitely generated projective modules with bounded homology and $P$ a bounded complex of arbitrary projective modules. Since the components of $M$ are finitely generated, each morphism $M \to P$ in the derived category
factors through a bounded complex $P'$ with finitely generated projective components. This yields the claim.

Since we do not assume that $A^e$ is noetherian, the $A$-bimodule $A$ will not, in general, belong to the singularity category $\Sigma(A')$. But it always belongs to the completed singularity category $\hat{\Sigma}(A')$. We define the singular Hochschild cohomology of $A$ to be the graded algebra with components

$$HH^0_{\hat{\Sigma}}(A, A) = \text{Hom}_{\hat{\Sigma}(A')}(A, \Sigma^n A), \ n \in \mathbb{Z}.$$

**Theorem 2.3.** Even if $A^e$ is non noetherian, there is a canonical isomorphism of graded algebras between the singular Hochschild cohomology of $A$ and the Hochschild cohomology of the dg singularity category $\Sigma_{dg}(A)$.

Let $P$ be a right bounded complex of projective $A^e$-modules. For $q \in \mathbb{Z}$, let $\sigma_{\geq q}P$ and $\sigma_{\leq q}P$ denote its stupid truncations:

$$\sigma_{\geq q}P : \ldots \to 0 \to P^q \to P^{q+1} \to \ldots$$

$$\sigma_{\leq q}P : \ldots \to P^q \to P^{q-1} \to \ldots \to 0 \to \ldots$$

so that we have a triangle

$$\sigma_{\geq q}P \to P \to \sigma_{\leq q}P \to \Sigma \sigma_{\geq q}P.$$

We have a direct system

$$P = \sigma_{\leq 0}P \to \sigma_{\leq -1}P \to \sigma_{\leq -2}P \to \ldots \to P_{\leq q} \to \ldots.$$

**Lemma 2.4.** Let $L \in \mathcal{D}^b(\text{Mod} A^e)$. We have a canonical isomorphism

$$\text{colim} \text{Hom}_{\mathcal{D}A^e}(L, \sigma_{\leq q}P) \xrightarrow{\sim} \text{Hom}_{\hat{\Sigma}(A')}(L, P).$$

In particular, if $P$ is a projective resolution of $A$ over $A^e$, we have

$$\text{colim} \text{Hom}_{\mathcal{D}A^e}(A, \Sigma^n \sigma_{\leq q}P) \xrightarrow{\sim} \text{Hom}_{\hat{\Sigma}(A')}(A, \Sigma^n A), \ n \in \mathbb{Z}.$$

**Proof.** Clearly, if $Q$ is a bounded complex of projective modules, each morphism $Q \to P$ in the derived category $\mathcal{D}A^e$ factors through $\sigma_{\geq q}P \to P$ for some $q \ll 0$. This shows that the morphisms $P \to \sigma_{\leq q}P$ form a cofinal subcategory in the category of morphisms $P \to P'$ whose cylinder is a bounded complex of projective modules. Whence the claim.

2.5. **Proof of Theorem 2.3.** We refer to [18, 20, 27] for foundational material on dg categories. We will follow the terminology of [20] and use the model category structure on the category of dg categories constructed in [25]. For a dg category $A$, denote by $X \mapsto Y(X)$ the dg Yoneda functor and by $\mathcal{D}A$ the derived category. We write $A^e$ for the enveloping dg category $A \otimes_k A^{op}$ and $I_A$ for the identity bimodule

$$I_A : (X, Y) \mapsto A(X, Y).$$

By definition, the Hochschild cohomology of $A$ is the graded endomorphism algebra of $I_A$ in the derived category $\mathcal{D}(A^e)$. In the case of the algebra $A$, the identity bimodule is the $A$-bimodule $A$. Recall that if $F : A \to B$ is a fully faithful dg functor, the restriction $F_* : \mathcal{D}B \to \mathcal{D}A$ is a localization functor admitting fully faithful left and right adjoint functors $F^*$ and $F'$ given respectively by

$$F^* : M \mapsto M \otimes_A F^*B \quad \text{and} \quad F' : N \mapsto \text{RHom}_A(B_F, N),$$

where $F^*B = B(?, F^-)$ and $B_F = B(F?, -)$.
Let $\mathcal{M}_0 = \mathcal{C}_{dgr}^{-,b}(\text{proj} A)$ denote the dg category of right bounded complexes of finitely generated projective $A$-modules with bounded homology. Notice that the morphism complexes of $\mathcal{M}_0$ have terms which involve infinite products of projective $A$-modules so that in general, the morphism complexes of $\mathcal{M}_0$ will not be cofibrant over $k$. Let $\mathcal{M} \to \mathcal{M}_0$ be a cofibrant resolution of $\mathcal{M}_0$. We assume, as we may, that the quasi-equivalence $\mathcal{M} \to \mathcal{M}_0$ is the identity on objects. Notice that the morphism complexes of $\mathcal{M}$ are cofibrant over $k$ so that we have $\mathcal{M} \otimes_k \mathcal{M}^{op} \to \mathcal{M} \otimes \mathcal{M}^{op}$. Let $\mathcal{P} \subset \mathcal{M}$ be the full dg subcategory of $\mathcal{M}$ formed by the bounded complexes of finitely generated projective $A$-modules. Let $\mathcal{S}$ denote the dg quotient $\mathcal{M}/\mathcal{P}$. We assume, as we may, that $\mathcal{S}$ is cofibrant. In the homotopy category of dg categories, we have an isomorphism between $\mathcal{S}g_{dgr}(A)$ and $\mathcal{S} = \mathcal{M}/\mathcal{P}$. Let $B$ be the dg endomorphism algebra of $A$ considered as an object of $\mathcal{P} \subset \mathcal{M}$. Notice that we have a quasi-isomorphism $B \to A$ and that both $B$ and $A$ are cofibrant over $k$. We view $B$ as a dg category with one object whose endomorphism algebra is $B$. We have the obvious inclusion and projection dg functors

$$B \overset{i}{\longrightarrow} \mathcal{M} \overset{p}{\longrightarrow} \mathcal{S}.$$ 

Consider the fully faithful dg functors

$$B \otimes B^{op} \overset{1 \otimes i}{\longrightarrow} B \otimes \mathcal{M}^{op} \overset{i \otimes 1}{\longrightarrow} \mathcal{M} \otimes \mathcal{M}^{op}.$$ 

The restriction along $G = 1 \otimes i$ admits the left adjoint $G^*$ given by

$$G^* : X \mapsto \mathcal{M}_i \overset{L}{\otimes}_B X,$$

and the restriction along $F = i \otimes 1$ admits the fully faithful left and right adjoints $F^*$ and $F^\dagger$ given by

$$F^* : Y \mapsto Y \overset{L}{\otimes}_B \mathcal{M} \quad \text{and} \quad F^\dagger : Y \mapsto \text{RHom}_B(\mathcal{M}_i, Y).$$

Since $F^*$ and $F^\dagger$ are the two adjoints of a localization functor, we have a canonical morphism

$$F^* \to F^\dagger.$$

**Lemma 2.6.** If $P$ is an arbitrary sum of copies of $B^k$, the morphism

$$F^* G^*(P) \to F^\dagger G^*(P)$$

is invertible.

**Proof.** Let $P$ be the direct sum of copies of $B^k$ indexed by a set $J$. Since $F^*$ and $G^*$ commute with (arbitrary) coproducts, the left hand side is the dg module

$$\bigoplus_J \mathcal{M}(i^?, -) \overset{L}{\otimes}_B (B \otimes B) \overset{L}{\otimes}_B \mathcal{M}(i^-, i-) = \bigoplus_J \mathcal{M}(B, -) \otimes \mathcal{M}(?, B),$$

The right hand side is the dg module

$$\text{RHom}_B(\mathcal{M}_i, \bigoplus_J \mathcal{M}(B \otimes B)) = \text{RHom}_B(\mathcal{M}_i, \bigoplus_J \mathcal{M}(B, -) \otimes B).$$

Let us evaluate the canonical morphism at $(M, L) \in \mathcal{M} \otimes \mathcal{M}^{op}$. We find the canonical morphism

$$\bigoplus_J \mathcal{M}(B, L) \otimes \mathcal{M}(M, B) \to \text{RHom}_B(\mathcal{M}(B, M), \bigoplus_J \mathcal{M}(B, L) \otimes B).$$

We have quasi-isomorphisms

$$\mathcal{M}(B, L) \otimes \mathcal{M}(M, B) \to \mathcal{M}_0(A, L) \otimes \mathcal{M}(M, B) \to L \otimes \mathcal{M}(M, B) \to L \otimes \text{Hom}_A(M, A)$$
because $M(M, B)$ and $L$ are cofibrant over $k$. Now the equivalence $\mathcal{D}(B) \xrightarrow{\sim} \mathcal{D}(A)$ takes $M(B, L) \otimes B$ to $M(B, L) \otimes A \xrightarrow{\sim} L \otimes A$. We have an quasi-isomorphism of dg $B$-modules $M(B, M) \xrightarrow{\sim} M_\emptyset(A, M) = M$ and so the equivalence $\mathcal{D}(B) \xrightarrow{\sim} \mathcal{D}(A)$ takes $M(B, M)$ to $M$. Whence an isomorphism

$$\text{RHom}_B(M(B, M), \bigoplus_j M(B, L) \otimes B) \xrightarrow{\sim} \text{RHom}_A(M, \bigoplus_j L \otimes A) = \text{Hom}_A(M, \bigoplus_j L \otimes A).$$

Thus, we have to show that the canonical morphism

$$\bigoplus_j L \otimes \text{Hom}_A(M, A) \rightarrow \text{Hom}_A(M, \bigoplus_j L \otimes A)$$

is a quasi-isomorphism. Recall that $L$ and $M$ are right bounded complexes of finitely generated projective modules with bounded homology. We fix $M$ and consider the morphism as a morphism of triangle functors with argument $L \in \mathcal{D}^b(\text{Mod} A)$. Then we are reduced to the case where $L$ is in $\text{Mod} A$. In this case, the morphism becomes an isomorphism of complexes because the components of $M$ are finitely generated projective. \(\square\)

Let us put $H = F^! G^* : \mathcal{D}(B^e) \rightarrow \mathcal{D}(M^e)$. Let us compute the image of the identity bimodule $B$ under $H$. We have

$$H(B) = F^!(M_i \otimes_B B) = F^!(M_i) = \text{RHom}_B(M_i, M_i)$$

and when we evaluate at $L, M$ in $M$, we find

$$H(B)(L, M) = \text{RHom}_B(M(i?, L), M(i?, M)) = \text{RHom}_B(M(B, L), M(B, M)).$$

We have seen in the above proof that the equivalence $\mathcal{D}(B) \xrightarrow{\sim} \mathcal{D}(A)$ takes $M(B, L)$ to $L$. Whence quasi-isomorphisms

$$H(B)(L, M) = \text{RHom}_B(M(B, L), M(B, M)) \xrightarrow{\sim} \text{RHom}_A(L, M) = \text{Hom}(L, M)$$

$$\leftarrow M(L, M).$$

Thus, the functor $H$ takes the identity bimodule $B$ to the identity bimodule $I_M$. Since $F^!$ and $G^*$ are fully faithful so is $H$. Denote by $N$ the image under the composition of $H$ with $\mathcal{D}(A^e) \xrightarrow{\sim} \mathcal{D}(B^e)$ of the closure of $\text{Proj} A^e$ under finite extensions. Then $H$ yields a fully faithful functor

$$\widetilde{Sg}(A^e) \rightarrow \mathcal{D}(M^e)/N$$

taking the bimodule $A$ to the identity bimodule $I_M$. Now notice that we have a Morita morphism of dg categories

$$S^e \xleftarrow{\sim} \frac{M \otimes M^\text{op}}{P \otimes M^\text{op} + M \otimes P^\text{op}}.$$

The functor $p^* : \mathcal{D}(M^e) \rightarrow \mathcal{D}(S^e)$ induces the quotient functor

$$\frac{\mathcal{D}(M \otimes M^\text{op})}{N} \xrightarrow{\mathcal{D}(M \otimes M^\text{op})} \frac{\mathcal{D}(P \otimes M^\text{op} + M \otimes P^\text{op})}{\mathcal{D}(P \otimes M^\text{op} + M \otimes P^\text{op})} = \mathcal{D}(S^e).$$

Since $p : M \rightarrow S$ is a localization, the image $p^*(I_M)$ is isomorphic to $I_S$. It suffices to show that $p^*$ induces bijections in the morphism spaces with target $I_M$

$$\text{Hom}_{\mathcal{D}(M^e)/N}(?, I_M) \xrightarrow{\text{Hom}_{\mathcal{D}(S^e)}(p^*(?), p^*(I_M))}.$$

For this, it suffices to show that $I_M$ is right orthogonal in $\mathcal{D}(M^e)/N$ on the images under the Yoneda functor of the objects in $P \otimes M^\text{op} + M \otimes P^\text{op}$. To show that $I_M$ is right orthogonal
on $Y(M \otimes \mathbb{P}^{op})$, it suffices to show that it is right orthogonal to an object $Y(M, B)$, $M \in M$. Now a morphism in $\mathcal{D}(\mathcal{M}^e)/\mathcal{N}$ is given by a diagram of $\mathcal{D}(\mathcal{M}^e)$ representing a left fraction

$$Y(M, B) \longrightarrow I'_M \longleftarrow I_M$$

where the cone over $I_M \to I'_M$ lies in $\mathcal{N}$. For each object $X$ of $\mathcal{D}M^e$, we have canonical isomorphisms

$$\text{Hom}_{\mathcal{D}M^e}(Y(M, B), X) = H^0(X(M, B)) = \text{Hom}_{\mathcal{D}M}(Y(M), X(?, B)).$$

Thus, the given fraction corresponds to a diagram in $\mathcal{D}(\mathcal{M})$ of the form

$$Y(M) \longrightarrow I'_M(?, B) \longleftarrow I_M(?, B) = M(?, B),$$

where the cone over $I_M(?, B) \to I'_M(?, B)$ is the image under $\mathcal{D}A \overset{\sim}{\to} \mathcal{D}B \to \mathcal{D}M$ of a bounded complex with projective components. Thus, the object $I'_M(?, B)$ is a direct factor of a finite extension of shifts of arbitrary coproducts $B$. Since $Y(M)$ is compact, the given morphism $Y(M) \to I'_M(?, M)$ must then factor through $Y(Q)$ for an object $Q$ of $\mathcal{P}$. This means that the given morphism $Y(M, B) \to I'_M$ factors through $Y(Q, B)$, which lies in $\mathcal{N}$. Thus, the given fraction represents the zero morphism of $\mathcal{D}(\mathcal{M}^e)/\mathcal{N}$, as was to be shown. The case of an object in $Y(\mathcal{P} \otimes \mathbb{M}^{op})$ is analogous. In summary, we have shown that the maps

$$\overline{\text{Sg}}(\mathcal{M}^e)(A, \Sigma^n A) \overset{H}{\longrightarrow} (\mathcal{D}(\mathcal{M}^e)/\mathcal{N})(I_M, \Sigma^n I_M) \overset{p'}{\longrightarrow} \mathcal{D}(\mathcal{S}^e)(I_S, \Sigma^n I_S)$$

are bijective, which implies the assertion on Hochschild cohomology.

3. Remark on a Possible Lift to the $B_\infty$-Level

Let $P \to A$ be a resolution of $A$ by projective $A$-$A$-bimodules. Let us assume for simplicity that $k$ is a field so that we can take $M = M_0$ and $B = A$. The proof in section 2 produces in fact isomorphisms in the derived category of $k$-modules

$$\text{colim} \text{RHom}_{A^e}(A, \sigma_{\leq q} P) \to \text{colim} \text{RHom}_{\mathcal{M}^e}(I_M, H\sigma_{\leq q} P) \to \text{colim} \text{RHom}_{\mathcal{S}^e}(I_S, p^* H\sigma_{\leq q} P) = \text{RHom}_{\mathcal{S}^e}(I_S, I_S).$$

For the bar resolution $P$, the truncation $\sigma_{\leq -q} P$ is canonically isomorphic to $\Sigma^q \Omega^q A$ so that the first complex carries a canonical $B_\infty$-structure constructed by Wang [29]. As explained in the introduction, it is classical that the last complex carries a canonical $B_\infty$-structure. It is not obvious to make the intermediate complexes explicit because the functor $H$, being a composition of a right adjoint with a left adjoint to a restriction functor, does not take cofibrant objects to cofibrant objects.

4. Proof of Theorem 1.3

By the Weierstrass preparation theorem, we may assume that $Q$ is a polynomial. Let $P_0 = k[x_1, \ldots, x_n]$ and $S = P_0/(Q)$. Then $S$ has isolated singularities but may have singularities other than the origin. Let $\mathfrak{m}$ be the maximal ideal of $P_0$ generated by the $x_i$ and let $R$ be the localization of $S$ at $\mathfrak{m}$. Now $R$ is local with an isolated singularity at $\mathfrak{m}$ and $A$ is isomorphic to the completion $\hat{R}$. By Theorem 3.2.7 of [14], in sufficiently high degrees $r$, the Hochschild cohomology of $S$ is isomorphic to the homology in degree $r$ of the complex

$$k[u] \otimes K(S, \partial_1 Q, \ldots, \partial_n Q),$$
where $u$ is of degree 2 and $K$ denotes the Koszul complex. Now $S$ is isomorphic to $K(P_0, Q)$ and so $K(S, \partial_1 Q, \ldots, \partial_n Q)$ is isomorphic to

$$K(P_0, Q, \partial_1 Q, \ldots, \partial_n Q).$$

Since $Q$ has isolated singularities, the $\partial_i Q$ form a regular sequence in $P_0$. So

$$K(P_0, Q, \partial_1 Q, \ldots, \partial_n Q)$$

is quasi-isomorphic to $K(M, Q)$, where $M = P_0/(\partial_1 Q, \ldots, \partial_n Q)$. Therefore, in high even degrees $2r$, the Hochschild cohomology of $S$ is isomorphic to

$$T = k[x_1, \ldots, x_n]/(Q, \partial_1 Q, \ldots, \partial_n Q)$$

as an $S$-module. Since $S$ and $S^e$ are noetherian, this implies that the Hochschild cohomology of $R$ in high even degrees is isomorphic to the localisation $T_m$. Since $R \otimes R$ is noetherian and Gorenstein (cf. Theorem 1.6 of [28]), by Theorem 6.3.4 of [4], the singular Hochschild cohomology of $R$ coincides with Hochschild cohomology in sufficiently high degrees. By Theorem 1.1, the Hochschild cohomology of $S_{dR}(R)$ is isomorphic to the singular Hochschild cohomology of $R$ and thus isomorphic to $T_m$ in high even degrees. Since $R$ is a hypersurface, the dg category $S_{dR}(R)$ is isomorphic, in the homotopy category of dg categories, to the underlying differential $\mathbb{Z}$-graded category of the differential $\mathbb{Z}/2$-graded category of matrix factorizations of $Q$, cf. [9], [24] and Theorem 2.49 of [3]. Thus, it is 2-periodic and so is its Hochschild cohomology. It follows that the zeroth Hochschild cohomology of $S_{dR}(R)$ is isomorphic to $T_m$ as an algebra. The completion functor $\otimes_R \hat{R}$ yields an embedding $S_g(R) \to S_g(A)$ through which $S_g(A)$ identifies with the idempotent completion of the triangulated category $S_g(R)$, cf. Theorem 5.7 of [7]. Therefore, the corresponding dg functor $S_{dR}(R) \to S_{dR}(A)$ induces an equivalence in the derived categories and an isomorphism in Hochschild cohomology. So we find an isomorphism

$$HH^0(S_{dR}(A), S_{dR}(A)) \xrightarrow{\sim} T_m.$$

Since $Q \in k[x_1, \ldots, x_n]^m$ has an isolated singularity at the origin, we have an isomorphism

$$T_m \xrightarrow{\sim} k[[x_1, \ldots, x_n]]/(Q, \partial_1 Q, \ldots, \partial_n Q)$$

with the Tyurina algebra of $A = P/(Q)$. Now by the Mather–Yau theorem [22], more precisely by its formal version [13, Prop. 2.1], in a fixed dimension, the Tyurina algebra determines $A$ up to isomorphism.

Notice that the Hochschild cohomology of the dg category of matrix factorizations considered as a differential $\mathbb{Z}/2$-graded category is different: As shown by Dyckerhoff [7], it is isomorphic to the Milnor algebra $P/(\partial_1 Q, \ldots, \partial_n Q)$ in even degree and vanishes in odd degree.

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