A New Approach of Ordered Exponential in NMR: the Path-Sum

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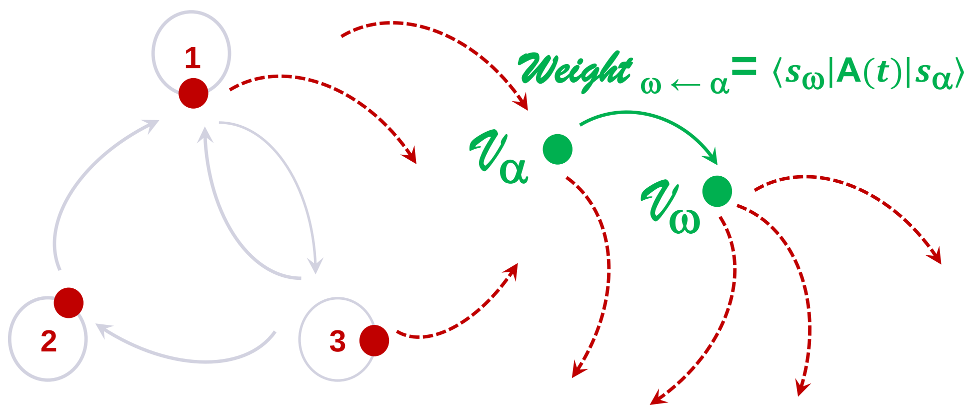

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General context – The evolution operator U(t)

Dyson time-ordering operator

$$\mathbf{U(t',t) = OE[-i\,H(t',t)]} = T\,\mathbf{exp}\left(-\mathbf{i}\int_t^{t'} \mathbf{H}(\tau)\mathbf{d}\tau\right)$$

$$\mathbf{U}(\tau_c) = exp\left(-i\tau_C \sum_{n=0}^{\infty} \overline{H^{(n)}}\right)$$

Magnus

$$\overline{\hat{H}} = {}^{(0)}\hat{H} - \frac{1}{2}\sum_{n\neq 0}\frac{\left[{}^{(-n)}\hat{H},\, {}^{(n)}\hat{H}\right]}{n\omega_m} + \frac{1}{2}\sum_{n\neq 0}\frac{\left[\left[{}^{(n)}\hat{H},\, {}^{(0)}\hat{H}\right],\, {}^{(-n)}\hat{H}\right]}{(n\omega_m)^2}$$

Floquet

$$+\frac{1}{3}\sum_{k,n\neq 0}\frac{\left[{}^{(n)}\hat{H},\left[\hat{H},\, {}^{(-n-k)}\hat{H}\right]\right]}{kn\omega_m^2} + \cdots$$

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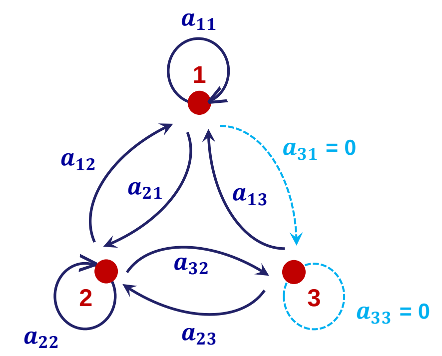
Outline

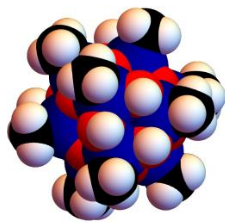
- Basic results of algebraic graph theory



- Path-Sum applied to Ordered Exponential (OE)

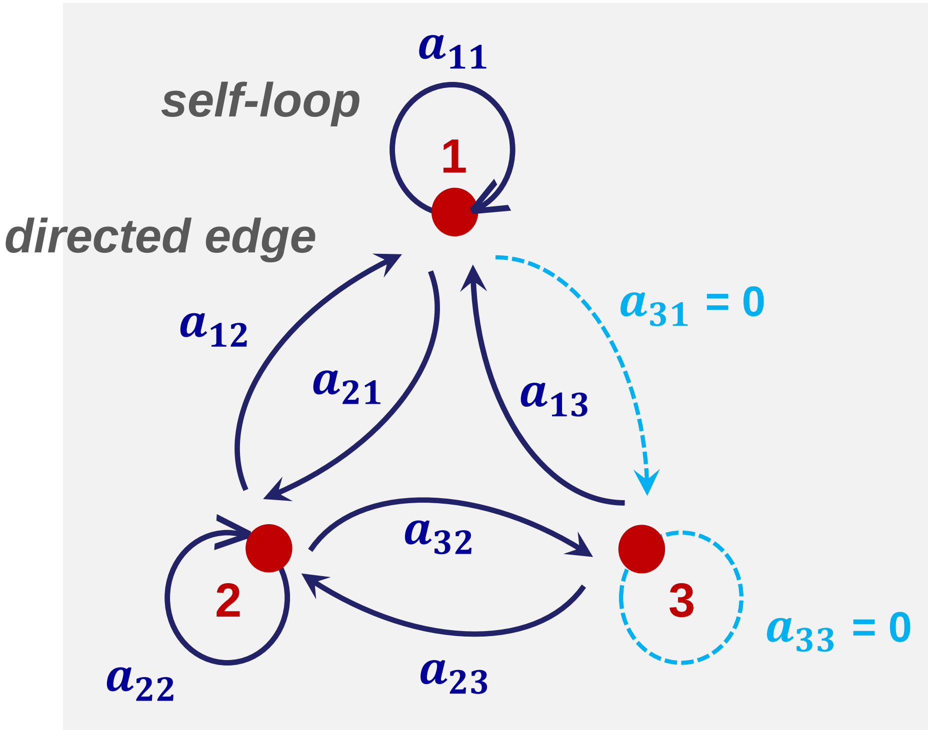
$$\mathbf{OE[A]}(t',t) = \begin{pmatrix} \int_t^{t'} G_{K_2,11}(t',\tau)d\tau & OE_{12}(t',t) \\ OE_{21}(t',t) & \int_t^{t'} G_{K_2,22}(t',\tau)d\tau \end{pmatrix}$$

- Applications:
  - ► Circularly polarized excitation
  - ► Linearly polarized excitation, Bloch-Siegert (BS) effect
  - ► N spins: homonuclear dipolar Hamiltonian, H_D



Basic results of algebraic graph theory

$\mathcal{G}$ = ($\mathcal{V}$ertex set, $\mathcal{E}$dge set)



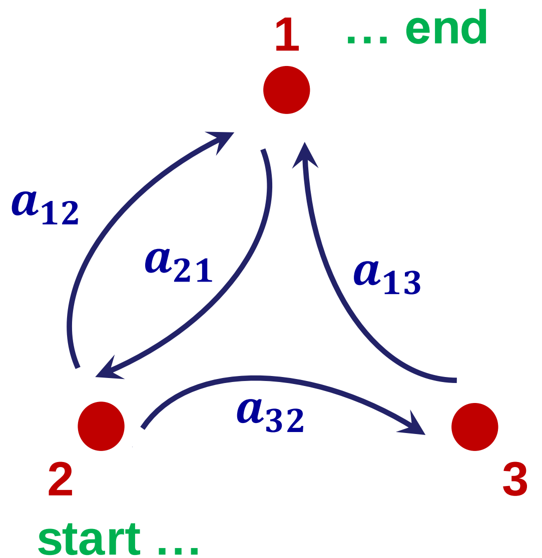
Adjacency *finite* matrix $\mathbf{A}_{\mathcal{G}}$

$$\mathbf{A}_{\mathcal{G}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix}$$

⬇

entry: ***weight*** on a ***directed edge***

ex.: ***walk*** $\mathcal{W}_{1\leftarrow 2}$ (from $\mathcal{V}_2$ **to** $\mathcal{V}_1$) of ***length* 4**



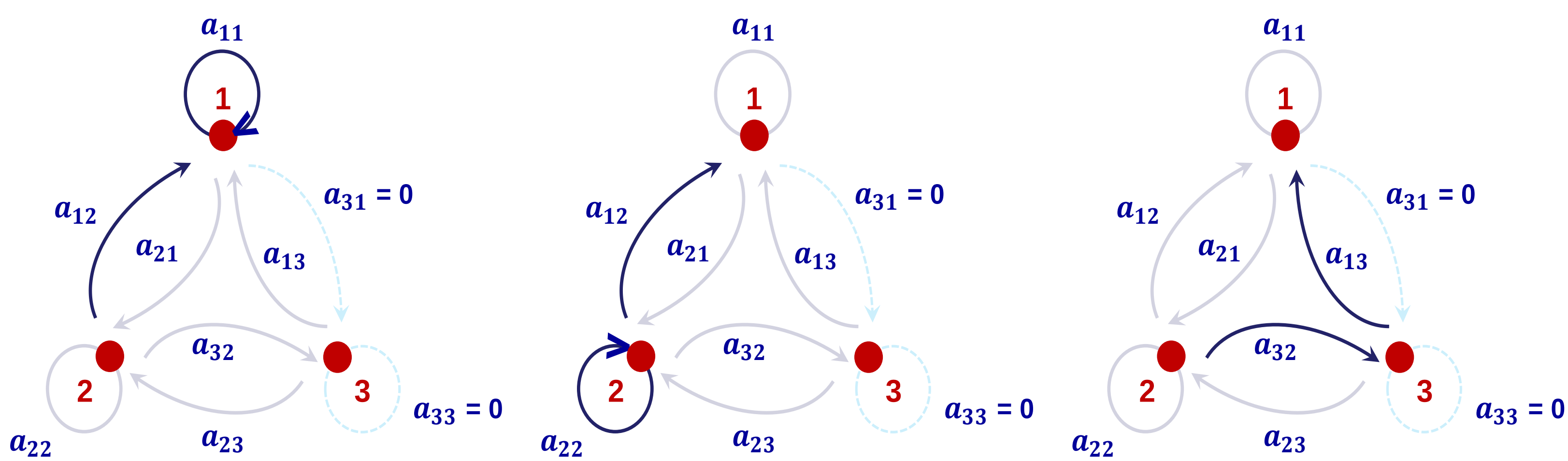
Basic results of algebraic graph theory

the ***powers*** of the **Adjacency** matrix $\mathbf{A}_{\mathcal{G}}$ on a graph $\mathcal{G}$ generate
ALL ***weighted*** **WALKS** $\mathcal{W}$ on $\mathcal{G}$

$$\mathbf{A}_{\mathcal{G}}^{2} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix}^{2} = \begin{pmatrix} & \blacksquare & \\ \vdots & \ddots & \vdots \\ & \cdots & \end{pmatrix}$$

$\blacksquare$: $a_{11} \times a_{12} + a_{12} \times a_{22} + a_{13} \times a_{32}$

$\mathcal{W}$ of ***length* 2** from v_2 to v_1 (**1** ← **2**)



$$\Sigma = a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32}$$

N. Biggs, in: *Algebraic Graph Theory* (1993)

Basic results of algebraic graph theory

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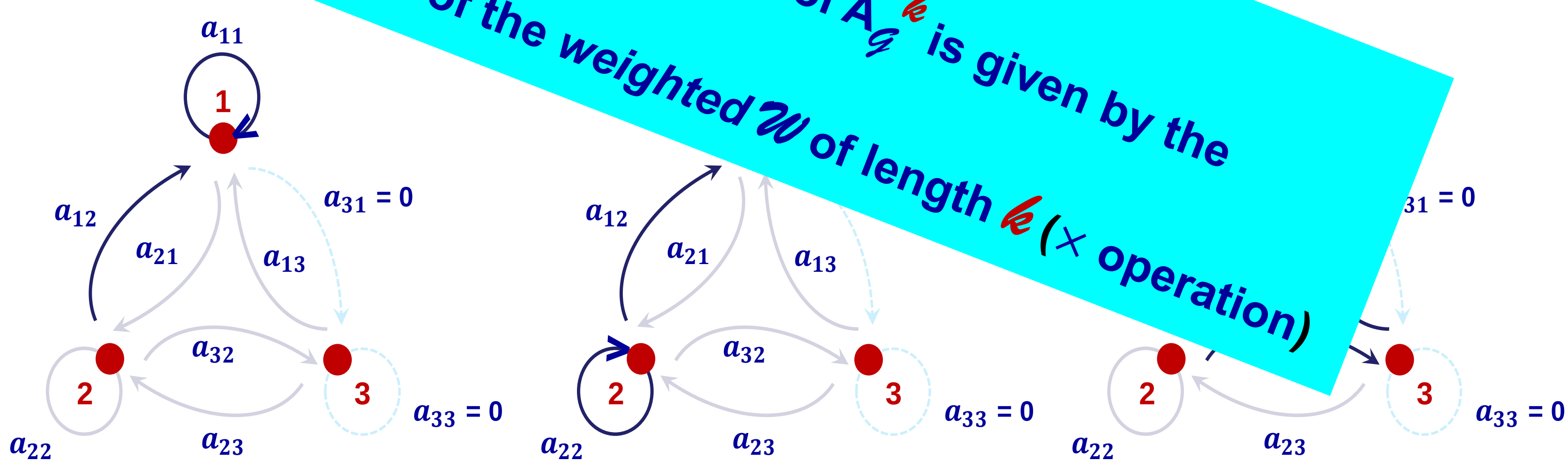
$$\mathbf{A}_{\mathcal{G}}^{2} = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ 0 & a_{3\cdots} & \cdots \end{pmatrix}$$

to keep in mind:
each element of $\mathbf{A}_{\mathcal{G}}^{k}$ is given by the sum of the weighted $\mathcal{W}$ of length k (× operation)



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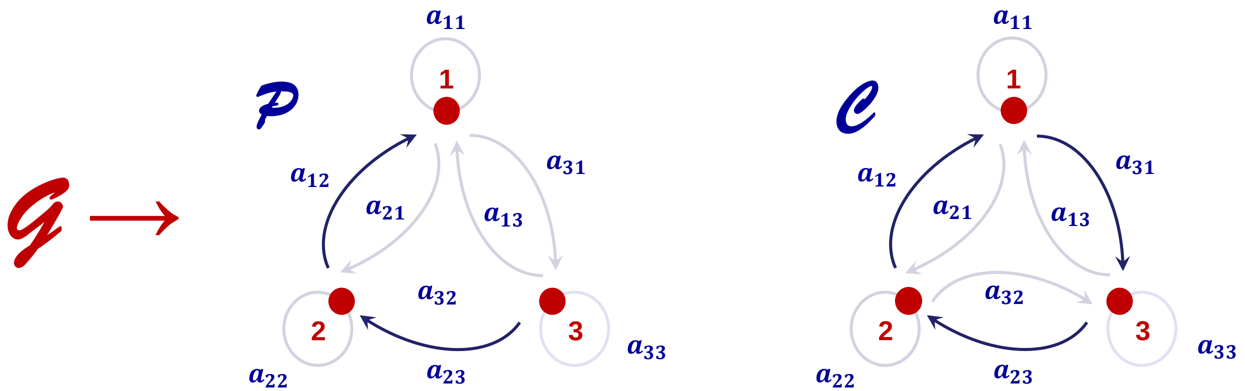
Path-Sum

◊ ***simple path*** $\mathcal{P}$ (self avoiding walk): $\mathcal{W}$ whose $\mathcal{V}$ are all **distinct**

◊ ***simple cycle*** $\mathcal{C}$ (self avoiding polygon): $\mathcal{W}$ whose **endpoints** are **identical** and **intermediate** $\mathcal{V}$ are all **distinct** and different from the endpoints



« Fundamental Theorem of Arithmetic » on $\mathcal{G}$ **(P.-L. Giscard, 2012)**

► $\mathcal{W}$ factor ***uniquely*** into ***prime*** elements, *i.e.* ***simple paths*** and ***simple cycles***

► if $\mathcal{G}$ is ***finite*** the number of primes is ***finite***

► resummation of all $\mathcal{W}$ involves a ***finite*** number of operations: ***sum*** *on* ***simple paths*** and ***continuous fraction*** *of* ***simple cycles*** with vertex removal

Power series of $A_{\mathcal{g}}$

ex.: $\exp[A_{\mathcal{g}}] = \sum_{k=0}^{\infty} \frac{1}{n!} A_{\mathcal{g}}^{k}$

$$(A_{\mathcal{g}})^{k} = \begin{pmatrix} & \cdots & \\ \vdots & (A_{\mathcal{g}})^{k}_{\omega\alpha} & \vdots \\ & \cdots & \end{pmatrix}$$

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$$F(A_{\mathcal{G}})_{\omega\alpha} = \sum_{k=0}^{\infty} c_k \sum_{\mathcal{W}_{\mathcal{G},\,\alpha\omega;\,k}} a_{\omega h_k} \cdots \times a_{h_3 h_2} \times a_{h_2 \alpha}$$

power series of $A_{\mathcal{G}}$

all *weighted* walks $\mathcal{W}$ **from** $\mathcal{V}_{\alpha}$ to $\mathcal{V}_{\omega}$ of length k

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power series of $A_{\mathcal{G}}$ — all *weighted* walks $\mathcal{W}$ **from** $\mathcal{V}_{\alpha}$ to $\mathcal{V}_{\omega}$ of length k

Path-Sum

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power series of $A_{\mathcal{G}}$

all *weighted* walks $\mathcal{W}$ **from** $\mathcal{V}_\alpha$ to $\mathcal{V}_\omega$ of length k

Path-Sum

$$F(A_{\mathcal{G}})_{\omega\alpha} = \sum_{\mathcal{P}_{\mathcal{G},\alpha\omega;\ell}} f(a_{\omega\omega}) \times a_{\omega\mu_\ell} \cdots f(a_{\mu_2\mu_2}) a_{\mu_2\alpha} \times f(a_{\alpha\alpha})$$

sum on the ***finite*** set of ***simple paths*** $\mathcal{P}$ of **length** ℓ

edge weight

effective $\mathcal{V}$ weight

sum over the ***finite*** set of ***simple cycles*** $\mathcal{C}$

(***continued fraction*** of ***finite breadth***)

$$\mathbf{OE}[\mathbf{A}_{\mathcal{G}}](t',t) = \begin{pmatrix} \cdots \\ \langle s_\omega|\mathbf{OE}[\mathbf{A}_{\mathcal{G}}](t',t)|s_\alpha\rangle \\ \cdots \end{pmatrix}$$

$$\mathbf{A}_{\mathcal{G}}(t) = \begin{pmatrix} \cdots \\ \langle s_\omega|\mathbf{A}(t)|s_\alpha\rangle \\ \cdots \end{pmatrix}$$

Σ ALL *weighted* walks $\omega \leftarrow \alpha$ on $\mathbf{A}_{\mathcal{G}}$

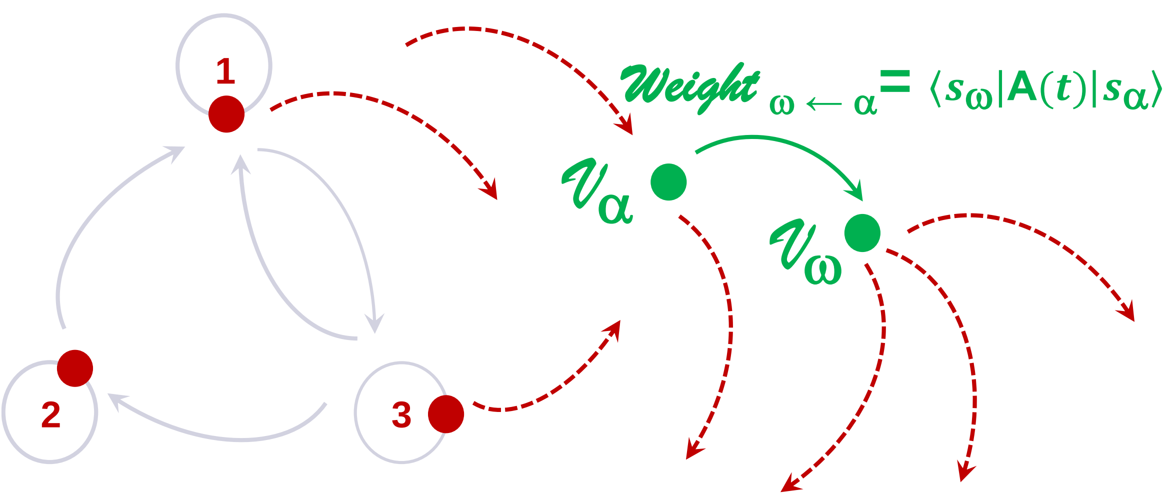
but using $*$–product

$$(f * g) = \int_t^{t'} f(t',\tau)\,g(\tau,t)\,d\tau$$

instead of $\times$

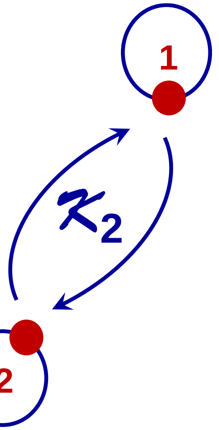
Path-Sum

$\mathbf{A}_{\mathcal{G}}$



An example: 2 × 2 matrix

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}$$

$\mathcal{G} =$ 

Path-Sum

$$\mathbf{OE[A]}(t',t) = \begin{pmatrix} \int_t^{t'} G_{K_2,11}(t',\tau)d\tau & OE_{12}(t',t) \\ OE_{21}(t',t) & \int_t^{t'} G_{K_2,22}(t',\tau)d\tau \end{pmatrix}$$

- ► ***entry*** → solving an equation with ***analytical tools***
- ► ***finite*** number of operations → ***unconditional convergence***
- ► ***non perturbative*** formulation of OE
- ► ***scalability***

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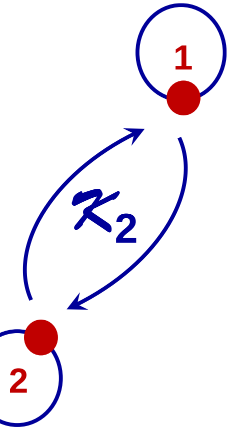
$\mathcal{G} =$



Path-Sum

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An example: 2 × 2 matrix

$$(f \ast g) = \int_t^{t'} f(t',\tau) g(\tau,t) d\tau$$

OE[A](*t'*,*t*) = $\begin{pmatrix} \int_t^{t'} G_{K_2,11}(t',\tau)d\tau & OE_{12}(t',t) \\ OE_{21}(t',t) & \int_t^{t'} G_{K_2,22}(t',\tau)d\tau \end{pmatrix}$

$a_{ij}(t)$

$$[1_\ast - (\ast \quad \ast \quad \ast \cdots)]^{\ast-1} = \sum_{n \geq 0} (\ast \quad \ast \quad \ast \cdots)^{\ast n}$$

Neumann series (analytical)

linear Volterra (2nd kind) (numerical)

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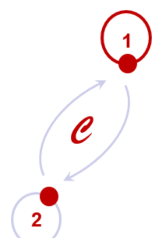
Neumann series (analytical)

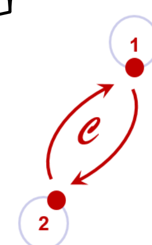
linear Volterra (2nd kind) (numerical)

sum on *simple cycles*

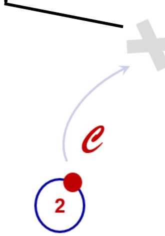
$$G_{K_2,11} = [1_* - a_{11} - a_{12} * G_{K_2\backslash\{1\},22} * a_{21}]^{*-1}$$

$\mathcal{K}_2$



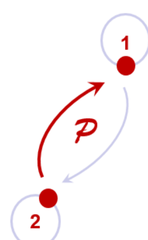


$$G_{K_2\backslash\{1\},22} = [1_* - a_{22}]^{*-1}$$



$$OE_{12}(t',t) = \int_t^{t'} G_{K_2\backslash\{2\},11} * a_{12} * G_{K_2,22}(t',\tau)d\tau$$

sum on *simple paths*



- **END !**
- ***finite*** sum on simple $\mathcal{P}$

- **END** of the ***continued fraction* !**
- ***finite*** sum on $\mathcal{C}$

Summary (partial)

► … take a ***finite*** matrix $\mathbf{A}_{\mathcal{G}}\mathbf{(t)}$ associated to $\mathcal{G}$ (Hermitian or not, periodic or not…)

► each entry of $\mathbf{A}_{\mathcal{G}}^{k}$ is given is given by a ***finite*** number of operations by using Path-Sum (with $\times$ product)

► each entry of **OE[A**$_{\mathcal{G}}$](t',t)] is given is given by a ***finite*** number of operations by using Path-Sum (with $*$ –product and $[\mathbf{1}_* - (* \;\; * \;\; * \cdots)]^{*-\mathbf{1}}$)

Summary (partial)

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► each entry of **OE[A**$_{\mathcal{G}}$](t',t)] is given is given by a ***finite*** number of operations by using Path-Sum (with $*$ −product and $[1_* - (* \;\; * \;\; * \cdots)]^{*-1}$)

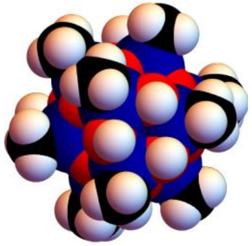
- the ***matrix*** nature of the problem is ***fully replaced*** when working on ***entries***
- or, one can keep it partially… → **PARTITIONS** (***scalability***) ←
- the ***convergence*** of the **Neumann** series (***analytical***) is ***superexponential***
- a convenient (***numerical***) approach: linear **Volterra** equations (***2nd kind***)

Outline

- Basic results of algebraic graph theory

- Path-Sum applied to the ordered exponential (OE)

- **Applications:**
  - ► **Circularly polarized excitation**
  - ► **Linearly polarized excitation, Bloch-Siegert (BS) effect**
  - ► **N spins homonuclear dipolar Hamiltonian, H_D**



Applications – Circularly polarized excitation (test model)

$$\mathbf{H}(t) = \begin{pmatrix} \frac{\omega_0}{2} & \beta e^{-i\omega t} \\ \beta e^{i\omega t} & -\frac{\omega_0}{2} \end{pmatrix}, [\mathbf{H}(t'), \mathbf{H}(t)] \neq 0$$

$$\mathbf{H}(t) = \frac{1}{2}\omega_0 \boldsymbol{\sigma}_\mathbf{z} + \beta[\boldsymbol{\sigma}_\mathbf{x}\cos(\omega t) + \boldsymbol{\sigma}_\mathbf{y}\sin(\omega t)]$$

$$[1_* - (* \quad * \quad * \cdots)]^{*-1}$$

Path-Sum

$$G_{K_2,11}(t) = \left(1_* - \frac{\omega_0}{2i} + \frac{i\beta^2}{\Delta}\left(e^{-i\Delta(t'-t)} - 1\right)\right)^{*-1}$$

OE entry

Neumann series

$$OE[-i\mathbf{H}](t)_{11} = 1 + \sum_{n=0}^{\infty} \frac{(-it\,\beta^2/\Delta)^{n+1}}{(n+1)!} \sum_{k=0}^{n+1} \binom{n+1}{k} \left(\frac{\Delta\omega_0}{2\beta^2} - 1\right)^k {}_2F_1\left(-k, -k+n+1; -n-1; \frac{\Delta^2}{\frac{\Delta\omega_0}{2} - \beta^2}\right)$$

Gauss hypergeometric

$$OE[-i\mathbf{H}](t)$$

$$\begin{pmatrix} e^{-\frac{1}{2}it\left(\Delta+\frac{\omega_0}{2}\right)}\left(\cos(\alpha t/2) + \frac{i}{\alpha}\left(\Delta - \frac{\omega_0}{2}\right)\sin(\alpha t/2)\right) & -\frac{2i\beta}{\alpha}e^{-\frac{1}{2}it\left(\Delta+\frac{\omega_0}{2}\right)}\sin(\alpha t/2) \\ -\frac{2i\beta}{\alpha}e^{\frac{1}{2}it\left(\Delta+\frac{\omega_0}{2}\right)}\sin(\alpha t/2) & e^{\frac{1}{2}it\left(\Delta+\frac{\omega_0}{2}\right)}\left(\cos(\alpha t/2) - \frac{i}{\alpha}\left(\Delta - \frac{\omega_0}{2}\right)\sin(\alpha t/2)\right) \end{pmatrix}$$

$$\mathbf{U}(t) = \exp\left(-\frac{1}{2}i\omega t\boldsymbol{\sigma}_\mathbf{z}\right)\exp(-it\left(\frac{1}{2}(\omega_0 - \omega)\boldsymbol{\sigma}_\mathbf{z} + \beta\boldsymbol{\sigma}_\mathbf{x}\right))$$

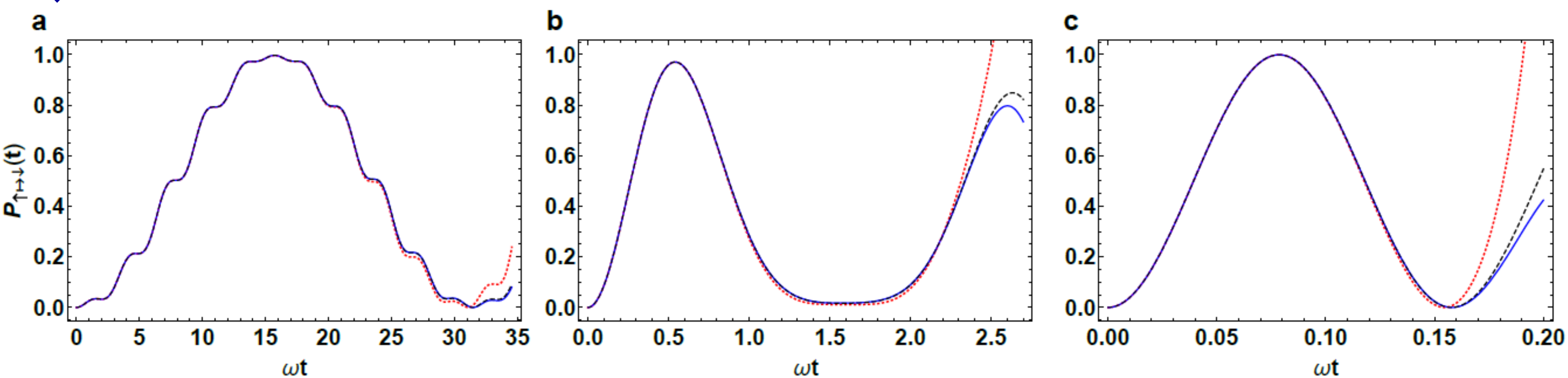
Applications – Linearly polarized excitation, Bloch-Siegert (BS) effect

$$\mathbf{H}(t) = \frac{1}{2}\omega_0 \boldsymbol{\sigma}_\mathbf{z} + 2\beta\, \boldsymbol{\sigma}_\mathbf{x} \cos(\omega t)$$

$$\mathbf{H}(t) = \begin{pmatrix} \frac{\omega_0}{2} & 2\beta\cos(\omega t) \\ 2\beta\cos(\omega t) & -\frac{\omega_0}{2} \end{pmatrix}$$

P(t) transition probability

$\boldsymbol{\omega = \omega_0}$ **or** $\boldsymbol{\omega \neq \omega_0}$



$\beta/\omega = 1/10$ $\beta/\omega = 3/2$ $\beta/\omega = 10$

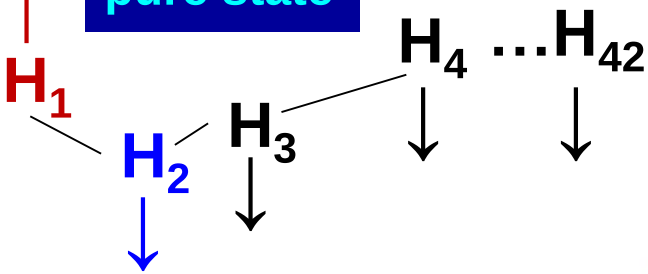
► **analytical** expression with **few orders** of the **Neumann series**

P.L. Giscard, C. Bonhomme, *to be submitted*

Applications – N spin systems, homonuclear dipolar Hamiltonian, H_D

t = 0

pure state



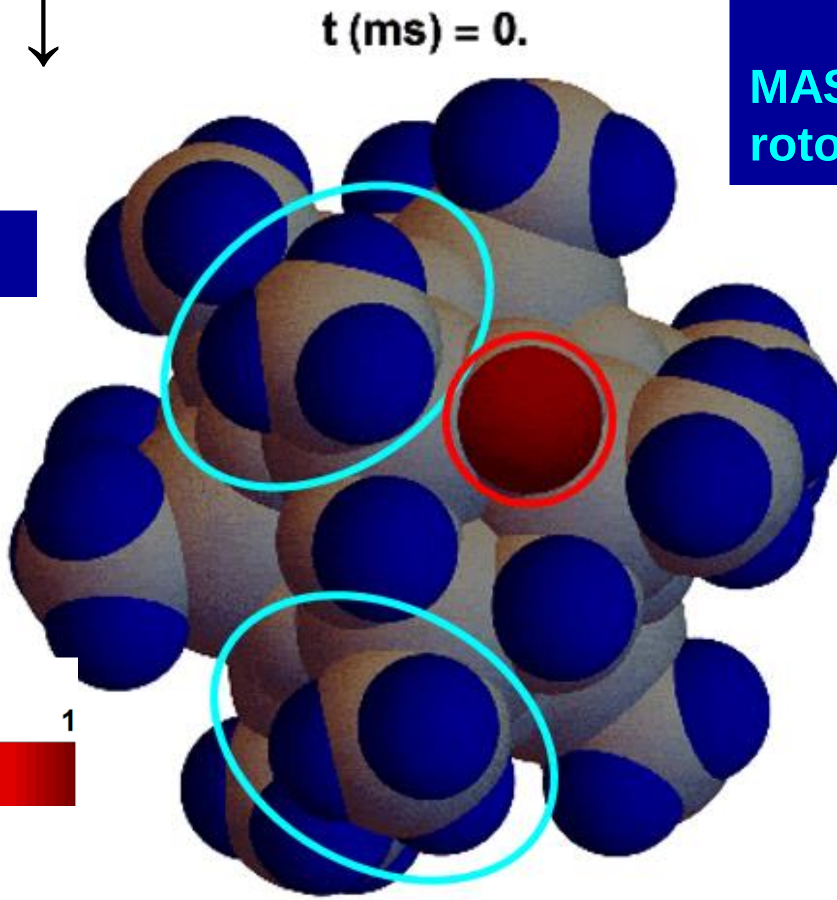
analytical expression

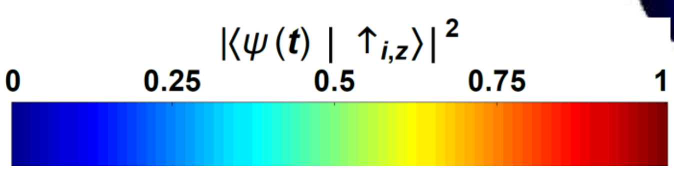
Coll.: F. Ribot, France

$(CH_3)_{12}(OH)_6Sn_{12}$

42 protons
« rigid » CH_3

MAS 10 kHz
rotor period 0.1 ms





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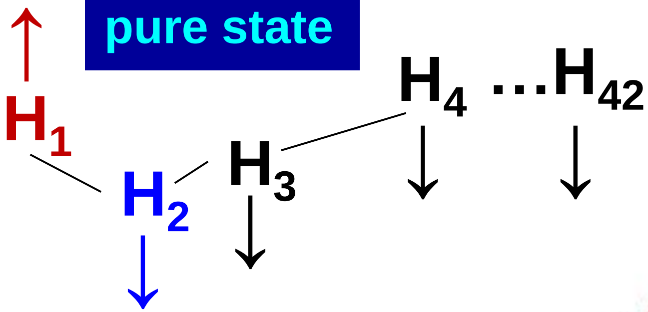
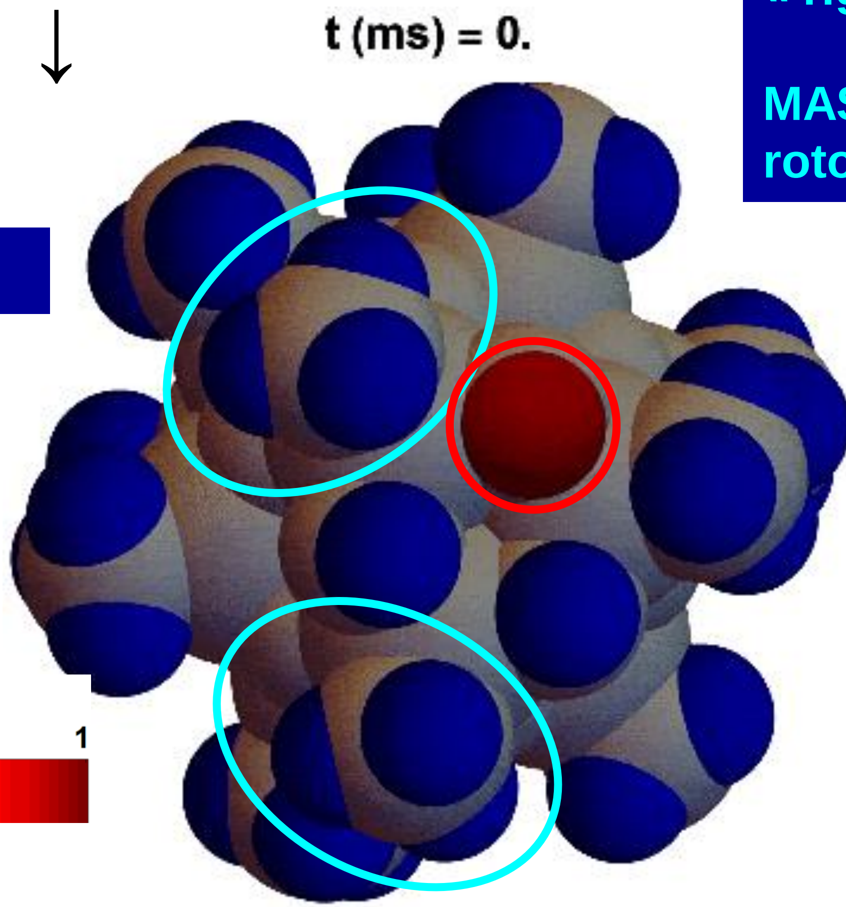
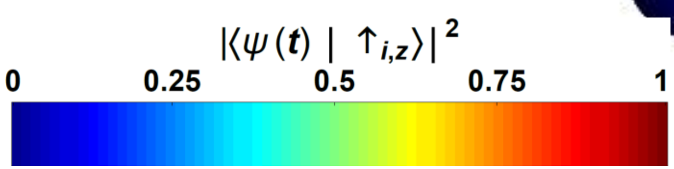


analytical expression





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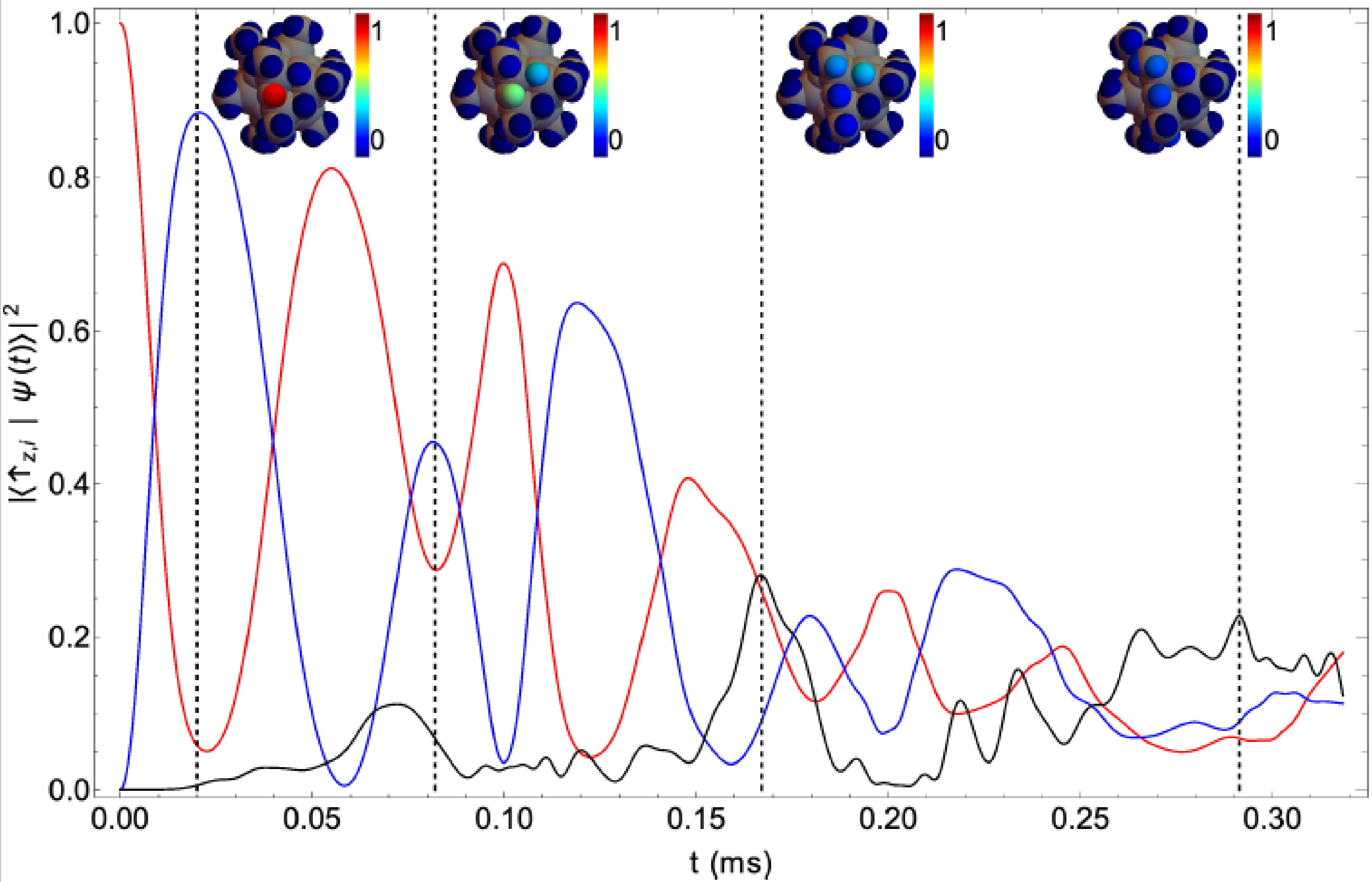
Applications – N spin systems, homonuclear dipolar Hamiltonian, H_D



Conclusions and acknowledgments

Path-Sum

- **a new approach**
- **analytical expression for U(t)**
- **unconditional convergence**
- **non perturbative formulation**
- **scalable to large spin systems**
- **other theory/applications to come…**

(very) warm thanks to P.-L. Giscard

Ass. Pr. in Calais, France

Liouville laboratory

Algebraic Combinatorials

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Post doctoral position available in Paris: on NMR instrumentation & DNP

To go further – Path-Sum *vs* other methods

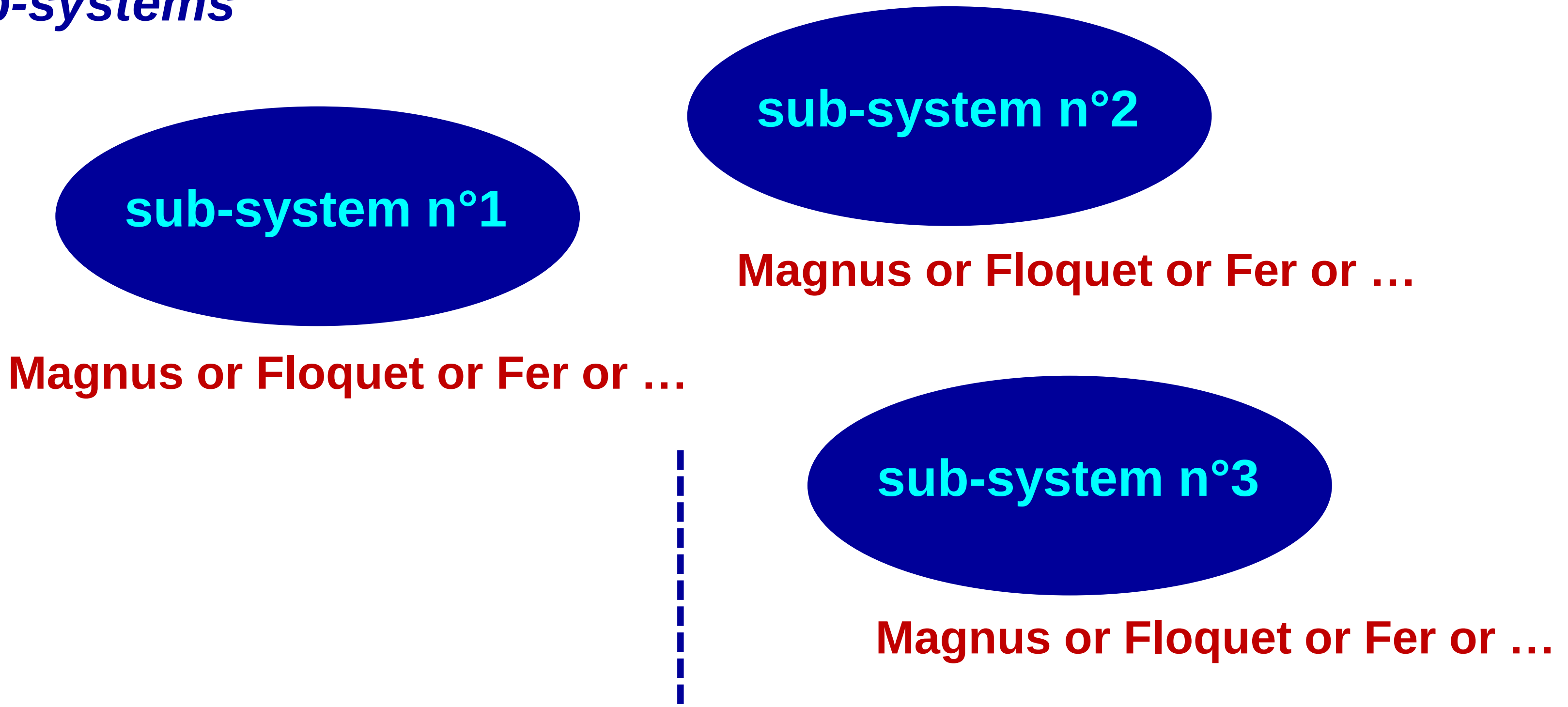
► main goal → get an ***exact*** form for ***U(t)***

► **FLOQUET** | **ZASSENHAUS** / **FER/TROTTER-SUZUKI** | **MAGNUS** | **PATH-SUM**

► usually:  on ***H(t)*** → choice in **FLOQUET** | **ZASSENHAUS** / **FER/TROTTER-SUZUKI** | **MAGNUS**

► **PATH-SUM** is ***exact*** and **PARTITIONS** allow to ***choose the dimension*** of the of the working space from ***H(t)*** to ***U(t)***

Take a **partition** of a spin system in a set of (***smaller***, ***independent***) ***sub-systems***



the ***exact*** evolution of the ***entire*** spin system as functions of the evolutions of the ***isolated sub-systems*** is given by **Path-Sum**

(though **non contiguous blocks** in H(t) matrix!)

To go further – WHY does Path-Sum work?

► the ***EXACT*** result is given by a ***FINITE*** number of terms

► the ***matrix*** nature of the problem is ***fully replaced*** when working on ***entries***

► or, one can keep it partially… → **PARTITIONS**

► ***hard*** work → $[\mathbf{1}_* - (* \; * \; * \cdots)]^{*-1}$

► hopefully: the ***Neumann series*** give the analytical solution at any order with unconditional convergence (not to be "found" … just apply a "recipe")

► the ***convergence*** of the Neumann series is ***superexponential***

► a ***convenient*** numerical approach: linear ***Volterra*** equations (***2nd kind***)

$$D_x^2\, u + \left[\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right] D_x u + \left[\frac{\alpha\beta\, x - q}{x(x-1)(x-a)}\right] u = 0$$

ex.: the best obtainable solution for the ***general 2 × 2 matrix*** (closed form for the ***confluent Heun's special functions***) (see Q. Xie, 2018)

To go further – Exponential explosions

► **1st** explosion: related to the ***size of H(t)*** with ***many-body*** systems (Q nature)

► **2nd** explosion: related to the ***time*** needed to isolate the ***primes*** ($\mathcal{G}$ nature)

Lanczos–Path-Sum (numerical) ***fixes*** the **2nd** explosion:

Idea behind: initial H(t) → time dependent ***tridiagonal matrix***

expectations: to reach ***excellent convergence*** with the breadth of the continued fraction and why not ?... "Circumvent" the **1st** explosion

P.-L. Giscard *et al.*, 2019, *in preparation*

To go further – Complexity theory

► for ***finite*** $\mathcal{G}$ **:** the decomposition of $\mathcal{W}$ in ***primes*** (e.g. ***simple paths*** & ***cycles***) for the ◘ (***nested***) operation ***exists and is unique***

► to determine the existence of a prime of ***length L*** is ***NP-complete*** (***no*(?)** algorithm with polynomial complexity)

► to ***count*** them is ***#P-complete*** (the same but for counting problems)

► to count them for a fixed ***length L*** is ***#W[1]-complete*** (same as *#P-complete* but with parameters, such as *L*, taken into account)

► **BUT**: for ***sparse*** $\mathcal{G}$: counting becomes ***polynomial*** in the max degree of $\mathcal{G}$ **!**

see: **P. L. Giscard *et al.*, *Algorithmica*, 2019**

To go further – Mathematical conditions on A(t) for Path-Sum to be valid -1

► fundamentally: $\mathcal{R}_{esolvent}[\mathbf{A(t)}]_{\ast\ \text{product}} = \frac{\mathbf{d}}{\mathbf{dt}}\mathbf{OE}[\mathbf{A}(\mathbf{t})] \rightarrow$ **Path-Sum**

► each ***entry*** of A(t) must be ***bounded*** on [0,t], a ***bounded*** interval of time

► if the entries are ***not bounded***, Path-Sum still work … but perhaps the Neumann series will ***not converge***

► ***continuity*** is ***not*** necessary

► ***if continuity***: Volterra equations are much ***easier*** to handle

► A(t) can be Hermitian ***or not***, periodic ***or not …*** and entries can be: matrices, quaternions, octonions, division rings…

To go further – Mathematical conditions on A(t) for Path-Sum to be valid -2

► ***finite*** A(t): ***sufficient*** condition for ***finite breadth*** of the continued fraction

► ***NOT*** a ***necessary*** condition: ex. a ***finite*** number of ***simples cycles*** in an ***infinite*** matrix

► in some cases, Path-Sum can still be applied on ***infinite matrices***: ***strong symmetry***, e.g. invariance by translation (soluble ***non-linear*** Volterra equations)

In other words:

■ ***infinity*** of cycles … but ***self-similar*** like in a ***fractal***

■ the corresponding continued fraction is of ***finite breadth***

To go further – Taylor… or Neumann series?

► take one entry: $f(t) = \mathrm{OE}[A(t)]_{\mathbf{ij}}$

► ***Taylor*** series: expansion in t^n *i.e.* $f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n$

ex.: $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n = 1 + t + t^2 + \cdots + t^n + \cdots$ with **r = 1** (***radius of CV***)

► ***Neumann*** series: uses the $*$ –**product**, i.e. $f(t) = \sum_{n=0}^{\infty} f^{*n}$

each order contains functions represented by intinite Taylor series

r = ∞ **(!)** with ***uniform*** & ***superexponential*** CV

To go further – N spins starting with a pure state

► starting with a ***pure state*** with ***1*** up-spin (total: **N**, ***any geometry***)

Path-sum contains all ***N-order correlations***

→ **if** $\boldsymbol{\omega_{rot} = 0}$

all terms of the Neumann series are ***explicitly*** known

→ **if** $\boldsymbol{\omega_{rot} \neq 0}$

still ***analytical*** up to the CV of the series to the solution

► starting with a ***pure state*** with ***4 or 5*** up-spin is still tractable

(*i.e.* no exponential explosion)

To go further – Pure state *vs* partial polarization

► **Pure state**: if ***k* up-spins** over N and ***k* << *N*** → space of states dim. ≈ N^k

(suppression of the exponential explosion)

► **Partial polarization**: a ***cut-off*** is needed → if $\left| \frac{int_{i,j}}{intV} \leq \frac{1}{cut\text{-}off} \right|$ then $int_{i,j} = 0$

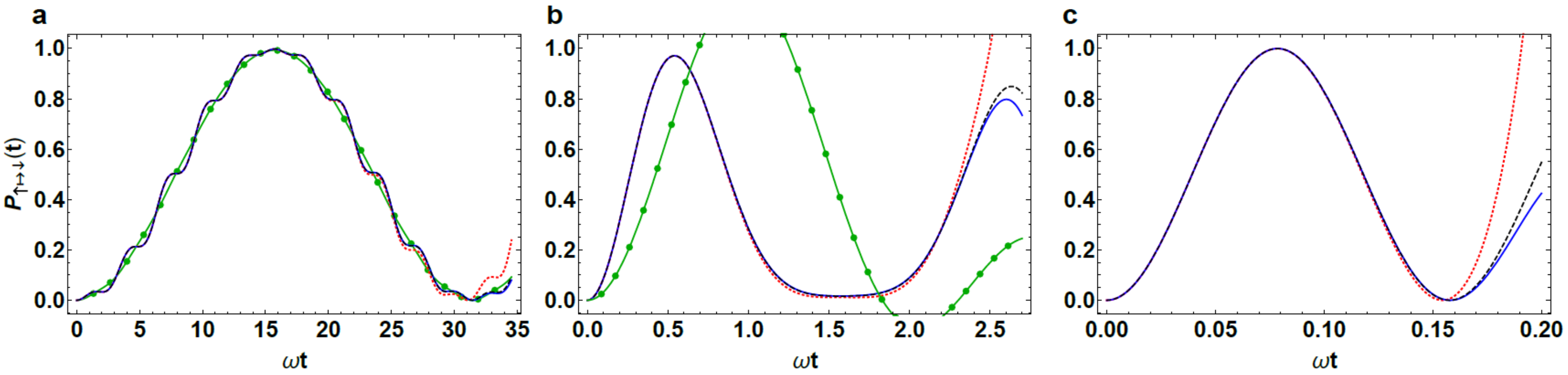
cut-off **:** « high » for ***chains*** but decreases for more **« *dense* »** spin systems

next target: to extend **Path-Sum** to **mixed states** *via* a **decomposition on pure states**

To go further – Path-Sum *vs* Floquet theory for Bloch-Siegert effect

P(t) transition probability

$\omega = \omega_0$ or $\omega \neq \omega_0$



$\beta/\omega = 1/10$ $\beta/\omega = 3/2$ $\beta/\omega = 10$

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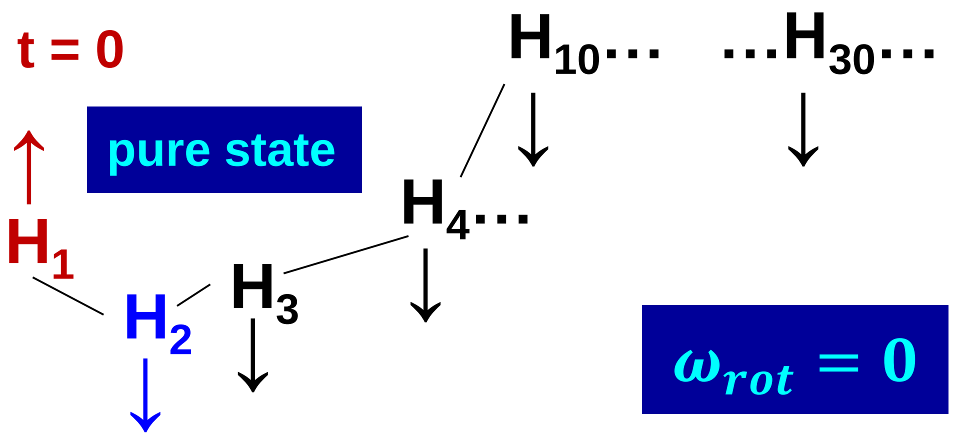
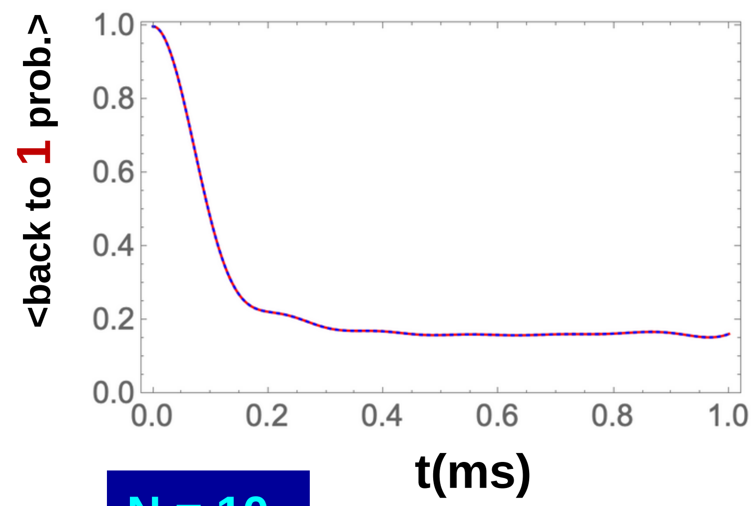
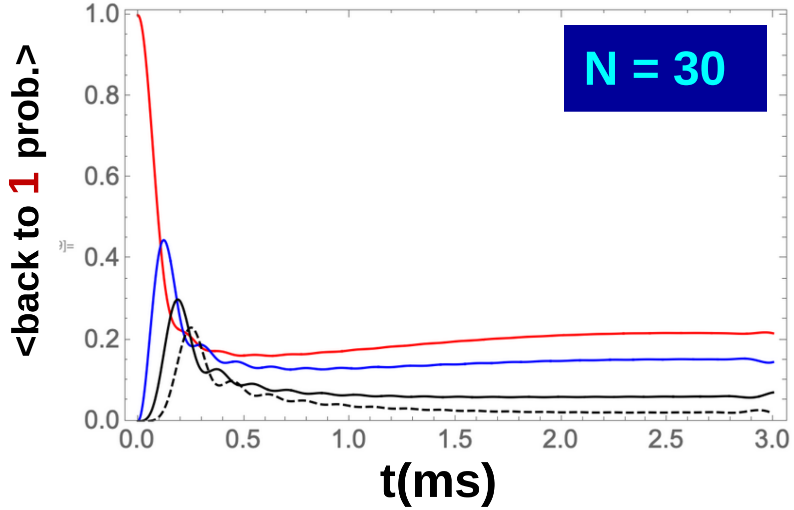
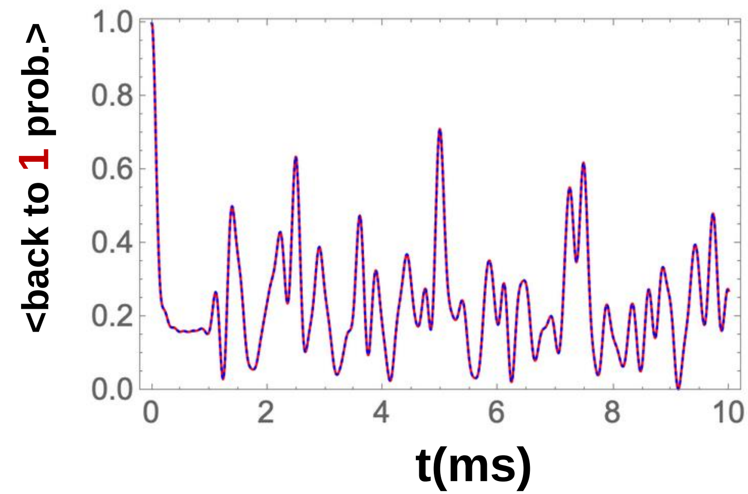
To go further – N spin chains and H_D





N = 10

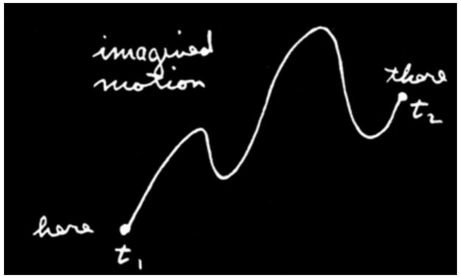




P.L. Giscard, C. Bonhomme, *to be submitted*

To go further – Liouvillian space, Feynman paths and diagrams

► extension of Path-Sum in the Liouvillian space is possible using the ***adjoint operator*** of H(t)



*« With application to quantum mechanics, path integrals suffer most grievously from a serious defect. They do not permit a discussion of spin operators or other such operators in a simple and lucid way » (**R.P. Feynman**)*

► Path-sum can be used starting from the ***Lagrangian*** with ***action*** as ***weight*** on a given $\mathcal{W}$

► Path-sum can be used starting from the ***Hamiltonian*** with ***energy*** as ***weight*** on a given $\mathcal{W}$

► ***Feynman diagrams***: $\mathcal{W}$ of $\mathcal{G}$ in the state space (but ***continuous***)

► Path-sum performs a formal **re-summation** of an infinite number of $\mathcal{W}$, *i.e.* Feynman diagrams !