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SINGULAR HOCHSCHILD COHOMOLOGY VIA THE SINGULARITY CATEGORY

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Abstract. We show that the singular Hochschild cohomology (=Tate–Hochschild cohomology) of an algebra A is isomorphic, as a graded algebra, to the Hochschild cohomology of the differential graded enhancement of the singularity category of A. The existence of such an isomorphism is suggested by recent work of Zhengfang Wang.

1. Introduction

Let k be a commutative ring. We write $\otimes$ for $\otimes_k$. Let A be a right noetherian (non commutative) k-algebra projective over k. The *stable derived category* or *singularity category* of A is defined as the Verdier quotient

$$\mathsf{Sg}(A) = \mathcal{D}^b(\mathsf{mod}\, A)/\, \mathsf{per}(A)$$

of the bounded derived category of finitely generated (right) A-modules by the *perfect derived category* $\mathsf{per}(A)$, *i.e.* the full subcategory of complexes quasi-isomorphic to bounded complexes of finitely generated projective modules. It was introduced by Buchweitz in an unpublished manuscript [4] in 1986 and rediscovered, in its scheme-theoretic variant, by Orlov in 2003 [24]. Notice that it vanishes when A is of finite global dimension and thus measures the degree to which A is 'singular', a view confirmed by the results of [24].

Let us suppose that the enveloping algebra $A^e = A \otimes A^{op}$ is also right noetherian. In analogy with Hochschild cohomology, in view of Buchweitz' theory, it is natural to define the *Tate–Hochschild cohomology* or *singular Hochschild cohomology* of A to be the graded algebra with components

$$HH^n_{sg}(A, A) = \mathsf{Hom}_{\mathsf{Sg}(A^e)}(A, \Sigma^n A)\ ,\ n \in \mathbb{Z}\,,$$

where Σ denotes the suspension (=shift) functor. It was studied for example in [10, 2, 23] and more recently in [31, 32, 30, 33, 29, 5]. Wang showed in [31] that, like Hochschild cohomology [11], singular Hochschild cohomology carries a structure of Gerstenhaber algebra. Now recall that the Gerstenhaber algebra structure on Hochschild cohomology is a small part of much richer higher structure on the Hochschild cochain complex $C(A, A)$ itself, namely the structure of a B_∞-algebra in the sense of Getzler–Jones [12, 5.2] given by the brace operations [1, 16]. In [29], Wang improves on [31] by defining a singular Hochschild cochain complex $C_{sg}(A, A)$ and endowing it with a B_∞-structure which in particular yields the Gerstenhaber algebra structure on $HH^*_{sg}(A, A)$.

Using [17] Lowen–Van den Bergh showed in [21, Theorem 4.4.1] that the Hochschild cohomology of A is isomorphic to the Hochschild cohomology of the canonical differential graded (=dg) enhancement of the (bounded or unbounded) derived category of A and that the isomorphism lifts to the B_∞-level (cf. Corollary 7.6 of [26] for a related statement). Together with the complete structural analogy between Hochschild and singular

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Hochschild cohomology described above, this suggests the question whether the singular Hochschild cohomology of A is isomorphic to the Hochschild cohomology of the canonical dg enhancement $\mathsf{Sg}_{dg}(A)$ of the singularity category $\mathsf{Sg}(A)$ (note that such an enhancement exists by the construction of $\mathsf{Sg}(A)$ as a Verdier quotient [19, 6]). Chen–Li–Wang show in [5] that this does hold at the level of Gerstenhaber algebras when A is the radical square zero algebra associated with a finite quiver without sources or sinks. Our main result is the following.

Theorem 1.1. *There is a canonical isomorphism of graded algebras between the singular Hochschild cohomology of A and the Hochschild cohomology of the dg singularity category $\mathsf{Sg}_{dg}(A)$.*

Conjecture 1.2. *The isomorphism of the theorem lifts to an isomorphism*

$$C_{sg}(A, A) \xrightarrow{\sim} C(\mathsf{Sg}_{dg}(A), \mathsf{Sg}_{dg}(A))$$

in the homotopy category of B_∞-algebras.

Notice that the B_∞-structure on Hochschild cohomology of dg categories is preserved (up to quasi-isomorphism) under Morita equivalences, cf. [17].

Let us mention an application of Theorem 1.1 obtained in joint work with Zheng Hua. Suppose that k is algebraically closed of characteristic 0 and let P the power series algebra $k[[x_1, \ldots, x_n]]$.

Theorem 1.3 ([15])**.** *Suppose that $Q \in P$ has an isolated singularity at the origin and $A = P/(Q)$. Then A is determined up to isomorphism by its dimension and the dg singularity category $\mathsf{Sg}_{dg}(A)$.*

In [8, Theorem 8.1], Efimov proves a related but different reconstruction theorem: He shows that if Q is a polynomial, it is determined, up to a formal change of variables, by the differential $\mathbb{Z}/2$-graded endomorphism algebra E of the residue field in the differential $\mathbb{Z}/2$-graded singularity category together with a fixed isomorphism between H^*B and the exterior algebra $\Lambda(k^n)$.

In section 2, we generalize Theorem 1.1 to the non noetherian setting and prove the generalized statement. We comment on a possible lift of this proof to the B_∞-level in section 3. We prove Theorem 1.3 in section 4.

2. Generalization and proof

2.1. **Generalization to the non noetherian case.** We assume that A is an arbitrary k-algebra projective as a k-module. Its singularity category $\mathsf{Sg}(A)$ is defined as the Verdier quotient $\mathcal{H}^{-,b}(\mathsf{proj}\, A)/\mathcal{H}^b(\mathsf{proj}\, A)$ of the homotopy category of right bounded complexes of finitely generated projective A-modules by its full subcategory of bounded complexes of finitely generated projective A-modules. Notice that when A is right noetherian, this is equivalent to the definition given in the introduction.

The (partially) *completed singularity category* $\widehat{\mathsf{Sg}}(A)$ is defined as the Verdier quotient of the bounded derived category $\mathcal{D}^b(\mathsf{Mod}\, A)$ of all right A-modules by its full subcategory consisting of all complexes quasi-isomorphic to bounded complexes of arbitrary projective modules.

Lemma 2.2. *The canonical functor $\mathsf{Sg}(A) \to \widehat{\mathsf{Sg}}(A)$ is fully faithful.*

Proof. Let M be a right bounded complex of finitely generated projective modules with bounded homology and P a bounded complex of arbitrary projective modules. Since the components of M are finitely generated, each morphism $M \to P$ in the derived category factors through a bounded complex P' with finitely generated projective components. This yields the claim. $\surd$

Since we do not assume that A^e is noetherian, the A-bimodule A will not, in general, belong to the singularity category $\mathsf{Sg}(A^e)$. But it always belongs to the completed singularity category $\widehat{\mathsf{Sg}}(A^e)$. We define the singular Hochschild cohomology of A to be the graded algebra with components

$$HH^n_{sg}(A, A) = \mathsf{Hom}_{\widehat{\mathsf{Sg}}(A^e)}(A, \Sigma^n A)\ ,\ n \in \mathbb{Z}.$$

Theorem 2.3. *Even if A^e is non noetherian, there is a canonical isomorphism of graded algebras between the singular Hochschild cohomology of A and the Hochschild cohomology of the dg singularity category $\mathsf{Sg}_{dg}(A)$.*

Let P be a right bounded complex of projective A^e-modules. For $q \in \mathbb{Z}$, let $\sigma_{>q}P$ and $\sigma_{\leq q}P$ denote its stupid truncations:

$$\sigma_{>q}P:\ \ldots \longrightarrow 0 \longrightarrow P^{q+1} \longrightarrow P^{q+1} \longrightarrow \ldots$$

$$\sigma_{\leq q}P:\ \ldots \longrightarrow P^{q-1} \longrightarrow P^{q} \longrightarrow 0 \longrightarrow \ldots$$

so that we have a triangle

$$\sigma_{>q}P \longrightarrow P \longrightarrow \sigma_{\leq q}P \longrightarrow \Sigma\sigma_{>q}P.$$

We have a direct system

$$P = \sigma_{\leq 0}P \longrightarrow \sigma_{\leq -1}P \longrightarrow \sigma_{\leq -2}P \longrightarrow \ldots \longrightarrow P_{\leq q} \longrightarrow \ldots .$$

Lemma 2.4. *Let $L \in \mathcal{D}^b(\mathsf{Mod}\, A^e)$. We have a canonical isomorphism*

$$\mathsf{colim}\ \mathsf{Hom}_{\mathcal{D}A^e}(L, \sigma_{\leq q}P) \xrightarrow{\sim} \mathsf{Hom}_{\widehat{\mathsf{Sg}}(A^e)}(L, P).$$

In particular, if P is a projective resolution of A over A^e, we have

$$\mathsf{colim}\ \mathsf{Hom}_{\mathcal{D}A^e}(A, \Sigma^n\sigma_{\leq q}P) \xrightarrow{\sim} \mathsf{Hom}_{\widehat{\mathsf{Sg}}(A^e)}(A, \Sigma^n A)\ ,\ n \in \mathbb{Z}.$$

Proof. Clearly, if Q is a bounded complex of projective modules, each morphism $Q \to P$ in the derived category $\mathcal{D}A^e$ factors through $\sigma_{>q}P \to P$ for some $q \ll 0$. This shows that the morphisms $P \to \sigma_{\leq q}P$ form a cofinal subcategory in the category of morphisms $P \to P'$ whose cylinder is a bounded complex of projective modules. Whence the claim. $\surd$

2.5. **Proof of Theorem 2.3.** We refer to [18, 20, 27] for foundational material on dg categories. We will follow the terminology of [20] and use the model category structure on the category of dg categories constructed in [25]. For a dg category $\mathcal{A}$, denote by $X \mapsto Y(X)$ the dg Yoneda functor and by $\mathcal{D}\mathcal{A}$ the derived category. We write $\mathcal{A}^e$ for the enveloping dg category $\mathcal{A} \overset{L}{\otimes}_k \mathcal{A}^{op}$ and $I_{\mathcal{A}}$ for the *identity bimodule*

$$I_{\mathcal{A}} : (X, Y) \mapsto \mathcal{A}(X, Y).$$

By definition, the Hochschild cohomology of $\mathcal{A}$ is the graded endomorphism algebra of $I_{\mathcal{A}}$ in the derived category $D(\mathcal{A}^e)$. In the case of the algebra A, the identity bimodule is the A-bimodule A. Recall that if $F : \mathcal{A} \to \mathcal{B}$ is a fully faithful dg functor, the restriction $F_* : \mathcal{D}\mathcal{B} \to \mathcal{D}\mathcal{A}$ is a localization functor admitting fully faithful left and right adjoint functors F^* and $F^!$ given respectively by

$$F^* : M \mapsto M \overset{L}{\otimes}_{\mathcal{A}}\, {}_F\mathcal{B} \quad \text{and} \quad F^! : N \mapsto \mathsf{RHom}_{\mathcal{A}}(\mathcal{B}_F, N)\ ,$$

where ${}_F\mathcal{B} = \mathcal{B}(?, F-)$ and $\mathcal{B}_F = \mathcal{B}(F?, -)$.

Let $\mathcal{M}_0 = \mathcal{C}^{-,b}_{dg}(\mathsf{proj}\, A)$ denote the dg category of right bounded complexes of finitely generated projective A-modules with bounded homology. Notice that the morphism complexes of $\mathcal{M}_0$ have terms which involve infinite products of projective A-modules so that in general, the morphism complexes of $\mathcal{M}_0$ will not be cofibrant over k. Let $\mathcal{M} \to \mathcal{M}_0$ be a cofibrant resolution of $\mathcal{M}_0$. We assume, as we may, that the quasi-equivalence $\mathcal{M} \to \mathcal{M}_0$ is the identity on objects. Notice that the morphism complexes of $\mathcal{M}$ are cofibrant over k so that we have $\mathcal{M} \overset{L}{\otimes}_k \mathcal{M}^{op} \xrightarrow{\sim} \mathcal{M} \otimes \mathcal{M}^{op}$. Let $\mathcal{P} \subset \mathcal{M}$ be the full dg subcategory of $\mathcal{M}$ formed by the bounded complexes of finitely generated projective A-modules. Let $\mathcal{S}$ denote the dg quotient $\mathcal{M}/\mathcal{P}$. We assume, as we may, that $\mathcal{S}$ is cofibrant. In the homotopy category of dg categories, we have an isomorphism between $\mathsf{Sg}_{dg}(A)$ and $\mathcal{S} = \mathcal{M}/\mathcal{P}$. Let B be the dg endomorphism algebra of A considered as an object of $\mathcal{P} \subset \mathcal{M}$. Notice that we have a quasi-isomorphism $B \to A$ and that both B and A are cofibrant over k. We view B as a dg category with one object whose endomorphism algebra is B. We have the obvious inclusion and projection dg functors

$$B \xrightarrow{i} \mathcal{M} \xrightarrow{p} \mathcal{S}.$$

Consider the fully faithful dg functors

$$B \otimes B^{op} \xrightarrow{\mathbf{1}\otimes i} B \otimes \mathcal{M}^{op} \xrightarrow{i\otimes \mathbf{1}} \mathcal{M} \otimes \mathcal{M}^{op} .$$

The restriction along $G = \mathbf{1} \otimes i$ admits the left adjoint G^* given by

$$G^* : X \mapsto \mathcal{M}_i \overset{L}{\otimes}_B X ,$$

and the restriction along $F = i \otimes \mathbf{1}$ admits the fully faithful left and right adjoints F^* and $F^!$ given by

$$F^* : Y \mapsto Y \overset{L}{\otimes}_B {}_i\mathcal{M} \quad \text{and} \quad F^! : Y \mapsto \mathsf{RHom}_B(\mathcal{M}_i, Y).$$

Since F^* and $F^!$ are the two adjoints of a localization functor, we have a canonical morphism $F^* \to F^!$.

Lemma 2.6. *If P is an arbitrary sum of copies of B^e, the morphism*

$$F^*G^*(P) \to F^!G^*(P)$$

is invertible.

Proof. Let P be the direct sum of copies of B^e indexed by a set J. Since F^* and G^* commute with (arbitrary) coproducts, the left hand side is the dg module

$$\bigoplus_J \mathcal{M}(i?, -) \overset{L}{\otimes}_B (B \otimes B) \overset{L}{\otimes}_B \mathcal{M}(?, i-) = \bigoplus_J \mathcal{M}(B, -) \otimes \mathcal{M}(?, B),$$

The right hand side is the dg module

$$\mathsf{RHom}_B(\mathcal{M}_i, \mathcal{M}_i \overset{L}{\otimes}_B (\bigoplus_J B \otimes B)) = \mathsf{RHom}_B(\mathcal{M}_i, \bigoplus_J \mathcal{M}(B, -) \otimes B).$$

Let us evaluate the canonical morphism at $(M, L) \in \mathcal{M} \otimes \mathcal{M}^{op}$. We find the canonical morphism

$$\bigoplus_J \mathcal{M}(B, L) \otimes \mathcal{M}(M, B) \to \mathsf{RHom}_B(\mathcal{M}(B, M), \bigoplus_j \mathcal{M}(B, L) \otimes B).$$

We have quasi-isomorphisms

$$\mathcal{M}(B, L) \otimes \mathcal{M}(M, B) \to \mathcal{M}_0(A, L) \otimes \mathcal{M}(M, B) \to L \otimes \mathcal{M}(M, B) \to L \otimes \mathsf{Hom}_A(M, A)$$

because $\mathcal{M}(M,B)$ and L are cofibrant over k. Now the equivalence $\mathcal{D}(B) \xrightarrow{\sim} \mathcal{D}(A)$ takes $\mathcal{M}(B,L)\otimes B$ to $\mathcal{M}(B,L)\otimes A \xrightarrow{\sim} L\otimes A$. We have an quasi-isomorphism of dg B-modules $\mathcal{M}(B,M) \xrightarrow{\sim} \mathcal{M}_0(A,M) = M$ and so the equivalence $\mathcal{D}(B) \xrightarrow{\sim} \mathcal{D}(A)$ takes $\mathcal{M}(B,M)$ to M. Whence an isomorphism

$$\mathsf{RHom}_B(\mathcal{M}(B,M), \bigoplus_J \mathcal{M}(B,L)\otimes B) \xrightarrow{\sim} \mathsf{RHom}_A(M, \bigoplus_J L\otimes A) = \mathsf{Hom}_A(M, \bigoplus_J L\otimes A).$$

Thus, we have to show that the canonical morphism

$$\bigoplus_J L\otimes \mathsf{Hom}_A(M,A) \to \mathsf{Hom}_A(M, \bigoplus_J L\otimes A)$$

is a quasi-isomorphism. Recall that L and M are right bounded complexes of finitely generated projective modules with bounded homology. We fix M and consider the morphism as a morphism of triangle functors with argument $L \in \mathcal{D}^b(\mathsf{Mod}\, A)$. Then we are reduced to the case where L is in $\mathsf{Mod}\, A$. In this case, the morphism becomes an isomorphism of complexes because the components of M are finitely generated projective. $\surd$

Let us put $H = F^! G^* : \mathcal{D}(B^e) \to \mathcal{D}(\mathcal{M}^e)$. Let us compute the image of the identity bimodule B under H. We have

$$H(B) = F^!(\mathcal{M}_i \overset{L}{\otimes}_B B) = F^!(\mathcal{M}_i) = \mathsf{RHom}_B(\mathcal{M}_i, \mathcal{M}_i)$$

and when we evaluate at L, M in $\mathcal{M}$, we find

$$H(B)(L,M) = \mathsf{RHom}_B(\mathcal{M}(i?,L), \mathcal{M}(i?,M)) = \mathsf{RHom}_B(\mathcal{M}(B,L), \mathcal{M}(B,M)).$$

We have seen in the above proof that the equivalence $\mathcal{D}(B) \xrightarrow{\sim} \mathcal{D}(A)$ takes $\mathcal{M}(B,L)$ to L. Whence quasi-isomorphisms

$$\begin{aligned} H(B)(L,M) = \mathsf{RHom}_B(\mathcal{M}(B,L), \mathcal{M}(B,M)) &\xrightarrow{\sim} \mathsf{RHom}_A(L,M) = \mathsf{Hom}(L,M) \\ &\xleftarrow{\sim} \mathcal{M}(L,M). \end{aligned}$$

Thus, the functor H takes the identity bimodule B to the identity bimodule $I_{\mathcal{M}}$. Since $F^!$ and G^* are fully faithful so is H. Denote by $\mathcal{N}$ the image under the composition of H with $\mathcal{D}(A^e) \xrightarrow{\sim} \mathcal{D}(B^e)$ of the closure of $\mathsf{Proj}\, A^e$ under finite extensions. Then H yields a fully faithful functor

$$\widehat{\mathsf{Sg}}(A^e) \to \mathcal{D}(\mathcal{M}^e)/\mathcal{N}$$

taking the bimodule A to the identity bimodule $I_{\mathcal{M}}$. Now notice that we have a Morita morphism of dg categories

$$\mathcal{S}^e \xleftarrow{\sim} \frac{\mathcal{M}\otimes\mathcal{M}^{op}}{\mathcal{P}\otimes\mathcal{M}^{op} + \mathcal{M}\otimes\mathcal{P}^{op}}.$$

The functor $p^* : \mathcal{D}(\mathcal{M}^e) \to \mathcal{D}(\mathcal{S}^e)$ induces the quotient functor

$$\frac{\mathcal{D}(\mathcal{M}\otimes\mathcal{M}^{op})}{\mathcal{N}} \longrightarrow \frac{\mathcal{D}(\mathcal{M}\otimes\mathcal{M}^{op})}{\mathcal{D}(\mathcal{P}\otimes\mathcal{M}^{op}+\mathcal{M}\otimes\mathcal{P}^{op})} = \mathcal{D}(\mathcal{S}^e)\ .$$

Since $p : \mathcal{M} \to \mathcal{S}$ is a localization, the image $p^*(I_{\mathcal{M}})$ is isomorphic to $I_{\mathcal{S}}$. It suffices to show that p^* induces bijections in the morphism spaces with target $I_{\mathcal{M}}$

$$\mathsf{Hom}_{\mathcal{D}(\mathcal{M}^e)/\mathcal{N}}(?, I_{\mathcal{M}}) \longrightarrow \mathsf{Hom}_{\mathcal{D}(\mathcal{S}^e)}(p^*(?), p^*(I_{\mathcal{M}}))\ .$$

For this, it suffices to show that $I_{\mathcal{M}}$ is right orthogonal in $\mathcal{D}(\mathcal{M}^e)/\mathcal{N}$ on the images under the Yoneda functor of the objects in $\mathcal{P}\otimes\mathcal{M}^{op}+\mathcal{M}\otimes\mathcal{P}^{op}$. To show that $I_{\mathcal{M}}$ is right orthogonal on $Y(\mathcal{M}\otimes\mathcal{P}^{op})$, it suffices to show that it is right orthogonal to an object $Y(M,B)$, $M\in\mathcal{M}$. Now a morphism in $\mathcal{D}(\mathcal{M}^e)/\mathcal{N}$ is given by a diagram of $\mathcal{D}(\mathcal{M}^e)$ representing a left fraction

$$Y(M,B)\longrightarrow I'_{\mathcal{M}}\longleftarrow I_{\mathcal{M}}$$

where the cone over $I_{\mathcal{M}}\to I'_{\mathcal{M}}$ lies in $\mathcal{N}$. For each object X of $\mathcal{D}\mathcal{M}^e$, we have canonical isomorphisms

$$\mathsf{Hom}_{\mathcal{D}\mathcal{M}^e}(Y(M,B),X)=H^0(X(M,B))=\mathsf{Hom}_{\mathcal{D}\mathcal{M}}(Y(M),X(?,B)).$$

Thus, the given fraction corresponds to a diagram in $\mathcal{D}(\mathcal{M})$ of the form

$$Y(M)\longrightarrow I'_{\mathcal{M}}(?,B)\longleftarrow I_{\mathcal{M}}(?,B)=\mathcal{M}(?,B)\ ,$$

where the cone over $I_{\mathcal{M}}(?,B)\to I'_{\mathcal{M}}(?,B)$ is the image under $\mathcal{D}A\xrightarrow{\sim}\mathcal{D}B\to\mathcal{D}\mathcal{M}$ of a bounded complex with projective components. Thus, the object $I'_{\mathcal{M}}(?,B)$ is a direct factor of a finite extension of shifts of arbitrary coproducts B. Since $Y(M)$ is compact, the given morphism $Y(M)\to I'_{\mathcal{M}}(?,M)$ must then factor through $Y(Q)$ for an object Q of $\mathcal{P}$. This means that the given morphism $Y(M,B)\to I'_{\mathcal{M}}$ factors through $Y(Q,B)$, which lies in $\mathcal{N}$. Thus, the given fraction represents the zero morphism of $\mathcal{D}(\mathcal{M}^e)/\mathcal{N}$, as was to be shown. The case of an object in $Y(\mathcal{P}\otimes\mathcal{M}^{op})$ is analogous. In summary, we have shown that the maps

$$\widehat{\mathsf{Sg}}(A^e)(A,\Sigma^nA)\xrightarrow{H}(\mathcal{D}(\mathcal{M}^e)/\mathcal{N})(I_{\mathcal{M}},\Sigma^nI_{\mathcal{M}})\xrightarrow{p^*}\mathcal{D}(\mathcal{S}^e)(I_{\mathcal{S}},\Sigma^nI_{\mathcal{S}})$$

are bijective, which implies the assertion on Hochschild cohomology.

3. Remark on a possible lift to the B_∞-level

Let $P\to A$ be a resolution of A by projective A-A-bimodules. Let us assume for simplicity that k is a field so that we can take $\mathcal{M}=\mathcal{M}_0$ and $B=A$. The proof in section 2 produces in fact isomorphisms in the derived category of k-modules

$$\begin{aligned}\mathsf{colim}\,\mathsf{RHom}_{A^e}(A,\sigma_{\leq q}P)&\to\mathsf{colim}\,\mathsf{RHom}_{\mathcal{M}^e}(I_{\mathcal{M}},H\sigma_{\leq q}P)\\&\to\mathsf{colim}\,\mathsf{RHom}_{\mathcal{S}^e}(I_{\mathcal{S}},p^*H\sigma_{\leq q}P)\\&=\mathsf{RHom}_{\mathcal{S}^e}(I_{\mathcal{S}},I_{\mathcal{S}}).\end{aligned}$$

For the bar resolution P, the truncation $\sigma_{\leq -q}P$ is canonically isomorphic to $\Sigma^q\Omega^qA$ so that the first complex carries a canonical B_∞-structure constructed by Wang [29]. As explained in the introduction, it is classical that the last complex carries a canonical B_∞-structure. It is not obvious to make the intermediate complexes explicit because the functor H, being a composition of a *right adjoint* with a left adjoint to a restriction functor, does not take cofibrant objects to cofibrant objects.

4. Proof of Theorem 1.3

By the Weierstrass preparation theorem, we may assume that Q is a polynomial. Let $P_0=k[x_1,\ldots,x_n]$ and $S=P_0/(Q)$. Then S has isolated singularities but may have singularities other than the origin. Let $\mathfrak{m}$ be the maximal ideal of P_0 generated by the x_i and let R be the localization of S at $\mathfrak{m}$. Now R is local with an isolated singularity at $\mathfrak{m}$ and A is isomorphic to the completion $\widehat{R}$. By Theorem 3.2.7 of [14], in sufficiently high degrees r, the Hochschild cohomology of S is isomorphic to the homology in degree r of the complex

$$k[u]\otimes K(S,\partial_1Q,\ldots,\partial_nQ)\ ,$$

where u is of degree 2 and K denotes the Koszul complex. Now S is isomorphic to $K(P_0, Q)$ and so $K(S, \partial_1 Q, \dots, \partial_n Q)$ is isomorphic to

$$K(P_0, Q, \partial_1 Q, \dots, \partial_n Q).$$

Since Q has isolated singularities, the $\partial_i Q$ form a regular sequence in P_0. So

$$K(P_0, Q, \partial_1 Q, \dots, \partial_n Q)$$

is quasi-isomorphic to $K(M, Q)$, where $M = P_0/(\partial_1 Q, \dots, \partial_n Q)$. Therefore, in high even degrees $2r$, the Hochschild cohomology of S is isomorphic to

$$T = k[x_1, \dots, x_n]/(Q, \partial_1 Q, \dots, \partial_n Q)$$

as an S-module. Since S and S^e are noetherian, this implies that the Hochschild cohomology of R in high even degrees is isomorphic to the localisation $T_{\mathfrak{m}}$. Since $R \otimes R$ is noetherian and Gorenstein (cf. Theorem 1.6 of [28]), by Theorem 6.3.4 of [4], the singular Hochschild cohomology of R coincides with Hochschild cohomology in sufficiently high degrees. By Theorem 1.1, the Hochschild cohomology of $\mathsf{Sg}_{dg}(R)$ is isomorphic to the singular Hochschild cohomology of R and thus isomorphic to $T_{\mathfrak{m}}$ in high even degrees. Since R is a hypersurface, the dg category $\mathsf{Sg}_{dg}(R)$ is isomorphic, in the homotopy category of dg categories, to the underlying differential $\mathbb{Z}$-graded category of the differential $\mathbb{Z}/2$-graded category of matrix factorizations of Q, cf. [9], [24] and Theorem 2.49 of [3]. Thus, it is 2-periodic and so is its Hochschild cohomology. It follows that the zeroth Hochschild cohomology of $\mathsf{Sg}_{dg}(R)$ is isomorphic to $T_{\mathfrak{m}}$ as an algebra. The completion functor $? \otimes_R \widehat{R}$ yields an embedding $\mathsf{Sg}(R) \to \mathsf{Sg}(A)$ through which $\mathsf{Sg}(A)$ identifies with the idempotent completion of the triangulated category $\mathsf{Sg}(R)$, cf. Theorem 5.7 of [7]. Therefore, the corresponding dg functor $\mathsf{Sg}_{dg}(R) \to \mathsf{Sg}_{dg}(A)$ induces an equivalence in the derived categories and an isomorphism in Hochschild cohomology. So we find an isomorphism

$$HH^0(\mathsf{Sg}_{dg}(A), \mathsf{Sg}_{dg}(A)) \xrightarrow{\sim} T_{\mathfrak{m}}.$$

Since $Q \in k[x_1, \dots, x_n]_{\mathfrak{m}}$ has an isolated singularity at the origin, we have an isomorphism

$$T_{\mathfrak{m}} \xrightarrow{\sim} k[[x_1, \dots, x_n]]/(Q, \partial_1 Q, \dots, \partial_n Q)$$

with the Tyurina algebra of $A = P/(Q)$. Now by the Mather–Yau theorem [22], more precisely by its formal version [13, Prop. 2.1], in a fixed dimension, the Tyurina algebra determines A up to isomorphism.

Notice that the Hochschild cohomology of the dg category of matrix factorizations considered as a differential $\mathbb{Z}/2$-graded category is different: As shown by Dyckerhoff [7], it is isomorphic to the Milnor algebra $P/(\partial_1 Q, \dots, \partial_n Q)$ in even degree and vanishes in odd degree.

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