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Abstract. We investigate Sarnak's conjecture on the Möbius function in the special case when the test function is the indicator of the set of integers for which a real additive function assumes a given value.

Keywords: Sarnak's conjecture, Möbius function, complexity, additive functions, concentration of additive functions, Halász mean value theorem, mean values of multiplicative functions.

1. Introduction and statements of results

According to a general pseudo-randomness principle related to a famous conjecture of Chowla [1] and recently considered by Sarnak [7], the Möbius function μ does not correlate with any function ξ of low complexity. In other words,

$$\sum_{n\leqslant x}\mu(n)\xi(n)=o\Big(\sum_{n\leqslant x}|\xi(n)|\Big)\qquad(x\to\infty). \tag{1·1}$$

There are many ways of constructing functions of low complexity. Sarnak and others use return times of sampling sequences of a dynamical system, which leads to a natural measure of the complexity. Here we propose to follow another path by selecting the test-function as the indicator of the set of those integers where a real additive function assumes a given value. It is known since Halász [5] that

$$Q(x;f):=\sup_{m\in\mathbb{R}}\sum_{\substack{n\leqslant x\\ f(n)=m}}1\ll\frac{x}{\sqrt{1+E(x)}} \tag{1·2}$$

where we have put

$$E(x):=\sum_{\substack{p\leqslant x\\ f(p)\neq 0}}\frac{1}{p}\cdot$$

Here and in the sequel, the letter p denotes a prime number.

The estimate (1·2) is known to be optimal in this generality since the two sides achieve the same order of magnitude when $f(n)$ is equal to the total number of prime factors of n, counted with or without multiplicity.

As a first investigation of the above described problem, we would like to show that

$$Q(x;f,\mu):=\sup_{m\in\mathbb{R}}\Bigg|\sum_{\substack{n\leqslant x\\ f(n)=m}}\mu(n)\Bigg|$$

is generically smaller than the right-hand side of (1·2). Of course we have to avoid the case when $f(p)$ is constant, for then $\mu(n)$ does not oscillate on the set of squarefree integers n with $f(n)=m$. Therefore we seek an estimate which coincides with (1·2) when $f(p)$ is close to a constant and which has smaller order of magnitude otherwise.

When $f(p)$ is restricted to assume the values 0 or 1 only, we thus expect a significant improvement over (1·2) when

$$F(x):=\sum_{p\leqslant x}\frac{1-f(p)}{p} \tag{1·3}$$

is large. Indeed, in this simple case we obtain the following estimate.

Theorem 1.1. *Let f denote a real additive arithmetic function such that $f(p)\in\{0,1\}$ for all p. Then, with the above notation and $c=(2\pi-4)/(3\pi-2)\approx 0.30751$, we have*

$$Q(x;f,\mu)\ll\frac{x\{1+F(x)\}\mathrm{e}^{-cF(x)}}{\sqrt{1+E(x)}}\cdot \tag{1·4}$$

For simplicity, let us retain in the sequel the hypothesis $f(p)\in\{0,1\}$.[1] Under the assumption that $F(x)$, as defined in (1·3) above, grows sufficiently slowly, we may prove an estimate that is valid for each m in a large range around the mean, and so may be stated in the exact frame of Sarnak's conjecture.

Let us denote by $N_m(x;f)$ the number of squarefree integers not exceeding x such that $f(n)=m$. It follows from results of Halász [3], [4], and Sárközy [6] that, given any $\kappa\in]0,1[$, we have

$$N_m(x;f)\asymp x\frac{E(x)^m}{m!}\mathrm{e}^{-E(x)}\qquad\big(\kappa E(x)\leqslant m\leqslant E(x)/\kappa\big). \tag{1·5}$$

Moreover, Halász announced (see [2], p. 312) the possibility to obtain, in the same range for m, an asymptotic formula for $N_m(x;f)$, a result which actually follows, as shown in [10], from a general effective mean value estimate for multiplicative functions established in the same work—see below.

This supports the hope to obtain an asymptotic formula for

$$N_m(x;f,\mu):=\sum_{\substack{n\leqslant x\\ f(n)=m}}\mu(n)$$

which directly compares to (1·5). In view of (1·1), we may assume with no loss of generality that f is strongly additive. We obtain the following result. Here and in the sequel we let $\log_k$ denote the k-fold iterated logarithm.

Theorem 1.2. *Let $\kappa\in]0,1[$ and let f denote a strongly additive function such that $f(p)\in\{0,1\}$ for all primes p. Assume furthermore that*

$$F(x):=\sum_{p\leqslant x}\frac{1-f(p)}{p}\ll\log_3 x\qquad(x\to\infty) \tag{1·6}$$

$$\sum_{\exp\{(\log x)/(\log_2 x)^D\}<p\leqslant y}\frac{\{1-f(p)\}\log p}{p}\ll\frac{(\log y)}{(\log_2 x)^{c_0}}\qquad\big(x^{1/(\log_2 x)^D}<y\leqslant x\big) \tag{1·7}$$

where D and c_0 are positive constants. Provided D is sufficiently large and uniformly in the range $\kappa E(x)\leqslant m\leqslant E(x)/\kappa$, we have

$$N_m(x;f,\mu)=(-1)^mN_m(x;f)\bigg\{\lambda_f\mathrm{e}^{-2F(x)}+O\bigg(\frac{1}{(\log_2 x)^b}\bigg)\bigg\}, \tag{1·8}$$

with

$$\lambda_f:=\prod_{f(p)=0}\frac{1-1/p}{1+1/p}\mathrm{e}^{2/p},\quad b:=\tfrac12\min\{1,c_0\kappa/(4-\kappa)\}. \tag{1·9}$$

1. All our results could be straightforwardly adapted to case when $f(p)$ is restricted to a fixed, finite set, or even to a set of moderate size depending on x.

To fix ideas, note that a strongly additive function f such that $f(p) \in \{0,1\}$ satisfies hypotheses (1·6) and (1·7) as soon as

$$\sum_{p\leqslant y}\{1-f(p)\}\log p \ll \frac{y}{(\log_2 y)^{\max(1,c_0)}}\cdot$$

The proof of Theorem 1.2 rests on the following recent result of the second author [10] (theorem 1.4), for the statement of which we introduce further notation. We let $\mathcal{M}(A,B)$ designate the class of those complex-valued multiplicative functions g such that

$$\max_p |g(p)| \leqslant A, \qquad \sum_{p,\,\nu\geqslant 2}\frac{|g(p^\nu)|\log p^\nu}{p^\nu} \leqslant B, \tag{1·10}$$

and, for $\mathfrak{b}\in\mathbb{R}$, we write

$$\beta_0 = \beta_0(\mathfrak{b},A) := 1 - \frac{\sin(2\pi\mathfrak{b}/A)}{2\pi\mathfrak{b}/A}\cdot \tag{1·11}$$

Moreover, given a complex-valued function g, we put $w_g := 1$ if g is real, $w_g := \frac12$ otherwise, and write

$$M(x;g) := \sum_{n\leqslant x} g(n), \qquad Z(x,g) := \sum_{p\leqslant x}\frac{g(p)}{p}\cdot$$

Theorem 1.3 ([10]). *Let*

$$\mathfrak{a}\in]0,\tfrac14], \quad \mathfrak{b}\in[\mathfrak{a},\tfrac12[, \quad \mathfrak{h}:=(1-\mathfrak{b})/\mathfrak{b}, \quad A\geqslant 2\mathfrak{b}, \quad B>0, \quad \beta:=\beta_0(\mathfrak{b},A),$$
$$x\geqslant 2, \quad 1/\sqrt{\log x}<\varepsilon\leqslant\tfrac12,$$

and let the multiplicative functions g, r, such that $r\in\mathcal{M}(x;2A,B)$, $|g|\leqslant r$, satisfy the conditions

$$\sum_{p\leqslant x}\frac{r(p)-\Re e\, g(p)}{p} \leqslant \tfrac12\beta\mathfrak{b}\log(1/\varepsilon), \tag{1·12}$$

$$\sum_{x^\varepsilon<p\leqslant y}\frac{\{r(p)-\Re e\, g(p)\}^{\mathfrak{h}}\log p}{p} \ll \varepsilon^{\delta\mathfrak{h}}\log y \qquad (x^\varepsilon<y\leqslant x), \tag{1·13}$$

with $\delta\in[\mathfrak{a},\frac13\beta\mathfrak{b}]$, and

$$\min_{x^\varepsilon<p\leqslant x} r(p) \geqslant 4\mathfrak{b}. \tag{1·14}$$

We then have

$$M(x;g) = M(x;r)\prod_p \frac{\sum_{p^\nu\leqslant x} g(p^\nu)/p^\nu}{\sum_{p^\nu\leqslant x} r(p^\nu)/p^\nu} + O\Big(\frac{x\,\varepsilon^{w_g\delta}\mathrm{e}^{Z(x;r)-\mathfrak{c}Z(x;|g|-g)}}{\log x}\Big) \tag{1·15}$$

where $\mathfrak{c}:=\mathfrak{b}/A$. The implicit constant in (1·15) *depends at most upon A, B, $\mathfrak{a}$, and $\mathfrak{b}$.*

2. Proof of Theorem 1.1

As noted by Halász [5], we may assume that f is integer-valued. (Note, however, that a slight modification of his construction is needed to ensure that changing the range of f does not create new coincidences.) With this reduction, we plainly have

$$Q(x;f,\mu) \leqslant \int_{-1/2}^{1/2} |M(x;\vartheta)|\,\mathrm{d}\vartheta$$

with

$$M(x;\vartheta) := \sum_{n\leqslant x}\mu(n)\mathrm{e}^{2\pi i\vartheta f(n)}.$$

From Corollary III.4.12 in [8], we get, uniformly for $\vartheta \in \mathbb{R}$, $T \geqslant 1$, $x \geqslant 1$,

$$M(x;\vartheta) \ll \frac{x\{1+m(x;\vartheta,T)\}}{\mathrm{e}^{m(x;\vartheta,T)}} + \frac{x}{T}, \tag{2·1}$$

where we have put

$$m(x;\vartheta,T) := \min_{|\tau|\leqslant T} \sum_{p\leqslant x} \frac{1+\cos(2\pi\vartheta f(p)-\tau\log p)}{p}\cdot$$

We select $T := \log x$, so that the second term on the right of (2·1) is negligible compared to the upper bound in (1·4). Let h_ϑ defined by

$$h_\vartheta(t) := 1 + \min\{\cos(t), \cos(2\pi\vartheta - t)\} \qquad (t \in \mathbb{R}),$$

so that

$$s_\vartheta := \frac{1}{2\pi}\int_{-\pi}^{\pi} h_\vartheta(t)\,\mathrm{d}t = 1 - \frac{2}{\pi}\big|\sin\big(\pi\vartheta\big)\big| \qquad (\vartheta \in [-\tfrac12, \tfrac12]),$$

and, for suitable $\tau \in [-T,T]$,

$$m(x;\vartheta,T) \geqslant \sum_{p\leqslant x} \frac{h_\vartheta(\tau\log p)}{p}\cdot$$

The right-hand side may be estimated via partial summation as made explicit in lemma III.4.13 of [8]. For any $w \in [2,x]$ and $\vartheta \in [-\frac12, \frac12]$, we have

$$\sum_{w<p\leqslant x} \frac{h_\vartheta(\tau\log p)}{p} = s_\vartheta \log\Big(\frac{\log x}{\log w}\Big) + O\bigg(\frac{1}{w\log x} + \frac{1+|\tau|}{\mathrm{e}^{\sqrt{\log w}}}\bigg). \tag{2·2}$$

If $1 \leqslant |\tau| \leqslant T$, we select $w := (\log_2 x)^2$ to obtain

$$m(x;\vartheta,T) \geqslant s_\vartheta \log_2 x + O(\log_3 x).$$

Next, set

$$\log v := (\log x)\exp\bigg\{-\frac{2\cos^2(\pi\vartheta)E(x)+2F(x)}{2+s_\vartheta}\bigg\}.$$

If $1/\log v < |\tau| \leqslant 1$, we put $w := v$ in (2·2) and get

$$\sum_{v<p\leqslant x} \frac{h_\vartheta(\tau\log p)}{p} \geqslant \frac{2s_\vartheta\cos^2(\pi\vartheta)}{2+s_\vartheta}E(x) + \frac{2s_\vartheta}{2+s_\vartheta}F(x) + O(1).$$

And finally, if $|\tau| \leqslant 1/\log v$, we have trivially

$$\begin{aligned}\sum_{p\leqslant v}\frac{1+\cos(2\pi\vartheta f(p)-\tau\log p)}{p} &= \sum_{p\leqslant v}\frac{1+\cos(2\pi\vartheta f(p))}{p} + O(1)\\ &= (1+\cos(2\pi\vartheta))\sum_{\substack{p\leqslant v\\ f(p)=1}}\frac1p + 2\sum_{\substack{p\leqslant v\\ f(p)=0}}\frac1p + O(1)\\ &\geqslant 2\cos^2(\pi\vartheta)E(x) + 2F(x) - 2\log\Big(\frac{\log x}{\log v}\Big) + O(1)\\ &\geqslant \frac{2s_\vartheta\cos^2(\pi\vartheta)}{2+s_\vartheta}E(x) + \frac{2s_\vartheta}{2+s_\vartheta}F(x) + O(1).\end{aligned}$$

Therefore, we get in all cases

$$\begin{aligned}m(x;\vartheta,T) &\geqslant \frac{2s_\vartheta\cos^2(\pi\vartheta)}{2+s_\vartheta}E(x) + \frac{2s_\vartheta}{2+s_\vartheta}F(x) + O(1)\\ &\geqslant c\cos^2(\pi\vartheta)E(x) + cF(x) + O(1).\end{aligned} \tag{2·3}$$

Integrating over ϑ immediately yields the result stated. □

3. Proof of Theorem 1.2

Let us introduce the multiplicative function $g(n) := \mu(n)z^{f(n)}$ with $z := -\varrho e^{2\pi i\vartheta}$, $|\vartheta| \leqslant \frac{1}{2}$, $\kappa \leqslant \varrho \leqslant 1/\kappa$. Put $r(n) := \mu(n)^2\varrho^{f(n)}$. From (2·3), we see that, with c as in the statement of Theorem 1.1,

$$\sum_{p\leqslant x} \frac{r(p) - \Re e\,(g(p)/p^{i\tau})}{p} \geqslant c\varrho\sin^2(\pi\vartheta)E(x) + c\varrho F(x) + O(1) \qquad (|\tau| \leqslant T := \log x).$$

We may therefore apply Corollary 2.1 of [10] to get

$$M(x;g) \ll M(x;r)\bigg\{e^{-c\varrho E(x)\sin^2(\pi\vartheta)-c\varrho F(x)}\log_2 x + \frac{1}{(\log x)^{\kappa}}\bigg\}. \tag{3·1}$$

With the aim of applying Cauchy's formula to detect $N_m(x;f,\mu)$, we next seek an estimate for $M(x;g)$ when ϑ is small, namely

$$|\vartheta| \leqslant \vartheta_0 := K\sqrt{\frac{\log_3 x}{\log_2 x}},$$

where K is a large constant—actually any $K > 1/\sqrt{4\kappa c}$ will do. We have

$$\sum_{p\leqslant x} \frac{r(p) - \Re e\, g(p)}{p} = \varrho(1-\cos 2\pi\vartheta)E(x) + 2\varrho F(x) \leqslant 2\varrho\pi^2\vartheta^2 + 2\varrho F(x),$$

hence condition (1·12) is plainly fulfilled with $\varepsilon := |\vartheta|^{2/\delta} + (\log_2 x)^{-c_0/(\mathfrak{h}\delta)}$ provided δ is chosen sufficiently small in terms of $\mathfrak{b}$, κ and K. Next, for $x^\varepsilon < y \leqslant x$, we have

$$\begin{aligned}\sum_{x^\varepsilon<p\leqslant y} \frac{\{r(p) - \Re e\, g(p)\}^{\mathfrak{h}}\log p}{p} &\ll \varrho\vartheta^{2\mathfrak{h}}\log y + \varrho \sum_{\substack{x^\varepsilon<p\leqslant y\\ f(p)=0}} \frac{\log p}{p}\\ &\ll_\kappa \left\{\varepsilon^{\delta\mathfrak{h}} + (\log_2 x)^{-c_0}\right\}\log y,\end{aligned}$$

so hypothesis (1·13) is also verified. Since (1·14) holds trivially on selecting $\mathfrak{b} := \kappa/4$, and hence $\mathfrak{h} = 4/\kappa - 1$, we conclude that (1·15) is valid. We obtain, with $\mathfrak{c} := \kappa\mathfrak{b}$,

$$M(x;g) = M(x;r)\prod_{\substack{p\leqslant x\\ f(p)=1}} \frac{1-z/p}{1+\varrho/p}\prod_{\substack{p\leqslant x\\ f(p)=0}} \frac{1-1/p}{1+1/p} + O\bigg(\frac{x\varepsilon^{\delta/2}e^{Z(x;r)-\mathfrak{c}Z(x;r-g)}}{\log x}\bigg). \tag{3·2}$$

Now, appealing for instance to theorem 1.1 of [9], we observe that

$$M(x;r) \asymp \frac{xe^{Z(x;r)}}{\log x}$$

and so we may rewrite (3·2) as

$$M(x;g) = M(x;r)\bigg\{\lambda_f e^{-(z+\varrho)E(x)-2F(x)} + O\Big(\Big(|\vartheta| + (\log_2 x)^{-c_0\mathfrak{h}/2}\Big)e^{-c_1\vartheta^2 E(x)-c_1F(x)}\Big)\bigg\},$$

valid for $|\vartheta| \leqslant \vartheta_0$ and some constant $c_1 > 0$. Integrating on the circle $|z| = \varrho := m/E(x)$ and taking (3·1) into account, we readily obtain in the stated range for m,

$$\begin{aligned} N_m(x;f,\mu) &= (-1)^m \int_{-1/2}^{1/2} \mathrm{e}^{-2i\pi\vartheta m} \varrho^{-m} M(x;g)\,\mathrm{d}\vartheta \\ &= (-1)^m \lambda_f M(x;r) \frac{E(x)^m}{m!\,\mathrm{e}^m} \Big\{ \mathrm{e}^{-2F(x)} + O\Big(\frac{\mathrm{e}^{-c_2 F(x)}}{(\log_2 x)^b} \Big) \Big\}, \end{aligned} \tag{3·3}$$

with $c_2 := \min(c_1, c\kappa)$. Since, by a straightforward variant of corollary 2.4 of [10][2],

$$N_m(x;f) = M(x;r) \frac{E(x)^m}{m!\,\mathrm{e}^m} \Big\{ 1 + O\Big(\frac{1}{\sqrt{\log_2 x}} \Big) \Big\},$$

we reach the required conclusion.

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2. Applied to $\omega(n;E)$ instead of $\Omega(n;E)$ with the notation of [10].