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Difference equations, uniform quasiclassical asymptotics and Airy functions

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We consider the second order difference equation $\psi(z+h)+\psi(z-h)+v(z)\psi(z)=0$ where z is a complex variable, $h>0$ is a parameter, and v is an analytic function. As $h\to 0$ the analytic solutions to this equation have quasiclassical behavior. In this note we describe their uniform asymptotics in neighborhoods of simple turning points, the neighborhoods being independent of h.

1 Introduction

We study analytic solutions to the second order difference equation

$$\psi(z+h)+\psi(z-h)+v(z)\psi(z)=0 \qquad (1)$$

where z is a complex variable, $h>0$ is a parameter, and v is an analytic function. We describe their asymptotics as $h\to 0$.

Note that the parameter h appears to be a standard quasiclassical parameter. Indeed, formally, $\psi(z+h)=\sum_{l=0}^{\infty}\frac{h^l}{l!}\frac{d^l\psi}{dz^l}(z)=e^{h\frac{d}{dz}}\psi(z)$, and h can be regarded as a small parameter in front of the derivative.

Difference equations in the complex plane in the quasiclassical regime arise in many fields of mathematical physics. For example, they arise when studying an electron in a crystal placed in a constant magnetic field (e.g., [1, 2]), wave scattering by thin wedges (e.g., [3]), one-dimensional quasiperiodic differential Schrödinger equations with two frequencies, one being small with respect to the other (e.g., [4]). Similar problems arise in the theory of orthogonal polynomials (e.g., [5]).

The quasiclassical asymptotics of analytic solutions to ordinary differential equations in the complex plane are well-known (e.g., [6, 7]). The method is often called *complex WKB method*. Its analog for difference equations was developed in [8, 9, 10]. This note is devoted to uniform asymptotic formulas describing analytic solutions to (1) in neighborhoods of simple turning points (see the definition below), the neiborhoods being independent of h.

2 A very short introduction to the complex WKB method

Below, U denotes the domain of analyticity of v.

2.1 Complex momentum

The main analytic object of the method is the *complex momentum* p defined by the formula

$$2\cos p+v(z)=0.$$

It is a multivalued analytic function. The branch points of p are called *turning points*. At turning points, one has $\cos p(z)\in\{\pm 1\}$ and $v(z)\in\{\pm 2\}$.

We call a set $D\subset U$ regular if $v(z)\neq\pm 2$ in D.

2.2 The main theorem of the complex WKB method

As in the case of differential equations, the main geometric notion of the complex WKB method is the one of *canonical domain*. The reader can find definitions and details in [9, 10]. Here, we note only that canonical domains are regular simply connected domains independent of h. The following two theorems hold.

Theorem 1 *Any regular point belongs to a canonical domain.*

This statement is essentially Lemma 5.2 from [11].

Theorem 2 ([9, 10]) *Let $K\subset U$ be a canonical domain, $z_0\in K$, and p be a branch of the complex*

momentum analytic in K. Suppose that either K is a bounded canonical domain or v is a trigonometric polynomial. Then, there exist solutions $\psi_\pm$ to (1) *analytic in K and such that as $h \to 0$*

$$\psi_\pm(z) = \frac{\exp\left(\pm\frac{i}{h}\int_{z_0}^{z} p(z)\,dz + o(1)\right)}{\sqrt{\sin(p(z))}}, \quad z \in K. \quad (2)$$

This asymptotic representation is locally uniform.

We note that, at a branch point, one has $\cos p(z) \in \{\pm 1\}$, and so, $\sin p(z) = 0$. Therefore, formula (2) can not be valid near a branch point.

For the equation $-h^2\psi''(z) + v(z)\psi(z) = 0$, formula (2) is replaced with $\psi_\pm(z) = \frac{1}{\sqrt{p(z)}}\, e^{\pm\frac{i}{h}\int_{z_0}^{z} p(z)\,dz + o(1)}$, where the complex momentum is defined by the relation $p^2 + v(z) = 0$, i.e., as for (1), by the symbol of the equation.

3 A warm up

Here, we define objects and recall some facts needed to formulate our results.

3.1 Complex momentum near a branch point

Below, $z_0 \in U$ is a branch point of the complex momentum p.

We note that as $\cos(p(z_0)) \in \{\pm 1\}$, one has either $p(z_0) = 0 \mod 2\pi$ or $p(z_0) = \pi \mod 2\pi$.

If $v'(z_0) \neq 0$, p is analytic in $\tau = \sqrt{z - z_0}$ in a neighborhood of zero, and

$$p(z) = p(z_0) + k_1\tau + \ldots, \quad \tau \to 0, \qquad k_1 \neq 0.$$

In this case, the branch point z_0 is called *simple*.

In this paper we assume that

- in U, there is a single branch point z_0 of the complex momentum and it is simple; for the sake of simplicity, we assume that U is a disk, and that z_0 is its center;
- $p(z_0) = 0 \mod 2\pi$.

Note that, the case $p(z_0) = \pi \mod 2\pi$ can obtained by a simple algebraic transformation from the case $p(z_0) = 0 \mod 2\pi$.

3.2 Function ζ

We cut U from z_0 along, say, the line $\{z - z_0 \leq 0\}$ and denote the thus obtained domain by U'. In U', we fix p, an analytic branch of the complex momentum and ζ, an analytic branch of the function

$$z \mapsto \left(\frac{3}{2i}\int_{z_0}^{z} (p(z) - p(z_0))\,dz\right)^{\frac{2}{3}}.$$

The function ζ is analytic in U. One has $\zeta(z_0) = 0$, and $\zeta'(z_0) \neq 0$. Possibly reducing U somewhat, we can and do assume that

- ζ is analytic isomorphism of U onto its image.

3.3 Basic facts on Airy functions

The equation

$$w''(\zeta) = \zeta w(\zeta), \qquad \zeta \in \mathbb{C},$$

is the Airy equation. Its solutions are called *Airy functions*.

Let $|\arg z| < 2\pi/3$. There exists an Airy function Ai that admits the following asymptotic representations as $|z| \to \infty$ (see [12])

$$\mathrm{Ai}\,(z) = \frac{\exp\left(-\frac{2}{3}z^{\frac{3}{2}} + o(1)\right)}{2\sqrt{\pi}\, z^{\frac{1}{4}}},$$

$$\mathrm{Ai}\,(-z) = \frac{\cos\left(\frac{2}{3}z^{\frac{3}{2}} - \frac{\pi}{4} + o(1)\right)}{\sqrt{\pi}\, z^{\frac{1}{4}}}\,(1 + o(1)),$$

where we use the analytic branches of $z \to z^{\frac{3}{2}}$ and $z \to z^{\frac{1}{4}}$ that are positive for $z > 0$.

Let $\omega = e^{2\pi i/3}$. For $j \in \mathbb{Z}_3$, we define $w_j(\cdot) = \mathrm{Ai}(\omega^j\cdot)$. The functions $(w_j)_{j\in\{1,2\}}$ are two more Airy functions.

4 Analytic solutions in a complex neighborhood of a branch point

4.1 Asymptotic solutions

For a function f defined in U and a point $z \in U$ s.t. $z \pm h \in U$, we set

$$Hf(z) = f(z+h) + f(z-h) + v(z)f(z). \quad (3)$$

We let

$$g(z) := \frac{\sinh\left(\sqrt{\zeta(z)}\zeta'(z)\right)}{\sqrt{\zeta(z)}}, \qquad z \in U,$$

where the determination of the square roots in the denominator and the numerator are the same. The definition of g is independent of the choice of this

determination, and g is analytic and does not vanish in U. We further denote by A_0 a branch of the function $z \to \frac{1}{\sqrt{g(z)}}$ that is analytic in U. One has

Theorem 3 *There exist $(A_l)_{l\in\mathbb{N}}$ and $(B_l)_{l\in\mathbb{N}}$, analytic functions on U, such that the following holds. Let w be one of the Airy functions $\{w_j;\ j \in \mathbb{Z}_3\}$. Pick $L \in \mathbb{N} \cup \{0\}$ and put*

$$W(z) = h^{\frac{1}{3}} w \sum_{l=0}^{L} h^l A_l(z) + h^{\frac{2}{3}} w' \sum_{l=1}^{L} h^l B_l(z),$$
$$w = w\left(\zeta(z)/h^{\frac{2}{3}}\right), \qquad w' = w'\left(\zeta(z)/h^{\frac{2}{3}}\right). \quad (4)$$

One has the locally uniform estimates

$$H(W) = O\left(h^{L+2+\frac{1}{3}} w\right) + O\left(h^{L+2+\frac{2}{3}} w'\right).$$

The coefficient A_0 was defined just above Theorem 3. There are simple induction formulas for the other coefficients.

We call the formal expression

$$h^{\frac{1}{3}} w \sum_{l=0}^{\infty} h^l A_l(z) + h^{\frac{2}{3}} w' \sum_{l=1}^{\infty} h^l B_l(z) \qquad (5)$$

an *asymptotic solution* to (1).

For the differential equation $-\psi''(z)+v(z)\psi(z) = 0$, in a neighborhood of a simple turning point (a point where $v(z) = 0$ and $v'(z) \neq 0$), there are asymptotic solutions of the form (5) (with different coefficients A_l, B_l and a different function ζ), see, e.g., [6, 7]. To justify the Ansatz (5) in the case of the difference equation requires to develop original techniques. In particular, one has to derive asymptotic formulas of the form

$$\begin{aligned} w(h^{-\frac{2}{3}}\zeta(z+h)) = h^{\frac{1}{3}}\, f(z)\, w(h^{-\frac{2}{3}}\zeta(z)) \\ + h^{\frac{2}{3}}\, g(z)\, w'(h^{-\frac{2}{3}}\zeta(z)) + \dots \end{aligned} \qquad (6)$$

where $f(z) = \cosh(\sqrt{\zeta(z)}\zeta'(z))$ and the dots denote smaller order terms. Trying to obtain this formula using Taylor expansions for the left hand side leads to a series containing an infinite number of infinite subsequences of terms of same order. So, one has to find an effective way to resum these sequences. To get formulas similar to (6), instead of resumming Taylor series, we use tools of the complex analysis.

4.1.1 Exact solutions

Let us turn to exact analytic solutions to (1). The main result of this note is

Theorem 4 *There exists $\overset{\circ}{U} \subset U$ a neighborhood of z_0 independent of h such that the following holds. Pick $L \in \mathbb{N}$ and $w \in \{w_j;\ j \in \mathbb{Z}_3\}$. Let W be the function constructed in Theorem 3 for these L and w. For sufficiently small h, there exists a solution ψ to equation (1) that is analytic in $\overset{\circ}{U}$ and, in $\overset{\circ}{U}$, admits the uniform asymptotic representation*

$$\psi(z) = W(z) + O(h^{L+1+\frac{1}{3}} w) + O(h^{L+1+\frac{2}{3}} w'),$$

where w and w' are defined in (4).

Let us briefly explain the idea of the proof of this result. First, using the approximate solutions from Theorem 3, we construct a parametrix R, i.e., an operator such that, for suitable functions f, one has $HRf = f + Df$, where H is defined in (3), and D is a small operator. The operator D appears to be a singular integral operator. To estimate its norm, one has to deal with geometric objects natural for complex WKB method. This allows to prove Theorem 4 for some special subdomains of U. Then, one studies the thus constructed solutions in larger domains.

We note that, as equation (1) is non-local in z, the ideas underlying the analysis of (1) are quite different from the ideas used to get similar results for differential equations (see, e.g., [6, 7]).

4.2 Related results

In [13] and [14] the authors essentially studied equations of the form

$$Y_{k+1} = M(hk)Y_k, \quad k \in \mathbb{Z}, \qquad (7)$$

with a small positive h and a $(n \times n)$-matrix function M defined on $\mathbb{R}$. If

$$Y(x+h) = M(x)Y(x), \quad x \in \mathbb{R}, \qquad (8)$$

then the sequence $Y_k = Y(kh)$, $k \in \mathbb{Z}$, satisfies (7). Equation (1) restricted to $\mathbb{R}$ is equivalent to (8) with $M(x) = \begin{pmatrix} -v(x) & -1 \\ 1 & 0 \end{pmatrix}$, and the points x where the two eigenvalues of this matrix coalesce are branch points of the complex momentum.

The short note [13] is mainly devoted to the case when all the eigenvalues of the matrix M in (7) are distinct. In [14] the author constructed asymptotic solutions to (7) in a small as $h \to 0$ neighborhood of a point where two eigenvalues of $M(x)$ coincide.

In [15], the authors considered the equations

$$\sum_{j=0}^{J} a_j(hk,h)\, y_{k+j} = 0, \quad k \in \mathbb{Z}. \tag{9}$$

We note that this class includes the difference Schrödinger equations

$$y_{k+1} + y_{k-1} + v(hk)y_k = 0, \quad k \in \mathbb{Z}.$$

The authors described the asymptotics of solutions to (9) for hk in a small (as $h \to 0$) neighborhood of a point where $v(x) \in \{\pm 1\}$.

There is a series of papers by J.S. Geronimo and coauthors, see, e.g. [5] and references therein, devoted to uniform asymptotic formulas for solutions to the equation $a_{k+1}\psi_{k+1} + b_k\psi_k + a_k\psi_{k-1} = x\psi_k$, $k \in \mathbb{Z}$, where x is the spectral parameter, the coefficients $(a_n)_n$ are positive and $(b_n)_n$ are real.

We also mention paper [16] where the authors constructed approximate solutions to a difference equation using the Maslov canonical operator.

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