Complex WKB method for difference equations with meromorphic coefficients

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We discuss one-dimensional difference Schrödinger equations in the complex plane. To study their solutions in the quasiclassical limit, V. Buslaev, A. Fedotov and E. Shchetka developed an analog of the complex WKB method. We consider the equation with a potential having a simple pole and study the asymptotics of its solutions in a neighborhood of this pole.

1 Introduction

We consider the difference equation

$$\psi(z+h) + \psi(z-h) + v(z)\psi(z) = 0, \qquad (1)$$

where z is the complex variable, v is a given analytic or meromorphic function, and $h > 0$ is the translation parameter.

One encounters such equations, for example, in solid state physics when studying, say, an electron in a crystal in a weak magnetic field. The translation parameter h is proportional to the magnetic flux through the periodicity cell. In solid state physics, one has $v(z) = w(z) - E$, where the function w is called the potential and the parameter E is called the spectral parameter. The equation with $w(z) = \cos(z)$ is the famous Harper equation, see, e.g., [1]; the equation with $w(z) = \tan(z)$ is a close relative of the well-known Maryland equation introduced by D. Grempel, S. Fishman and R. Prange in [2].

Difference equations with the small translation parameters arise also when studying the scattering of waves by wedge-shaped domains in the framework of the Sommerfeld–Malyuzhinets method. The translation parameters appear to be proportional to the angles of the wedges, see [6].

We study the asymptotics of solutions to (1) as $h \to 0$. Since formally $\exp\left(h\frac{d}{dz}\right)\Psi(z) = \Psi(z+h)$, the parameter h in (1) can be regarded as a small parameter in front of the derivative and, thus, appears to be a standard quasiclassical parameter.

To study the one-dimensional differential equations in the quasiclassical limit, one uses the classical complex WKB method, see [3]. To study one-dimensional difference equations with analytic coefficients, in papers [4,5] their authors developed an analog of the complex WKB method.

In this paper, we consider the case of meromorphic v. To be more precise, we assume that B_0 is a neighborhood of $z = 0$ (here and below a neighborhood of a point is an open disc centered at this point), and that v is analytic in $B_0 \setminus \{0\}$ and has a simple pole at zero. Let ψ be a solution to (1) analytic in $B_0 \setminus \mathbb{R}_+$ ($\mathbb{R}_+$ denotes the set of positive real numbers). Equation (1) implies that $\psi(z) = -\psi(z-2h) - v(z-h)\psi(z-h)$. Therefore, for sufficiently small h, ψ can be meromorphically continued into B_0. After that, it can have poles at the points $z = h, 2h, 3h\ldots$

When h is small, these points become close one to another. We describe the quasiclassical asymptotics in B_0 of solutions to (1) having poles at $z = h, 2h, 3h\ldots$

2 A brief introduction to the complex WKB method

Let us briefly describe the main constructions of the complex WKB method for the difference equation (1) with an analytic coefficient v. We note that formally this equation can be written in the form

$$(2\cos\hat{p} + v(z))\psi(z) = 0, \quad \hat{p} = -ih\frac{d}{dz}\,. \qquad (2)$$

We define *the complex momentum* p by the formula

$$2\cos\, p(z) + v(z) = 0.$$

It is an analytic multivalued function. Its branch points satisfy the relations $\pm 2 + v(z) = 0$. We call a subset D of the domain of analyticity of v *regular*, if $v(z) \neq \pm 2$ in D.

Let D be a simply connected regular domain, and let p be a branch of the complex momentum analytic in D. All the other branches of p that are analytic in D are of the form $\pm p(z) + 2\pi m$, $m \in \mathbb{Z}$.

The complex momentum is the main analytic object of the complex WKB method. In terms of the complex momentum, one defines the *canonical domains* that are the main geometric objects of the method. The definition of the canonical domains can be found in [4,5]. Here, we note that the canonical domains are regular and simply connected, and that one has the following assertion.

Theorem 1 *Any regular point is contained in a canonical domain.*

This statement is an analog of Lemma 5.3 from [7]. The principle result of the method is

Theorem 2 ([5]) *Let $K \subset \mathbb{C}$ be a canonical domain, let $z_0 \in K$, and let p be a branch of the complex momentum analytic in K. For sufficiently small h, there exists ψ, a solution to* (1)*, analytic in K and such that in K as $h \to 0$*

$$\psi(z) = \frac{e^{\frac{i}{h}\int_{z_0}^{z} p(z)\,dz + o(1)}}{\sqrt{\sin(p(z))}}\,. \qquad (3)$$

The asymptotics is locally uniform in z.

3 A continuation principle

Let z_0 be a regular point, and let V_0 be its regular neighborhood. Assume that there is ψ, a solution to (1) that is analytic in V_0 and admits in V_0 the uniform asymptotic representation (3).

There exist general statements (Continuation principles) allowing to describe the asymptotics of ψ outside of V_0. One of them is

Theorem 3 (The rectangle lemma) *Let $z_1 \in V_0$. Consider the straight line $L = \{z \in \mathbb{C} : \operatorname{Im} z = \operatorname{Im} z_1\}$. Let $z_2 \in L$, $\operatorname{Re} z_2 > \operatorname{Re} z_1$, and let the segment $[z_1, z_2] = \{z \in L : \operatorname{Re} z_1 \leq \operatorname{Re} z \leq \operatorname{Re} z_2\}$ be regular. If $\operatorname{Im} p(z) < 0$ along $[z_1, z_2]$, then the asymptotic representation* (3) *is valid and uniform in an independent of h regular neighborhood of $[z_1, z_2]$.*

This theorem is an analog of Lemma 5.1 from [7]. It roughly says that the asymptotic formula (3) stays valid along a horizontal line as long as ψ exponentially grows.

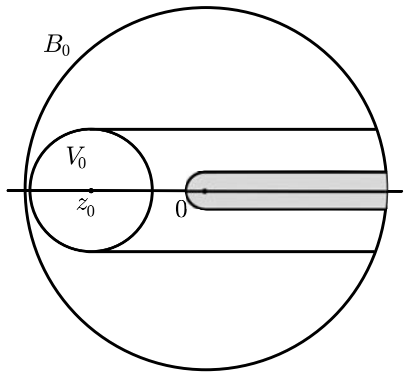

Figure 1: B_0, V_0, the δ-neighborhood of $\mathbb{R}_+$ (the δ-neighborhood is shaded in).

4 Typical quasiclassical formulation of the problem

For the sake of simplicity, we additionally assume that $v(z) \in \mathbb{R}$ for $z \in \mathbb{R}$. In this case, if B_0 is chosen sufficiently small, then: (1) in B_0, $v(z) \in \mathbb{R}$ only for $z \in \mathbb{R}$; (2) the set $B_0 \setminus \{0\}$ is regular. We assume that B_0 has these two properties.

Let p be a branch of the complex momentum analytic in $B_0 \setminus \mathbb{R}_+$. For B_0 under consideration, $\operatorname{Im} p(z)$ does not vanish in $B_0 \setminus \mathbb{R}_+$. For the sake of definiteness, we assume that it is negative.

In B_0 we pick $z_0 < 0$, see Fig. 1. In view of Theorem 1, there is a regular neighborhood $V_0 \subset B_0$ of z_0 such that there exists a solution ψ to equation (1) that is analytic in V_0 and admits representation (3) there.

Let $\delta > 0$ be sufficiently small. Let B_δ be the domain B_0 without the δ-neighborhood of $\mathbb{R}_+$. As $\operatorname{Im} p(z) < 0$ in $B_0 \setminus \mathbb{R}_+$, then by Theorem 3 asymptotics (3) is valid and uniform in B_δ to the right from V_0. The problem is to describe ψ in the δ-neighborhood of $\mathbb{R}_+$.

5 The main result

Below, instead of saying that a function can be analytically continued into a domain, we say that it is analytic in this domain.

The complex momentum p has a logarithmic branch point at zero. One can check the following assertion.

Lemma 1 *In $B_0 \setminus \mathbb{R}_+$ we fix analytic branches of* $\ln$ *and* p*. The function $z \mapsto p(z) - i\ln z$ is analytic in B_0. The function $z \mapsto z \sin p(z)$ is analytic and does not vanish in B_0.*

For $z \in B_0 \setminus \mathbb{R}_+$, we set

$$U_0(z) = \sqrt{\frac{h}{-2\pi z \sin p(z)}} \times \exp\left\{\frac{z}{h}\ln\frac{1}{h} + \frac{i}{h}\int_0^z (p(z) - i\ln(-z))\,dz\right\}.$$

Here p, $\sqrt{\sin p}$, $z \mapsto \ln(-z)$ and $z \mapsto \sqrt{-z}$ are functions analytic in $B_0 \setminus \mathbb{R}_+$. In V_0, p and $\sqrt{\sin p}$ coincide with p and $\sqrt{\sin p}$ in (3). The functions $z \mapsto \ln(-z)$ and $z \mapsto \sqrt{-z}$ satisfy the conditions $\ln(-z)|_{z=-1} = 0$ and $\sqrt{-z}|_{z=-1} = 1$. By Lemma 1, U_0 is analytic in B_0.

Our main result is

Theorem 4 *Let $\delta > 0$ be sufficiently small. In the δ-neighborhood of $\mathbb{R}_+$, as $h \to 0$, the solution ψ admits the uniform asymptotic representation*

$$\psi(z) = \Gamma\left(1 - \frac{z}{h}\right) U_0(z)\, e^{\frac{i}{h}\int_{z_0}^0 p\,dz + o(1)}. \quad (4)$$

So, the special function describing the asymptotic behavior of ψ near the poles generated by the pole of the potential at $z = 0$ is the Euler Γ-function.

Near the point $z = 0$, the formula (4) can not be simplified. For large values of $|z/h|$, the Γ-function in (4) can be replaced with its asymptotics. To be more precise, let us pick $\epsilon > 0$. By means of the asymptotic formula

$$\Gamma(1+\zeta) = \sqrt{2\pi\zeta}\, e^{\zeta(\ln\zeta - 1) + o(1)}, \ |\zeta| \to \infty, \quad (5)$$

that is uniform in the sector $|\arg\zeta| \le \pi - \epsilon$, one checks that if $|\arg z - \pi| \le \pi - \epsilon$ and, say, $|z| \ge \delta/2$, representation (4) turns into (3).

Now let us discuss the case where $|z| \ge \delta/2$ and $|\arg z| \le \epsilon$. In this case, to simplify (4), first we use the relation $\Gamma(1-\zeta) = \frac{\pi}{\sin(\pi\zeta)} \frac{1}{\Gamma(\zeta)}$ and, next, the asymptotic representation (5). This leads to the uniform asymptotic representation

$$\psi(z) = \frac{e^{\frac{i}{h}\int_{z_0}^z p(z)\,dz + o(1)}}{(1 - e^{-2\pi i z/h})\sqrt{\sin(p(z))}}, \quad h \to 0,$$

where p, $z \mapsto \int_0^z p(z)\,dz$ and $\sqrt{\sin(p)}$ are obtained by analytic continuation in the anticlockwise direction from $B_0 \setminus \mathbb{R}_+$ into the sector under consideration.

6 The ideas of the proof

Let us note that the function

$$f(z) = \psi(z)/(\Gamma(1 - z/h)U_0(z))$$

is analytic in a neighborhood of zero independent of h, say, a disc of radius $\delta > 0$. Therefore, to prove (4), due to the maximum principle, it suffices to check that $f(z) = 1 + o(1)$ for, say, $|z| = \delta/2$. This is done by means of a rather standard asymptotic computation made using the complex WKB method for difference equations.

7 Acknowledgments

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