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Abstract

We show that the intrinsic equations of Koiter's model of a linearly elastic shell can be derived from the intrinsic formulation of the three-dimensional equations of a linearly elastic shell, by using an appropriate a priori assumption regarding the three-dimensional strain tensor fields appearing in these equations. To this end, we recast in particular the Dirichlet boundary conditions satisfied by any admissible displacement field as boundary conditions satisfied by the covariant components of the corresponding strain tensor field expressed in the natural curvilinear coordinates of the shell. Then we show that, when restricted to strain tensor fields satisfying a specific a priori assumption, these new boundary conditions reduce to those of the intrinsic equations of Koiter's model of a linearly elastic shell. *To cite this article: P.G. Ciarlet, C. Mardare, C. R. Acad. Sci. Paris, Ser. I 3xx (20xx).*

Résumé

Justification asymptotique des équations intrinsèques du modèle de coques linéairement élastiques de Koiter. Nous établissons que les équations intrinsèques du modèle de coques linéairement élastiques de Koiter peuvent étre déduites de la formulation intrinsèque des équations tridimensionnelles d'une coque linéairement élastique en faisant une hypothèse a priori appropriée sur les champs de tenseurs de déformation tridimensionnels apparaissant dans ces équations. A cette fin, nous reformulons en particulier les conditions au bord de Dirichlet satisfaites par tout champ de déplacements admissible comme des conditions au bord satisfaites par les composantes covariantes du champ de tenseurs de déformations exprimées en fonction des coordonnées curvilignes naturelles de la coque. Nous montrons ensuite que, lorsqu'elles sont restreintes aux champs de tenseurs de déformations satisfaisant une hypothèse a priori spécifique, les nouvelles conditions au bord se ramènent à celles des équations intrinsèques du modèle de coques linéairement élastiques de Koiter. *Pour citer cet article : P.G. Ciarlet, C. Mardare, C. R. Acad. Sci. Paris, Ser. I 3xx (20xx).*

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1. Geometry of the reference configuration of a shell

Greek indices and exponents vary in the set $\{1, 2\}$, Latin indices and exponents vary in the set $\{1, 2, 3\}$, and the summation convention for repeated indices and exponents is used. The three-dimensional Euclidean space is denoted $\mathbb{E}^3$ and the inner product, the vector product, and the norm, in $\mathbb{E}^3$ are respectively denoted $\cdot$, $\wedge$, and $|\cdot|$. Given any integer $n > 1$, the space of all real $n \times n$ symmetric matrices is denoted $\mathbb{S}^n$. Given any open subset Ω of $\mathbb{R}^n$, $n \geq 1$, and any integer $m \geq 0$, the notation $\mathcal{C}^m(\overline{\Omega}; \mathbb{E}^3)$ denotes the space of vector-valued fields in $\mathbb{E}^3$ with components in $\mathcal{C}^m(\overline{\Omega})$. Similar definitions hold for the spaces $\mathcal{C}^m(\overline{\Omega}; \mathbb{S}^n)$ and $H^1(\Omega; \mathbb{R}^3)$. A generic point in $\mathbb{R}^2$ is denoted $y = (y_\alpha)$ and partial derivatives of the first and second order are denoted $\partial_\alpha := \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$.

Let $\omega \subset \mathbb{R}^2$ be a non-empty connected open set whose boundary is of class $\mathcal{C}^3$ (in the sense of [10]), and let $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{E}^3$ be an *immersion* of class $\mathcal{C}^4$, that is, a mapping $\boldsymbol{\theta} \in \mathcal{C}^4(\overline{\omega}; \mathbb{E}^3)$ such that the two vector fields

$$\boldsymbol{a}_\alpha := \partial_\alpha \boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$$

are linearly independent at each point $y \in \overline{\omega}$. Then $S = \boldsymbol{\theta}(\overline{\omega})$ is a *surface with boundary* in $\mathbb{E}^3$,

$$\boldsymbol{a}_3 := \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$$

is a unit normal vector field along S, the three vector fields $\boldsymbol{a}_i$ form the *covariant bases* along S, and the three vector fields $\boldsymbol{a}^i$, defined by the relations

$$\boldsymbol{a}^i \cdot \boldsymbol{a}_j = \delta^i_j \text{ in } \overline{\omega},$$

form the *contravariant bases* along S.

The covariant and contravariant components of the *first fundamental form* associated with the immersion $\boldsymbol{\theta}$ are respectively denoted and defined by

$$a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta \in \mathcal{C}^3(\overline{\omega}) \quad \text{and} \quad a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta \in \mathcal{C}^3(\overline{\omega}),$$

the covariant and mixed components of the *second fundamental form* associated with the immersion $\boldsymbol{\theta}$ are respectively denoted and defined by

$$b_{\alpha\beta} := \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 \in \mathcal{C}^2(\overline{\omega}) \quad \text{and} \quad b^\alpha_\beta := a^{\alpha\sigma} b_{\sigma\beta} \in \mathcal{C}^2(\overline{\omega}),$$

the *Christoffel symbols* (of the second kind) associated with the immersion $\boldsymbol{\theta}$ are denoted and defined by

$$\Gamma^\sigma_{\alpha\beta} := \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}^\sigma \in \mathcal{C}^2(\overline{\omega}),$$

the mixed components of the *Riemann curvature tensor field* associated with the immersion $\boldsymbol{\theta}$ are denoted and defined by

$$R^{\psi}_{\cdot\alpha\sigma\varphi} := \partial_\sigma \Gamma^\psi_{\alpha\varphi} - \partial_\varphi \Gamma^\psi_{\alpha\sigma} + \Gamma^\beta_{\alpha\varphi}\Gamma^\psi_{\beta\sigma} - \Gamma^\beta_{\alpha\sigma}\Gamma^\psi_{\beta\varphi} \in \mathcal{C}^1(\overline{\omega}),$$

and the *area element* along the surface S is denoted and defined by $\sqrt{a}\, \mathrm{d}y$, where

$$a := \det(a_{\alpha\beta}) \in \mathcal{C}^3(\overline{\omega}).$$

The above assumptions on ω and $\boldsymbol{\theta}$ imply that the boundary $\gamma := \partial\omega$ of ω, resp. the boundary $\boldsymbol{\theta}(\gamma)$ of S, is a curve, or a finite union of curves if γ is not connected, of class $\mathcal{C}^3$ in $\mathbb{R}^2$, resp. in $\mathbb{E}^3$. For definiteness, these curves are oriented by the *inner* normal vector field to the boundary of ω; thus, if

$$\boldsymbol{\nu}(y) := \nu_\alpha(y)\boldsymbol{a}^\alpha(y) = \nu^\alpha(y)\boldsymbol{a}_\alpha(y) \in \mathbb{E}^3$$

designates the unique *unit normal vector* to the curve $\boldsymbol{\theta}(\gamma)$ at the point $\boldsymbol{\theta}(y)$ that is contained in the tangent plane to S at $\boldsymbol{\theta}(y)$ and whose orientation is such that its covariant components

$$(\nu_\alpha(y)) \in \mathbb{R}^2$$

form an *inner* normal vector to the curve γ, then

$$(\tau^\alpha(y)) \in \mathbb{R}^2, \text{ where } \tau^1(y) := \nu_2(y) \text{ and } \tau^2(y) := -\nu_1(y)$$

is a positively-oriented *tangent vector* to the curve γ at $y \in \gamma$, and

$$\boldsymbol{\tau}(y) := \tau^\alpha(y)\boldsymbol{a}_\alpha(y) \in \mathbb{E}^3$$

is the positively-oriented *unit tangent vector* to the curve $\boldsymbol{\theta}(\gamma)$ at $\boldsymbol{\theta}(y)$.

Then the three vectors

$$\boldsymbol{\tau}(y),\ \boldsymbol{\nu}(y),\ \boldsymbol{a}_3(y)$$

form the *Darboux frame* at the point $\boldsymbol{\theta}(y)$, $y \in \gamma$, of the curve $\boldsymbol{\theta}(\gamma)$, and the three scalars

$$\begin{aligned}
\kappa_g(y) &:= \partial_\tau \boldsymbol{\tau}(y) \cdot \boldsymbol{\nu}(y) = -\boldsymbol{\tau}(y) \cdot \partial_\tau \boldsymbol{\nu}(y),\\
\kappa_n(y) &:= \partial_\tau \boldsymbol{\tau}(y) \cdot \boldsymbol{a}_3(y) = -\boldsymbol{\tau}(y) \cdot \partial_\tau \boldsymbol{a}_3(y),\\
\tau_g(y) &:= \partial_\tau \boldsymbol{\nu}(y) \cdot \boldsymbol{a}_3(y) = -\boldsymbol{\nu}(y) \cdot \partial_\tau \boldsymbol{a}_3(y),
\end{aligned}$$

where the notation $\partial_\tau \boldsymbol{\tau}(y)$ denotes the derivative at $\boldsymbol{\theta}(y)$ of the vector field $\boldsymbol{\tau} \circ \boldsymbol{\theta}^{-1}$ with respect to the arclength abscissa along the curve $\boldsymbol{\theta}(\gamma)$, respectively designate the *geodesic curvature*, the *normal curvature*, and the *geodesic torsion*, of the curve $\boldsymbol{\theta}(\gamma)$ at $\boldsymbol{\theta}(y)$.

Let $\varepsilon > 0$ be a small enough parameter, so that the extension $\boldsymbol{\Theta} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{E}^3)$ of the immersion $\boldsymbol{\theta} \in \mathcal{C}^4(\overline{\omega}; \mathbb{E}^3)$ to the three-dimensional domain $\overline{\Omega} \subset \mathbb{R}^3$, defined by

$$\boldsymbol{\Theta}(x) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ for all } x = (y, x_3) \in \overline{\Omega}, \text{ where } \Omega := \omega \times]-\varepsilon, \varepsilon[,$$

is itself an *immersion* at each point $x \in \overline{\Omega}$ (see Theorem 4.1-1 in [2] for the proof of the existence of such a parameter ε). Let the notation ∂_i designate the partial differential operators $\partial_\alpha := \partial/\partial y_\alpha$ for $i = \alpha$ and $\partial_3 := \partial/\partial x_3$ for $i = 3$. Then, for each $x \in \overline{\Omega}$, the three vectors

$$\boldsymbol{g}_i(x) := \partial_i \boldsymbol{\Theta}(x)$$

form the *covariant basis* at $\boldsymbol{\Theta}(x) \in \mathbb{E}^3$. Its dual basis is formed by the three vectors $\boldsymbol{g}^i(x) \in \mathbb{E}^3$, which are defined as the unique solution to the equations

$$\boldsymbol{g}^i(x) \cdot \boldsymbol{g}_j(x) = \delta^i_j,$$

and which form the *contravariant basis* at $\boldsymbol{\Theta}(x) \in \mathbb{E}^3$.

The covariant and contravariant components of the *metric tensor field* associated with the immersion $\boldsymbol{\Theta}$ are respectively denoted and defined by

$$g_{ij} := \boldsymbol{g}_i \cdot \boldsymbol{g}_j \in \mathcal{C}^2(\overline{\Omega}) \text{ and } g^{ij} := \boldsymbol{g}^i \cdot \boldsymbol{g}^j \in \mathcal{C}^2(\overline{\Omega}),$$

the *Christoffel symbols* (of the second kind) associated with the immersion $\boldsymbol{\Theta}$ are denoted and defined by

$$G^k_{ij} := \partial_i \boldsymbol{g}_j \cdot \boldsymbol{g}^k \in \mathcal{C}^1(\overline{\Omega}),$$

and the *volume element* inside the three-dimensional manifold $\boldsymbol{\Theta}(\overline{\Omega})$ is denoted and defined by $\sqrt{g}\,\mathrm{d}x$, where

$$g := \det(g_{ij}) \in \mathcal{C}^2(\overline{\Omega}).$$

The image $\boldsymbol{\Theta}(\Gamma) \subset \mathbb{E}^3$ by the immersion $\boldsymbol{\Theta}$ of the lateral face $\Gamma := \gamma \times]-\varepsilon, \varepsilon[$ of the cylinder Ω is a surface, or a finite union of surfaces if γ is not connected, of class $\mathcal{C}^3$ in $\mathbb{E}^3$. The *tangent plane* to the surface $\boldsymbol{\Theta}(\Gamma)$ at each point $\boldsymbol{\Theta}(x)$, $x = (y, x_3) \in \Gamma$, is spanned by the two vectors

$$\boldsymbol{g}_3(x) := \partial_3 \boldsymbol{\Theta}(x) \text{ and } \boldsymbol{t}(x) := t^\alpha(x) \boldsymbol{g}_\alpha(x),$$

where the coefficients $t^\alpha(x)$ are defined by

$$t^\alpha(x) := \tau^\alpha(y) \text{ for all } x = (y, x_3) \in \Gamma.$$

Then

$$\boldsymbol{n}(x) = n^\alpha(x)\boldsymbol{g}_\alpha(x) := \frac{\boldsymbol{g}_3(x) \wedge \boldsymbol{t}(x)}{|\boldsymbol{g}_3(x) \wedge \boldsymbol{t}(x)|} \in \mathbb{E}^3$$

is a *unit normal vector* at the point $\boldsymbol{\Theta}(x)$ to the surface $\boldsymbol{\Theta}(\Gamma)$, oriented is such a way that the three vectors

$$\boldsymbol{t}(x),\ \boldsymbol{n}(x),\ \boldsymbol{g}_3(x),$$

form in this order a positively-oriented basis in $\mathbb{E}^3$. Note that this basis can be seen as an extension of the Darboux frames associated with the curve γ to frames along the surface Γ, since

$$\boldsymbol{t}(y,0) = \boldsymbol{\tau}(y),\ \boldsymbol{n}(y,0) = \boldsymbol{\nu}(y), \text{ and } \boldsymbol{g}_3(y,0) = \boldsymbol{a}_3(y), \text{ for all } y \in \gamma.$$

In the rest of this Note, we consider a *shell* with *reference configuration* $\boldsymbol{\Theta}(\overline{\Omega})$, assumed to be a *natural state* (i.e., stress-free), whose *middle surface* $S = \boldsymbol{\theta}(\overline{\omega})$ and (constant) *thickness* $2\varepsilon > 0$ satisfy the above assumptions. We assume that the shell is made of a *linearly elastic material* with *Lamé constants*

$$\lambda \geq 0 \text{ and } \mu > 0,$$

and that it is subjected to a *homogeneous boundary condition of place* on a portion $\boldsymbol{\Theta}(\Gamma_0)$ of its lateral face, where $\Gamma_0 := \gamma_0 \times]-\varepsilon, \varepsilon[$ and $\gamma_0 \subset \gamma$ is a *non-empty relatively open subset of the boundary of* ω. Finally, we assume that the shell is subjected to applied *body forces* whose densities per unit volume in the reference configuration is a vector field

$$f^i \boldsymbol{g}_i : \Omega \to \mathbb{E}^3, \text{ where } f^i \in L^2(\Omega).$$

Note that applied *surface forces* with non-zero densities on the upper and lower faces $\boldsymbol{\Theta}(\omega \times \{+\varepsilon\})$ and $\boldsymbol{\Theta}(\omega \times \{-\varepsilon\})$ of the shell could be also considered, at the expense of minor modifications of the ensuing analysis, but for simplicity they will not be considered here.

2. Classical and intrinsic formulations of Koiter's model of a linearly elastic shell

Consider a linearly elastic shell that satisfies the assumptions of Sect. 1. Define the space

$$\boldsymbol{V}(\omega) := \{\boldsymbol{\eta} := (\eta_i) \in \mathcal{C}^2(\overline{\omega}) \times \mathcal{C}^2(\overline{\omega}) \times \mathcal{C}^3(\overline{\omega});\ \eta_i = \partial_\alpha \eta_3 = 0 \text{ on } \gamma_0\},$$

and the functional $\mathrm{j_K} : \boldsymbol{V}(\omega) \to \mathbb{R}$ by

$$\mathrm{j_K}(\boldsymbol{\eta}) := \int_\omega a^{\alpha\beta\sigma\tau}\Big\{\frac{\varepsilon}{2}\gamma_{\sigma\tau}(\boldsymbol{\eta})\gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{6}\rho_{\sigma\tau}(\boldsymbol{\eta})\rho_{\alpha\beta}(\boldsymbol{\eta})\Big\}\sqrt{a}\,\mathrm{d}y - \int_\omega p^i \eta_i \sqrt{a}\,\mathrm{d}y \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{V}(\omega),$$

where

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}), \qquad p^i := \int_{-\varepsilon}^{\varepsilon} f^i(\cdot, x_3)\,\mathrm{d}x_3,$$

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}\big(\partial_\alpha(\eta_i \boldsymbol{a}^i) \cdot \boldsymbol{a}_\beta + \partial_\beta(\eta_i \boldsymbol{a}^i) \cdot \boldsymbol{a}_\alpha\big), \qquad \rho_{\alpha\beta}(\boldsymbol{\eta}) := \big(\partial_{\alpha\beta}(\eta_i \boldsymbol{a}^i) - \Gamma^\sigma_{\alpha\beta}\partial_\sigma(\eta_i \boldsymbol{a}^i)\big) \cdot \boldsymbol{a}_3,$$

respectively denote the contravariant components of the *two-dimensional elasticity tensor* of the elastic material constituting the shell, the contravariant components of the *density of the resulting applied forces per unit area along the middle surface* S of the shell, and the covariant components of the *linearized change of metric, and of curvature, tensor fields* associated with the displacement field $\eta_i \boldsymbol{a}^i$ of the surface S.

Then, according to the landmark paper by Koiter (Ref. [9]), re-interpreted here in its "modern" formulation, the unknown *displacement field* $\eta_i \boldsymbol{a}^i$ *of the middle surface* of the shell is such that the vector field $\boldsymbol{\eta} = (\eta_i)$ should be the unique minimizer of the extension by continuity of the functional $\mathrm{j_K}$ over the completion of the space $\boldsymbol{V}(\omega)$ with respect to the norm

$$(\eta_i) \in \boldsymbol{V}(\omega) \ \to \ \sum_\alpha \|\eta_\alpha\|_{H^1(\omega)} + \|\eta_3\|_{H^2(\omega)}.$$

Remark 1 The definition of the space $\boldsymbol{V}(\omega)$ as a subspace of the space $\mathcal{C}^2(\overline{\omega}) \times \mathcal{C}^2(\overline{\omega}) \times \mathcal{C}^3(\overline{\omega})$ is a deliberate choice, meant to simplify the ensuing analysis by using classical function spaces, instead of Sobolev spaces as would have been the case had we chosen the space $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$. □

An *intrinsic formulation* of Koiter's model of a linearly elastic shell consists in replacing the above unknown $\boldsymbol{\eta}$ by an appropriate *"measure of strain"*. Since the linear mapping

$$\boldsymbol{\eta} \in \boldsymbol{V}(\omega) \to ((\gamma_{\alpha\beta}(\boldsymbol{\eta})), (\rho_{\alpha\beta}(\boldsymbol{\eta}))) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2) \times \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$$

is *one-to-one*, as a consequence of the well-known *infinitesimal rigid displacement lemma on a surface* (see, e.g., [1]), the above pair of matrix fields provides an instance of such a "measure of strain". More specifically, let

$$\mathbb{V}(\omega) := \{\big((\gamma_{\alpha\beta}(\boldsymbol{\eta})), (\rho_{\alpha\beta}(\boldsymbol{\eta}))\big);\ \boldsymbol{\eta} \in \boldsymbol{V}(\omega)\}$$

denote the image of the space $\boldsymbol{V}(\omega)$ under the linear mapping above. Then the mapping $\mathcal{F}_\omega : \boldsymbol{V}(\omega) \to \mathbb{V}(\omega)$ defined by

$$\mathcal{F}_\omega(\boldsymbol{\eta}) := ((\gamma_{\alpha\beta}(\boldsymbol{\eta})), (\rho_{\alpha\beta}(\boldsymbol{\eta}))) \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{V}(\omega),$$

is *one-to-one*, and *onto*, so that its inverse

$$\mathcal{G}_\omega := \mathcal{F}_\omega^{-1}$$

is well defined. Hence the unknown $\boldsymbol{\eta}$ can be replaced in the classical formulation of Koiter's model of a linearly elastic shell by the pair of matrix fields

$$((c_{\alpha\beta}), (r_{\alpha\beta})) := \mathcal{F}_\omega(\boldsymbol{\eta}).$$

In this fashion, the corresponding *intrinsic formulation* of Koiter's model of a linearly elastic shell asserts that the *new unknown* $((c_{\alpha\beta}), (r_{\alpha\beta})) \in \mathbb{V}(\omega)$ is the unique minimizer of the functional

$$\mathrm{j}_\mathrm{K}^\sharp := \mathrm{j_K} \circ \mathcal{G}_\omega : \mathbb{V}(\omega) \to \mathbb{R}.$$

Of course, just as in the classical formulation, such a minimizer can be found provided the functional $\mathrm{j}_\mathrm{K}^\sharp$ is extended by continuity to the completion of the space $\mathbb{V}(\omega)$ with respect to an appropriate norm.

The functional space $\mathbb{V}(\omega)$ appearing in the above intrinsic formulation of Koiter's model has been *explicitly characterised* by the authors in [7], where it was shown that

$$\mathbb{V}(\omega) = \mathbb{V}^\sharp(\omega),$$

where the space $\mathbb{V}^\sharp(\omega)$ is defined by

$$\begin{aligned}\mathbb{V}^\sharp(\omega) := \big\{((c_{\alpha\beta}), (r_{\alpha\beta})) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2) \times \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2);\ & S_{\beta\alpha\sigma\varphi} = 0 \text{ and } S_{3\alpha\sigma\varphi} = 0 \text{ in } \omega, \\ c_{\alpha\beta}\tau^\alpha\tau^\beta = 0 \ \text{ and } \ c_{\alpha\beta|\sigma}\tau^\alpha \left(2\nu^\beta\tau^\sigma - \tau^\beta\nu^\sigma\right) + \kappa_g c_{\alpha\beta}\nu^\alpha\nu^\beta &= 0 \ \text{ on } \gamma_0, \\ r_{\alpha\beta}\tau^\alpha\tau^\beta = 0 \ \text{ and } \ r_{\alpha\beta}\tau^\alpha\nu^\beta - c_{\alpha\beta}\nu^\alpha\left(2\kappa_n\tau^\beta + \tau_g\nu^\beta\right) &= 0 \ \text{ on } \gamma_0\big\},\end{aligned}$$

the functions $\tau^\alpha, \nu^\alpha, \kappa_g, \kappa_n, \tau_g$ being defined as in Sect. 1 along the curve $\boldsymbol{\theta}(\gamma_0)$, and the distributions $S_{\beta\alpha\sigma\varphi} \in \mathcal{D}'(\omega)$ and $S_{3\alpha\sigma\varphi} \in \mathcal{D}'(\omega)$ being defined in terms of the functions $c_{\alpha\beta}$ and $r_{\alpha\beta}$ by

$$S_{\beta\alpha\sigma\varphi} := c_{\sigma\alpha|\beta\varphi} + c_{\varphi\beta|\alpha\sigma} - c_{\varphi\alpha|\beta\sigma} - c_{\sigma\beta|\alpha\varphi} + R^{\psi}_{\cdot\alpha\sigma\varphi} c_{\beta\psi} - R^{\psi}_{\cdot\beta\sigma\varphi} c_{\alpha\psi} - b_{\varphi\alpha} r_{\sigma\beta} - b_{\sigma\beta} r_{\varphi\alpha} + b_{\sigma\alpha} r_{\varphi\beta} + b_{\varphi\beta} r_{\sigma\alpha},$$
$$S_{3\alpha\sigma\varphi} := b^{\psi}_{\sigma}(c_{\alpha\psi|\varphi} + c_{\varphi\psi|\alpha} - c_{\varphi\alpha|\psi}) - b^{\psi}_{\varphi}(c_{\alpha\psi|\sigma} + c_{\sigma\psi|\alpha} - c_{\sigma\alpha|\psi}) - r_{\sigma\alpha|\varphi} + r_{\varphi\alpha|\sigma},$$

where

$$c_{\alpha\beta|\sigma} := \partial_\sigma c_{\alpha\beta} - \Gamma^{\varphi}_{\alpha\sigma} c_{\varphi\beta} - \Gamma^{\varphi}_{\beta\sigma} c_{\alpha\varphi} \quad \text{and} \quad c_{\alpha\beta|\sigma\varphi} := \partial_\varphi c_{\alpha\beta|\sigma} - \Gamma^{\psi}_{\alpha\varphi} c_{\psi\beta|\sigma} - \Gamma^{\psi}_{\beta\varphi} c_{\alpha\psi|\sigma} - \Gamma^{\psi}_{\sigma\varphi} c_{\alpha\beta|\psi}$$

denote the usual *covariant derivatives* of first and second order of the tensor field $c_{\alpha\beta}$ along the surface S.

Note that the above distributions satisfy the symmetry relations

$$S_{3\alpha\sigma\varphi} = -S_{3\alpha\varphi\sigma} \quad \text{and} \quad S_{\beta\alpha\sigma\varphi} = S_{\sigma\varphi\beta\alpha} = -S_{\sigma\varphi\alpha\beta},$$

which in turn imply that only three of them, e.g. S_{1212}, S_{3112}, and S_{3212}, are independent.

The *main objective* of this Note is to prove that the above characterisation of the space $\mathbb{V}(\omega)$ can be deduced from the intrinsic formulation of the three-dimensional equations of a linearly elastic shell by using an *appropriate a priori assumption regarding the three-dimensional strain tensor fields*, according to which the covariant components e_{i3} of the admissible three-dimensional strain tensor fields must vanish in Ω; see the definition of the space $\mathbb{V}_H(\Omega)$ in Theorem 2.

3. Classical and intrinsic formulations of the three-dimensional equations of a linearly elastic shell

Consider a *linearly elastic shell* that satisfies the assumptions of Sect. 1. Define the space

$$\boldsymbol{V}(\Omega) := \{\boldsymbol{v} := (v_i) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{R}^3);\ v_i = 0 \text{ on } \Gamma_0\}$$

and the functional $\mathcal{J} : \boldsymbol{V}(\Omega) \to \mathbb{R}$ by

$$\mathcal{J}(\boldsymbol{v}) := \frac{1}{2}\int_\Omega A^{ijk\ell} \varepsilon_{k\ell}(\boldsymbol{v}) \varepsilon_{ij}(\boldsymbol{v}) \sqrt{g}\,\mathrm{d}x - \int_\Omega f^i v_i \sqrt{g}\,\mathrm{d}x \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega),$$

where

$$A^{ijk\ell} := \lambda g^{ij} g^{k\ell} + \mu(g^{ik} g^{j\ell} + g^{i\ell} g^{jk}) \quad \text{and} \quad \varepsilon_{ij}(\boldsymbol{v}) := \frac{1}{2}\big(\partial_i(v_k \boldsymbol{g}^k)\cdot \boldsymbol{g}_j + \partial_j(v_k \boldsymbol{g}^k)\cdot \boldsymbol{g}_i\big)$$

respectively denote the contravariant components of the *three-dimensional elasticity tensor field* of the elastic material constituting the shell, and the covariant components of the *linearized change of metric tensor field*, also known as the *linearized strain tensor field*, associated with the *displacement field* $v_i \boldsymbol{g}^i$ of the shell.

Then the *classical formulation* of the three-dimensional equations of a linearly elastic shell in curvilinear coordinates (Ref. [1]) asserts that the unknown *displacement field* $v_i \boldsymbol{g}^i$ *of the shell* is such that the vector field $\boldsymbol{v} = (v_i)$ should be the unique minimizer of the extension by continuity of the functional $\mathcal{J}$ over the completion of the space $\boldsymbol{V}(\Omega)$ with respect to the norm

$$(v_i) \in \boldsymbol{V}(\Omega) \ \to \ \sum_i \|v_i\|_{H^1(\Omega)}.$$

An *intrinsic formulation* of the above equations consists in replacing the above unknown $\boldsymbol{v}$ by an appropriate *"measure of strain"*. Since the linear mapping

$$\boldsymbol{v} \in \boldsymbol{V}(\Omega) \to (\varepsilon_{ij}(\boldsymbol{v})) \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3)$$

is *one-to-one*, as a consequence of the well-known *infinitesimal rigid displacement lemma* (see, e.g., [1]), the above matrix field provides an instance of such a "measure of strain". More specifically, let

$$\mathbb{V}(\Omega) := \{(\varepsilon_{ij}(\boldsymbol{v}))\,;\ \boldsymbol{v} \in \boldsymbol{V}(\Omega)\}$$

denote the image of the space $\boldsymbol{V}(\Omega)$ under the linear mapping above. Then the mapping $\mathcal{F}_\Omega : \boldsymbol{V}(\Omega) \to \mathbb{V}(\Omega)$ defined by

$$\mathcal{F}_\Omega(\boldsymbol{\eta}) := (\varepsilon_{ij}(\boldsymbol{v})) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega),$$

is *one-to-one* and *onto*, so that its inverse

$$\mathcal{G}_\Omega := \mathcal{F}_\Omega^{-1}$$

is well defined. Hence the unknown $\boldsymbol{v}$ can be replaced in the classical formulation of the three-dimensional equations of a linearly elastic shell by the matrix field

$$(e_{ij}) := \mathcal{F}_\Omega(\boldsymbol{v}).$$

In this fashion, the corresponding *intrinsic formulation of the three-dimensional equations of a linearly elastic shell* asserts that the *new unknown* $(e_{ij}) \in \mathbb{V}(\Omega)$ is the unique minimizer of the functional

$$\mathcal{J}^\sharp := \mathcal{J} \circ \mathcal{G}_\Omega : \mathbb{V}(\Omega) \to \mathbb{R}.$$

Of course, just as in the classical formulation, such a minimizer can be found provided the functional $\mathcal{J}^\sharp$ is extended by continuity to the completion of the space $\mathbb{V}(\Omega)$ with respect to an appropriate norm.

The next theorem, which constitutes the *first main result* of this Note, *explicitly characterises* the space $\mathbb{V}(\Omega)$ appearing in the above intrinsic formulation of the three-dimensional equations of a linearly elastic shell in the curvilinear coordinates associated with the immersion $\boldsymbol{\Theta}$ defining the reference configuration of the shell. Note that the functions $t^i\|_j$, $e_{ij\|k}$, and $e_{ij\|hk}$, appearing in this theorem are nothing but the usual *covariant derivatives* of vector and tensor fields (the functions $G^i_{j\ell}$ denote the Christoffel symbols defined in Sect 1).

Theorem 1 *Consider a linearly elastic shell that satisfies the assumptions of* Sect. 1. *Assume in addition that ω is simply-connected and that γ_0 is connected. Define the spaces*

$$\mathbb{V}(\Omega) := \{(e_{ij}) \in \mathcal{C}^1(\overline{\Omega};\mathbb{S}^3);\ e_{ij} = \varepsilon_{ij}(\boldsymbol{v}),\ \boldsymbol{v} = (v_i) \in \mathcal{C}^2(\overline{\Omega};\mathbb{R}^3);\ v_i = 0 \text{ on } \Gamma_0\},$$

$$\begin{aligned}\mathbb{V}^\sharp(\Omega) := \big\{(e_{ij}) \in \mathcal{C}^1(\overline{\Omega};\mathbb{S}^3);\ & e_{ij\|k\ell} + e_{\ell k\|ji} - e_{ik\|j\ell} - e_{\ell j\|ki} = 0 \text{ in } \Omega,\\ & e_{\alpha\beta}t^\alpha t^\beta = e_{\alpha 3}t^\alpha = e_{3\beta}t^\beta = e_{33} = 0 \text{ on } \Gamma_0,\\ & 2e_{\alpha\beta\|\sigma}t^\alpha n^\beta t^\sigma - e_{\alpha\beta\|\sigma}t^\alpha t^\beta n^\sigma + e_{\alpha\beta}n^\alpha n^\beta t^\sigma\|_\varphi n_\sigma t^\varphi = 0 \text{ on } \Gamma_0,\\ & e_{\alpha 3\|\beta}n^\alpha t^\beta + e_{\alpha\beta\|3}t^\alpha n^\beta - e_{\alpha 3\|\beta}t^\alpha n^\beta + e_{\alpha\beta}n^\alpha n^\beta t^\sigma\|_3 n_\sigma = 0 \text{ on } \Gamma_0,\\ & 2e_{\alpha 3\|3}n^\alpha - e_{33\|\alpha}n^\alpha = 0 \text{ on } \Gamma_0\},\end{aligned}$$

where

$$t^i\|_j := \partial_j t^i + G^i_{j\ell}t^\ell, \quad e_{ij\|k} := \partial_k e_{ij} - G^\ell_{ik}e_{\ell j} - G^\ell_{kj}e_{i\ell},$$
$$e_{ij\|hk} := \partial_k e_{ij\|h} - G^\ell_{ik}e_{\ell j\|h} - G^\ell_{jk}e_{i\ell\|h} - G^\ell_{hk}e_{ij\|\ell}.$$

Then

$$\mathbb{V}(\Omega) = \mathbb{V}^\sharp(\Omega).$$

□

Proof. Let $\tilde{\boldsymbol{\theta}} \in \mathcal{C}^3(\tilde{\omega};\mathbb{E}^3)$ be a local chart of $\boldsymbol{\Theta}(\Gamma_0)$ defined over an open subset $\tilde{\omega}$ of $\mathbb{R}^2$ and let $\tilde{\varepsilon} > 0$ be a small enough parameter such that the mapping $\tilde{\boldsymbol{\Theta}} : \tilde{\omega}\times]-\tilde{\varepsilon},\tilde{\varepsilon}[\to \mathbb{E}^3$ defined by

$$\tilde{\boldsymbol{\Theta}}(\tilde{y},\tilde{x}_3) := \tilde{\boldsymbol{\theta}}(\tilde{y}) + \tilde{x}_3\tilde{\boldsymbol{a}}_3(\tilde{y}) \text{ for all } (\tilde{y},\tilde{x}_3) \in \tilde{\omega}\times]-\tilde{\varepsilon},\tilde{\varepsilon}[, \quad \text{where } \tilde{\boldsymbol{a}}_3 := \frac{\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2}{|\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2|} \text{ and } \tilde{\boldsymbol{a}}_\alpha := \frac{\partial\tilde{\boldsymbol{\theta}}}{\partial\tilde{y}_\alpha},$$

is itself an immersion of class $\mathcal{C}^2$ (see again Theorem 4.1-1 in [2]).

A matrix field $(e_{ij}) \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3)$ belongs to the space $\mathbb{V}(\Omega)$ if and only if there exists a vector field $\boldsymbol{v} = (v_i) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{R}^3)$ such that $e_{ij} = \varepsilon_{ij}(\boldsymbol{v})$ in Ω and $v_i = 0$ on Γ_0. Then the equality $\mathbb{V}(\Omega) = \mathbb{V}^\sharp(\Omega)$ will be proved by combining the following three results.

First, by Theorems 5.1 and 6.1 in [4] (which can be applied because Ω is simply-connected, as a consequence of the assumption that ω is simply-connected), a matrix field $(e_{ij}) \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3)$ satisfies $e_{ij} = \varepsilon_{ij}(\boldsymbol{v})$ in $\overline{\Omega}$ for some vector field $\boldsymbol{v} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{R}^3)$ if and only if it satisfies the *Saint Venant compatibility conditions in curvilinear coordinates*, viz.,

$$e_{ij\|k\ell} + e_{\ell k\|ji} - e_{ik\|j\ell} - e_{\ell j\|ki} = 0 \text{ in } \Omega.$$

Second, by Theorem 6.1 in [5] (which can be applied because Γ_0 is connected, as a consequence of the assumption that γ_0 is connected), a vector field $\boldsymbol{v} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{R}^3)$ satisfies the boundary condition

$$\boldsymbol{v} + \boldsymbol{r} = \boldsymbol{0} \text{ on } \Gamma_0 \text{ for some vector field } \boldsymbol{r} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{R}^3) \text{ such that } \varepsilon_{ij}(\boldsymbol{r}) = 0 \text{ in } \Omega$$

if and only if

$$\tilde{e}_{\alpha\beta} = 0 \text{ and } \tilde{e}_{\alpha 3\|\beta} + \tilde{e}_{\beta 3\|\alpha} - \tilde{e}_{\alpha\beta\|3} + \tilde{b}_{\alpha\beta}\tilde{e}_{33} = 0 \text{ on } \tilde{\omega} \times \{0\},$$

where $\tilde{e}_{ij}$ are the covariant components of the tensor field $e_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3)$ associated with the immersion $\tilde{\boldsymbol{\Theta}}$, and $\tilde{b}_{\alpha\beta} \in \mathcal{C}^0(\tilde{\omega})$ are the covariant components of the *second fundamental form* associated with the immersion $\tilde{\boldsymbol{\theta}}$.

Third, a series of long, but otherwise straightforward, computations show that the functions $\tilde{e}_{ij}$ satisfy the above boundary conditions on $\tilde{\omega} \times \{0\}$ if and only if the corresponding functions e_{ij} satisfy the following boundary conditions on Γ_0:

$$\begin{aligned} e_{\alpha\beta}t^\alpha t^\beta = e_{\alpha 3}t^\alpha = e_{3\beta}t^\beta = e_{33} &= 0 \text{ on } \Gamma_0, \\ 2e_{\alpha\beta\|\sigma}t^\alpha n^\beta t^\sigma - e_{\alpha\beta\|\sigma}t^\alpha t^\beta n^\sigma + e_{\alpha\beta}n^\alpha n^\beta t^\sigma\|_\varphi n_\sigma t^\varphi &= 0 \text{ on } \Gamma_0, \\ e_{\alpha 3\|\beta}n^\alpha t^\beta + e_{\alpha\beta\|3}t^\alpha n^\beta - e_{\alpha 3\|\beta}t^\alpha n^\beta + e_{\alpha\beta}n^\alpha n^\beta t^\sigma\|_3 n_\sigma &= 0 \text{ on } \Gamma_0, \\ 2e_{\alpha 3\|3}n^\alpha - e_{33\|\alpha}n^\alpha &= 0 \text{ on } \Gamma_0. \end{aligned}$$

□

Remark 2 The above boundary conditions on Γ_0, that are satisfied by the covariant components e_{ij} of the linearized strain tensor field associated with a displacement field of a *shell*, generalize similar boundary conditions that are satisfied by the covariant components e_{ij} of the linearized strain tensor field associated with a displacement field of a *plate*, identified previously by the authors (see Lemmas 3 and 4 in [6]). To see this, observe that the immersion $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbb{E}^3$ defining the reference configuration of a plate is the identity mapping restricted to the set $\overline{\Omega}$, in which case

$$e_{ij\|k} = \partial_k e_{ij} \quad \text{in } \overline{\Omega},$$

and

$$\begin{aligned} t^\alpha(\cdot, x_3) = \tau^\alpha \quad &\text{and} \quad n^\alpha(\cdot, x_3) = \nu^\alpha \text{ on } \Gamma, \\ t^\sigma\|_3 = 0 \quad \text{and} \quad t^\sigma\|_\varphi n_\sigma t^\varphi &= n_\sigma \partial_\tau t^\sigma = \kappa(n_\sigma n^\sigma) = \kappa \quad \text{on } \Gamma, \end{aligned}$$

where $\kappa : \gamma \to \mathbb{R}$ denotes the *curvature* along the planar curve γ.

4. Canonical isomorphism between the spaces $\mathbb{V}^\sharp(\omega)$ and $\mathbb{V}^\sharp(\Omega)$

Define the linear mappings $\mathcal{H} : \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3) \to \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2) \times \mathcal{C}^0(\overline{\omega}; \mathbb{S}^2)$ and $\mathcal{K} : \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2) \times \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2) \to \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3)$ by letting

$$\mathcal{H}((e_{ij})) := \big((e_{\alpha\beta}(\cdot,0)),(-\partial_3 e_{\alpha\beta}(\cdot,0))\big) \text{ for all } (e_{ij}) \in \mathcal{C}^1(\overline{\Omega};\mathbb{S}^3),$$

$$\mathcal{K}((c_{\alpha\beta}),(r_{\alpha\beta})) := (e_{ij}) \text{ for all } ((c_{\alpha\beta}),(r_{\alpha\beta})) \in \mathcal{C}^1(\overline{\omega};\mathbb{S}^2)\times\mathcal{C}^1(\overline{\omega};\mathbb{S}^2),$$

where

$$e_{\alpha\beta}(\cdot,x_3) := c_{\alpha\beta} - x_3 r_{\alpha\beta} + x_3^2\Big(\frac{1}{2}(b_\alpha^\sigma r_{\sigma\beta} + b_\beta^\sigma r_{\alpha\sigma}) - b_\alpha^\sigma b_\beta^\varphi c_{\sigma\varphi}\Big) \text{ in } \overline{\Omega},$$

$$e_{i3} = e_{3i} := 0 \text{ in } \overline{\Omega}.$$

Note that the mapping $\mathcal{K}$ is *one-to-one* and that the range of the mapping $\mathcal{H}$ contains the space $\mathcal{C}^1(\overline{\omega};\mathbb{S}^2)\times\mathcal{C}^1(\overline{\omega};\mathbb{S}^2)$.

To begin with, we show that there exists a natural isomorphism between the space $\mathbb{V}(\omega)$ found in the *intrinsic formulation of Koiter's model* (Sect 2) and a subspace $\mathbb{V}_H(\Omega) \subset \mathbb{V}(\Omega)$ of the space $\mathbb{V}(\Omega)$ found in the *intrinsic formulation of the three-dimensional equations of a linearly elastic shell* (Sect. 3).

In this respect, note that the definition of the subspace $\mathbb{V}_H(\Omega)$ exactly corresponds to an a priori assumption used by Koiter (Ref. [9]), according to which the covariant components e_{i3} of the three-dimensional strain tensor fields inside a shell must vanish.

Theorem 2 *Consider a linearly elastic shell that satisfies the assumptions of* Sect. 1. *Define the spaces*

$$\mathbb{V}(\omega) := \{((c_{\alpha\beta}),(r_{\alpha\beta})) \in \mathcal{C}^1(\overline{\omega};\mathbb{S}^2)\times\mathcal{C}^1(\overline{\omega};\mathbb{S}^2);\ c_{\alpha\beta} = \gamma_{\alpha\beta}(\boldsymbol{\eta}),\ r_{\alpha\beta} = \rho_{\alpha\beta}(\boldsymbol{\eta}),$$
$$\boldsymbol{\eta} = (\eta_i) \in \mathcal{C}^2(\overline{\omega})\times\mathcal{C}^2(\overline{\omega})\times\mathcal{C}^3(\overline{\omega});\ \eta_i = \partial_\alpha\eta_3 = 0 \text{ on } \gamma_0\},$$

$$\mathbb{V}(\Omega) := \{(e_{ij}) \in \mathcal{C}^1(\overline{\Omega};\mathbb{S}^3);\ e_{ij} = \varepsilon_{ij}(\boldsymbol{v}),\ \boldsymbol{v} = (v_i) \in \mathcal{C}^2(\overline{\Omega};\mathbb{R}^3);\ v_i = 0 \text{ on } \Gamma_0\},$$

$$\mathbb{V}_H(\Omega) := \{(e_{ij}) \in \mathbb{V}(\Omega);\ e_{i3} = 0 \text{ in } \Omega\} \subset \mathcal{C}^1(\overline{\Omega};\mathbb{S}^3).$$

Then the mapping $\mathcal{H} : \mathbb{V}_H(\Omega) \to \mathbb{V}(\omega)$ *is a well-defined bijection and its inverse is the mapping* $\mathcal{K} : \mathbb{V}(\omega) \to \mathbb{V}_H(\Omega)$.

Proof. If ε is small enough, Lemmas 4 and 5 in [8] show that the mapping $\boldsymbol{\mathcal{F}}$ that associates with any vector field (η_i) in the space

$$\{(\eta_i) \in H^1(\omega)\times H^1(\omega)\times H^2(\omega);\ \eta_i = \partial_\alpha\eta_3 = 0 \text{ on } \gamma_0\}$$

the vector field $(v_i) : \overline{\Omega} \to \mathbb{R}^3$ whose components are given by

$$v_\alpha(\cdot,x_3) := \eta_\alpha - x_3(\partial_\alpha\eta_3 + 2b_\alpha^\sigma\eta_\sigma) + x_3^2 b_\alpha^\beta(\partial_\beta\eta_3 + b_\beta^\sigma\eta_\sigma) \text{ and } v_3(\cdot,x_3) := \eta_3 \text{ in } \overline{\Omega},$$

is *one-to-one*, and *onto* the space

$$\{\boldsymbol{v} := (v_i) \in H^1(\Omega;\mathbb{R}^3);\ \varepsilon_{i3}(\boldsymbol{v}) = 0 \text{ in } \Omega,\ v_i = 0 \text{ on } \Gamma_0\}.$$

Given any $(e_{ij}) \in \mathbb{V}_H(\Omega)$, there exists $\boldsymbol{v} = (v_i) \in \mathcal{C}^2(\overline{\Omega};\mathbb{R}^3)$ such that $v_i = 0$ on Γ_0 and $\varepsilon_{i3}(\boldsymbol{v}) = e_{i3} = 0$ in $\overline{\Omega}$. Hence there exists a (unique) vector field $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega)\times H^1(\omega)\times H^2(\omega)$ such that $\eta_i = \partial_\alpha\eta_3 = 0$ on γ_0 and $\boldsymbol{\mathcal{F}}(\boldsymbol{\eta}) = \boldsymbol{v}$. Consequently, $\eta_\alpha \in \mathcal{C}^2(\overline{\omega})$, $\eta_3 \in \mathcal{C}^3(\overline{\omega})$, and

$$\varepsilon_{\alpha\beta}(\boldsymbol{v})(\cdot,x_3) = \gamma_{\alpha\beta}(\boldsymbol{\eta}) - x_3\rho_{\alpha\beta}(\boldsymbol{\eta}) + x_3^2\Big(\frac{1}{2}(b_\alpha^\sigma\rho_{\sigma\beta}(\boldsymbol{\eta}) + b_\beta^\sigma\rho_{\alpha\sigma}(\boldsymbol{\eta})) - b_\alpha^\sigma b_\beta^\varphi c_{\sigma\varphi}\Big) \text{ in } \overline{\Omega},$$

which, combined with the relations $e_{\alpha\beta} = \varepsilon_{\alpha\beta}(\boldsymbol{v})$ in Ω, further implies that $e_{\alpha\beta}(\cdot,0) = \gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\partial_3 e_{\alpha\beta}(\cdot,0) = -\rho_{\alpha\beta}(\boldsymbol{\eta})$ in ω.

This shows that the mapping $\mathcal{H} : \mathbb{V}_H(\Omega) \to \mathbb{V}(\omega)$ is well-defined, linear, and surjective. That it is in addition injective follows from the infinitesimal rigid displacement lemma on a surface and the boundary conditions satisfied be the vector field (η_i) defined as above in terms of the vector field (v_i). That the inverse of $\mathcal{H} : \mathbb{V}_H(\Omega) \to \mathbb{V}(\omega)$ is the mapping $\mathcal{K} : \mathbb{V}(\omega) \to \mathbb{V}_H(\Omega)$ is clear. □

An immediate consequence of Theorems 1 and 2 is that, if ω is simply-connected and γ_0 is connected, then $\mathbb{V}_H(\Omega) = \mathbb{V}_H^\sharp(\Omega)$, where

$$\mathbb{V}_H^\sharp(\Omega) := \{(e_{ij}) \in \mathbb{V}^\sharp(\Omega);\ e_{i3} = 0 \text{ in } \Omega\} \subset \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3).$$

It remains to prove that the mapping $\mathcal{K}$ maps the subspace $\mathbb{V}^\sharp(\omega) \subset \mathcal{C}^1(\omega; \mathbb{S}^2) \times \mathcal{C}^1(\omega; \mathbb{S}^2)$ onto the subspace $\mathbb{V}_H^\sharp(\Omega) \subset \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3)$. We divide the proof of this result into two distinct lemmas, which are also of interest by themselves. The distributions $S_{\alpha\beta\sigma\varphi}$ and $S_{3\beta\sigma\varphi}$ appearing below are defined in terms of the functions $c_{\alpha\beta}$ and $r_{\alpha\beta}$ as in Sect. 2.

Lemma 1 *Consider a linearly elastic shell that satisfies the assumptions of* Sect. 1. *Assume in addition that ω is simply-connected. Let $(e_{ij}) \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3)$ be a matrix field that satisfies $e_{i3} = e_{3i} = 0$ in $\overline{\Omega}$. Then (e_{ij}) satisfies the Saint Venant compatibility conditions*

$$e_{ij\|k\ell} + e_{\ell k\|ji} - e_{ik\|j\ell} - e_{\ell j\|ki} = 0 \text{ in } \Omega$$

if and only if there exist two matrix fields $(c_{\alpha\beta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$ and $(r_{\alpha\beta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$ that satisfy the Saint Venant compatibility conditions on a surface, that is,

$$S_{\alpha\beta\sigma\varphi} = 0 \text{ and } S_{3\beta\sigma\varphi} = 0 \text{ in } \omega,$$

such that

$$e_{\alpha\beta}(\cdot, x_3) := c_{\alpha\beta} - x_3 r_{\alpha\beta} + x_3^2\Big(\frac{1}{2}(b_\alpha^\sigma r_{\sigma\beta} + b_\beta^\sigma r_{\alpha\sigma}) - b_\alpha^\sigma b_\beta^\varphi c_{\sigma\varphi}\Big) \text{ in } \overline{\Omega}.$$

Proof. First, let $(e_{ij}) \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3)$ be a matrix field that satisfies

$$e_{i3} = e_{3i} = 0 \text{ in } \overline{\Omega}$$

and

$$e_{ij\|k\ell} + e_{\ell k\|ji} - e_{ik\|j\ell} - e_{\ell j\|ki} = 0 \text{ in } \Omega.$$

Then Theorem 6.1 in [4] shows that there exists a vector field $\boldsymbol{v} = (v_i) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{R}^3)$ such that $e_{ij} = \varepsilon_{ij}(\boldsymbol{v})$ in $\overline{\Omega}$. Since $\varepsilon_{i3}(\boldsymbol{v}) = 0$ in Ω by assumption, Lemma 5 in [8] further shows that there exists a vector field $\boldsymbol{\eta} = (\eta_i) \in \mathcal{C}^2(\overline{\omega}) \times \mathcal{C}^2(\overline{\omega}) \times \mathcal{C}^3(\overline{\omega})$ such that

$$v_\alpha(\cdot, x_3) = \eta_\alpha - x_3(\partial_\alpha\eta_3 + 2b_\alpha^\sigma\eta_\sigma) + x_3^2 b_\alpha^\beta(\partial_\beta\eta_3 + b_\beta^\sigma\eta_\sigma) \text{ and } v_3(\cdot, x_3) = \eta_3 \text{ in } \overline{\Omega}.$$

Let $c_{\alpha\beta} := \gamma_{\alpha\beta}(\boldsymbol{\eta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$ and $r_{\alpha\beta} := \rho_{\alpha\beta}(\boldsymbol{\eta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$, where the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ are defined in terms of $\boldsymbol{\eta}$ as in Sect. 2. Then Theorem 4.1 in [3] shows that the corresponding distributions $S_{\alpha\beta\sigma\varphi}$ and $S_{3\beta\sigma\varphi}$ satisfy

$$S_{\alpha\beta\sigma\varphi} = 0 \text{ and } S_{3\beta\sigma\varphi} = 0 \text{ in } \omega,$$

and Lemma 2 in [8] shows that the functions $e_{\alpha\beta} = \varepsilon_{\alpha\beta}(\boldsymbol{v})$ satisfy

$$e_{\alpha\beta}(\cdot, x_3) = c_{\alpha\beta} - x_3 r_{\alpha\beta} + x_3^2\Big(\frac{1}{2}(b_\alpha^\sigma r_{\sigma\beta} + b_\beta^\sigma r_{\alpha\sigma}) - b_\alpha^\sigma b_\beta^\varphi c_{\sigma\varphi}\Big) \text{ in } \overline{\Omega}.$$

Second, let $(e_{ij}) \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3)$ be a matrix field that satisfies

$$e_{i3} = e_{3i} = 0 \text{ in } \overline{\Omega}$$

and

$$e_{\alpha\beta}(\cdot, x_3) = c_{\alpha\beta} - x_3 r_{\alpha\beta} + x_3^2\Big(\frac{1}{2}(b_\alpha^\sigma r_{\sigma\beta} + b_\beta^\sigma r_{\alpha\sigma}) - b_\alpha^\sigma b_\beta^\varphi c_{\sigma\varphi}\Big) \text{ in } \overline{\Omega},$$

where the matrix fields $(c_{\alpha\beta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$ and $(r_{\alpha\beta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$ satisfy

$$S_{\alpha\beta\sigma\varphi} = 0 \text{ and } S_{3\beta\sigma\varphi} = 0 \text{ in } \omega.$$

Then Theorem 5.1 in [3] shows that there exists a vector field $\boldsymbol{\eta} = (\eta_i) \in \mathcal{C}^2(\overline{\omega}) \times \mathcal{C}^2(\overline{\omega}) \times \mathcal{C}^3(\overline{\omega})$ such that

$$c_{\alpha\beta} = \gamma_{\alpha\beta}(\boldsymbol{\eta}) \ \text{ and } \ r_{\alpha\beta} = \rho_{\alpha\beta}(\boldsymbol{\eta}) \ \text{ in } \overline{\omega}.$$

Let the vector field $\boldsymbol{v} = (v_i) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{R}^3)$ be defined by

$$v_\alpha(\cdot, x_3) = \eta_\alpha - x_3(\partial_\alpha \eta_3 + 2b_\alpha^\sigma \eta_\sigma) + x_3^2 b_\alpha^\beta(\partial_\beta \eta_3 + b_\beta^\sigma \eta_\sigma) \ \text{ and } \ v_3(\cdot, x_3) = \eta_3 \ \text{ in } \overline{\Omega}.$$

Then Lemma 2 in [8], combined with the above expressions of the functions e_{ij}, shows that

$$\varepsilon_{ij}(\boldsymbol{v}) = e_{ij} \ \text{ in } \overline{\Omega},$$

and Theorem 5.1 in [4] next shows that the matrix field (e_{ij}) satisfies the equations

$$e_{ij\|k\ell} + e_{\ell k\|ji} - e_{ik\|j\ell} - e_{\ell j\|ki} = 0 \text{ in } \Omega.$$

□

Lemma 2 *Consider a linearly elastic shell that satisfies the assumptions of* Sect. 1. *Assume in addition that ω is simply-connected and that γ_0 is connected. With each pair of matrix fields $(c_{\alpha\beta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$ and $(r_{\alpha\beta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$ that satisfy*

$$S_{\alpha\beta\sigma\varphi} = 0 \ \text{ and } \ S_{3\beta\sigma\varphi} = 0 \ \text{ in } \omega,$$

associate the matrix field $(e_{ij}) \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3)$ defined by

$$\begin{aligned} e_{\alpha\beta}(\cdot, x_3) &:= c_{\alpha\beta} - x_3 r_{\alpha\beta} + x_3^2\Big(\frac{1}{2}(b_\alpha^\sigma r_{\sigma\beta} + b_\beta^\sigma r_{\alpha\sigma}) - b_\alpha^\sigma b_\beta^\varphi c_{\sigma\varphi}\Big) \ \text{ in } \overline{\Omega}, \\ e_{i3} = e_{3i} &:= 0 \ \text{ in } \overline{\Omega}. \end{aligned}$$

Then the matrix field (e_{ij}) satisfies the boundary conditions

$$\begin{aligned} e_{\alpha\beta} t^\alpha t^\beta = e_{\alpha 3} t^\alpha = e_{3\beta} t^\beta = e_{33} &= 0 \text{ on } \Gamma_0, \\ 2e_{\alpha\beta\|\sigma} t^\alpha n^\beta t^\sigma - e_{\alpha\beta\|\sigma} t^\alpha t^\beta n^\sigma + e_{\alpha\beta} n^\alpha n^\beta t^\sigma\|_\varphi n_\sigma t^\varphi &= 0 \text{ on } \Gamma_0, \\ e_{\alpha 3\|\beta} n^\alpha t^\beta + e_{\alpha\beta\|3} t^\alpha n^\beta - e_{\alpha 3\|\beta} t^\alpha n^\beta + e_{\alpha\beta} n^\alpha n^\beta t^\sigma\|_3 n_\sigma &= 0 \text{ on } \Gamma_0, \\ 2e_{\alpha 3\|3} n^\alpha - e_{33\|\alpha} n^\alpha &= 0 \text{ on } \Gamma_0, \end{aligned}$$

if and only if the matrix fields $(c_{\alpha\beta})$ and $(r_{\alpha\beta})$ satisfy the boundary conditions

$$\begin{aligned} c_{\alpha\beta}\tau^\alpha\tau^\beta = 0 \ \text{ and } \ c_{\alpha\beta|\sigma}\tau^\alpha(2\nu^\beta\tau^\sigma - \tau^\beta\nu^\sigma) + \kappa_g c_{\alpha\beta}\nu^\alpha\nu^\beta = 0 \ \text{ on } \gamma_0, \\ r_{\alpha\beta}\tau^\alpha\tau^\beta = 0 \ \text{ and } \ r_{\alpha\beta}\tau^\alpha\nu^\beta - c_{\alpha\beta}\nu^\alpha(2\kappa_n\tau^\beta + \tau_g\nu^\beta) = 0 \ \text{ on } \gamma_0. \end{aligned}$$

Proof. The three vector fields $\boldsymbol{\tau} = \tau^\alpha \boldsymbol{a}_\alpha$, $\boldsymbol{\nu} = \nu^\alpha \boldsymbol{a}_\alpha$, and $\boldsymbol{a}_3$, of the Darboux frames along the curve $\boldsymbol{\theta}(\gamma)$ and their respective extensions $\boldsymbol{t} = t^\alpha \boldsymbol{g}_\alpha$, $\boldsymbol{n} = n^\alpha \boldsymbol{g}_\alpha$, and $\boldsymbol{g}_3$, along the surface $\boldsymbol{\Theta}(\Gamma)$ are related to each other by the relations

$$\begin{aligned} \boldsymbol{t}(y, x_3) &= (1 - x_3\kappa_n(y))\,\boldsymbol{\tau}(y) - x_3\tau_g(y)\,\boldsymbol{\nu}(y), \\ \boldsymbol{n}(y, x_3) &= \big((1 - x_3\kappa_n(y))^2 + (x_3\tau_g(y))^2\big)^{-1/2}\Big(x_3\tau_g(y)\,\boldsymbol{\tau}(y) + (1 - x_3\kappa_n(y))\,\boldsymbol{\nu}(y)\Big), \\ \boldsymbol{g}_3(y, x_3) &= \boldsymbol{a}_3(y), \end{aligned}$$

for all $(y, x_3) \in \gamma \times [-\varepsilon, \varepsilon]$, where $\kappa_g, \kappa_n, \tau_g$ denote respectively the geodesic curvature, the normal curvature, and the geodesic torsion, along the curve $\boldsymbol{\theta}(\gamma)$ defined in Sect. 2.

Besides, the vector fields of the Darboux frames along the curve $\boldsymbol{\theta}(\gamma)$ satisfy the equations:

$$\partial_\tau \boldsymbol{\tau} = \kappa_g \boldsymbol{\nu} + \kappa_n \boldsymbol{a}_3, \quad \partial_\tau \boldsymbol{\nu} = -\kappa_g \boldsymbol{\tau} + \tau_g \boldsymbol{a}_3, \text{ and } \partial_\tau \boldsymbol{a}_3 = -\kappa_n \boldsymbol{\tau} - \tau_g \boldsymbol{\nu}, \text{ on } \gamma.$$

Let the matrix fields $(c_{\alpha\beta})$ and $(r_{\alpha\beta})$ be given that satisfy $S_{\alpha\beta\sigma\varphi} = 0$ and $S_{3\beta\sigma\varphi} = 0$ in $\mathcal{D}'(\omega)$, and let the matrix field (e_{ij}) be defined in terms of $(c_{\alpha\beta})$ and $(r_{\alpha\beta})$ as in the statement of the lemma. Then a series of long, and rather technical, computations based on the above relations shows that the following four assertions hold: First,

$$2e_{\alpha 3\|3} n^\alpha - e_{33\|\alpha} n^\alpha = 0 \text{ on } \Gamma_0.$$

Second,

$$e_{\alpha\beta} t^\alpha t^\beta = e_{\alpha 3} t^\alpha = e_{3\beta} t^\beta = e_{33} = 0 \text{ on } \Gamma_0$$

if and only if the following equations are simultaneously satisfied:

$$\begin{aligned} c_{\alpha\beta}\tau^\alpha\tau^\beta &= 0 \text{ on } \gamma_0,\\ r_{\alpha\beta}\tau^\alpha\tau^\beta - \kappa_n(c_{\alpha\beta}\tau^\alpha\tau^\beta) &= 0 \text{ on } \gamma_0,\\ \tau_g\big(r_{\alpha\beta}\tau^\alpha\nu^\beta - c_{\alpha\beta}\nu^\alpha(2\kappa_n\tau^\beta + \tau_g\nu^\beta)\big) &= 0 \text{ on } \gamma_0. \end{aligned}$$

Third,

$$2e_{\alpha\beta\|\sigma} t^\alpha n^\beta t^\sigma - e_{\alpha\beta\|\sigma} t^\alpha t^\beta n^\sigma + e_{\alpha\beta} n^\alpha n^\beta t^\sigma \|_\varphi n_\sigma t^\varphi = 0 \text{ on } \Gamma_0$$

if and only if

$$\big((1 - x_3\kappa_n(y))^2 + (x_3\tau_g(y))^2\big)^{1/2}\big(c_{\alpha\beta|\sigma}\tau^\alpha(2\nu^\beta\tau^\sigma - \tau^\beta\nu^\sigma) + \kappa_g c_{\alpha\beta}\nu^\alpha\nu^\beta\big) = 0 \text{ on } \gamma_0.$$

Fourth,

$$e_{\alpha 3\|\beta} n^\alpha t^\beta + e_{\alpha\beta\|3} t^\alpha n^\beta - e_{\alpha 3\|\beta} t^\alpha n^\beta + e_{\alpha\beta} n^\alpha n^\beta t^\sigma \|_3 n_\sigma = 0 \text{ on } \Gamma_0$$

if and only if

$$(1 - x_3\kappa_n(y))^2\big((1 - x_3\kappa_n(y))^2 + (x_3\tau_g(y))^2\big)^{-1/2}\big(r_{\alpha\beta}\tau^\alpha\nu^\beta - c_{\alpha\beta}\nu^\alpha(2\kappa_n\tau^\beta + \tau_g\nu^\beta)\big) = 0 \text{ on } \gamma_0.$$

The lemma then follows by combining the above four assertions. □

We are now in a position to establish the *second main result* of this Note. The functions κ_g, κ_n, and τ_g, resp. the distributions $S_{\beta\alpha\sigma\varphi}$ and $S_{3\alpha\sigma\varphi}$, are defined as in Sect. 1, resp. Sect 2.

Theorem 3 *Consider a linearly elastic shell that satisfies the assumptions of* Sect. 1. *Assume in addition that ω is simply-connected and that γ_0 is connected. Define the spaces*

$$\begin{aligned} \mathbb{V}(\omega) := \{&((c_{\alpha\beta}), (r_{\alpha\beta})) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2) \times \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2);\ c_{\alpha\beta} = \gamma_{\alpha\beta}(\boldsymbol{\eta}),\ r_{\alpha\beta} = \rho_{\alpha\beta}(\boldsymbol{\eta}),\\ &\boldsymbol{\eta} = (\eta_i) \in \mathcal{C}^2(\overline{\omega}) \times \mathcal{C}^2(\overline{\omega}) \times \mathcal{C}^3(\overline{\omega});\ \eta_i = \partial_\alpha\eta_3 = 0 \text{ on } \gamma_0\}, \end{aligned}$$

$$\begin{aligned} \mathbb{V}^\sharp(\omega) := \big\{&((c_{\alpha\beta}), (r_{\alpha\beta})) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2) \times \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2);\ S_{\beta\alpha\sigma\varphi} = 0 \text{ and } S_{3\alpha\sigma\varphi} = 0 \text{ in } \omega,\\ &c_{\alpha\beta}\tau^\alpha\tau^\beta = 0 \text{ and } c_{\alpha\beta|\sigma}\tau^\alpha(2\nu^\beta\tau^\sigma - \tau^\beta\nu^\sigma) + \kappa_g c_{\alpha\beta}\nu^\alpha\nu^\beta = 0 \text{ on } \gamma_0,\\ &r_{\alpha\beta}\tau^\alpha\tau^\beta = 0 \text{ and } r_{\alpha\beta}\tau^\alpha\nu^\beta - c_{\alpha\beta}\nu^\alpha(2\kappa_n\tau^\beta + \tau_g\nu^\beta) = 0 \text{ on } \gamma_0\big\}. \end{aligned}$$

Then

$$\mathbb{V}(\omega) = \mathbb{V}^\sharp(\omega).$$

Proof. Since ω is simply-connected and γ_0 is connected, Theorems 1 and 2 show that a pair of matrix fields $((c_{\alpha\beta}), (r_{\alpha\beta}))$ belongs to the space $\mathbb{V}(\omega)$ if and only if there exists a (unique) matrix field $(e_{ij}) \in \mathbb{V}_H^\sharp(\Omega)$ such that

$$c_{\alpha\beta} = e_{\alpha\beta}(\cdot, 0) \text{ and } r_{\alpha\beta} = -\partial_3 e_{\alpha\beta}(\cdot, 0) \text{ in } \omega,$$

where

$$\mathbb{V}^\sharp_H(\Omega) = \big\{(e_{ij}) \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3);\ e_{i3} = 0 \text{ in } \overline{\Omega},\ e_{ij\|k\ell} + e_{\ell k\|ji} - e_{ik\|j\ell} - e_{\ell j\|ki} = 0 \text{ in } \Omega,$$
$$e_{\alpha\beta}t^\alpha t^\beta = e_{\alpha 3}t^\alpha = e_{3\beta}t^\beta = e_{33} = 0 \text{ on } \Gamma_0,$$
$$2e_{\alpha\beta\|\sigma}t^\alpha n^\beta t^\sigma - e_{\alpha\beta\|\sigma}t^\alpha t^\beta n^\sigma + e_{\alpha\beta}n^\alpha n^\beta t^\sigma\|_\varphi n_\sigma t^\varphi = 0 \text{ on } \Gamma_0,$$
$$e_{\alpha 3\|\beta}n^\alpha t^\beta + e_{\alpha\beta\|3}t^\alpha n^\beta - e_{\alpha 3\|\beta}t^\alpha n^\beta + e_{\alpha\beta}n^\alpha n^\beta t^\sigma\|_3 n_\sigma = 0 \text{ on } \Gamma_0,$$
$$2e_{\alpha 3\|3}n^\alpha - e_{33\|\alpha}n^\alpha = 0 \text{ on } \Gamma_0\}.$$

Then the conclusion follows from Lemmas 1 and 2, which together show that

$$\mathbb{V}^\sharp_H(\Omega) = \{\mathcal{K}((c_{\alpha\beta}), (r_{\alpha\beta}));\ ((c_{\alpha\beta}), (r_{\alpha\beta})) \in \mathbb{V}^\sharp(\omega)\},$$

where $\mathcal{K}$ is the mapping defined at the beginning of Sect 4 and $\mathbb{V}^\sharp(\omega)$ is defined in the statement of the theorem. □

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