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A remark on Sarnak’s conjecture

Régis de la Bretèche & Gérald Tenenbaum

Abstract. We investigate Sarnak’s conjecture on the Möbius function in the special case when the test function is the indicator of the set of integers for which a real additive function assumes a given value.

Keywords: Sarnak’s conjecture, Möbius function, complexity, additive functions, concentration of additive functions, Halász mean value theorem, mean values of multiplicative functions.

1. Introduction and statements of results

According to a general pseudo-randomness principle related to a famous conjecture of Chowla [1] and recently considered by Sarnak [7], the Möbius function $\mu$ does not correlate with any function $\varphi$ of low complexity. In other words,

\[
\sum_{n \leq x} \mu(n) \xi(n) = o\left(\sum_{n \leq x} |\xi(n)|\right) \quad (x \to \infty).
\]

(1.1)

There are many ways of constructing functions of low complexity. Sarnak and others use return times of sampling sequences of a dynamical system, which leads to a natural measure of the complexity. Here we propose to follow another path by selecting the test-function as the indicator of the set of those integers where a real additive function assumes a given value. It is known since Halász [5] that

\[
Q(x; f, \mu) := \sup_{m \in \mathbb{R}} \left\{ \sum_{n \leq x, f(n) = m} 1 \ll \frac{x}{\sqrt{1 + E(x)}} \right\}
\]

where we have put

\[
E(x) := \sum_{p \leq x, f(p) \neq 0} \frac{1}{p}.
\]

(1.2)

Here and in the sequel, the letter $p$ denotes a prime number.

The estimate (1.2) is known to be optimal in this generality since the two sides achieve the same order of magnitude when $f(n)$ is equal to the total number of prime factors of $n$, counted with or without multiplicity.

As a first investigation of the above described problem, we would like to show that

\[
Q(x; f, \mu) := \sup_{m \in \mathbb{R}} \left| \sum_{n \leq x, f(n) = m} \mu(n) \right|
\]

is generically smaller than the right-hand side of (1.2). Of course we have to avoid the case when $f(p)$ is constant, for then $\mu(n)$ does not oscillate on the set of squarefree integers $n$ with $f(n) = m$. Therefore we seek an estimate which coincides with (1.2) when $f(p)$ is close to a constant and which has smaller order of magnitude otherwise.

When $f(p)$ is restricted to assume the values 0 or 1 only, we thus expect a significant improvement over (1.2) when

\[
F(x) := \sum_{p \leq x} \frac{1 - f(p)}{p}
\]

is large. Indeed, in this simple case we obtain the following estimate.
Theorem 1.1. Let $f$ denote a real additive arithmetic function such that $f(p) \in \{0, 1\}$ for all $p$. Then, with the above notation and $c = (2\pi - 4)/(3\pi - 2) \approx 0.30751$, we have

\begin{equation}
Q(x; f, \mu) \ll \frac{x(1 + F(x))e^{-cF(x)}}{\sqrt{1 + E(x)}}.
\end{equation}

For simplicity, let us retain in the sequel the hypothesis $f(p) \in \{0, 1\}$. Under the assumption that $F(x)$, as defined in (1.3) above, grows sufficiently slowly, we may prove an estimate that is valid for each $m$ in a large range around the mean, and so may be stated in the exact frame of Sarnak’s conjecture.

Let us denote by $N_m(x; f)$ the number of squarefree integers not exceeding $x$ such that $f(n) = m$. It follows from results of Halász [3], [4], and Sárkőzy [6] that, given any $\kappa \in [0, 1]$, we have

\begin{equation}
N_m(x; f) \asymp x \frac{E(x)^m}{m!} e^{-E(x)} \quad (\kappa E(x) \leq m \leq E(x)/\kappa).
\end{equation}

Moreover, Halász announced (see [2], p. 312) the possibility to obtain, in the same range for $m$, an asymptotic formula for $N_m(x; f)$, a result which actually follows, as shown in [10], from a general effective mean value estimate for multiplicative functions established in the same work—see below.

This supports the hope to obtain an asymptotic formula for

$$N_m(x; f, \mu) := \sum_{\substack{n \leq x \\quad \mu(n)}} \mu(n)$$

which directly compares to (1.5). In view of (1.1), we may assume with no loss of generality that $f$ is strongly additive. We obtain the following result. Here and in the sequel we let $\log_k$ denote the $k$-fold iterated logarithm.

Theorem 1.2. Let $\kappa \in [0, 1]$ and let $f$ denote a strongly additive function such that $f(p) \in \{0, 1\}$ for all primes $p$. Assume furthermore that

\begin{equation}
(1 - f(p)/p) \ll \log_3 x \quad (x \to \infty)
\end{equation}

\begin{equation}
\frac{1 - f(p)}{p} \log_{\log_2 x} y \ll \frac{(\log y)\log_{\log_2 x^{c_0}}(x^{1/(\log_2 x)^D})}{\log_{\log_2 x}} \quad (x^{1/(\log_2 x)^D} < y \leq x)
\end{equation}

where $D$ and $c_0$ are positive constants. Provided $D$ is sufficiently large and uniformly in the range $\kappa E(x) \leq m \leq E(x)/\kappa$, we have

\begin{equation}
N_m(x; f, \mu) = (-1)^mN_m(x; f)\left\{\lambda_f e^{-2F(x)} + O\left(\frac{1}{(\log_2 x)^b}\right)\right\},
\end{equation}

with

\begin{equation}
\lambda_f := \prod_{f(p) = 0} \frac{1 - 1/p}{1 + 1/p} e^{2/p}, \quad b := \frac{1}{2} \min\{1, c_0\kappa/(4 - \kappa)\}.
\end{equation}

1. All our results could be straightforwardly adapted to case when $f(p)$ is restricted to a fixed, finite set, or even to a set of moderate size depending on $x$. 
A remark on Sarnak’s conjecture

To fix ideas, note that a strongly additive function $f$ such that $f(p) \in \{0, 1\}$ satisfies hypotheses (1.6) and (1.7) as soon as

$$\sum_{p \leq y} \{1 - f(p)\} \log p \ll \frac{y}{(\log_2 y)^{\max(1, c_0)}}.$$  

The proof of Theorem 1.2 rests on the following recent result of the second author [10] (theorem 1.4), for the statement of which we introduce further notation. We let $M(A, B)$ designate the class of those complex-valued multiplicative functions $g$ such that

$$\max_p |g(p)| \leq A, \quad \sum_{p, \nu \geq 2} \frac{|g(p^\nu)| \log p^\nu}{p^\nu} \leq B,$$

and, for $b \in \mathbb{R}$, we write

$$\beta_0 = \beta_0(b, A) := 1 - \frac{\sin(2\pi b/A)}{2\pi b/A}.$$  

Moreover, given a complex-valued function $g$, we put $w_g := 1$ if $g$ is real, $w_g := \frac{1}{2}$ otherwise, and write

$$M(x; g) := \sum_{n \leq x} g(n), \quad Z(x, g) := \sum_{p \leq x} \frac{g(p)}{p}.$$  

**Theorem 1.3 ([10]).** Let

$$a \in \left[0, \frac{1}{4}\right], \quad b \in \left[a, \frac{1}{2}\right], \quad h := (1 - b)/b, \quad A \geq 2b, \quad B > 0, \quad \beta := \beta_0(b, A), \quad x \geq 2, \quad 1/\sqrt{\log x} < \varepsilon \leq \frac{1}{2},$$

and let the multiplicative functions $g, r$, such that $r \in M(x; 2A, B), |g| \leq r$, satisfy the conditions

$$\sum_{p \leq x} \frac{r(p) - \Re g(p)}{p} \leq \frac{1}{2} \beta b \log(1/\varepsilon),$$

$$\sum_{x^z < p \leq y} \frac{(r(p) - \Re g(p))^b \log p}{p} \ll \varepsilon^b \log y \quad (x^z < y \leq x),$$

with $\delta \in \left[a, \frac{1}{2} \beta b\right]$, and

$$\min_{x^z < p \leq x} r(p) \geq 4b.$$  

We then have

$$M(x; g) = M(x; r) \prod_p \frac{\sum_{p^\nu \leq x} g(p^\nu)/p^\nu}{\sum_{p^\nu \leq x} r(p^\nu)/p^\nu} + O\left(\frac{x \varepsilon^\delta \log y}{\log x}\right)$$

where $\varepsilon := b/A$. The implicit constant in (1.15) depends at most upon $A$, $B$, $a$, and $b$.

**2. Proof of Theorem 1.1**

As noted by Halász [5], we may assume that $f$ is integer-valued. (Note, however, that a slight modification of his construction is needed to ensure that changing the range of $f$ does not create new coincidences.) With this reduction, we plainly have

$$Q(x; f; \mu) \leq \int_{-1/2}^{1/2} |M(x; \vartheta)| \, d\vartheta$$

with

$$M(x; \vartheta) := \sum_{n \leq x} \mu(n) e^{2\pi i \vartheta n}.$$  

From Corollary III.4.12 in [8], we get, uniformly for $\vartheta \in \mathbb{R}$, $T \geq 1$, $x \geq 1$,

\begin{equation}
M(x; \vartheta) \ll \frac{x \{1 + m(x; \vartheta, T)\}}{e^{m(x; \vartheta, T)}} + \frac{x}{T},
\end{equation}

where we have put

\[m(x; \vartheta, T) := \min_{|\tau| \leq T} \sum_{p \leq x} \frac{1 + \cos(2\pi \vartheta f(p) \tau \log p)}{p} .\]

We select $T := \log x$, so that the second term on the right of (2.1) is negligible compared to the upper bound in (1.4). Let $h_{\vartheta}$ defined by

\[h_{\vartheta}(t) := 1 + \min\{\cos(t), \cos(2\pi \vartheta - t)\} \quad (t \in \mathbb{R}),\]

so that

\[s_{\vartheta} := \frac{1}{2\pi} \int_{-\pi}^{\pi} h_{\vartheta}(t) \, dt = 1 - \frac{2}{\pi} \sin(\pi \vartheta) \quad (\vartheta \in [-\frac{1}{2}, \frac{1}{2}] ),\]

and, for suitable $\tau \in [-T, T]$,

\[m(x; \vartheta, T) \geq \sum_{p \leq x} \frac{h_{\vartheta}(\tau \log p)}{p} .\]

The right-hand side may be estimated via partial summation as made explicit in lemma III.4.13 of [8]. For any $w \in [2, x]$ and $\vartheta \in [-\frac{1}{2}, \frac{1}{2}]$, we have

\begin{equation}
\sum_{w \leq p \leq x} \frac{h_{\vartheta}(\tau \log p)}{p} = s_{\vartheta} \log \left( \frac{\log x}{\log w} \right) + O \left( \frac{1}{w \log x} + \frac{1 + |\tau|}{e^{\sqrt{\log w}}} \right) .
\end{equation}

If $1 \leq |\tau| \leq T$, we select $w := (\log_2 x)^2$ to obtain

\[m(x; \vartheta, T) \geq s_{\vartheta} \log_2 x + O(\log_3 x) .\]

Next, set

\[\log v := (\log x) \exp \left\{ -\frac{2 \cos^2(\pi \vartheta) E(x) + 2 F(x)}{2 + s_{\vartheta}} \right\} .\]

If $1/\log v < |\tau| \leq 1$, we put $w := v$ in (2.2) and get

\[\sum_{w \leq p \leq x} \frac{h_{\vartheta}(\tau \log p)}{p} \geq \frac{2 s_{\vartheta} \cos^2(\pi \vartheta)}{2 + s_{\vartheta}} E(x) + \frac{2 s_{\vartheta}}{2 + s_{\vartheta}} F(x) + O(1) .\]

And finally, if $|\tau| \leq 1/\log v$, we have trivially

\[\sum_{p \leq v} \frac{1 + \cos(2\pi \vartheta f(p) \tau \log p)}{p} = \sum_{p \leq v} \frac{1 + \cos(2\pi \vartheta f(p) \tau \log p)}{p} + O(1)
\]

\[= (1 + \cos(2\pi \vartheta)) \sum_{p \leq v} \frac{1}{p} + 2 \sum_{p \leq v} \frac{1}{p} + O(1)
\]

\[\geq 2 \cos^2(\pi \vartheta) E(x) + 2 F(x) - 2 \log \left( \frac{\log x}{\log v} \right) + O(1)
\]

\[\geq \frac{2 s_{\vartheta} \cos^2(\pi \vartheta)}{2 + s_{\vartheta}} E(x) + \frac{2 s_{\vartheta}}{2 + s_{\vartheta}} F(x) + O(1) .\]

Therefore, we get in all cases

\begin{equation}
m(x; \vartheta, T) \geq \frac{2 s_{\vartheta} \cos^2(\pi \vartheta)}{2 + s_{\vartheta}} E(x) + \frac{2 s_{\vartheta}}{2 + s_{\vartheta}} F(x) + O(1)
\end{equation}

\[\geq c \cos^2(\pi \vartheta) E(x) + c F(x) + O(1).
\]

Integrating over $\vartheta$ immediately yields the result stated. \hfill \Box
3. Proof of Theorem 1.2

Let us introduce the multiplicative function \( g(n) := \mu(n)z^{\ell(n)} \) with \( z := -e^{2\pi i \vartheta}, |\vartheta| \leq \frac{1}{2}, \) \( \kappa \leq \theta \leq 1/\kappa. \) Put \( r(n) := \mu(n)^2\rho^{\ell(n)}. \) From (2.3), we see that, with \( c \) as in the statement of Theorem 1.1,

\[
\sum_{p \leq x} \frac{r(p) - 2\Re g(p)p^{1/2}}{p} \geq cq \sin^2(\pi \vartheta)E(x) + cqF(x) + O(1) \quad (|\tau| \leq T := \log x).
\]

We may therefore apply Corollary 2.1 of [10] to get

\[
M(x; g) \ll M(x; r) \left\{ e^{-cqE(x)} \sin^2(\pi \vartheta) - c\theta F(x) \log x + \frac{1}{(\log x)^\kappa} \right\}.
\]

With \( \vartheta \) small enough in terms of \( \kappa, x \), and so we may rewrite (3.12) is plainly fulfilled with \( \varepsilon := |\vartheta|^{2/\delta} + (\log x)^{-\theta/(\delta \vartheta)} \) provided \( \delta \) is chosen sufficiently small in terms of \( b, \kappa \) and \( x \). Next, for \( x^2 < y \leq x \), we have

\[
\sum_{x^2 < p \leq y} \frac{\{r(p) - 2\Re g(p)\}b \log p}{p} \ll b^\delta \log y + b \sum_{x^2 < p \leq y} \frac{\log p}{p} \ll b \delta \log y + b^\delta y,
\]

so hypothesis (1.13) is also verified. Since (1.14) holds trivially on selecting \( b := \kappa/4 \), and hence \( b = 4/\kappa - 1 \), we conclude that (1.15) is valid. We obtain, with \( c := \kappa b \),

\[
M(x; g) = M(x; r) \prod_{p \leq x, f(p) = 1} \frac{1 - z/p}{1 + \varepsilon/p} \prod_{p \leq x, f(p) = 0} \frac{1 - 1/p}{1 + 1/p} + O \left( \frac{x^2 e^{2(x; r)} - \varepsilon e^{2(x; r) - \varepsilon}}{\log x} \right).
\]

Now, appealing for instance to theorem 1.1 of [9], we observe that

\[
M(x; r) \asymp \frac{x e^{2(x; r)}}{\log x}
\]

and so we may rewrite (3.2) as

\[
M(x; g) = M(x; r) \left\{ \lambda_r e^{-(z+\varepsilon)E(x) - 2F(x)} + O \left( (|\vartheta| + (\log x)^{-\theta/2} \varepsilon) e^{-\varepsilon \theta^2 E(x) - \varepsilon_1 F(x)} \right) \right\},
\]
valid for $|\theta| \leq \vartheta_0$ and some constant $c_1 > 0$. Integrating on the circle $|z| = \varrho := m/E(x)$ and taking (3-1) into account, we readily obtain in the stated range for $m$,

$$N_m(x; f, \mu) = (-1)^m \int_{-1/2}^{1/2} e^{-2i\pi \varrho m} \varrho^{-m} M(x; \rho) d\varrho$$

(3-3)

$$= (-1)^m \lambda_f M(x; r) \frac{E(x)^m}{m! e^m} \left\{ e^{-2F(x)} + O\left( \frac{e^{-c_2 F(x)}}{\log_2 x} \right) \right\},$$

with $c_2 := \min(c_1, \kappa)$. Since, by a straightforward variant of corollary 2.4 of [10]($^2$),

$$N_m(x; f) = M(x; r) \frac{E(x)^m}{m! e^m} \left\{ 1 + O\left( \frac{1}{\sqrt{\log_2 x}} \right) \right\},$$

we reach the required conclusion.

References


2. Applied to $\omega(n; E)$ instead of $\Omega(n; E)$ with the notation of [10].