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Measure-independent anomaly of nonlocality

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We show that any Bell local state, with a hidden nonlocality that can be revealed by local filtering, is more, or equally, entangled than nonlocal states. More precisely, it can be deterministically transformed into a nonlocal state, by local operations and classical communication. For such a state, there is a clear anomaly of nonlocality, for any measures of entanglement and nonlocality. Moreover, we prove that the hidden nonlocality of any bipartite state more, or equally, entangled than nonlocal states, can be revealed by local operations and the sending of two one-bit messages, one in each direction. For some particular states, one bit of communication is even enough.

I. INTRODUCTION

Bell nonlocality and quantum entanglement are two distinct notions, whose relation is not straightforward [1–4]. Entanglement is a quantum resource [5–10] which cannot be generated by deterministic state transformations involving only local operations and classical communication (LOCC) [11, 12]. In other words, a state is necessarily changed into a less, or equally, entangled state, by such a LOCC transformation. Thus, any proper measure of entanglement cannot increase under LOCC operations [12, 13]. The states with vanishing entanglement, are the separable states, which are the mixtures of product states [1]. Any of them, can be reached by LOCC, from any state.

Whereas entanglement is defined strictly within the framework of quantum mechanics, this is not the case for Bell nonlocality. The latter concerns the outcomes of local measurements, quantum or not, performed by distant observers. Their joint probabilities are said to be Bell local, if they can be reproduced by a hidden-variable model, in which a measurement outcome is determined only by the corresponding measurement, and the hidden variables, and does not depend on any other measurement [3]. As well known, when a set of probabilities is Bell local, it satisfies Bell inequalities, such as the Clauser-Horne-Shimony-Holt (CHSH) inequality [14, 15]. In the opposite case, it is said nonlocal, and violates such an inequality.

For a separable state, the joint probabilities of local measurement results, are always Bell local. But this is not specific to separable density operators. Some entangled states have this property [1, 2]. A state for which there are measurements violating a Bell inequality, is said to be nonlocal. A pure state is nonlocal if and only if it is entangled [16–19]. Nevertheless, even for pure states, the relation between quantum entanglement and Bell nonlocality, is not obvious. For the simplest composite system, consisting of two two-level systems, the more entangled a pure state is, the more it can violate the CHSH inequality [20, 21]. But the situation is less clear for other Bell inequalities [22–26], or other measures of nonlocality [27–30], which are not maximum for maximally entangled states. However, it can be argued that they are

not correct measures of nonlocality, and that, hence, no "anomaly of nonlocality" can be evidenced using them [31].

Different procedures have been proposed to violate a Bell inequality with a Bell local entangled state, and hence, in some sense, reveal its hidden nonlocality. Such a violation can, for instance, be obtained by using several copies of the same state [32–34], by performing local pre-measurements and selecting specific outcomes [35–37], by combining these two approaches [38], or by using more sophisticated techniques [39–41]. In these scenarios, the density operator whose nonlocality is tested, is not the Bell local state of interest. The corresponding transformations change it into a state which is nonlocal, and possibly also more entangled. For example, several copies of a density operator, constitute a state more, or equally, entangled than this density operator, since the former is transformed into the latter, by local partial traces, which are LOCC operations. Thus, the above-mentioned procedures are not helpful in the understanding of a potential anomaly of nonlocality.

In this paper, we show that there exist Bell local states which are more, or equally, entangled than nonlocal ones. In this case, the anomaly of nonlocality is manifest, since any nonlocality measure increases in going from such a Bell local state to a corresponding nonlocal one, whereas entanglement measures do not. The outline of the paper is as follows. We first recall, in Sec.II, what are Bell nonlocality, LOCC operations, and local filtering operations. In Sec.III, we prove our main result, namely, that a bipartite state which can be changed into a nonlocal one, by local filtering, is more, or equally, entangled than nonlocal states. In this section, we also discuss a particular state, which is Bell local, and as entangled as a nonlocal state. In Sec.IV, we address the issue of the amount of classical communication, required to transform, by LOCC, a given bipartite state into a nonlocal state. We show that, whenever such a transformation is possible, two observers can achieve it with the transmission of only one bit from one observer to the other, and one bit in the opposite direction. Moreover, for some states, a single bit suffices. Finally, in Sec.V, we summarize our results and discuss an important question they raise.

II. PRELIMINARIES

In this section, we introduce the notions used throughout the paper, namely, the local filtering [3, 35–37], the LOCC-based ordering of quantum states [11, 12], and the Bell nonlocality [3, 14, 15].

A. Local filtering and LOCC operations

In this paper, we consider two kinds of quantum state transformations. For two systems, say A and B, whose Hilbert spaces are, respectively, \mathcal{H}_A and \mathcal{H}_B , a local filtering operation, described by the operators M on \mathcal{H}_A , and N on \mathcal{H}_B , changes the state ρ , of A and B, into

$$\hat{\rho} = M \otimes N \rho M^\dagger \otimes N^\dagger / p, \quad (1)$$

where $p = \text{tr}(M^\dagger M \otimes N^\dagger N \rho) \in (0, 1]$ [3]. The operators M and N are such that $M^\dagger M \leq I_A$ and $N^\dagger N \leq I_B$, where I_A and I_B are the identity operators of A and B, respectively. This filtering transformation is stochastic. It is achieved by performing measurements on A and B, and selecting specific outcomes. The state $\hat{\rho}$ is obtained with probability p .

The other transformations of interest for our purpose, are the LOCC operations. A state ρ is more, or equally, entangled than another one ρ' , if and only if there is a LOCC map Λ , such that $\rho' = \Lambda(\rho)$ [12]. Such a transformation is deterministic, i.e., it gives ρ' , from ρ , with probability unity. It is a composition of local partial traces, and of one-way LOCC operations of the form

$$\Lambda(\rho) = \sum_i (F_i \otimes I_C) \rho (F_i \otimes I_C)^\dagger \otimes |i\rangle\langle i|, \quad (2)$$

where C is A or B, the linear maps $F_i : \mathcal{H}_D \rightarrow \mathcal{H}_E$, with D the other system, B or A, are such that $\sum_i F_i^\dagger F_i = I_D$, and $|i\rangle$ are orthonormal states of an ancillary system, close to C, see Appendix. The system E can be D itself, a subsystem of it, or a system of which D is a subsystem. The transformation (2) involves a measurement on one of the systems, by an observer, and the sending, to another observer, of the outcome i , which is recorded using the ancilla. If i has two possible values, only one bit is exchanged between the two observers. Operations of the form of eq.(2), play an essential role in what follows.

B. Bell nonlocality

Let us denote $p(ij|kl)$ the probability of the outcomes i and j , of measurements, indexed by k and l , performed on systems A and B, respectively. The set $\{p(ij|kl)\}_{i,j,k,l}$ is Bell local if and only if these probabilities can be written as

$$p(ij|kl) = \int d\lambda q(\lambda) p(i|k\lambda) p(j|l\lambda), \quad (3)$$

where λ denotes hidden variables, q a probability density function, and $p(i|k\lambda)$ a probability distribution of the outcome i of the measurement k [3].

Within the framework of quantum mechanics, a measurement, on A, with m_k outcomes, is described by a set of positive operators, $\mathbf{A}_k = \{A_{i|k}\}_{i=1}^{m_k}$, such that $\sum_{i=1}^{m_k} A_{i|k} = I_A$, and joint probabilities of measurements on A and B, are given by

$$p(ij|kl) = \text{tr}(\rho A_{i|k} \otimes B_{j|l}), \quad (4)$$

for the state ρ of A and B, where $B_{j|l}$ are the operators describing the measurement l on B [42]. The state ρ is nonlocal if there are measurements \mathbf{A}_k and \mathbf{B}_l such that $\{p(ij|kl)\}_{i,j,k,l}$ does not satisfy eq.(3). If, on the contrary, the probabilities (4) can be written in the form of eq.(3), for any measurements \mathbf{A}_k and \mathbf{B}_l , ρ is a Bell local state.

III. REVEALING NONLOCALITY BY LOCC

Some Bell local states can be changed into nonlocal ones, by local filtering [35–37]. For such a state ρ , we show below that there are nonlocal states less, or equally, entangled than ρ . We then consider a particular Bell local state, first studied in Ref.[37], which is as entangled as a nonlocal state.

A. Main result

Proposition 1. If a bipartite state can be changed into a nonlocal state, by local filtering, then two observers can deterministically transform it into a nonlocal state, using local operations, and the transmission of one bit from one observer to the other, and one bit in the opposite direction.

Proof. Consider a composite system AB, consisting of systems A and B, and a state ρ , of AB, such that there are operators M and N for which the state (1) is nonlocal. Since $M^\dagger M \leq I_A$ and $N^\dagger N \leq I_B$, there exist operators \tilde{M} and \tilde{N} such that $M^\dagger M + \tilde{M}^\dagger \tilde{M} = I_A$, and $N^\dagger N + \tilde{N}^\dagger \tilde{N} = I_B$. The LOCC operation given by eq.(2) with orthonormal states $|i\rangle$ of a two-level system B' , $F_0 = |0'\rangle M$, and $F_1 = |1'\rangle \tilde{M}$, where $|i'\rangle$ are orthonormal states of a two-level system A' , transforms ρ into

$$\rho_1 = R_0 \otimes M \rho M^\dagger \otimes S_0 + R_1 \otimes \tilde{M} \rho \tilde{M}^\dagger \otimes S_1, \quad (5)$$

where $R_i = |i'\rangle\langle i'|$, $S_i = |i\rangle\langle i|$, and the short-hand notation $\tilde{M} = M \otimes I_B$ is used. A similar operation, with N and \tilde{N} , in place of M and \tilde{M} , respectively, and two more two-level systems, A'' and B'' , transforms ρ_1 into

$$\rho_2 = \sum_{i=0}^3 P_i \otimes K_i \rho K_i^\dagger \otimes Q_i, \quad (6)$$

where P_i (Q_i) are four projectors of system $A'A''$ ($B'B''$), summing to $I_{A'A''}$ ($I_{B'B''}$), $K_0 = M \otimes N$, $K_1 = M \otimes \tilde{N}$, $K_2 = \tilde{M} \otimes N$, and $K_3 = \tilde{M} \otimes \tilde{N}$.

The filtered state $\hat{\rho}$, given by eq.(1), is nonlocal, by assumption. Thus, there are measurements \mathbf{A}_k and \mathbf{B}_l such that the vector \mathbf{p} , termed behavior [43], whose components are the probabilities $p(ij|kl)$, given by eq.(4) with the density operator $\hat{\rho}$, does not satisfy eq.(3). We define the positive operators, on $\mathcal{H}_{A'A''A}$,

$$\tilde{A}_{i|k} = P_0 \otimes A_{i|k} + \delta_{i,r_k} (I_{A'A''} - P_0) \otimes I_A, \quad (7)$$

where $r_k \in \{1, \dots, m_k\}$, with m_k the number of outcomes of \mathbf{A}_k . The set $\tilde{\mathbf{A}}_k = \{\tilde{A}_{i|k}\}_{i=1}^{m_k}$ constitutes a measurement, on $A'A''A$, since $\sum_{i=1}^{m_k} \tilde{A}_{i|k} = I_{A'A''A}$. Similarly, from a measurement \mathbf{B}_l , on B, with n_l outcomes, a measurement $\tilde{\mathbf{B}}_l$, on $BB'B''$, involving an integer $s_l \in \{1, \dots, n_l\}$, can be defined. The probabilities $\tilde{p}(ij|kl) = \text{tr}(\rho_2 \tilde{A}_{i|k} \otimes \tilde{B}_{j|l})$ can be written as

$$\tilde{p}(ij|kl) = pp(ij|kl) + (1-p)d_\lambda(ij|kl), \quad (8)$$

where $p = \text{tr}(M^\dagger M \otimes N^\dagger N \rho)$, $d_\lambda(ij|kl) = \delta_{i,r_k} \delta_{j,s_l}$, and $\lambda = (r_1, r_2, \dots, s_1, s_2, \dots)$.

The behavior $\tilde{\mathbf{p}}$ is Bell local if and only if it belongs to the compact convex polytope

$$\mathcal{L} = \left\{ \sum_\lambda q_\lambda \mathbf{d}_\lambda : q_\lambda \geq 0, \sum_\lambda q_\lambda = 1 \right\}, \quad (9)$$

where the sums run over all λ [3, 44]. There is a particular λ such that $\tilde{\mathbf{p}} = p\mathbf{p} + \bar{p}\mathbf{d}_\lambda \notin \mathcal{L}$, where $\bar{p} = 1-p$. This can be seen as follows. Assume that, for any λ , $p\mathbf{p} + \bar{p}\mathbf{d}_\lambda \in \mathcal{L}$. This implies, together with the convexity of \mathcal{L} , that $p\mathbf{p} + \bar{p}\mathbf{p}' \in \mathcal{L}$, for any Bell local behavior $\mathbf{p}' = \sum_\lambda q_\lambda \mathbf{d}_\lambda$. This gives a sequence of elements of \mathcal{L} , $[1 - \bar{p}^n]\mathbf{p} + \bar{p}^n\mathbf{p}'$, that converges to \mathbf{p} , which is not possible since \mathcal{L} is closed, and $\mathbf{p} \notin \mathcal{L}$. In conclusion, there are measurements $\tilde{\mathbf{A}}_k$ and $\tilde{\mathbf{B}}_l$ for which the corresponding behavior $\tilde{\mathbf{p}}$ is not in \mathcal{L} , and hence ρ_2 is nonlocal. The two-stage LOCC transformation $\rho \mapsto \rho_1 \mapsto \rho_2$ involves the sending of two one-bit messages, one in each direction. \square

The exchange and storage of classical information play a crucial role in the above transformation leading to a nonlocal state. To see it, consider the state ρ_3 obtained from the state (6), by tracing out the systems B' and A'' , used to record the bits exchanged between the two observers. This density operator can be written as $\rho_3 = \Lambda_B \circ \Lambda_A(\rho)$, where Λ_A and Λ_B are local operations. Thus, it is Bell local if ρ is [2].

B. Example

As an example, we consider the state, of two three-level systems, A and B,

$$\rho = p|\psi\rangle\langle\psi| + pM \otimes \tilde{N} + q\tilde{M} \otimes N + 4q\tilde{M} \otimes \tilde{N}, \quad (10)$$

where $|\psi\rangle = (|0\rangle|0'\rangle + |1\rangle|1'\rangle)/\sqrt{2}$, with orthonormal states $|i\rangle$ of A, and $|i'\rangle$ of B, $q = (1 - 3p)/6$, $p \leq 1/18$, $M = |0\rangle\langle 0| + |1\rangle\langle 1|$, and $\tilde{M} = |2\rangle\langle 2|$. The operators N and \tilde{N} , on \mathcal{H}_B , are given by similar expressions. This state has been shown to be Bell local [37]. The corresponding filtered state (1) is $|\psi\rangle\langle\psi|$, which maximally violates the CHSH inequality. It is attained with probability p .

The state (6), obtained by LOCC from ρ , is here

$$\rho_2 = pP_0 \otimes |\psi\rangle\langle\psi| \otimes Q_0 + pP_1 \otimes M \otimes \tilde{N} \otimes Q_1 + qP_2 \otimes \tilde{M} \otimes N \otimes Q_2 + 4qP_3 \otimes \tilde{M} \otimes \tilde{N} \otimes Q_3, \quad (11)$$

It results, from the above proof, that it is nonlocal. This can be shown directly as follows. We find

$$\langle A_1(B_1 + B_2) + A_2(B_1 - B_2) \rangle = 2p(\sqrt{2} - 1) + 2 \geq 2,$$

where $A_k B_l = A_k \otimes B_l$, and $\langle \dots \rangle = \text{tr}(\rho_2 \dots)$, for the dichotomic observables

$$A_k = P_0 \otimes (\sigma_k + \tilde{M}) + \tilde{P} \otimes I_A, \\ B_l = ([\sigma'_1 + (3 - 2l)\sigma'_2]/\sqrt{2} + \tilde{N}) \otimes Q_0 + I_B \otimes \tilde{Q},$$

where $\sigma_1 = |0\rangle\langle 0| - |1\rangle\langle 1|$, $\sigma_2 = |0\rangle\langle 1| + |1\rangle\langle 0|$, σ'_1 and σ'_2 are defined similarly for system B, $\tilde{P} = I_{A'A''} - P_0$, and $\tilde{Q} = I_{B'B''} - Q_0$. That is to say, ρ_2 violates the CHSH inequality. Other measurements may lead to a larger violation. It is also possible that other Bell inequalities could be more appropriate. The local operations consisting in tracing out the systems $A'A''$ and $B'B''$, change the state (11) back into the state (10). They are thus equally entangled, whereas one is Bell local and the other is nonlocal.

IV. AMOUNT OF COMMUNICATION REQUIRED TO REVEAL NONLOCALITY

We have seen above that the hidden nonlocality of some bipartite states, can be revealed by sending two one-bit messages, one in each direction. One can wonder whether this can be achieved with less communication. Some exchange of information is necessary, since local operations alone cannot transform a Bell local state into a nonlocal one [2]. We show below that one bit of communication is enough for some states. We then prove that, for any state more, or equally, entangled than nonlocal states, it can be done with only one bit per direction.

A. One bit is the minimum

The proof below applies to Bell local states that can be changed into nonlocal states, with only one local filter. Let us first show that there exist such states. Consider a Bell local state ρ , such that the filtered state (1) is nonlocal, and the state $\rho' \propto M \otimes I_B \rho M^\dagger \otimes I_B$, obtained, from

ρ , by applying only the filter described by M . The latter is transformed into the nonlocal state (1), by the filter described by N . If ρ' is nonlocal, then the hidden nonlocality of ρ can be revealed with one local filter. If, on the contrary, ρ' is Bell local, then its hidden nonlocality can be revealed with one local filter.

Proposition 2. If a bipartite state can be changed into a nonlocal state, with one local filter, then it can be deterministically transformed into a nonlocal state, with local operations, and one bit of communication.

Proof. Consider two systems A and B, and a state ρ of these systems, such that there is an operator M on \mathcal{H}_A , for which the state $\rho' = M \otimes I_B \rho M^\dagger \otimes I_B / p$, where $p = \text{tr}(M^\dagger M \otimes I_B \rho)$, is nonlocal. There exists \tilde{M} such that $M^\dagger M + \tilde{M}^\dagger \tilde{M} = I_A$. Since ρ' is nonlocal, there are measurements \mathbf{A}_k , with m_k outcomes, and \mathbf{B}_l , with n_l outcomes, such that the behavior \mathbf{p} , whose components are the probabilities $p(ij|kl)$, given by eq.(4) with the density operator ρ' , is nonlocal, i.e., does not satisfy eq.(3). Let us introduce two two-level systems A' and B' , and define the positive operators, on $\mathcal{H}_{A'A}$,

$$\tilde{A}_{i|k} = |0'\rangle\langle 0'| \otimes A_{i|k} + \delta_{i,r_k} |1'\rangle\langle 1'| \otimes I_A,$$

where $|i'\rangle$ are orthonormal states of A' , and $r_k \in \{1, \dots, m_k\}$, and similar ones, $\tilde{B}_{j|l}$, on $\mathcal{H}_{B'B'}$, with an integer of $\{1, \dots, n_l\}$, and orthonormal states of B' . The set $\tilde{\mathbf{A}}_k = \{\tilde{A}_{i|k}\}_{i=1}^{m_k}$ constitutes a measurement on $A'A$, and $\tilde{\mathbf{B}}_l$ on $B'B'$. The probabilities $\tilde{p}(ij|kl) = \text{tr}(\rho_1 \tilde{A}_{i|k} \otimes \tilde{B}_{j|l})$, where ρ_1 is given by eq.(5), can be cast into the form of eq.(8), with p defined above. Since \mathbf{p} is nonlocal, there is a deterministic behavior \mathbf{d}_λ , such that $\tilde{\mathbf{p}} = p\mathbf{p} + (1-p)\mathbf{d}_\lambda$ is nonlocal, see the proof of proposition 1, and hence ρ_1 is nonlocal. The LOCC transformation $\rho \mapsto \rho_1$ involves the sending of only one one-bit message. \square

B. One bit per direction is enough

Proposition 3. A bipartite state is more, or equally, entangled than a nonlocal state, if and only if it can be deterministically transformed into a nonlocal state, with local operations, and two one-bit messages, one in each direction.

Proof. Consider a state ρ , of systems A and B, such that there is a LOCC operation Λ , for which $\rho' = \Lambda(\rho)$ is nonlocal. The map Λ is separable [11, 45], i.e., $\rho' = \sum_i q_i \omega_i$ with the states $\omega_i = M_i \otimes N_i \rho M_i^\dagger \otimes N_i^\dagger / q_i$, where the operators M_i and N_i , on \mathcal{H}_A and \mathcal{H}_B , respectively, are such that $\sum_i M_i^\dagger M_i \otimes N_i^\dagger N_i = I_{AB}$, and $q_i = \text{tr}(M_i^\dagger M_i \otimes N_i^\dagger N_i \rho)$. Since ρ' is nonlocal, there are measurements \mathbf{A}_k and \mathbf{B}_l such that the corresponding behavior \mathbf{p} is nonlocal. It can be written as $\mathbf{p} = \sum_i q_i \mathbf{p}_i$,

where \mathbf{p}_i is the behavior for the state ω_i , and the measurements \mathbf{A}_k and \mathbf{B}_l . Since the set \mathcal{L} of the Bell local behaviors, given by eq.(9), is convex, $\mathbf{p} \notin \mathcal{L}$, and $\sum_i q_i = 1$, there is i such that $\mathbf{p}_i \notin \mathcal{L}$.

The corresponding operators M_i and N_i obey $M_i^\dagger M_i \leq I_A$, and $N_i^\dagger N_i \leq I_B$, see Appendix. Thus, there are \tilde{M} and \tilde{N} such that $M_i^\dagger M_i + \tilde{M}^\dagger \tilde{M} = I_A$, and $N_i^\dagger N_i + \tilde{N}^\dagger \tilde{N} = I_B$. As shown in the proof of proposition 1, there is a LOCC operation, involving the sending of two one-bit messages, one in each direction, that transforms ρ into the state ρ_2 , given by eq.(6) with $K_0 = M_i \otimes N_i$, $K_1 = M_i \otimes \tilde{N}$, $K_2 = \tilde{M} \otimes N_i$, and $K_3 = \tilde{M} \otimes \tilde{N}$. Since $\mathbf{p}_i \notin \mathcal{L}$, there are measurements $\tilde{\mathbf{A}}_k$, given by eq.(7), and $\tilde{\mathbf{B}}_l$, defined similarly from \mathbf{B}_l , such that the behavior of components $\text{tr}(\rho_2 \tilde{A}_{i|k} \otimes \tilde{B}_{j|l})$, is nonlocal, and hence ρ_2 is nonlocal, see the proof of proposition 1.

The converse follows directly from the definition of the entanglement ordering. \square

V. CONCLUSION

In summary, we have shown that there are states which are Bell local, but more, or equally, entangled than nonlocal ones. They are those with a hidden nonlocality that can be revealed by local filtering. For these states, there is a clear anomaly of nonlocality, for any measures of entanglement and nonlocality. We have also proved that any state more, or equally, entangled than nonlocal ones, can be changed into a nonlocal state, with local operations and only two bits of communication, one in each direction. For some particular states, a single bit is even enough.

A natural question arising from these results, is whether all entangled states are more, or equally, entangled than nonlocal states. In other words, do all Bell local entangled states have a hidden nonlocality that can be revealed by LOCC? It has been shown recently that the answer to the similar question for local filtering, is negative [46]. But this does not imply a negative answer for LOCC. Due to our last result, this issue can be addressed by considering only LOCC operations involving the sending of a single one-bit message per direction.

APPENDIX: ONE-WAY LOCC DECOMPOSITION

In this Appendix, we show how any LOCC operation can be obtained from a sequence of one-way LOCC maps of the form of eq.(2). Any LOCC transformation of a state ρ of a bipartite system $A_1 B_1$, can be written as $\Lambda(\rho) = \sum_i K_i \rho K_i^\dagger$, where $\mathbf{i} = (i_1, \dots, i_{2n})$, i_r runs from 1 to d_r , and

$$\begin{aligned} K_{\mathbf{i}} &= \left(M_{i_{2n-1}}^{(2n-1)} \otimes N_{i_{2n}}^{(2n)} \right) \dots \left(M_{i_1}^{(1)} \otimes N_{i_2}^{(2)} \right), \\ &= M_{i_{2n-1}}^{(2n-1)} \dots M_{i_1}^{(1)} \otimes N_{i_{2n}}^{(2n)} \dots N_{i_2}^{(2)}, \end{aligned}$$

with $\mathbf{i}_r = (i_1, \dots, i_r)$. The linear maps $M_{\mathbf{i}_{2r-1}}^{(2r-1)} : \mathcal{H}_{A_r} \rightarrow \mathcal{H}_{A_{r+1}}$ satisfy

$$\sum_{i_{2r-1}=1}^{d_{2r-1}} \left(M_{\mathbf{i}_{2r-2}, i_{2r-1}}^{(2r-1)} \right)^\dagger M_{\mathbf{i}_{2r-2}, i_{2r-1}}^{(2r-1)} = I_{A_r},$$

and the operators $N_{\mathbf{i}_{2r}}^{(2r)} : \mathcal{H}_{B_r} \rightarrow \mathcal{H}_{B_{r+1}}$ obey similar relations [11]. We remark that the above equality gives, for any $|\psi\rangle \in \mathcal{H}_{A_r}$, $\langle \psi | (M_{\mathbf{i}_{2r-1}}^{(2r-1)})^\dagger M_{\mathbf{i}_{2r-1}}^{(2r-1)} | \psi \rangle \leq \langle \psi | \psi \rangle$, and hence, for any $|\psi\rangle \in \mathcal{H}_{A_1}$,

$$\langle \psi | \left(M_{\mathbf{i}_{2n-1}}^{(2n-1)} \dots M_{\mathbf{i}_1}^{(1)} \right)^\dagger M_{\mathbf{i}_{2n-1}}^{(2n-1)} \dots M_{\mathbf{i}_1}^{(1)} | \psi \rangle \leq \langle \psi | \psi \rangle.$$

Let us introduce the systems A'_r and B'_r , of Hilbert space dimension d_r , where $r \in \{1, \dots, 2n\}$, and the composite systems $A^{[r]} = A'_1 \dots A'_{2r-1} A_{r+1}$, and $B^{[r]} = B_{r+1} B'_1 \dots B'_{2r}$. We denote by $P_{i_r}^{(r)}$ projectors such that $\sum_{i_r=1}^{d_r} P_{i_r}^{(r)} = I_{A'_r}$, and $Q_{i_r}^{(r)}$ similar projectors for B'_r , and define the one-way LOCC operations $\Lambda_1, \dots, \Lambda_{2n}$ by

$$\begin{aligned} \Lambda_1(\rho_1) &= \sum_{i_1} P_{i_1}^{(1)} \otimes \left(M_{i_1}^{(1)} \otimes I_{B_1} \right) \rho_1 \left(M_{i_1}^{(1)} \otimes I_{B_1} \right)^\dagger \otimes Q_{i_1}^{(1)} \\ \Lambda_{2r}(\rho_{2r}) &= \sum_{i_{2r}} P_{i_{2r}}^{(2r)} \otimes \left(I_{A^{[r]}} \otimes N_{\mathbf{i}_{2r}}^{(2r)} \otimes Q_{i_{2r-1}} \right) \rho_{2r} \left(I_{A^{[r]}} \otimes N_{\mathbf{i}_{2r}}^{(2r)} \otimes Q_{i_{2r-1}} \right)^\dagger \otimes Q_{i_{2r}}^{(2r)} \\ \Lambda_{2r+1}(\rho_{2r+1}) &= \sum_{i_{2r+1}} P_{i_{2r+1}}^{(2r+1)} \otimes \left(P_{i_{2r}} \otimes M_{\mathbf{i}_{2r+1}}^{(2r+1)} \otimes I_{B^{[r]}} \right) \rho_{2r+1} \left(P_{i_{2r}} \otimes M_{\mathbf{i}_{2r+1}}^{(2r+1)} \otimes I_{B^{[r]}} \right)^\dagger \otimes Q_{i_{2r+1}}^{(2r+1)}, \end{aligned}$$

where $P_{i_r} = P_{i_r}^{(r)} \otimes P_{i_{r-1}}^{(r-1)} \otimes \dots \otimes P_{i_1}^{(1)}$, i_r runs from 1 to d_r , ρ_1 is a state of $A_1 B_1$, ρ_{2r} of $A^{[r]} B_r B'_1 \dots B'_{2r-1}$, and ρ_{2r+1} of $A'_1 \dots A'_{2r} A_{r+1} B^{[r]}$. The LOCC map $\Phi =$

$\Lambda_{2n} \circ \dots \circ \Lambda_1$ transforms a state ρ of $A_1 B_1$, into $\Phi(\rho) = \sum_{\mathbf{i}} P_{\mathbf{i}} \otimes K_{\mathbf{i}} \rho K_{\mathbf{i}}^\dagger \otimes Q_{\mathbf{i}}$. Tracing out the ancillary systems A'_r and B'_r , gives $\Lambda(\rho)$.

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