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Abstract

We consider an abstract evolution equation with linear damping, a nonlinear term of Duffing type, and a small forcing term. The abstract problem is inspired by some models for damped oscillations of a beam subject to external loads or magnetic fields, and shaken by a transversal force.

The main feature is that very natural choices of the boundary conditions lead to equations whose linear part involves two operators that do not commute.

We extend to this setting the results that are known in the commutative case, namely that for asymptotically small forcing terms all solutions are eventually close to the three equilibrium points of the unforced equation, two stable and one unstable.

1 Introduction

Let us consider the partial differential equation

$$u_{tt} + \delta u_t + k_1 u_{xxxx} + k_2 u_{xx} - k_3 \left(\int_0^1 u_x^2 \, dx \right) u_{xx} = f(t, x) \tag{1.1}$$

in the strip $(t, x) \in [0, +\infty) \times [0, 1]$, where δ, k_1, k_2, k_3 are positive constants, and $f : [0, +\infty) \times [0, 1] \to \mathbb{R}$ is a given function (forcing term).

Equation (1.1) was derived as a model for the motion of a beam in different physical systems, for example

- in [7] the beam is buckled by an external load k_2, and shaken by a transverse displacement (depending only on time, in that model),

- in [9] (the so-called magneto-elastic cantilever beam) the beam is clamped vertically at the upper end, and suspended at the other end between two magnets secured to a base, and the whole system is shaken by an external force transversal to the beam.

Equation (1.1) may be seen as an abstract evolution problem in a Hilbert space, but the precise setting depends on the boundary conditions.

The "commutative" case   Let us consider equation (1.1) with boundary conditions

$$u(t, x) = u_{xx}(t, x) = 0 \qquad \forall (t, x) \in [0, +\infty) \times \{0, 1\}, \tag{1.2}$$

physically corresponding to "hinged ends". In this case (1.1) may be seen as an abstract evolution problem of the form

$$u'' + \delta u' + k_1 A^2 u - k_2 A u + k_3 |A^{1/2} u|^2 A u = f(t) \tag{1.3}$$

in the Hilbert space $H := L^2((0, 1))$, where $Au = -u_{xx}$ with domain

$$D(A) := \left\{ u \in H^2((0, 1)) : u(0) = u(1) = 0 \right\}.$$

Up to changing the unknown and the operator according to the rules

$$u(t) \rightsquigarrow \alpha u(\beta t), \qquad A \rightsquigarrow \gamma A$$

for suitable values of α, β, γ, we can assume that three of the four constants in (1.3) are equal to 1, and we end up with an equation of the form

$$u'' + u' + A^2 u - \lambda A u + |A^{1/2} u|^2 A u = f(t), \tag{1.4}$$

depending only on one positive parameter λ. We point out that with these choices it turns out that, up to constants, $A^2 u = u_{xxxx}$ with domain

$$D(A^2) := \left\{ u \in H^4((0, 1)) : u(0) = u_{xx}(0) = u(1) = u_{xx}(1) = 0 \right\}.$$

Equation (1.4) can be considered more generally whenever A is a coercive selfadjoint operator and $A^{-1} : H \to H$ is compact. In this case H admits a countable orthonormal system $\{e_n\}$ made of the eigenvectors of A. The theory has been done when the first eigenvalues λ_1 of A is simple. The behavior of solutions to (1.4) depends on the position of λ with respect to the eigenvalues of A. When $\lambda < \lambda_1$ the operator $A^2 - \lambda A$ is positive, the functional

$$\mathcal{E}_{A,A}(u) := \frac{1}{2}|Au|^2 - \frac{\lambda}{2}|A^{1/2}u|^2 + \frac{1}{4}|A^{1/2}u|^4 \qquad (1.5)$$

is convex and has a unique minimum point at the origin, and the trivial solution $u(t) \equiv 0$ is the unique stationary solution of equation (1.4) in the unforced case $f(t) \equiv 0$. If $f(t)$ is asymptotically small enough, then all solutions are asymptotic to each other as $t \to +\infty$, and lie eventually in a neighborhood of the origin whose radius depends on the asymptotic size of the forcing term. We refer to [1, 5, 4, 8] for significant results in the convex case

The case with $\lambda \in (\lambda_1, \lambda_2)$ was investigated in [6]. Now the operator $A^2 - \lambda A$ is negative in the direction spanned by e_1, and positive in the orthogonal space. The functional $\mathcal{E}_{A,A}(u)$ has three stationary points: the origin, which is no longer a minimum point, and two minimum points of the form $\pm \sigma_0 e_1$, with $\sigma_0 = (\lambda - \lambda_1)^{1/2} \lambda_1^{-1/2}$.

As a consequence, in the unforced case $f(t) \equiv 0$ equation (1.1) has three stationary solutions: the trivial solution $u(t) \equiv 0$, which is now unstable, and the two stable solutions of the form $u(t) \equiv \pm \sigma_0 e_1$, corresponding to the minimum points of the functional $\mathcal{E}_{A,A}(u)$. In the forced case with an external force that is asymptotically small enough, all solutions fall eventually in a neighborhood of one of the three stationary points, within a distance depending on the asymptotic size of the forcing, and any two solutions that are eventually close to the same stationary point are actually asymptotic to each other.

When $\lambda > \lambda_2$, the number of stationary points of the functional increases, as well as the number of stationary solutions to (1.4) in the unforced case. This regime has not been investigated explicitly, but the same approach as in [6] is likely to work when all eigenvalues are simple, i.e. $0 < \lambda_1 < \lambda_2 < \ldots$ or more generally as long as we do not cross a multiple eigenvalue.

We conclude this paragraph by mentioning two more sets of boundary conditions that lead to commutative operators.

- The periodic boundary conditions

$$u(t,0) = u(t,1), \qquad u_x(t,0) = u_x(t,1), \qquad \forall t \geq 0,$$

$$u_{xx}(t,0) = u_{xx}(t,1), \qquad u_{xxx}(t,0) = u_{xxx}(t,1), \qquad \forall t \geq 0,$$

in which case the operator A acts again as $Au = -u_{xx}$, but now with domain

$$D(A) := \left\{ u \in H^2((0,1)) : u(0) = u(1), \ u_x(0) = u_x(1) \right\}.$$

- Boundary conditions such as

$$u(t,0) = u_{xx}(t,0) = 0, \qquad u_x(t,1) = u_{xxx}(t,1) = 0 \qquad \forall t \geq 0.$$

Indeed such a case can be easily reduced to (1.2) after extending the solution to the interval $(0, 2)$ by means of a reflection with respect to $x = 1$.

The "non-commutative" case Let us consider now equation (1.1) with boundary conditions
$$u(t, x) = u_x(t, x) = 0 \qquad \forall (t, x) \in [0, +\infty) \times \{0, 1\},$$
physically corresponding to "clamped ends". After suitable variable changes, we end up with an abstract evolution problem of the form

$$u'' + u' + B^2 u - \lambda A u + |A^{1/2} u|^2 A u = f(t). \tag{1.6}$$

The Hilbert space H and the operator A are the same as before. Also the operator B^2 acts as $B^2 u = u_{xxxx}$ as in the previous case, but now with domain

$$D(B^2) := \left\{ u \in H^4((0, 1)) : u(0) = u_x(0) = u(1) = u_x(1) = 0 \right\}.$$

This makes a great difference, because A^2 and B^2 have now different eigenspaces, and hence they do not commute (note also that, with this choice of the domain $D(B^2)$, the operator B, defined as the square root of B^2, does not act as $-u_{xx}$).

Nevertheless, the functional has now the form

$$\mathcal{E}_{A,B}(u) := \frac{1}{2} |Bu|^2 - \frac{\lambda}{2} |A^{1/2} u|^2 + \frac{1}{4} |A^{1/2} u|^4, \tag{1.7}$$

which is qualitatively similar to (1.5). In particular, in Proposition 2.4 we show that there exist again two positive constants $\lambda_2 > \lambda_1 > 0$, which are now the two smallest eigenvalues of the operator $A^{-1} B^2$ (see Proposition 2.7 and the final appendix), with the following properties.

- When $\lambda < \lambda_1$ the operator $B^2 - \lambda A$ is positive, and the functional $\mathcal{E}_{A,B}(u)$ is convex with a unique minimum point at the origin.

- When $\lambda_1 < \lambda < \lambda_2$ the operator $B^2 - \lambda A$ is negative in a subspace of dimension one, and positive in the orthogonal subspace. In this regime the functional $\mathcal{E}_{A,B}(u)$ has three stationary points: the origin, which in no longer a minimum point, and two minimum points that are symmetric with respect to the origin.

- When $\lambda > \lambda_2$ the operator $B^2 - \lambda A$ is negative in a subspace of dimension at least two, and the functional $\mathcal{E}_{A,B}(u)$ has more than three stationary points.

In this paper we investigate the regime $\lambda \in (\lambda_1, \lambda_2)$, and we show that solutions to (1.6) have the same qualitative behavior as the solutions to the "commutative" model (1.4) in the corresponding regime.

We conclude this paragraph by mentioning that the non-commutative case is also the correct setting for dealing with boundary conditions such as

$$u(t, 0) = u_{xx}(t, 0) = u(t, 1) = u_x(t, 1) = 0 \qquad \forall t \geq 0,$$

physically corresponding to a beam hinged in $x = 0$, and clamped in $x = 1$. In this case A is the same operator as before, and $B^2 u = u_{xxxx}$ with domain

$$D(B^2) := \left\{ u \in H^4((0,1)) : u(0) = u_{xx}(0) = u(1) = u_x(1) = 0 \right\}.$$

Unfortunately, the cantilever beam with one free end described in [9] fits neither in the commutative, nor in the non-commutative setting. That model involves nonlinear boundary conditions in the free endpoint, and for this reason it deserves a distinct theory that we plan to investigate in the future.

Structure of the paper This paper is organized as follows. In section 2 we clarify the functional setting, we state a preliminary well-posedness result for (1.6) (Proposition 2.1), and then we state our main result (Theorem 2.5) concerning the existence of three different asymptotic regimes, and a simple consequence (Corollary 2.6). In section 3 we state four auxiliary propositions, where we concentrate the technical machinery of the paper. In section 4 we prove all the abstract properties of the operator $B^2 - \lambda A$ that we need in the paper. Section 5 is devoted to the proof of the four auxiliary propositions. Section 6 contains the proof of our main result. In section 7 we show that the beam equation (1.1) with clamped ends fits in our abstract framework. Finally, in the appendix we discuss the correct functional setting for the operator $A^{-1}B^2$ in the case where A and B do not commute necessarily.

2 Statements

Throughout this paper we always consider equation (1.6) with initial data

$$u(0) = u_0 \in D(B), \qquad u'(0) = u_1 \in H. \tag{2.1}$$

Well-posedness Rather classical techniques lead to the following well-posedness result under quite general assumptions on the operators A and B, and on the parameter λ.

Proposition 2.1. *Let H be a Hilbert space, let λ be a real number, let $f : \mathbb{R} \to H$ be a continuous function, and let A and B be two self-adjoint nonnegative linear operators on H with dense domains $D(B) \subseteq D(A)$.*
  Let us assume that there exists a positive constant μ_1 such that

$$|Bu|^2 \geq \mu_1 |Au|^2 \qquad \forall u \in D(B). \tag{2.2}$$

  Then the following statements hold true.

(1) (Global existence and uniqueness) *For every $(u_0, u_1) \in D(B) \times H$, problem (1.6)–(2.1) admits a unique global solution*

$$u \in C^0\left(\mathbb{R}, D(B)\right) \cap C^1\left(\mathbb{R}, H\right).$$

(2) *(Continuous dependence on initial data)* *Let* $\{(u_{0n}, u_{1n})\} \subseteq D(B) \times H$ *be any sequence with*

$$(u_{0n}, u_{1n}) \to (u_0, u_1) \qquad in \ D(B) \times H,$$

and let $u_n(t)$ *denote the solution to (1.6) with data* $u_n(0) = u_{0n}$ *and* $u'(0) = u_{1n}$. *Then for every* $T > 0$ *it turns out that*

$$u_n(t) \to u(t) \qquad uniformly \ in \ C^0\left([-T, T], D(B)\right),$$

$$u'_n(t) \to u'(t) \qquad uniformly \ in \ C^0\left([-T, T], H\right).$$

(3) *(Derivative of the energy)* *The classical energy*

$$E(t) := \frac{1}{2}|u'(t)|^2 + \frac{1}{2}|Bu(t)|^2 - \frac{\lambda}{2}|A^{1/2}u(t)|^2 + \frac{1}{4}|A^{1/2}u(t)|^4 \qquad (2.3)$$

is of class C^1, *and its time-derivative is*

$$E'(t) = -|u'(t)|^2 + \langle u'(t), f(t) \rangle \qquad \forall t \in \mathbb{R}.$$

Remark 2.2. For the sake of simplicity, in the statement of Proposition 2.1 above we assumed that the forcing term $f(t)$ is defined for every $t \in \mathbb{R}$. Of course, if $f(t)$ is defined only in the half-line $t \geq 0$, or in some interval $[0, T]$, then the solution $u(t)$ is defined for the same values of t.

In the case where $A = B$, it was proved that the asymptotic dynamics depend on the position of λ with respect to the spectrum of A. When the two operators A and B are different, and do not commute, it is not immediately clear which set will play the role of the spectrum of A. From the heuristic point of view, it is useful to consider first the finite dimensional case.

The finite dimensional case If $\dim H < \infty$, then the operators A and B are represented by two symmetric and positive matrices, and the square roots of the quadratic forms associated to B^2 and A define two equivalent norms on H. Moreover let us define

$$\Sigma = \{u \in H, \quad |A^{1/2}u|^2 = 1\}$$

and

$$\lambda_1 := \min_{u \in \Sigma} |Bu|^2 \qquad (2.4)$$

then any minimizer u of (2.4) satisfies

$$B^2 u = \lambda_1 A u. \qquad (2.5)$$

In particular, λ_1 is the smallest eigenvalue of the matrix $A^{-1}B^2$, and the set of minimizers of (2.4) spans the corresponding eigenspace. From the definition of λ_1 it follows also that the matrix $B^2 - \lambda A$ is positive for every $\lambda < \lambda_1$.

Now let us choose a minimizer $e_1 \in \Sigma$ of (2.4) and let us set

$$\Sigma_+ = \{u \in H, \quad |A^{1/2}u|^2 = 1, \ \langle u, Ae_1 \rangle = 0\}$$

and

$$\lambda_2 := \min_{u \in \Sigma_+} |Bu|^2 \geq \lambda_1. \tag{2.6}$$

Then it turns out that λ_2 is the second smallest eigenvalue of $A^{-1}B^2$, and the set of minimizers of (2.6) is the corresponding eigenspace. If the strict inequality $\lambda_2 > \lambda_1$ holds true (and this happens if and only if λ_1 is simple), then for every $\lambda \in (\lambda_1, \lambda_2)$ the matrix $B^2 - \lambda A$ has exactly one negative eigenvalue, while all remaining eigenvalues are positive. In this case one says that the negative inertia index of $B^2 - \lambda A$ is 1.

For $\lambda > \lambda_2$, the matrix $B^2 - \lambda A$ has at least two negative eigenvalues.

This process can be carried on, thus showing that the number of negative eigenvalues of $B^2 - \lambda A$ increases when λ crosses the eigenvalues of $A^{-1}B^2$.

Operators with gap condition In the following definition we extend to infinite dimensions the framework described above.

Definition 2.3 (Pairs of operators with gap condition)**.** Let H be a Hilbert space, and let $\lambda_1 < \lambda_2$ be two positive real numbers. We say that two operators A and B satisfy the (λ_1, λ_2) gap condition, and we write $(A, B) \in \mathcal{G}(H, \lambda_1, \lambda_2)$, if

- A and B are self-adjoint linear operators on H with dense domains $D(B) \subseteq D(A)$, and there exists a positive constant μ_1 for which (2.2) holds true.

- there exists a positive real number μ_2 such that

$$|A^{1/2}u|^2 \geq \mu_2|u|^2 \qquad \forall u \in D(A^{1/2}), \tag{2.7}$$

- there exists $e_1 \in D(B^2)$, with $|A^{1/2}e_1| = 1$, such that

$$B^2 e_1 = \lambda_1 Ae_1 \tag{2.8}$$

  and

$$|Bu|^2 \geq \lambda_2|A^{1/2}u|^2 \qquad \forall u \in \{x \in D(B) : \langle x, Ae_1 \rangle = 0\}. \tag{2.9}$$

In the following result we collect all the properties of this class of operators that we need in this paper.

Proposition 2.4 (Properties of pairs of operators with gap condition)**.** *Let H be a Hilbert space, let $\lambda_1 < \lambda_2$ be two positive real numbers, and let $(A, B) \in \mathcal{G}(H, \lambda_1, \lambda_2)$.*

Then the following statements hold true.

(1) For every $\lambda \in \mathbb{R}$ the operator $B^2 - \lambda A$ is self-adjoint as an unbounded linear operator in H with domain $D(B^2)$.

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(2) *For every $\lambda < \lambda_1$ the self-adjoint operator $B^2 - \lambda A$ is positive, namely*

$$|Bu|^2 - \lambda|A^{1/2}u|^2 > 0 \qquad \forall u \in D(B) \setminus \{0\}. \tag{2.10}$$

(3) *For every $\lambda \in (\lambda_1, \lambda_2)$ the operator $B^2 - \lambda A$ has negative inertia index equal to one. More precisely, there exist a positive real number λ_0, and an element $e_0 \in D(B^2)$ with $|e_0| = 1$ (both λ_0 and e_0 do depend also on λ), such that*

$$B^2 e_0 - \lambda A e_0 = -\lambda_0 e_0, \tag{2.11}$$

and there exists a positive constant μ_3 (again depending on λ) such that

$$|Bu|^2 - \lambda|A^{1/2}u|^2 \geq \mu_3|Bu|^2 \qquad \forall u \in \{x \in D(B) : \langle x, e_0 \rangle = 0\}. \tag{2.12}$$

(4) *For every $\lambda \in (\lambda_1, \lambda_2)$ the functional $\mathcal{E}_{A,B}(u)$ defined in (1.7) has three stationary points, namely the three solutions to the equation*

$$B^2 u - \lambda A u + |A^{1/2}u|^2 A u = 0.$$

These three solutions are 0 and $\pm\sigma_0 e_1$, where e_1 is the vector that appears in (2.8) and (2.9), and

$$\sigma_0 := \sqrt{\lambda - \lambda_1}. \tag{2.13}$$

(5) *If λ_2 is the largest constant for which (2.9) holds true, then for every $\lambda > \lambda_2$ the operator $B^2 - \lambda A$ has negative inertia index at least two, namely there exists a two-dimensional subspace $V \subseteq D(B)$ such that*

$$|Bv|^2 - \lambda|A^{1/2}v|^2 < 0 \qquad \forall v \in V \setminus \{0\}. \tag{2.14}$$

Main result   From now on, we always consider problem (1.6)–(2.1) under the following assumptions, which we briefly call *standard assumptions*:

- H is a Hilbert space,

- $0 < \lambda_1 < \lambda_2$ are real numbers,

- $(A, B) \in \mathcal{G}(H, \lambda_1, \lambda_2)$ is a pair of operators satisfying the gap condition,

- $\lambda \in (\lambda_1, \lambda_2)$ is a real number,

- $f : [0, +\infty) \to H$ is a bounded and continuous function.

Our main result is the following.

Theorem 2.5 (Asymptotic behavior of solutions with small external force). *Let us consider problem (1.6)–(2.1) under the standard assumptions presented above. Let σ_0 be the constant defined by (2.13).*

Then there exists two positive constants ε_0 and M_0, independent of the forcing term $f(t)$ and of the solution $u(t)$, for which the following statements hold true whenever

$$\limsup_{t \to +\infty} |f(t)| \leq \varepsilon_0. \tag{2.15}$$

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(1) (Alternative) *For every solution $u(t)$ to (1.6), there exists $\sigma \in \{-\sigma_0, 0, \sigma_0\}$ such that*

$$\limsup_{t \to +\infty} \left(|u'(t)| + |B(u(t) - \sigma e_1)| \right) \le M_0 \limsup_{t \to +\infty} |f(t)|. \qquad (2.16)$$

(2) (Asymptotic convergence) *If $u(t)$ and $v(t)$ are any two solutions to (1.6) satisfying (2.16) with the same $\sigma \in \{-\sigma_0, 0, \sigma_0\}$, then $u(t)$ and $v(t)$ are asymptotic to each other in the sense that*

$$\lim_{t \to +\infty} \left(|u'(t) - v'(t)| + |B(u(t) - v(t))| \right) = 0. \qquad (2.17)$$

(3) (Solutions with $\sigma = \pm\sigma_0$) *The set of initial data (u_0, u_1) for which the solution to (1.6)–(2.1) satisfies (2.16) with a given $\sigma \in \{-\sigma_0, \sigma_0\}$ is a nonempty open subset of $D(B) \times H$.*

(4) (Solutions with $\sigma = 0$) *The set of initial data (u_0, u_1) for which the solution to (1.6)–(2.1) satisfies (2.16) with $\sigma = 0$ is a nonempty closed subset of $D(B) \times H$.*

When there is no external force, or the external force vanishes in the limit, then all solutions tend to one of the three stationary points of the functional $\mathcal{E}_{A,B}(u)$.

Corollary 2.6 (Asymptotically unforced equation). *Let us consider problem (1.6)–(2.1) under the standard assumptions presented before Theorem 2.5.*
   Let us assume in addition that

$$\lim_{t \to +\infty} f(t) = 0 \qquad \text{in } H.$$

Then there exists $\sigma \in \{-\sigma_0, 0, \sigma_0\}$ such that

$$\lim_{t \to +\infty} \left(|u'(t)| + |B(u(t) - \sigma e_1|) \right) = 0.$$

Application to the beam equation with clamped ends   The abstract results stated in Theorem 2.5 and Corollary 2.6 can be applied to the beam equation (1.1) with clamped ends. To this end, it is enough to show that the relevant operators A and B fit in the framework of Definition 2.3.

Proposition 2.7 (Abstract setting for the beam with clamped ends). *Let us consider the Hilbert space $H = L^2((0,1))$, the self-adjoint positive unbounded operator A on H defined by $Au = -u_{xx}$ with domain*

$$D(A) := H^2((0,1)) \cap H_0^1((0,1)),$$

and the self-adjoint positive unbounded operator B on H such that $B^2 u = u_{xxxx}$ with domain

$$D(B^2) := H^4((0,1)) \cap H_0^2((0,1)).$$

   Then it turns out that $(A, B) \in \mathcal{G}(H, \lambda_1, \lambda_2)$ with $\lambda_1 = 4\pi^2$ and $\lambda_2 = 4\alpha_1^2$, where $\alpha_1 \in (\pi, 3\pi/2)$ is the smallest positive real solution to the equation $\tan \alpha = \alpha$.

3   Auxiliary results

In this section we state four auxiliary results that correspond to the key steps in the proof of Theorem 2.5.

To begin with, we introduce the operator

$$Ru := u - 2\langle u, e_0\rangle e_0 \qquad \forall u \in H, \tag{3.1}$$

where e_0 is a unit vector satisfying (2.11). We observe that R is the bounded operator on H such that $Re_0 = -e_0$ and $Rv = v$ for every v orthogonal to e_0. Then we set

$$\delta := \frac{1}{2}\left(1 + \frac{2}{\mu_2^{1/2}}|A^{1/2}e_0|\right)^{-1}, \tag{3.2}$$

where μ_2 is the constant that appears in (2.7), and we define the operator

$$Pu := u + \delta Ru \qquad \forall u \in H. \tag{3.3}$$

Finally, we choose a positive real number γ_0 such that

$$\gamma_0 \le \frac{1}{2(5+2\delta)}, \qquad \gamma_0 \le \frac{\delta}{(5+\delta)(1+\delta)} \min\left\{\mu_2^2\mu_1\mu_3, \lambda_0\right\}, \tag{3.4}$$

where μ_1, λ_0 and μ_3 are the constants that appear in (2.2), (2.11) and (2.12), respectively.

For every solution $u(t)$ to (1.6), we consider the classical energy $E(t)$ defined by (2.3), and the modified energy

$$F(t) := E(t) + 2\gamma_0\langle Pu(t), u'(t)\rangle + \gamma_0\langle Pu(t), u(t)\rangle. \tag{3.5}$$

In the first result we prove that the energy $F(t)$ of solutions to (1.6) is bounded for t large in terms of the norm of the forcing term. As a consequence, all solutions are bounded in $D(B) \times H$.

Proposition 3.1 (Ultimate bound on solutions). *Let us consider problem (1.6)–(2.1) under the standard assumptions presented before Theorem 2.5. Let $F(t)$ be the energy defined in (3.5).*

Then there exists a positive constant M_1, independent of the forcing term $f(t)$ and of the solution $u(t)$, such that

$$\limsup_{t\to+\infty} F(t) \le M_1 \limsup_{t\to+\infty} |f(t)|^2, \tag{3.6}$$

and there exist two positive constants M_2 and M_3, again independent of $f(t)$ and $u(t)$, such that

$$\limsup_{t\to+\infty}\left(|u'(t)|^2 + |Bu(t)|^2\right) \le M_2 + M_3\limsup_{t\to+\infty} |f(t)|^2. \tag{3.7}$$

In the second result we deal with solutions $u(t)$ such that $|u(t)|$ is eventually smaller than a universal constant. We show that the norm of these solutions in the energy space is asymptotically bounded by the norm in H of the forcing term.

Proposition 3.2 (Solutions in the unstable regime). *Let us consider problem (1.6)–(2.1) under the standard assumptions presented before Theorem 2.5.*

Then there exist two positive constants β_0 and M_4, independent of the forcing term $f(t)$ and of the solution $u(t)$, for which the following implication is true:

$$
\boxed{\begin{aligned} &\limsup_{t \to +\infty} |f(t)| \leq 1 \\ &\limsup_{t \to +\infty} |u(t)| \leq \beta_0 \end{aligned}} \implies \boxed{\limsup_{t \to +\infty} \left(|u'(t)| + |Bu(t)| \right) \leq M_4 \limsup_{t \to +\infty} |f(t)|} \tag{3.8}
$$

In the third result we deal with solutions that, *at a given time*, are close to one of the stable stationary points of the functional (1.7). Here "close to a stationary point" means that the energy $E(t)$ is negative at the given time. We show that these solutions lie eventually in a neighborhood of the same stationary point, within a distance depending on the norm in H of the forcing term.

Proposition 3.3 (Solutions in the stable regime). *Let us consider problem (1.6)–(2.1) under the standard assumptions presented before Theorem 2.5. Let e_1 be the vector that appears in (2.8) and (2.9), and let $E(t)$ be the energy defined in (2.3).*

Then for every $\eta > 0$ there exist two constants $\varepsilon_1 > 0$ and $M_5 > 0$, independent of the forcing term $f(t)$ and of the solution $u(t)$, for which the following implication is true:

$$
\boxed{\begin{aligned} &\exists T_0 \geq 0 \ such \ that \\ &\sup_{t \geq T_0} |f(t)| \leq \varepsilon_1, \ E(T_0) < -\eta \end{aligned}}
$$

$$
\Downarrow \tag{3.9}
$$

$$
\boxed{\begin{aligned} &\exists \sigma \in \{\sigma_0, -\sigma_0\} \ such \ that \\ &\limsup_{t \to +\infty} \left(|u'(t)| + |B(u(t) - \sigma e_1)| \right) \leq M_5 \limsup_{t \to +\infty} |f(t)| \end{aligned}}
$$

Moreover, when the assumptions in the upper box of (3.9) are satisfied, it turns out that

$$
\langle u(t), Ae_1 \rangle \neq 0 \qquad \forall t \geq T_0, \tag{3.10}
$$

and as a consequence the sign of σ coincides with the sign of $\langle u(T_0), Ae_1 \rangle$.

In the last result we show that any two solutions to (1.6) that are close enough to the same stationary point of the functional (1.7) are actually asymptotic to each other.

Proposition 3.4 (Close solutions are asymptotic to each other). *Let us consider problem (1.6)–(2.1) under the standard assumptions presented before Theorem 2.5.*

Then there exists $r_0 > 0$ with the following property: if $u(t)$ and $v(t)$ are two solutions to (1.6), and there exists $\sigma \in \{-\sigma_0, 0, \sigma_0\}$ such that

$$
\limsup_{t \to +\infty} \left(|u'(t)| + |B(u(t) - \sigma e_1)| + |v'(t)| + |B(v(t) - \sigma e_1)| \right) \leq r_0, \tag{3.11}
$$

then $u(t)$ and $v(t)$ are asymptotic to each other in the sense of (2.17).

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4 Some linear algebra in infinite dimensions

A fundamental tool in [6] was considering the components of a solution $u(t)$ with respect to the eigenspaces of A. Due to the presence of two operators, in this paper we are forced to consider different decompositions of the Hilbert space H in different parts of the proof. In this section we introduce the decompositions that we need in the sequel, we state their basic properties, and we prove the linear algebra results of Proposition 2.4.

4.1 Decomposition of the space in the stable regime

Let e_1 be the vector that appears in (2.8) and (2.9). We consider the decomposition

$$H := W_- \oplus W_+, \tag{4.1}$$

where W_- is the one-dimensional subspace spanned by e_1, and W_+ is the subspace orthogonal to Ae_1. We point out that this is a direct sum, in general not orthogonal in the sense of H (but orthogonality is true in the sense of $D(A^{1/2})$ for vectors belonging to $D(A^{1/2})$, both projections remaining in $D(A^{1/2})$). Every vector $u \in H$ can be written in a unique way in the form

$$u = \alpha e_1 + w,$$

where $\alpha \in \mathbb{R}$ and $w \in W_+$ are given by

$$\alpha := \langle u, Ae_1 \rangle \qquad \text{and} \qquad w := u - \alpha e_1.$$

Due to (2.8), it turns out that

$$\langle w, B^2 e_1 \rangle = \lambda_1 \langle w, Ae_1 \rangle = 0 \qquad \forall w \in W_+, \tag{4.2}$$

from which it follows that

$$|A^{1/2}u|^2 = \alpha^2 + |A^{1/2}w|^2 \tag{4.3}$$

and

$$|Bu|^2 = \alpha^2 |Be_1|^2 + |Bw|^2 = \lambda_1 \alpha^2 + |Bw|^2 \tag{4.4}$$

for every $u \in D(B)$. Moreover, from (2.9) we deduce that

$$|Bw|^2 = \frac{\lambda_2 - \lambda}{\lambda_2} |Bw|^2 + \frac{\lambda}{\lambda_2} |Bw|^2 \geq \frac{\lambda_2 - \lambda}{\lambda_2} |Bw|^2 + \lambda |A^{1/2}w|^2$$

for every $w \in D(B) \cap W_+$, and in particular

$$|Bw|^2 - \lambda |A^{1/2}w|^2 \geq \frac{\lambda_2 - \lambda}{\lambda_2} |Bw|^2 \qquad \forall w \in D(B) \cap W_+. \tag{4.5}$$

From (4.3), (4.4), and (2.13) we deduce that, if $\lambda > \lambda_1$,

$$\begin{aligned}
|Bu|^2 - \lambda |A^{1/2}u|^2 &= (\lambda_1 - \lambda)\alpha^2 + |Bw|^2 - \lambda |A^{1/2}w|^2 \\
&= -\sigma_0^2 \alpha^2 + |Bw|^2 - \lambda |A^{1/2}w|^2.
\end{aligned} \tag{4.6}$$

4.2 Proof of Proposition 2.4

Statement (1) We have to prove that the operator $B^2 - \lambda A$ is self-adjoint with domain $D(B^2)$. To this end, it is enough to prove that the operator $B^2u - \lambda Au + mu$ is symmetric and maximal monotone with domain $D(B^2)$ for some real number m (here we exploit that a symmetric maximal monotone operator is self-adjoint, and that the sum of a self-adjoint operator and a bounded symmetric operator is again self-adjoint).

The symmetry is trivial, and therefore we can limit ourselves to check monotonicity and maximality.

- We claim that $B^2u - \lambda Au + mu$ is monotone when m is large enough. If $\lambda \leq 0$ the conclusion is true even with $m = 0$. If $\lambda > 0$ we exploit the inequality

$$|A^{1/2}u|^2 = \langle Au, u \rangle \leq \frac{\varepsilon}{2}|Au|^2 + \frac{1}{2\varepsilon}|u|^2 \qquad \forall u \in D(A) \quad \forall \varepsilon > 0,$$

and from (2.2) we deduce that

$$\begin{aligned}
\langle B^2u - \lambda Au + mu, u \rangle &= |Bu|^2 - \lambda|A^{1/2}u|^2 + m|u|^2 \\
&\geq \mu_1|Au|^2 - \frac{\lambda\varepsilon}{2}|Au|^2 - \frac{\lambda}{2\varepsilon}|u|^2 + m|u|^2.
\end{aligned}$$

At this point it is enough to choose $\varepsilon = 2\mu_1\lambda^{-1}$ and $m \geq \frac{\lambda}{2\varepsilon}$.

- We claim that $B^2u - \lambda Au + mu$ is surjective from $D(B^2)$ to H when m is large enough, namely that for every $f \in H$ there exists $u \in D(B^2)$ such that

$$B^2u - \lambda Au + mu = f. \tag{4.7}$$

To this end, we exploit a fixed point technique. For every $v \in D(A)$, the equation

$$B^2u + mu = \lambda Av + f \tag{4.8}$$

has a unique solution $u = T(v) \in D(B^2) \subseteq D(A)$. This defines a function $T : D(A) \to D(A)$. Any fixed point of T lies in $D(B^2)$, and is a solution to (4.7). Let v_1 and v_2 be in $D(A)$, and let us set $u_1 = T(v_1)$ and $u_2 = T(v_2)$. From (4.8) we obtain that

$$\begin{aligned}
|B(u_2 - u_1)|^2 + m|u_2 - u_1|^2 &= \lambda\langle A(v_2 - v_1), u_2 - u_1 \rangle \\
&\leq \lambda|A(v_2 - v_1)| \cdot |u_2 - u_1| \\
&\leq \frac{\lambda^2}{2m}|A(v_2 - v_1)|^2 + \frac{m}{2}|u_2 - u_1|^2.
\end{aligned}$$

Recalling (2.2), when m is large enough we deduce that

$$|A(u_2 - u_1)|^2 + |u_2 - u_1|^2 \leq \frac{1}{2}|A(v_2 - v_1)|^2,$$

and hence T is a contraction in the Hilbert space $D(A)$. This proves that the operator $B^2u - \lambda Au + mu$ is also maximal with domain $D(B^2)$.

Remark 4.1. The sum of a maximal monotone operator L_1 and a monotone operator L_2 dominated by L_1, with the same (or larger) domain, is again maximal monotone (see [3, Proposition 2.10]). Therefore, we can give an alternative proof by writing

$$B^2 u - \lambda A u + m u = \frac{2}{3} B^2 u + \frac{1}{3} B^2 u - \lambda A u + m u,$$

and applying the abstract result with $L_1 u := \frac{2}{3} B^2 u$ and $L_2 u := \frac{1}{3} B^2 u - \lambda A u + m u$. It is enough to check that L_2 is monotone and dominated by L_1, and this can be done as we did in the first item of the previous proof.

Statement (2)   Let us write any element $u \in H$ in the form $u = \alpha e_1 + w$ according to the direct sum (4.1). From (4.3), (4.4) and (2.9) it follows that

$$
\begin{aligned}
|Bu|^2 - \lambda |A^{1/2} u|^2 &= (\lambda_1 - \lambda)\alpha^2 + |Bw|^2 - \lambda |A^{1/2} w|^2 \\
&\geq (\lambda_1 - \lambda)\alpha^2 + (\lambda_2 - \lambda)|A^{1/2} w|^2 \\
&\geq 0,
\end{aligned}
$$

with strict inequality if either $\alpha \neq 0$ or $w \neq 0$. This proves (2.10).

Statement (3) – Computation of the negative inertia index   For every $\lambda > \lambda_1$, the operator $B^2 - \lambda A$ is negative in the one-dimensional subspace of H generated by e_1.

We claim that, if $\lambda < \lambda_2$, the same operator cannot be negative, or even just less than or equal to 0, in any subspace of H of dimension at least two. Indeed, any such subspace contains a vector $v \neq 0$ with $\langle v, A e_1 \rangle = 0$, and for this vector it turns out that

$$|Bv|^2 - \lambda |A^{1/2} v|^2 > |Bv|^2 - \lambda_2 |A^{1/2} v|^2 \geq 0,$$

where the second inequality follows from (2.9).

Statement (3) – Existence of an eigenvector with negative eigenvalue   According to the spectral theory (see for example [10, Theorem VIII.4]), we can identify H with $L^2(M, \mu)$ for some measure space (M, μ) in such a way that under this identification the operator $B^2 - \lambda A$ becomes a multiplication operator. This means that there exists a measurable function $\lambda(\xi)$ in M with the property that, if $\widehat{u}(\xi) \in L^2(M, \mu)$ corresponds to some $u \in H$, then $\lambda(\xi)\widehat{u}(\xi)$ corresponds to $B^2 u - \lambda A u$.

Let us consider the set
$$N := \{\xi \in M : \lambda(\xi) < 0\}.$$

We claim that $\mu(N) > 0$, and $\lambda(\xi)$ is equal to some negative constant $-\lambda_0$ for almost every $\xi \in N$. If we prove these claims, then the vector $e_0 \in H$ that under the identification corresponds to the characteristic function of N is an eigenvector of $B^2 - \lambda A$ with eigenvalue $-\lambda_0$.

In order to prove that $\mu(N) > 0$ it is enough to observe that otherwise the operator $B^2 - \lambda A$ would be nonnegative in H.

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In order to prove that $\lambda(\xi)$ is essentially constant in N, let us assume that this is not the case. Then there exists a real number $\lambda_* < 0$ such that the two sets

$$N_1 := \{\xi \in N : \lambda(\xi) \leq \lambda_*\} \qquad \text{and} \qquad N_2 := \{\xi \in N : \lambda(\xi) > \lambda_*\}$$

have positive measure. In this case the two vectors u_1 and u_2 corresponding to the characteristic functions of N_1 and N_2 would be two orthogonal vectors that span a two-dimensional subspace of H where $B^2 - \lambda A$ is negative, and we already know that this is not possible when $\lambda \in (\lambda_1, \lambda_2)$.

Statement (3) – Estimate in the orthogonal space Let us prove that (2.12) holds true if we choose

$$\mu_3 < \min\left\{\frac{\lambda_2 - \lambda}{\lambda_2 + \lambda}, \frac{\lambda_0}{2\lambda|A^{1/2}e_0|^2 + \lambda_0}\right\}. \tag{4.9}$$

To this end, let us assume that it is not the case. Then there exists $v \in D(B)$ with

$$\langle v, e_0 \rangle = 0 \qquad \text{and} \qquad |Bv|^2 - \lambda|A^{1/2}v| < \mu_3|Bv|^2. \tag{4.10}$$

Let us set

$$\widehat{\lambda} := \frac{1 + \mu_3}{1 - \mu_3}\lambda, \tag{4.11}$$

and let us observe that $\widehat{\lambda} < \lambda_2$ because of the first request in the definition of μ_3.

Now we show that the operator $B^2 - \widehat{\lambda}A$ is less than or equal to 0 on the two-dimensional subspace of H spanned by e_0 and v, which we already shown to be absurd. To this end, we take a generic vector $u = \alpha e_0 + \beta v$, and with some computations we obtain that

$$
\begin{aligned}
|Bu|^2 - \widehat{\lambda}|A^{1/2}u|^2 &= |Bu|^2 - \lambda|A^{1/2}u|^2 - (\widehat{\lambda} - \lambda)|A^{1/2}u|^2 \\
&= -\lambda_0\alpha^2 + \beta^2\left(|Bv|^2 - \lambda|A^{1/2}v|^2\right) \\
&\quad - (\widehat{\lambda} - \lambda)\left(\alpha^2|A^{1/2}e_0|^2 + \beta^2|A^{1/2}v|^2 + 2\alpha\beta\langle A^{1/2}e_0, A^{1/2}v\rangle\right).
\end{aligned}
$$

Now from (4.9) and (4.10) we deduce that

$$|Bv|^2 < \frac{\lambda}{1 - \mu_3}|A^{1/2}v|^2 \qquad \text{and} \qquad \frac{\lambda_0}{|A^{1/2}e_0|^2} + (\widehat{\lambda} - \lambda) \geq \frac{4\lambda\mu_3}{1 - \mu_3},$$

and keeping (4.11) into account we conclude that

$$
\begin{aligned}
|Bu|^2 - \widehat{\lambda}|A^{1/2}u|^2 &\leq -\alpha^2\left\{\frac{\lambda_0}{|A^{1/2}e_0|^2} + (\widehat{\lambda} - \lambda)\right\}|A^{1/2}e_0|^2 - \beta^2\left\{\widehat{\lambda} - \frac{\lambda}{1 - \mu_3}\right\}|A^{1/2}v|^2 \\
&\quad + 2(\widehat{\lambda} - \lambda)|\alpha| \cdot |\beta| \cdot |A^{1/2}e_0| \cdot |A^{1/2}v| \\
&\leq -\frac{\lambda\mu_3}{1 - \mu_3}\left(2|\alpha| \cdot |A^{1/2}e_0| - |\beta| \cdot |A^{1/2}v|\right)^2 \\
&\leq 0.
\end{aligned}
$$

Statement (4)   Let us set $\mu := |A^{1/2}u|^2$, and let us look for nonzero solutions to equation

$$B^2 u - (\lambda - \mu)Au = 0. \tag{4.12}$$

Let us write as usual $u = \alpha e_1 + w$ according to the direct sum (4.1). Then equation (4.12) reduces to

$$\alpha B^2 e_1 - (\lambda - \mu)\alpha A e_1 + B^2 w - (\lambda - \mu)Aw = 0.$$

Due to (2.8), taking the scalar product of this equation with w we obtain that

$$|Bw|^2 = (\lambda - \mu)|A^{1/2}w|^2.$$

Since $\lambda - \mu < \lambda_2$, this is impossible if $w \neq 0$ because of (2.9). It follows that $u = \alpha e_1$ for some $\alpha \neq 0$, and

$$\alpha B^2 e_1 - (\lambda - \mu)\alpha A e_1 = 0.$$

Keeping (2.8) into account, we conclude that

$$\lambda_1 = \lambda - \mu = \lambda - |A^{1/2}u|^2 = \lambda - \alpha^2,$$

which implies that $\alpha = \pm\sigma_0$, with σ_0 given by (2.13).

Statement (5)   If λ_2 is the largest constant for which (2.9) holds true, then for every $\lambda > \lambda_2$ there exists a vector $v \neq 0$ (possibly depending on λ) such that $\langle v, Ae_1 \rangle = 0$ and $|Bv|^2 < \lambda|A^{1/2}v|^2$. At this point it turns out that (2.14) holds true in the two-dimensional subspace of H spanned by e_1 and v.

4.3   Decomposition of the space in the unstable regime

Let us consider the operator $L := B^2 - \lambda A$, which we know to be self-adjoint with domain $D(L) = D(B^2)$. Since $\lambda \in (\lambda_1, \lambda_2)$, from statement (3) of Proposition 2.4 we know that L has a negative eigenvalue $-\lambda_0$. Given a corresponding eigenvector e_0 with unit norm, we write H as a direct orthogonal sum

$$H := H_- \oplus H_+, \tag{4.13}$$

where H_- is the one-dimensional subspace spanned by e_0, and H_+ is the subspace orthogonal to e_0. In this way any vector $u \in H$ can be written in a unique way as the sum of a low-frequency component $u_- \in H_-$ and a high-frequency component $u_+ \in H_+$, where of course

$$u_- := \langle u, e_0 \rangle e_0 \quad \text{and} \quad u_+ := u - u_-.$$

We point out that H_- and H_+ are invariant subspaces for L, but they are not necessarily invariant spaces for A or B.

From (2.12) we know that L is a positive operator when restricted to H_+, and

$$|L^{1/2}u|^2 \geq \mu_3|Bu|^2 \qquad \forall u \in D(B) \cap H_+. \tag{4.14}$$

Since from (2.2) and (2.7) we know that

$$|Bu|^2 \geq \mu_1 |Au|^2 \geq \mu_1 \mu_2 |A^{1/2}u|^2 \geq \mu_1 \mu_2^2 |u|^2 \qquad \forall u \in D(B), \qquad (4.15)$$

from (4.14) we can derive estimates for $|u|$, $|A^{1/2}u|$ and $|Au|$ in terms of $|L^{1/2}u|$ for every $u \in D(B) \cap H_+$.

5 Proof of auxiliary results

5.1 Useful ultimate bounds

In this subsection we recall three results concerning ultimate bounds that are crucial in the sequel. For a proof we refer to [6, Section 4.1] and to the references quoted therein.

Lemma 5.1. *For every bounded function $\varphi : [0, +\infty) \to \mathbb{R}$ of class C^1 there exists a sequence $t_n \to +\infty$ of nonnegative real numbers such that*

$$\lim_{n \to +\infty} \varphi(t_n) = \limsup_{t \to +\infty} \varphi(t) \qquad and \qquad \lim_{n \to +\infty} \varphi'(t_n) = 0.$$

Lemma 5.2. *Let m be a positive real number, let $\psi : [0, +\infty) \to \mathbb{R}$ be a continuous function, and let $y \in C^2([0, +\infty), \mathbb{R})$ be a solution to*

$$y''(t) + y'(t) - my(t) = \psi(t).$$

Let us assume that both $\psi(t)$ and $y(t)$ are bounded.
Then it turns out that

$$\limsup_{t \to +\infty} |y(t)| \leq \frac{1}{m} \limsup_{t \to +\infty} |\psi(t)|,$$

$$\limsup_{t \to +\infty} |y'(t)| \leq 2 \limsup_{t \to +\infty} |\psi(t)|.$$

Lemma 5.3. *Let X be a Hilbert space, and let $\mathcal{L}$ be a self-adjoint linear operator on X with dense domain $D(\mathcal{L})$. Let us assume that there exists a constant $m > 0$ such that*

$$\langle \mathcal{L}x, x \rangle \geq m|x|^2 \qquad \forall x \in D(\mathcal{L}).$$

Let $\psi : [0, +\infty) \to X$ be a bounded continuous function, and let

$$y \in C^0\left([0, +\infty), D(\mathcal{L}^{1/2})\right) \cap C^1\left([0, +\infty), X\right)$$

be a solution to

$$y''(t) + y'(t) + \mathcal{L}y(t) = \psi(t).$$

Then it turns out that

$$\limsup_{t \to +\infty} \left(|y'(t)|^2 + |\mathcal{L}^{1/2}y(t)|^2\right) \leq 9 \max\left\{1, \frac{1}{m}\right\} \cdot \limsup_{t \to +\infty} |\psi(t)|^2.$$

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5.2 Useful estimates for functionals and energies

In this subsection we collect some identities and inequalities that are needed several times in the sequel.

Let R and P be the operators defined in (3.1) and (3.3). Let us consider the orthogonal direct sum (4.13). Then it turns out that

$$Ru = \begin{cases} -u & \text{if } u \in H_-, \\ u & \text{if } u \in H_+, \end{cases} \qquad \text{and} \qquad Pu = \begin{cases} (1-\delta)u & \text{if } u \in H_-, \\ (1+\delta)u & \text{if } u \in H_+. \end{cases}$$

As a consequence we obtain that

$$(1-\delta)|u|^2 \leq \langle Pu, u \rangle \leq (1+\delta)|u|^2 \qquad \forall u \in H, \tag{5.1}$$

and

$$(1-\delta)\langle Pu, u \rangle \leq |Pu|^2 \leq (1+\delta)\langle Pu, u \rangle \qquad \forall u \in H. \tag{5.2}$$

Let us consider the functional $\mathcal{E}_{A,B}(u)$ defined in (1.7). Since

$$\frac{1}{4}x^4 - \frac{\lambda}{2}x^2 \geq -\frac{\lambda^2}{4} \qquad \forall x \in \mathbb{R},$$

we obtain that

$$\mathcal{E}_{A,B}(u) \geq \frac{1}{2}|Bu|^2 - \frac{\lambda^2}{4} \qquad \forall u \in D(B), \tag{5.3}$$

and analogously

$$E(t) \geq \frac{1}{2}|u'(t)|^2 + \frac{1}{2}|Bu(t)|^2 - \frac{\lambda^2}{4}. \tag{5.4}$$

Let us write now u in the form $\alpha e_1 + w$ according to the direct sum (4.1). Then from (4.3), (4.6), and (4.5) we deduce that

$$\begin{aligned} \mathcal{E}_{A,B}(u) &= \frac{1}{4}\alpha^4 - \frac{1}{2}\sigma_0^2\alpha^2 + \frac{1}{2}|Bw|^2 - \frac{\lambda}{2}|A^{1/2}w|^2 \\ &\quad + \frac{1}{4}|A^{1/2}w|^4 + \frac{1}{2}\alpha^2 \cdot |A^{1/2}w|^2 \\ &\geq \frac{1}{4}\alpha^4 - \frac{1}{2}\sigma_0^2\alpha^2 + \frac{1}{2}\frac{\lambda_2 - \lambda}{\lambda_2}|Bw|^2. \end{aligned} \tag{5.5}$$

This shows in particular that

$$\mathcal{E}_{A,B}(u) \geq -\frac{1}{4}\sigma_0^4 \qquad \forall u \in D(B),$$

with equality if and only if $u = \pm\sigma_0 e_1$, and in addition

$$\mathcal{E}_{A,B}(u) + \frac{1}{4}\sigma_0^4 \geq \frac{1}{4}\left(\alpha^2 - \sigma_0^2\right)^2 + \frac{1}{2}\frac{\lambda_2 - \lambda}{\lambda_2}|Bw|^2. \tag{5.6}$$

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5.3 Proof of Proposition 3.1

Estimates on $F(t)$ from above and below Let us consider the energy $F(t)$ defined in (3.5). We prove that

$$F(t) \leq \frac{3}{4}|u'(t)|^2 + \frac{1}{2}|Bu(t)|^2 - \frac{\lambda}{2}|A^{1/2}u(t)|^2 + \frac{1}{4}|A^{1/2}u(t)|^4 + 2\gamma_0(1+\delta)|u(t)|^2 \quad (5.7)$$

and

$$F(t) \geq \frac{1}{4}|u'(t)|^2 + \frac{1}{2}|Bu(t)|^2 - \frac{\lambda}{2}|A^{1/2}u(t)|^2 + \frac{1}{4}|A^{1/2}u(t)|^4. \quad (5.8)$$

Indeed, from (5.2) it follows that

$$|2\langle Pu(t), u'(t)\rangle| \leq (1+\delta)|u'(t)|^2 + \frac{1}{1+\delta}|Pu(t)|^2 \leq (1+\delta)|u'(t)|^2 + \langle Pu(t), u(t)\rangle.$$

Plugging this inequality into (3.5) we deduce that

$$\begin{aligned}
F(t) &\leq \left(\frac{1}{2} + \gamma_0(1+\delta)\right)|u'(t)|^2 + \frac{1}{2}|Bu(t)|^2 - \frac{\lambda}{2}|A^{1/2}u(t)|^2 \\
&\quad + \frac{1}{4}|A^{1/2}u(t)|^4 + 2\gamma_0\langle Pu(t), u(t)\rangle
\end{aligned} \quad (5.9)$$

and

$$F(t) \geq \left(\frac{1}{2} - \gamma_0(1+\delta)\right)|u'(t)|^2 + \frac{1}{2}|Bu(t)|^2 - \frac{\lambda}{2}|A^{1/2}u(t)|^2 + \frac{1}{4}|A^{1/2}u(t)|^4. \quad (5.10)$$

Finally, we observe that (3.2) and (3.4) imply in particular that $\gamma_0(1+\delta) \leq 1/4$. At this point, (5.7) follows from (5.9) and (5.1), while (5.8) follows from (5.10).

Estimates for the operator R We show that

$$\delta|\langle Ru, Au\rangle| \leq \frac{1}{2}|A^{1/2}u|^2 \qquad \forall u \in D(A^{1/2}), \quad (5.11)$$

and

$$\langle Ru, (B^2 - \lambda A)u\rangle \geq \min\left\{\mu_2^2\mu_1\mu_3, \lambda_0\right\} \cdot |u|^2 \qquad \forall u \in D(B). \quad (5.12)$$

Indeed, since $|\langle u, e_0\rangle| \leq |u|$, from (2.7) it follows that

$$\begin{aligned}
|A^{1/2}Ru| &= \left|A^{1/2}u - 2\langle u, e_0\rangle A^{1/2}e_0\right| \\
&\leq |A^{1/2}u| + 2|u| \cdot |A^{1/2}e_0| \\
&\leq |A^{1/2}u| + \frac{2}{\mu_2^{1/2}}|A^{1/2}u| \cdot |A^{1/2}e_0|,
\end{aligned}$$

and therefore

$$|\langle Ru, Au\rangle| \leq |A^{1/2}Ru| \cdot |A^{1/2}u| \leq \left(1 + \frac{2}{\mu_2^{1/2}}|A^{1/2}e_0|\right)|A^{1/2}u|^2.$$

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At this point, (5.11) follows from (3.2).

In order to prove (5.12), we write u as $u_+ + u_-$ according to the decomposition (4.13). Since H_- and H_+ are invariant subspaces for $B^2 - \lambda A$, from (2.11), (2.12) and (4.15) it follows that

$$
\begin{aligned}
\langle Ru, (B^2 - \lambda A)u \rangle &= \langle u_+ - u_-, (B^2 - \lambda A)u_+ + (B^2 - \lambda A)u_- \rangle \\
&= \langle u_+, (B^2 - \lambda A)u_+ \rangle - \langle u_-, (B^2 - \lambda A)u_- \rangle \\
&\geq \mu_3 |Bu_+|^2 + \lambda_0 |u_-|^2 \\
&\geq \mu_3 \mu_2^2 \mu_1 |u_+|^2 + \lambda_0 |u_-|^2,
\end{aligned}
$$

which implies (5.12).

Differential inequality solved by $F(t)$ We prove that

$$
F'(t) \leq -4\gamma_0 F(t) + |f(t)|^2 \qquad \forall t \geq 0. \tag{5.13}
$$

To this end, we compute the time-derivative of $F(t)$ and we exploit (1.6) and (3.3). We obtain that

$$
\begin{aligned}
F'(t) &= -|u'(t)|^2 + 2\gamma_0 \langle Pu'(t), u'(t) \rangle \\
&\quad + \langle u'(t), f(t) \rangle + 2\gamma_0 \langle Pu(t), f(t) \rangle \\
&\quad - 2\gamma_0 \left(|Bu(t)|^2 - \lambda |A^{1/2}u(t)|^2 + |A^{1/2}u(t)|^4 \right) \\
&\quad - 2\gamma_0 \delta \langle Ru(t), (B^2 - \lambda A)u(t) \rangle - 2\gamma_0 \delta |A^{1/2}u(t)|^2 \cdot \langle Ru(t), Au(t) \rangle.
\end{aligned}
$$

Let L_1, L_2, L_3, L_4 denote the terms of the four lines. From (5.1) and the first condition in (3.4) we obtain that

$$
L_1 \leq -|u'(t)|^2 + 2\gamma_0(1+\delta)|u'(t)|^2 \leq -\left(\frac{1}{2} + 3\gamma_0 \right) |u'(t)|^2.
$$

From (5.1) and (5.2) we obtain that

$$
\begin{aligned}
L_2 &\leq \frac{1}{2}|u'(t)|^2 + \frac{1}{2}|f(t)|^2 + 2\gamma_0^2 |Pu(t)|^2 + \frac{1}{2}|f(t)|^2 \\
&\leq \frac{1}{2}|u'(t)|^2 + 2\gamma_0^2(1+\delta)^2 |u(t)|^2 + |f(t)|^2.
\end{aligned}
$$

Finally, from (5.12), (5.11) and the second condition in (3.4) we obtain that

$$
\begin{aligned}
L_4 &\leq -2\gamma_0 \delta \min\left\{ \mu_2^2 \mu_1 \mu_3, \lambda_0 \right\} \cdot |u(t)|^2 + \gamma_0 |A^{1/2}u(t)|^4 \\
&\leq -2\gamma_0^2(5+\delta)(1+\delta) \cdot |u(t)|^2 + \gamma_0 |A^{1/2}u(t)|^4.
\end{aligned}
$$

Plugging all these estimates into the expression for $F'(t)$, and keeping (5.7) into account, we deduce (5.13).

Conclusion Integrating the differential inequality (5.13), and letting $t \to +\infty$, we obtain (3.6) with $M_1 := (4\gamma_0)^{-1}$. Finally, from (5.8) and (5.3) we obtain that

$$|u'(t)|^2 + |Bu(t)|^2 \leq \lambda^2 + 4F(t) \qquad \forall t \geq 0,$$

and therefore (3.7) with $M_2 := \lambda^2$ and $M_3 := 1/\gamma_0$ is a consequence of (3.6). $\quad\square$

5.4 Proof of Proposition 3.2

Choice of parameters According to the direct orthogonal sum (4.13), any solution $u(t)$ to (1.6) is the sum of a low-frequency component $u_-(t) \in H_-$ and a high-frequency component $u_+(t) \in H_+$. Let L denote as usual the operator $B^2 - \lambda A$.

The high-frequency component is a solution to

$$u_+''(t) + u_+'(t) + Lu_+(t) + |A^{1/2}u_+(t)|^2(Au_+(t))_+ = \psi_1(t) + \psi_2(t), \qquad (5.14)$$

where

$$\begin{aligned} \psi_1(t) \quad &:= \quad -|A^{1/2}u_+(t)|^2(Au_-(t))_+ - 2\langle A^{1/2}u_+(t), A^{1/2}u_-(t)\rangle(Au_-(t))_+ \\ &\quad -2\langle A^{1/2}u_+(t), A^{1/2}u_-(t)\rangle(Au_+(t))_+ - |A^{1/2}u_-(t)|^2(Au_+(t))_+, \quad (5.15) \end{aligned}$$

and

$$\psi_2(t) := f_+(t) - |A^{1/2}u_-(t)|^2(Au_-(t))_+. \qquad (5.16)$$

The low-frequency component is a solution to

$$u_-''(t) + u_-'(t) - \lambda_0 u_-(t) = \psi_3(t), \qquad (5.17)$$

where

$$\psi_3(t) := f_-(t) - |A^{1/2}u(t)|^2(Au(t))_-. \qquad (5.18)$$

We recall that H_- and H_+ are not necessarily invariant subspaces for A, and for this reason we have to deal with terms of the form $(Au_\pm(t))_\pm$ in the previous equations.

Now let us set

$$\gamma_1 := \min\left\{\frac{1}{24}, \frac{\mu_2^2\mu_1\mu_3}{14}\right\}, \qquad (5.19)$$

let us consider the two constants

$$\Gamma_1 := \left(\frac{2}{\gamma_1}\frac{1}{\mu_2\mu_1\mu_3}\right)^{3/2}, \qquad \Gamma_2 := 2^{10}\,\Gamma_1 \cdot |Ae_0|^6,$$

and let us choose $\beta_0 > 0$ small enough so that

$$2 \cdot 24^2\beta_0^4 \cdot |Ae_0|^2 \leq \frac{\gamma_1}{2}\mu_1\mu_3, \qquad (5.20)$$

and

$$16\beta_0^2 \cdot |A^{1/2}e_0|^4 + 4\Gamma_2\beta_0^8 \cdot |A^{1/2}e_0| \leq \frac{\lambda_0}{2}, \qquad (5.21)$$

We claim that, whenever $f(t)$ and $u(t)$ satisfy the assumptions in the left-hand side of (3.8) with this value of β_0, the solution $u(t)$ satisfies the estimates in the right-hand side of (3.8) for a suitable constant M_4 independent of $u(t)$ and $f(t)$. In the sequel we always assume that $f(t)$ and $u(t)$ satisfy the estimates in the left-hand side of (3.8).

Estimate on right-hand sides From the assumptions in the left-hand side of (3.8) we deduce that there exists $t_0 \geq 0$ such that

$$|u(t)| \leq 2\beta_0 \qquad \forall t \geq t_0, \tag{5.22}$$

$$|f(t)| \leq 2 \qquad \forall t \geq t_0. \tag{5.23}$$

We prove that for every $t \geq t_0$ it turns out that

$$|\psi_1(t)| \leq 24\beta_0^2 \cdot |Ae_0| \cdot |Au_+(t)| \tag{5.24}$$

$$|\psi_2(t)| \leq |f_+(t)| + 4\beta_0^2 |Ae_0|^2 \cdot |u_-(t)|, \tag{5.25}$$

$$|\psi_3(t)| \leq |f_-(t)| + 16\beta_0^2 |A^{1/2}e_0|^4 \cdot |u_-(t)| + 4|A^{1/2}e_0| \cdot |A^{1/2}u_+(t)|^3. \tag{5.26}$$

To begin with, from (5.22) we deduce that for every $t \geq t_0$ it tuns out that

$$|u_-(t)|^2 \leq 4\beta_0^2, \tag{5.27}$$

$$|Au_-(t)|^2 \leq 4\beta_0^2 |Ae_0|^2, \tag{5.28}$$

$$|A^{1/2}u_-(t)|^2 = \langle u_-(t), Au_-(t) \rangle \leq 4\beta_0^2 |Ae_0|. \tag{5.29}$$

Exploiting (5.22), and (5.27) through (5.29), we estimate the four terms in the right-hand side of (5.15), which for the sake of shortness we denote by T_1, T_2, T_3, T_4. For the first term it turns out that

$$|T_1| \leq |u_+(t)| \cdot |Au_+(t)| \cdot |Au_-(t)| \leq 2\beta_0 \cdot |Au_+(t)| \cdot 2\beta_0 |Ae_0|.$$

For the second term it turns out that

$$|T_2| \leq 2|Au_+(t)| \cdot |u_-(t)| \cdot |Au_-(t)| \leq 2|Au_+(t)| \cdot 2\beta_0 \cdot 2\beta_0 |Ae_0|.$$

For the third term it turns out that

$$|T_3| \leq 2|u_+(t)| \cdot |Au_-(t)| \cdot |(Au_+(t))_+| \leq 4\beta_0 \cdot 2\beta_0 |Ae_0| \cdot |Au_+(t)|.$$

For the fourth term it turns out that

$$|T_4| \leq |A^{1/2}u_-(t)|^2 \cdot |(Au_+(t))_+| \leq 4\beta_0^2 |Ae_0| \cdot |Au_+(t)|.$$

Plugging the last four inequalities into (5.15) we deduce (5.24).
Let us consider now $\psi_2(t)$. From (5.29) we obtain that

$$\begin{aligned} \left| |A^{1/2}u_-(t)|^2 (Au_-(t))_+ \right| &\leq |A^{1/2}u_-(t)|^2 \cdot |Au_-(t)| \\ &\leq 4\beta_0^2 |Ae_0| \cdot |u_-(t)| \cdot |Ae_0|. \end{aligned}$$

Plugging this estimate into (5.16) we deduce (5.25).

As for $\psi_3(t)$, we observe that

$$(Au(t))_- = \langle Au(t), e_0 \rangle e_0 = \langle A^{1/2}u(t), A^{1/2}e_0 \rangle e_0,$$

and therefore

$$
\begin{aligned}
\left| |A^{1/2}u(t)|^2 (Au(t))_- \right| &\leq |A^{1/2}e_0| \cdot |A^{1/2}u(t)|^3 \\
&\leq |A^{1/2}e_0| \cdot \left(|A^{1/2}u_-(t)| + |A^{1/2}u_+(t)| \right)^3 \\
&\leq |A^{1/2}e_0| \cdot \left(4|u_-(t)|^3 \cdot |A^{1/2}e_0|^3 + 4|A^{1/2}u_+(t)|^3 \right) \\
&\leq 4|A^{1/2}e_0|^4 \cdot 4\beta_0^2 |u_-(t)| + 4|A^{1/2}e_0| \cdot |A^{1/2}u_+(t)|^3.
\end{aligned}
$$

Plugging this estimate into (5.18) we deduce (5.26).

Estimates on the high-frequency component We prove that

$$\limsup_{t\to+\infty} \left(|u'_+(t)|^2 + |L^{1/2}u_+(t)|^2 \right) \leq \frac{2}{\gamma_1} \limsup_{t\to+\infty} |\psi_2(t)|^2, \tag{5.30}$$

and

$$\limsup_{t\to+\infty} |A^{1/2}u_+(t)|^3 \leq 16\Gamma_1 \cdot \limsup_{t\to+\infty} |f(t)| + \Gamma_2 \beta_0^8 \cdot \limsup_{t\to+\infty} |u_-(t)|. \tag{5.31}$$

To this end, we consider the energy

$$
\begin{aligned}
F_+(t) &:= \frac{1}{2}|u'_+(t)|^2 + \frac{1}{2}|L^{1/2}u_+(t)|^2 + \frac{1}{4}|A^{1/2}u_+(t)|^4 \\
&\quad + 2\gamma_1 \langle u_+(t), u'_+(t) \rangle + \gamma_1 |u_+(t)|^2.
\end{aligned}
$$

Since

$$\left| 2\langle u_+(t), u'_+(t) \rangle \right| \leq |u_+(t)|^2 + |u'_+(t)|^2,$$

this energy can be estimated from below by

$$F_+(t) \geq \left(\frac{1}{2} - \gamma_1 \right) |u'_+(t)|^2 + \frac{1}{2}|L^{1/2}u_+(t)|^2, \tag{5.32}$$

and from above by

$$F_+(t) \leq \left(\frac{1}{2} + \gamma_1 \right) |u'_+(t)|^2 + \frac{1}{2}|L^{1/2}u_+(t)|^2 + \frac{1}{4}|A^{1/2}u_+(t)|^4 + 2\gamma_1 |u_+(t)|^2. \tag{5.33}$$

Let us compute the time-derivative of $F_+(t)$. Keeping (5.14) into account, we obtain that (for the sake of shortness, we omit here the explicit dependence on t)

$$
\begin{aligned}
F'_+ &= -(1 - 2\gamma_1)|u'_+|^2 - 2\gamma_1 |L^{1/2}u_+|^2 - 2\gamma_1 |A^{1/2}u_+|^4 \\
&\quad + \langle u'_+, \psi_1 \rangle + \langle u'_+, \psi_2 \rangle + 2\gamma_1 \langle u_+, \psi_1 \rangle + 2\gamma_1 \langle u_+, \psi_2 \rangle. \tag{5.34}
\end{aligned}
$$

We point out that in the computation we exploited identities such as

$$\langle u'_+(t), (Au_+(t))_+\rangle = \langle u'_+(t), Au_+(t)\rangle,$$

and

$$\langle u_+(t), (Au_+(t))_+\rangle = \langle u_+(t), Au_+(t)\rangle = |A^{1/2}u_+(t)|^2.$$

Now we estimate the terms in (5.34). As for the terms with $\psi_2(t)$, we simply observe that

$$\langle u'_+(t), \psi_2(t)\rangle \le \frac{1}{2}|u'_+(t)|^2 + \frac{1}{2}|\psi_2(t)|^2,$$

and

$$2\gamma_1\langle u_+(t), \psi_2(t)\rangle \le 2\gamma_1^2|u_+(t)|^2 + \frac{1}{2}|\psi_2(t)|^2.$$

In a similar way, for the terms with $\psi_1(t)$ we observe that

$$\langle u'_+(t), \psi_1(t)\rangle \le \frac{1}{4}|u'_+(t)|^2 + |\psi_1(t)|^2,$$

and

$$2\gamma_1\langle u_+(t), \psi_1(t)\rangle \le \gamma_1^2|u_+(t)|^2 + |\psi_1(t)|^2.$$

Plugging all these estimates into (5.34), and keeping (5.24) into account, we conclude that

$$
\begin{aligned}
F'_+(t) \;\le\; & -\left(\frac{1}{4} - 2\gamma_1\right)|u'_+(t)|^2 - 2\gamma_1|L^{1/2}u_+(t)|^2 - 2\gamma_1|A^{1/2}u_+(t)|^4 \\
& +3\gamma_1^2|u_+(t)|^2 + 2\cdot 24^2\beta_0^4 \cdot |Ae_0|^2 \cdot |Au_+(t)|^2 + |\psi_2(t)|^2
\end{aligned}
$$

for every $t \ge t_0$. Now we claim that

$$F'_+(t) \le -2\gamma_1 F_+(t) + |\psi_2(t)|^2 \qquad\qquad \forall t \ge t_0. \qquad (5.35)$$

To this end, keeping (5.33) into account, it is enough to show that

$$
\left(\frac{1}{4} - 3\gamma_1 - 2\gamma_1^2\right)|u'_+(t)|^2 + \frac{3\gamma_1}{2}|A^{1/2}u_+(t)|^4 + \gamma_1|L^{1/2}u_+(t)|^2
$$
$$
-7\gamma_1^2|u_+(t)|^2 - 2\cdot 24^2\beta_0^4\cdot|Ae_0|^2\cdot|Au_+(t)|^2 \;\ge\; 0. \qquad (5.36)
$$

The coefficient of $|u'_+(t)|^2$ is positive because $\gamma_1 \le 1/24$. Moreover, from (4.14), (4.15), and the second condition in (5.19) it turns out that

$$\frac{\gamma_1}{2}|L^{1/2}u_+(t)|^2 \ge \frac{\gamma_1}{2}\mu_2^2\mu_1\mu_3 \cdot |u_+(t)|^2 \ge 7\gamma_1^2|u_+(t)|^2,$$

while from (5.20) it follows that

$$\frac{\gamma_1}{2}|L^{1/2}u_+(t)|^2 \ge \frac{\gamma_1}{2}\mu_1\mu_3 \cdot |Au_+(t)|^2 \ge 2\cdot 24^2\beta_0^4 \cdot |Ae_0|^2 \cdot |Au_+(t)|^2.$$

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This completes the proof of (5.36), and therefore also of (5.35). Integrating this differential inequality we deduce that

$$\limsup_{t\to+\infty} F_+(t) \le \frac{1}{2\gamma_1} \limsup_{t\to+\infty} |\psi_2(t)|^2.$$

Since $\gamma_1 \le 1/4$, this inequality and (5.32) imply (5.30).

It remains to prove (5.31). As usual, from (4.14) and (4.15) we obtain that

$$|L^{1/2}u_+(t)|^2 \ge \mu_3\mu_1\mu_2|A^{1/2}u_+(t)|^2,$$

and therefore from (5.30) we deduce that

$$
\begin{aligned}
\limsup_{t\to+\infty} |A^{1/2}u_+(t)|^3 &\le \left(\frac{1}{\mu_2\mu_1\mu_3}\right)^{3/2} \limsup_{t\to+\infty} |L^{1/2}u_+(t)|^3 \\
&\le \Gamma_1 \limsup_{t\to+\infty} |\psi_2(t)|^3. \qquad (5.37)
\end{aligned}
$$

On the other hand, from (5.25), (5.27) and (5.23) we obtain that

$$
\begin{aligned}
|\psi_2(t)|^3 &\le 4|f(t)|^3 + 4\cdot 64\beta_0^6 \cdot |Ae_0|^6 \cdot |u_-(t)|^3 \\
&\le 16|f(t)| + 2^{10}\beta_0^8 \cdot |Ae_0|^6 \cdot |u_-(t)|
\end{aligned}
$$

for every $t \ge t_0$. Plugging this estimate into (5.37), and letting $t \to +\infty$, we obtain (5.31).

Estimates on the low-frequency component   We prove that

$$\limsup_{t\to+\infty} |u_-(t)| \le \frac{2\left(1 + 64\,\Gamma_1|A^{1/2}e_0|\right)}{\lambda_0} \limsup_{t\to+\infty} |f(t)|. \qquad (5.38)$$

To this end, we recall that $u_-(t)$ is a solution to (5.17). Since both the solution $u_-(t)$ and the right-hand side $\psi_3(t)$ are bounded for positive times, this equation fits in the framework of Lemma 5.2, from which we deduce that

$$\limsup_{t\to+\infty} |u_-(t)| \le \frac{1}{\lambda_0} \limsup_{t\to+\infty} |\psi_3(t)|. \qquad (5.39)$$

On the other hand, from (5.26) and (5.31) we know that

$$
\begin{aligned}
\limsup_{t\to+\infty} |\psi_3(t)| &\le \limsup_{t\to+\infty} |f(t)| + 16\beta_0^2|A^{1/2}e_0|^4 \cdot \limsup_{t\to+\infty} |u_-(t)| \\
&\quad + 4|A^{1/2}e_0| \cdot \limsup_{t\to+\infty} |A^{1/2}u_+(t)|^3 \\
&\le \left(1 + 64\,\Gamma_1|A^{1/2}e_0|\right) \limsup_{t\to+\infty} |f(t)| \\
&\quad + \left(16\beta_0^2 \cdot |A^{1/2}e_0|^4 + 4\Gamma_2\beta_0^8 \cdot |A^{1/2}e_0|\right) \limsup_{t\to+\infty} |u_-(t)|. \quad (5.40)
\end{aligned}
$$

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Plugging this inequality into (5.39), and keeping the smallness assumption (5.21) into account, we conclude that

$$\limsup_{t \to +\infty} |u_-(t)| \leq \frac{1 + 64\,\Gamma_1 |A^{1/2} e_0|}{\lambda_0} \limsup_{t \to +\infty} |f(t)| + \frac{1}{2} \limsup_{t \to +\infty} |u_-(t)|,$$

which implies (5.38).

Conclusion   We prove that $u(t)$ satisfies the estimate in the right-hand side of (3.8) for a suitable constant M_4. In the sequel $c_1, \ldots, c_5$ denote suitable constants, all independent of the solution and of the forcing term.

From (5.38) we know that

$$\limsup_{t \to +\infty} |u_-(t)| \leq c_1 \limsup_{t \to +\infty} |f(t)|. \tag{5.41}$$

Plugging this estimate into (5.40) we deduce that

$$\limsup_{t \to +\infty} |\psi_3(t)| \leq c_2 \limsup_{t \to +\infty} |f(t)|.$$

Applying again Lemma 5.2 to equation (5.17) we deduce that

$$\limsup_{t \to +\infty} |u'_-(t)| \leq 2 \limsup_{t \to +\infty} |\psi_3(t)| \leq c_3 \limsup_{t \to +\infty} |f(t)|.$$

Plugging (5.41) into (5.25) we obtain that

$$\limsup_{t \to +\infty} |\psi_2(t)| \leq c_4 \limsup_{t \to +\infty} |f(t)|,$$

and therefore from (5.30) we deduce that

$$\limsup_{t \to +\infty} \left(|u'_+(t)|^2 + |L^{1/2} u_+(t)|^2 \right) \leq c_5 \limsup_{t \to +\infty} |f(t)|^2.$$

Recalling (4.14), all these estimates imply the conclusion. $\square$

5.5   Proof of Proposition 3.3

Choice of parameters   We can assume, up to replacing η with a smaller positive real number, that

$$\eta < \frac{1}{4} \sigma_0^4. \tag{5.42}$$

Let us consider the inequality

$$\frac{1}{4} x^4 - \frac{1}{2} \sigma_0^2 x^2 < -\frac{\eta}{4}. \tag{5.43}$$

Due to (5.42), the number in the right-hand side is negative but larger than the minimum of the function in the left-hand side. Therefore, the set of solutions to this

inequality is the union of two disjoint intervals of the form (x_1, x_2) and $(-x_2, -x_1)$ for suitable real numbers $0 < x_1 < x_2 < \sigma_0 \sqrt{2}$.

Now let us choose $\gamma_2 > 0$ such that

$$\gamma_2 \leq \frac{1}{8}, \qquad \frac{2\gamma_2(1 + 2\gamma_2)}{\mu_2} \leq (2 - \gamma_2)(\lambda_2 - \lambda), \qquad (5.44)$$

$$\gamma_2 \left(\frac{\lambda^2}{2} + \frac{2\lambda^2}{\mu_2^2 \mu_1} + 4\sigma_0^2 \right) \leq \frac{\eta}{2}, \qquad (5.45)$$

and

$$\frac{\gamma_2}{2}\sigma_0^2 + \frac{2\gamma_2(1 + 2\gamma_2)}{\mu_2} \leq \sigma_0(2 - \gamma_2)x_1. \qquad (5.46)$$

Finally, let us choose $\varepsilon_1 > 0$ such that

$$\frac{\varepsilon_1^2}{2\gamma_2^2} < \frac{\eta}{4}. \qquad (5.47)$$

Estimate at time T_0   Let us write $u(t)$ in the form $\alpha(t)e_1 + w(t)$ according to the direct sum (4.1). We prove that

$$|\alpha(T_0)| > x_1. \qquad (5.48)$$

Indeed, from (5.5) we obtain that

$$E(t) \geq \frac{1}{2}|u'(t)|^2 + \frac{1}{4}\alpha^4(t) - \frac{1}{2}\sigma_0^2\alpha^2(t) + \frac{1}{2}\frac{\lambda_2 - \lambda}{\lambda_2}|Bw(t)|^2.$$

Setting $t = T_0$, from the assumption that $E(T_0) < -\eta$ we conclude that

$$\frac{1}{4}\alpha^4(T_0) - \frac{1}{2}\sigma_0^2\alpha^2(T_0) < -\eta.$$

Comparing with (5.43) we deduce (5.48).

Modified energy and basic estimates from above and below   Due to (5.48) and the symmetry of the problem, in the sequel we can assume, without loss of generality, that $\alpha(T_0) > x_1$. In this case we claim that the solution is eventually close to the stationary point $\sigma_0 e_1$, and for this reason we introduce the modified energy

$$S(t) := E(t) + \frac{1}{4}\sigma_0^4 + 2\gamma_2\langle u(t) - \sigma_0 e_1, u'(t)\rangle + \gamma_2|u(t) - \sigma_0 e_1|^2.$$

From the inequality

$$2\langle u(t) - \sigma_0 e_1, u'(t)\rangle \leq |u(t) - \sigma_0 e_1|^2 + |u'(t)|^2 \qquad (5.49)$$

we deduce that

$$S(t) \leq \left(\frac{1}{2} + \gamma_2 \right)|u'(t)|^2 + \frac{1}{2}|Bu(t)|^2 - \frac{\lambda}{2}|A^{1/2}u(t)|^2 + \frac{1}{4}|A^{1/2}u(t)|^4$$

$$+ \frac{1}{4}\sigma_0^4 + 2\gamma_2|u(t) - \sigma_0 e_1|^2, \qquad (5.50)$$

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and

$$S(t) \geq \left(\frac{1}{2} - \gamma_2 \right) |u'(t)|^2 + \frac{1}{2}|Bu(t)|^2 - \frac{\lambda}{2}|A^{1/2}u(t)|^2 + \frac{1}{4}|A^{1/2}u(t)|^4 + \frac{1}{4}\sigma_0^4.$$

If we write $u(t)$ in the usual form $\alpha(t)e_1 + w(t)$, and we keep (5.6) into account, the estimate from below implies that

$$S(t) \geq \left(\frac{1}{2} - \gamma_2 \right) |u'(t)|^2 + \frac{1}{4} \left(\alpha^2(t) - \sigma_0^2 \right)^2 + \frac{1}{2}\frac{\lambda_2 - \lambda}{\lambda_2}|Bw(t)|^2. \qquad (5.51)$$

Modified energy at time T_0   We prove that

$$S(T_0) < \frac{1}{4}\sigma_0^4 - \frac{\eta}{2}. \qquad (5.52)$$

Indeed, the energies $S(t)$ and $E(t)$ satisfy

$$\begin{aligned} S(t) &= E(t) + \frac{1}{4}\sigma_0^4 \\ &\quad + 2\gamma_2\langle u(t) - \sigma_0 e_1, u'(t)\rangle + \gamma_2|u(t) - \sigma_0 e_1|^2. \end{aligned} \qquad (5.53)$$

Let Λ denote the sum of the two terms in the last line. From (5.49) we know that

$$\Lambda \leq \gamma_2|u'(t)|^2 + 2\gamma_2|u(t) - \sigma_0 e_1|^2 \leq \gamma_2 \left(|u'(t)|^2 + 4|u(t)|^2 + 4\sigma_0^2 \right). \qquad (5.54)$$

Setting $t = T_0$, from (5.4) we obtain that

$$|u'(T_0)|^2 \leq 2E(T_0) + \frac{\lambda^2}{2} \leq \frac{\lambda^2}{2},$$

and similarly from (5.4) and (4.15) we obtain that

$$|u(T_0)|^2 \leq \frac{1}{\mu_2^2\mu_1}|Bu(T_0)|^2 \leq \frac{1}{\mu_2^2\mu_1}\left(2E(T_0) + \frac{\lambda^2}{2} \right) \leq \frac{1}{\mu_2^2\mu_1}\frac{\lambda^2}{2}. \qquad (5.55)$$

Replacing the last two inequalities into (5.54) we deduce that

$$\Lambda \leq \gamma_2 \left(\frac{\lambda^2}{2} + \frac{2\lambda^2}{\mu_2^2\mu_1} + 4\sigma_0^2 \right).$$

Plugging this estimate into (5.53), and keeping the smallness assumption (5.45) into account, we deduce (5.52).

Modified energy and potential well   We show that, for every $t \geq 0$, the following implication holds true:

$$S(t) < \frac{1}{4}\sigma_0^4 - \frac{\eta}{4} \qquad \Longrightarrow \qquad |\alpha(t)| > x_1. \qquad (5.56)$$

Indeed, from (5.51) we know that

$$S(t) \geq \frac{1}{4}\left(\alpha^2(t) - \sigma_0^2\right)^2,$$

and therefore the inequality in the left-hand side of (5.56) implies that

$$\frac{1}{4}\alpha^4(t) - \frac{1}{2}\sigma_0^2\alpha^2(t) < -\frac{\eta}{4}.$$

Comparing with (5.43), this implies that $|\alpha(t)| > x_1$.

Differential inequality in the potential-well   We prove that, for every $t \geq 0$, the following implication holds true:

$$\alpha(t) \geq x_1 \qquad \Longrightarrow \qquad S'(t) \leq -2\gamma_2^2 S(t) + |f(t)|^2. \qquad (5.57)$$

To begin with, we compute the time-derivative of $S(t)$, which turns out to be

$$\begin{aligned} S'(t) &= -(1 - 2\gamma_2)|u'(t)|^2 + \langle u'(t), f(t)\rangle + 2\gamma_2\langle u(t) - \sigma_0 e_1, f(t)\rangle \\ &\quad -2\gamma_2\langle u(t) - \sigma_0 e_1, B^2 u(t) - \lambda A u(t) + |A^{1/2}u(t)|^2 A u(t)\rangle \end{aligned}$$

From the usual inequalities

$$\langle u'(t), f(t)\rangle \leq \frac{1}{2}|u'(t)|^2 + \frac{1}{2}|f(t)|^2$$

and

$$2\gamma_2\langle u(t) - \sigma_0 e_1, f(t)\rangle \leq 2\gamma_2^2|u(t) - \sigma_0 e_1|^2 + \frac{1}{2}|f(t)|^2$$

we deduce that

$$\begin{aligned} S'(t) &\leq -\left(\frac{1}{2} - 2\gamma_2\right)|u'(t)|^2 + |f(t)|^2 + 2\gamma_2^2|u(t) - \sigma_0 e_1|^2 \\ &\quad -2\gamma_2\langle u(t), B^2 u(t) - \lambda A u(t) + |A^{1/2}u(t)|^2 A u(t)\rangle \\ &\quad +2\gamma_2\sigma_0\langle e_1, B^2 u(t) - \lambda A u(t) + |A^{1/2}u(t)|^2 A u(t)\rangle. \end{aligned}$$

Keeping (5.50) into account, inequality (5.57) is proved if we show that

$$\left(\frac{1}{2} - 2\gamma_2 - \gamma_2^2 - 2\gamma_2^3\right)|u'(t)|^2 - 2\gamma_2^2(1 + 2\gamma_2)|u(t) - \sigma_0 e_1|^2$$

$$+\gamma_2(2 - \gamma_2)\left(|Bu(t)|^2 - \lambda|A^{1/2}u(t)|^2\right) + \gamma_2\left(2 - \frac{\gamma_2}{2}\right)|A^{1/2}u(t)|^4$$

$$-2\gamma_2\sigma_0\langle e_1, B^2 u(t) - \lambda A u(t) + |A^{1/2}u(t)|^2 A u(t)\rangle - \frac{\gamma_2^2}{2}\sigma_0^4 \geq 0. \quad (5.58)$$

Now we write $u(t)$ in the usual form $\alpha(t)e_1 + w(t)$ according to the direct sum (4.1), and we estimate the terms in the left-hand side.

- The coefficient of $|u'(t)|^2$ is nonnegative because $\gamma_2 \leq 1/8$.

- For the second term we exploit (2.7) and (4.3), obtaining that

$$
\begin{aligned}
|u(t) - \sigma_0 e_1|^2 &\leq \frac{1}{\mu_2}|A^{1/2}(u(t) - \sigma_0 e_1)|^2 \\
&= \frac{1}{\mu_2}\left\{(\alpha(t) - \sigma_0)^2 + |A^{1/2}w(t)|^2\right\}.
\end{aligned}
$$

- For the third term we exploit (4.6), (4.5) and (2.9), and we obtain that

$$
\begin{aligned}
|Bu(t)|^2 - \lambda|A^{1/2}u(t)|^2 &= -\sigma_0^2\alpha^2(t) + |Bw(t)|^2 - \lambda|A^{1/2}w(t)|^2 \\
&\geq -\sigma_0^2\alpha^2(t) + (\lambda_2 - \lambda)|A^{1/2}w(t)|^2.
\end{aligned}
$$

- We expand the fourth term according to (4.3).

- Finally, from (4.2), (2.8) and (2.13) we know that

$$
\langle e_1, B^2 u(t) - \lambda A u(t)\rangle = \alpha(t)\lambda_1 - \alpha(t)\lambda = -\sigma_0^2\alpha(t),
$$

while from (4.3) we know that

$$
|A^{1/2}u(t)|^2 \cdot \langle e_1, Au(t)\rangle = \left(\alpha^2(t) + |A^{1/2}w(t)|^2\right)\cdot\alpha(t),
$$

and therefore the scalar product in the fifth term is equal to

$$
-\sigma_0^2\alpha(t) + \alpha^3(t) + \alpha(t)\cdot|A^{1/2}w(t)|^2.
$$

Keeping all these equalities and inequalities into account, we obtain that (5.58) holds true if we show that

$$
k_1(t)|A^{1/2}w(t)|^2 + k_2(t)|A^{1/2}w(t)|^4 + 2k_3(t)|A^{1/2}w(t)|^2 + k_4(t) \geq 0, \tag{5.59}
$$

where

$$
k_1(t) := (2 - \gamma_2)(\lambda_2 - \lambda) - \frac{2\gamma_2(1 + 2\gamma_2)}{\mu_2},
$$

$$
k_2(t) := 2 - \frac{\gamma_2}{2}, \qquad k_3(t) := \alpha(t)\left\{\left(2 - \frac{\gamma_2}{2}\right)\alpha(t) - \sigma_0\right\},
$$

and

$$
\begin{aligned}
k_4(t) &:= -\frac{2\gamma_2(1 + 2\gamma_2)}{\mu_2}(\alpha(t) - \sigma_0)^2 - (2 - \gamma_2)\sigma_0^2\alpha^2(t) + \left(2 - \frac{\gamma_2}{2}\right)\alpha^4(t) \\
&\quad + 2\sigma_0^3\alpha(t) - 2\sigma_0\alpha^3(t) - \frac{\gamma_2}{2}\sigma_0^4 \\
&= \left\{\left(2 - \frac{\gamma_2}{2}\right)\alpha^2(t) + \sigma_0(2 - \gamma_2)\alpha(t) - \frac{\gamma_2}{2}\sigma_0^2 - \frac{2\gamma_2(1 + 2\gamma_2)}{\mu_2}\right\}(\alpha(t) - \sigma_0)^2.
\end{aligned}
$$

Now we exploit the assumption that $\alpha(t) \geq x_1$. When this is the case, from the smallness assumptions (5.44) it follows that

$$k_1(t) \geq 0, \qquad k_2(t) \geq 1, \qquad k_3(t) \geq \alpha(t)(\alpha(t) - \sigma_0),$$

while from the smallness assumption (5.46) it follows that

$$k_4(t) \geq \alpha^2(t)(\alpha(t) - \sigma_0)^2.$$

As a consequence, the left-hand side of (5.59) is greater than or equal to

$$\left(|A^{1/2}w(t)|^2 + \alpha(t)(\alpha(t) - \sigma_0) \right)^2,$$

and therefore it is nonnegative in this regime. This completes the proof of (5.57).

Potential-well argument   We prove that

$$\alpha(t) > x_1 \qquad \forall t \geq T_0, \tag{5.60}$$

which is equivalent to (3.10). To this end, let us set

$$T_1 := \sup \left\{ t \geq T_0 : \alpha(\tau) > x_1 \text{ for every } \tau \in [T_0, t] \right\}.$$

We observe that T_1 is the supremum of an open set containing $t = T_0$ because we assumed that $\alpha(T_0) > 0$ after showing (5.48). It follows that T_1 is well-defined, greater than T_0, and it satisfies

$$\alpha(t) > x_1 \qquad \forall t \in [T_0, T_1). \tag{5.61}$$

If $T_1 = +\infty$, then (5.60) is proved. Let us assume by contradiction that $T_1 < +\infty$. Then the maximality of T_1 implies that $\alpha(T_1) = x_1$. On the other hand, from (5.61) it follows that (5.57) holds true for every $t \in [T_0, T_1]$. Integrating this differential inequality, and recalling that $|f(t)| \leq \varepsilon_1$ for every $t \geq T_0$, we deduce that

$$S(t) \leq S(T_0) + \frac{\varepsilon_1^2}{2\gamma_2^2} \qquad \forall t \in [T_0, T_1].$$

Keeping (5.52) and (5.47) into account, this implies that

$$S(T_1) < \frac{1}{4}\sigma_0^4 - \frac{\eta}{4},$$

which in turn implies that $|\alpha(T_1)| > x_1$ because of (5.56). This contradicts the fact that $\alpha(T_1) = x_1$, and completes the proof of (5.60).

Conclusion Since we have established (5.60), now we know that the differential inequality in (5.57) holds true for every $t \geq T_0$. Integrating this differential inequality, and letting $t \to +\infty$, we deduce that

$$\limsup_{t \to +\infty} S(t) \leq \frac{1}{2\gamma_2^2} \limsup_{t \to +\infty} |f(t)|^2. \tag{5.62}$$

Now we prove that there exists a constant Γ_3 such that

$$|u'(t)|^2 + |B(u(t) - \sigma_0 e_1)|^2 \leq \Gamma_3 \, S(t) \tag{5.63}$$

for all solutions $u(t)$ with $\alpha(t) \geq 0$. If we show this claim, then (5.62) implies the conclusion in the lower box of (3.9).

In order to prove (5.63), it is enough to observe that

$$\begin{aligned}
|B(u(t) - \sigma_0 e_1)|^2 &= (\alpha(t) - \sigma_0)^2 \, |Be_1|^2 + |Bw(t)|^2 \\
&= \lambda_1 \, (\alpha(t) - \sigma_0)^2 + |Bw(t)|^2 \\
&\leq \frac{(\alpha(t) + \sigma_0)^2}{\sigma_0^2} \cdot \lambda_1 \, (\alpha(t) - \sigma_0)^2 + |Bw(t)|^2 \\
&= \frac{\lambda_1}{\sigma_0^2} \left(\alpha^2(t) - \sigma_0^2 \right)^2 + |Bw(t)|^2,
\end{aligned} \tag{5.64}$$

where in the inequality we exploited the assumption that $\alpha(t) \geq 0$. At this point it is enough to observe that, due to (5.51), the energy $S(t)$ controls both $|u'(t)|^2$ and the terms in the right-hand side of (5.64), up to constants. $\square$

5.6 Proof of Proposition 3.4

Let $r(t) := u(t) - v(t)$ denote the difference between two solutions $u(t)$ and $v(t)$ to equation (1.6). This difference is a solution to equation

$$r''(t) + r'(t) + B^2 r(t) - \lambda A r(t) = g(t), \tag{5.65}$$

where

$$\begin{aligned}
g(t) &:= -|A^{1/2} u(t)|^2 Au(t) + |A^{1/2} v(t)|^2 Av(t) \\
&= -|A^{1/2} u(t)|^2 Ar(t) - \langle u(t) + v(t), Ar(t) \rangle Av(t).
\end{aligned} \tag{5.66}$$

Now we consider separately the unstable case $\sigma = 0$ and the stable cases $\sigma = \pm \sigma_0$. The constants $c_6, \ldots, c_{21}$ in the sequel are independent of $u(t)$ and $v(t)$.

Unstable case From (5.66) we deduce that

$$|g(t)| \leq \left(|A^{1/2} u(t)|^2 + |u(t) + v(t)| \cdot |Av(t)| \right) |Ar(t)|. \tag{5.67}$$

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On the other hand, from (3.11) with $\sigma = 0$ and (4.15) we know that

$$\limsup_{t \to +\infty} |A^{1/2}u(t)|^2 \leq c_6 r_0^2 \qquad \text{and} \qquad \limsup_{t \to +\infty} |u(t) + v(t)| \cdot |Av(t)| \leq c_7 r_0^2,$$

and therefore from (5.67) we obtain that

$$\limsup_{t \to +\infty} |g(t)| \leq c_8 r_0^2 \cdot \limsup_{t \to +\infty} |Ar(t)| \leq c_9 r_0^2 \cdot \limsup_{t \to +\infty} |Br(t)|. \tag{5.68}$$

Let L denote as usual the operator $B^2 - \lambda A$, let $r_-(t)$ and $r_+(t)$ denote the components of $r(t)$ with respect to the direct orthogonal sum (4.13), and let $g_-(t)$ and $g_+(t)$ denote the corresponding components of $g(t)$. As already observed, L is the differential of the functional (1.7) in the origin.

Since e_0 is an eigenvector of L with eigenvalue λ_0, the low-frequency component $r_-(t)$ is a solution to equation

$$r_-''(t) + r_-'(t) - \lambda_0 r_-(t) = g_-(t), \tag{5.69}$$

while the high-frequency component $r_+(t)$ is a solution to equation

$$r_+''(t) + r_+'(t) + Lr_+(t) = g_+(t). \tag{5.70}$$

Equation (5.69) is a scalar equation that fits in the framework of Lemma 5.2 with

$$y(t) := r_-(t), \qquad m := \lambda_0, \qquad \psi(t) := g_-(t).$$

Indeed, $r_-(t)$ is bounded because $u(t)$ and $v(t)$ are bounded, and for analogous reasons also $g_-(t)$ is bounded. As a consequence, from Lemma 5.2 we deduce that

$$\limsup_{t \to +\infty} \left(|r_-'(t)|^2 + |Br_-(t)|^2 \right) \leq c_{10} \limsup_{t \to +\infty} |g_-(t)|^2. \tag{5.71}$$

Now from (4.14) and (4.15) we know that

$$|L^{1/2}x|^2 \geq \mu_3 |Bx|^2 \geq \mu_2^2 \mu_1 \mu_3 |x|^2 \qquad \forall x \in H_+ \cap D(B),$$

and therefore equation (5.70) fits in the framework of Lemma 5.3 with

$$X := H_+, \qquad \mathcal{L} := L, \qquad y(t) := r_+(t), \qquad m := \mu_2^2 \mu_1 \mu_3, \qquad \psi(t) := g_+(t).$$

As a consequence, from Lemma 5.3 we deduce that

$$\begin{aligned}
\limsup_{t \to +\infty} \left(|r_+'(t)|^2 + |Br_+(t)|^2 \right) &\leq c_{11} \limsup_{t \to +\infty} \left(|r_+'(t)|^2 + |L^{1/2}r_+(t)|^2 \right) \\
&\leq c_{12} \limsup_{t \to +\infty} |g_+(t)|^2. \tag{5.72}
\end{aligned}$$

Since $|Br(t)|^2 \leq 2|Br_-(t)|^2 + 2|Br_+(t)|^2$ (we recall that H_- and H_+ are not necessarily invariant subspaces for B), from (5.71), (5.72), and (5.68) we conclude that

$$
\begin{aligned}
\limsup_{t \to +\infty} \left(|r'(t)|^2 + |Br(t)|^2\right) \ \leq\ & 2\limsup_{t \to +\infty} \left(|r'_-(t)|^2 + |Br_-(t)|^2\right) \\
& + 2\limsup_{t \to +\infty} \left(|r'_+(t)|^2 + |Br_+(t)|^2\right) \\
\leq\ & c_{13} \limsup_{t \to +\infty} |g(t)|^2 \\
\leq\ & c_{14} r_0^4 \cdot \limsup_{t \to +\infty} |Br(t)|^2 \\
\leq\ & c_{14} r_0^4 \cdot \limsup_{t \to +\infty} \left(|r'(t)|^2 + |Br(t)|^2\right).
\end{aligned}
$$

If r_0 is small enough, the coefficient $c_{14} r_0^4$ is less than 1. It follows that

$$
\lim_{t \to +\infty} \left(|r'(t)|^2 + |Br(t)|^2\right) = 0, \tag{5.73}
$$

which in turn is equivalent to (2.17).

Stable case We assume, without loss of generality, that $\sigma = \sigma_0$ (the case $\sigma = -\sigma_0$ being symmetric). In order to exploit the smallness of $u(t) - \sigma_0 e_1$ and $v(t) - \sigma_0 e_1$, with some algebra we rewrite (5.65) in the form

$$
r''(t) + r'(t) + B^2 r(t) - \lambda A r(t) + \sigma_0^2 A r(t) + 2\sigma_0^2 \langle r(t), Ae_1 \rangle Ae_1 = \widehat{g}(t), \tag{5.74}
$$

where

$$
\begin{aligned}
\widehat{g}(t) \ :=\ & -\left(|A^{1/2}(u(t) - \sigma_0 e_1)|^2 + 2\langle A(u(t) - \sigma_0 e_1), \sigma_0 e_1 \rangle\right) A r(t) \\
& - \langle u(t) + v(t), Ar(t) \rangle A(v(t) - \sigma_0 e_1) \\
& - \langle u(t) + v(t) - 2\sigma_0 e_1, Ar(t) \rangle \sigma_0 Ae_1.
\end{aligned}
$$

Due to (3.11) with $\sigma = \sigma_0$, the forcing term $\widehat{g}(t)$ satisfies

$$
\limsup_{t \to +\infty} |\widehat{g}(t)| \leq \left(c_{15} r_0 + c_{16} r_0^2\right) \limsup_{t \to +\infty} |Ar(t)|. \tag{5.75}
$$

Now we observe that (5.74) can be rewritten in the form

$$
r''(t) + r'(t) + C r(t) = \widehat{g}(t), \tag{5.76}
$$

where C is the linear operator on H (with domain $D(C) = D(B^2)$) defined by

$$
Cx = B^2 x - \lambda A x + \sigma_0^2 A x + 2\sigma_0^2 \langle x, Ae_1 \rangle Ae_1,
$$

which coincides with the differential of the functional (1.7) in $\sigma_0 e_1$.

We claim that there exists a positive constant c_{17} such that

$$
|Bx|^2 \leq c_{17} |C^{1/2} x|^2 \qquad \forall x \in D(B), \tag{5.77}
$$

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which implies in particular that

$$|C^{1/2}x|^2 \geq c_{18}|x|^2 \qquad \forall x \in D(B)$$

for a suitable positive constant c_{18}. To this end, we first observe that

$$|C^{1/2}x|^2 = |Bx|^2 - \lambda|A^{1/2}x|^2 + \sigma_0^2 \cdot |A^{1/2}x|^2 + 2\sigma_0^2 \langle x, Ae_1 \rangle^2.$$

Then we write x in the form $\alpha e_1 + w$ according to the direct sum (4.1), and from (4.6) and (4.5) we deduce that

$$\begin{aligned}
|Bx|^2 - \lambda|A^{1/2}x|^2 &= -\sigma_0^2\alpha^2 + |Bw|^2 - \lambda|A^{1/2}w|^2 \\
&\geq -\sigma_0^2\alpha^2 + \frac{\lambda_2 - \lambda}{\lambda_2}|Bw|^2,
\end{aligned}$$

and

$$\sigma_0^2 \cdot |A^{1/2}x|^2 + 2\sigma_0^2 \langle x, Ae_1 \rangle^2 = 3\sigma_0^2\alpha^2 + \sigma_0^2 \cdot |A^{1/2}w|^2.$$

From these inequalities it follows that

$$|C^{1/2}u|^2 \geq \frac{\lambda_2 - \lambda}{\lambda_2}|Bw|^2 + \sigma_0^2 \cdot |A^{1/2}w|^2 + 2\sigma_0^2\alpha^2. \tag{5.78}$$

On the other hand from (4.4) we know that

$$|Bx|^2 = \lambda_1\alpha^2 + |Bw|^2. \tag{5.79}$$

Comparing (5.78) and (5.79) we deduce (5.77). At this point, equation (5.76) fits in the framework of Lemma 5.3 with

$$X := H, \qquad \mathcal{L} := C, \qquad y(t) := r(t), \qquad m := c_{18}, \qquad \psi(t) := \widehat{g}(t).$$

As a consequence, from (5.77), Lemma 5.3, and (5.75) we deduce that

$$\begin{aligned}
\limsup_{t \to +\infty} \left(|r'(t)|^2 + |Br(t)|^2\right) &\leq c_{19} \limsup_{t \to +\infty} \left(|r'(t)|^2 + |C^{1/2}r(t)|^2\right) \\
&\leq c_{20} \limsup_{t \to +\infty} |\widehat{g}(t)|^2. \\
&\leq c_{20} \left(c_{15}r_0 + c_{16}r_0^2\right)^2 \cdot \limsup_{t \to +\infty} |Ar(t)|^2 \\
&\leq c_{21}(r_0^2 + r_0^4) \cdot \limsup_{t \to +\infty} \left(|r'(t)|^2 + |Br(t)|^2\right).
\end{aligned}$$

If r_0 is small enough, we obtain again (5.73), which in turn is equivalent to (2.17). $\quad\square$

6 Proof of Theorem 2.5

Choice of parameters   To begin with, we consider the constants β_0 and M_4 of Proposition 3.2. Then we apply Proposition 3.3 with

$$\eta := \min\left\{\frac{\gamma_0}{4}(1-\delta)\beta_0^2, \frac{1}{8}\sigma_0^4\right\}, \tag{6.1}$$

where δ and γ_0 are chosen in (3.2) and (3.4). From Proposition 3.3 we obtain two more constants ε_1 and M_5. We also consider the constant M_1 of Proposition 3.1, and the constant r_0 of Proposition 3.4.

With a little abuse of notation, we consider the function

$$E(u,v) := \frac{1}{2}|v|^2 + \frac{1}{2}|Bu|^2 - \frac{\lambda}{2}|A^{1/2}u|^2 + \frac{1}{4}|A^{1/2}u|^4,$$

defined for every $(u,v) \in D(B) \times H$. In this way the classical energy $E(t)$ defined in (2.3) is just $E(u(t), u'(t))$.

The function $E(u,v)$ is continuous in $D(B) \times H$, and

$$E(\sigma_0 e_1, 0) = -\frac{1}{4}\sigma_0^4 < -\eta.$$

As a consequence, there exists $\delta_1 > 0$ such that

$$|v| + |B(u - \sigma_0 e_1)| \leq \delta_1 \quad \Longrightarrow \quad E(u,v) < -\eta. \tag{6.2}$$

We can also assume that δ_1 is small enough so that

$$|B(u - \sigma_0 e_1)| \leq \delta_1 \quad \Longrightarrow \quad \langle u, Ae_1 \rangle > 0. \tag{6.3}$$

We claim that the conclusions of Theorem 2.5 hold true if we choose

$$M_0 := \max\{M_4, M_5\},$$

and we choose $\varepsilon_0 > 0$ such that

$$\varepsilon_0 \leq \min\left\{1, \frac{\varepsilon_1}{2}, \frac{r_0}{2M_0}, \frac{\delta_1}{2M_0}\right\}$$

and

$$\varepsilon_0^2 \leq \frac{\gamma_0(1-\delta)\beta_0^2}{2M_1}. \tag{6.4}$$

Alternative   Let us assume that (2.15) is satisfied, and let $u(t)$ be any solution to (1.6). Let us set

$$L := \limsup_{t \to +\infty} |u(t)|.$$

We observe that L is finite because of (3.7), and we distinguish two cases.

Let us assume that $L \leq \beta_0$. Since $\varepsilon_0 \leq 1$, we can apply Proposition 3.2, from which we deduce that in this case $u(t)$ satisfies (2.16) with $\sigma = 0$.

So it remain to consider the case $L > \beta_0$. In this case we claim that we are in the framework of Proposition 3.3 with η given by (6.1), namely there exists $T_0 \geq 0$ for which the two inequalities in the upper box of (3.9) are satisfied.

In order to check the first one, we observe that $\varepsilon_0 \leq \varepsilon_1/2$, and therefore from assumption (2.15) it follows that $|f(t)| \leq \varepsilon_1$ whenever t is large enough.

In order to check the second one, we consider the function $\varphi(t) := \langle u(t), Pu(t) \rangle$. Due to (5.1), the function $\varphi(t)$ is bounded from above and

$$L_1 := \limsup_{t \to +\infty} \varphi(t) > (1 - \delta)\beta_0^2.$$

As a consequence, from Lemma 5.1 we deduce that there exists a sequence $t_n \to +\infty$ such that

$$\varphi(t_n) = \langle u(t_n), Pu(t_n) \rangle \to L_1 \qquad \text{and} \qquad \varphi'(t_n) = \langle u'(t_n), Pu(t_n) \rangle \to 0.$$

Now we observe that $E(t) = F(t) - \gamma_0 \varphi(t) - 2\gamma_0 \varphi'(t)$. Setting $t = t_n$, and letting $n \to +\infty$, we obtain that

$$\begin{aligned} \limsup_{n \to +\infty} E(t_n) &= \limsup_{n \to +\infty} F(t_n) - \lim_{n \to +\infty} \varphi(t_n) \\ &\leq M_1 \varepsilon_0^2 - \gamma_0(1 - \delta)\beta_0^2 \\ &\leq -2\eta, \end{aligned}$$

where in the inequalities we have exploited (3.6), the smallness assumption (6.4), and our definition (6.1) of η.

This shows that the two inequalities in the upper box of (3.9) are satisfied if we choose $T_0 := t_n$ with n large enough. At this point, from the conclusion in the lower box of (3.9) we deduce that in this case $u(t)$ satisfies (2.16) with $\sigma = \pm\sigma_0$.

Asymptotic convergence Since $M_0 \varepsilon_0 \leq r_0/2$, any pair of solutions satisfying (2.16) with the same $\sigma \in \{-\sigma_0, 0, \sigma_0\}$ satisfies also (3.11) with the same σ. At this point, (2.17) follows from Proposition 3.4.

Solutions in the stable regime Let us consider the case $\sigma = \sigma_0$ (but the argument is symmetric when $\sigma = -\sigma_0$). We claim that, when $f(t)$ satisfies (2.15) with our choice of ε_0, the following characterization holds true:

> "a solution to (1.6) satisfies (2.16) with $\sigma = \sigma_0$ if and only if there exists $T_0 \geq 0$, possibly depending on the solution, for which the two inequalities in upper box of (3.9) hold true with η given by (6.1), and $\langle u(T_0), Ae_1 \rangle > 0$".

Let us prove this characterization. The "if part" is exactly Proposition 3.3. As for the "only if part", it is enough to show that (2.15) and (2.16) with $\sigma = \sigma_0$ imply that

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the two inequalities in the upper box of (3.9), and the further condition $\langle u(t), Ae_1 \rangle > 0$, hold true when T_0 is large enough.

The first one follows from (2.15) because $\varepsilon_0 \leq \varepsilon_1/2$.

For the second one we observe that (2.16) implies that

$$|u'(t)| + |B(u(t) - \sigma_0 e_1)| < 2M_0\varepsilon_0 \leq \delta_1 \tag{6.5}$$

when t is large enough, and hence from (6.2) we deduce that $E(t) < -\eta$ for the same values of t.

The further condition holds true when t is large enough because of (6.5) and (6.3).

Given the characterization, we can prove our conclusions. Indeed, due to the continuous dependence on initial data, the set of initial data (u_0, u_1) originating a solution $u(t)$ for which there required T_0 exists is an open set. In order to prove that it is nonempty, we choose $T_0 \geq 0$ such that $|f(t)| \leq 2\varepsilon_0 \leq \varepsilon_1$ for every $t \geq T_0$, and we consider the solution $u(t)$ to (1.6) with "initial" data

$$u(T_0) = \sigma_0 e_1, \qquad u'(T_0) = 0.$$

This solution fits in the assumptions of Proposition 3.3 because

$$E(T_0) = -\frac{1}{4}\sigma_0^4 < -\eta.$$

We note that the extra condition $\langle u(T_0), Ae_1 \rangle > 0$ guarantees that the set of initial data for which the solution satisfies (2.16) with $\sigma = \sigma_0$ is disjoint from the set of initial data for which the same relation is fulfilled with $\sigma = -\sigma_0$.

Solutions in the unstable regime   Due to the alternative of statement (1), the set of initial data originating a solution satisfying (2.16) with $\sigma = 0$ is the complement of the set of initial data giving rise to solutions with $\sigma = \pm\sigma_0$. Since that set is the union of two open sets, the complement is necessarily closed, and nonempty because the phase space $D(B) \times H$ is connected and cannot be represented as the union of two disjoint nonempty open sets. $\square$

7   The concrete case (proof of Proposition 2.7)

We need to check that the operators A and B satisfy all the requirements in Definition 2.3. It is a classical result that A is a self-adjoint operator, and it satisfies (2.7) with $\mu_2 := \pi^2$. Indeed, in this concrete case it turns out that $D(A^{1/2}) = H_0^1((0,1))$, and (2.7) reduces to

$$\int_0^1 u_x(x)^2\, dx \geq \pi^2 \int_0^1 u(x)^2\, dx \qquad \forall u \in H_0^1((0,1)),$$

which is Poincaré inequality.

The operator B is the square root of B^2, and its domain is

$$D(B) = H_0^2((0,1)) \subseteq H^2((0,1)) \cap H_0^1((0,1)) = D(A),$$

as required. Moreover, inequality (2.2) holds true with $\mu_1 := 1$ because

$$|Bu|^2 = \langle B^2 u, u \rangle = |Au|^2 = \int_0^1 u_{xx}(x)^2 \, dx \qquad \forall u \in H_0^2((0,1)).$$

So it remains to check that (2.8) and (2.9) hold true with the values of λ_1 and λ_2 given in the statement, and with a suitable e_1. To this end, we begin by investigating all nontrivial solutions to (2.8), and then we conclude the proof in two alternative ways.

Nontrivial solutions to equation (2.8)   We look for all pairs (λ, φ), where λ is a positive real number and φ is a smooth function that satisfies

$$\varphi_{xxxx}(x) + \lambda \varphi_{xx}(x) = 0 \qquad \forall x \in (0,1), \tag{7.1}$$

with boundary conditions

$$\varphi(0) = \varphi_x(0) = \varphi(1) = \varphi_x(1) = 0. \tag{7.2}$$

Setting $\sqrt{\lambda} =: 2\alpha$ for the sake of shortness, all solution to (7.1) are of the form

$$\varphi(x) = c_1 + c_2 \, x + c_3 \cos(2\alpha x) + c_4 \sin(2\alpha x).$$

From the boundary conditions in $x = 0$ we deduce that $c_1 + c_3 = c_2 + 2\alpha c_4 = 0$, so that we can restrict to solutions of the form

$$\varphi(x) = c_1 \left[1 - \cos(2\alpha x)\right] + c_4 \left[\sin(2\alpha x) - 2\alpha x\right].$$

The boundary conditions in $x = 1$ are now equivalent to the system

$$\begin{cases} [1 - \cos(2\alpha)] \cdot c_1 + [\sin(2\alpha) - 2\alpha] \cdot c_4 = 0, \\ 2\alpha \sin(2\alpha) \cdot c_1 + 2\alpha[\cos(2\alpha) - 1] \cdot c_4 = 0. \end{cases}$$

With some standard algebra, we can show that this system has a nontrivial solution if and only if

$$\sin^2 \alpha = \alpha \sin \alpha \cos \alpha.$$

The solutions to this equation are all solutions to $\sin \alpha = 0$, namely the values of the form $\alpha = k\pi$ (with k any positive integer), and all solutions to $\tan \alpha = \alpha$, which are an infinite sequence α_k, with one element in each interval of the form $(k\pi, (k + 1/2)\pi)$. Recalling that $\lambda = 4\alpha^2$, the required eigenvalues are those of the form $4\pi^2 k^2$, with corresponding eigenfunctions

$$\varphi_k(x) = 1 - \cos(2k\pi x),$$

and those of the form $4\alpha_k^2$, with corresponding eigenfunctions of the form

$$\varphi_k(x) = \alpha_k(1 - \cos(2\alpha_k x)) + \sin(2\alpha_k x) - 2\alpha_k x.$$

Conclusion through variational approach   We show that (2.8) and (2.9) hold true with $\lambda_1 = 4\pi^2$, and consequently e_1 equal to a suitable multiple of $1-\cos(2\pi x)$, and $\lambda_2 = 4\alpha_1^2$. To this end, we consider the minimum problem (2.4), which now reads as

$$\min\left\{\int_0^1 \varphi_{xx}(x)^2\,dx : \varphi \in H_0^2((0,1)) \text{ and } \int_0^1 \varphi_x(x)^2\,dx = 1\right\},$$

and the minimum problem (2.6), which now reads as

$$\min\left\{\int_0^1 \varphi_{xx}(x)^2\,dx : \varphi \in H_0^2((0,1)), \int_0^1 \varphi_x(x)^2\,dx = 1, \int_0^1 \sin(2\pi x)\varphi_x(x)\,dx = 0\right\}.$$

In both cases a standard application of the direct method in the calculus of variations shows that the minimum exists, and any minimizer satisfies (7.1) with boundary conditions (7.2), and λ equal to the minimum value. It follows that the two minimum values are the two smallest values of λ for which (7.1)–(7.2) has nontrivial solutions, and from the previous analysis we know that these values are exactly $4\pi^2$ and $4\alpha_1^2$. This proves (2.8) and (2.9) as required.

Conclusion through functional analytic approach   As discussed in section A.2 of the appendix, the operator $A^{-1}B^2$ is symmetric with compact inverse in $H_0^1((0,1))$, and therefore the eigenfunctions that we found above span a dense subspace of $L^2((0,1))$, and they are orthogonal with respect to the scalar product of $H_0^1((0,1))$. At this point (2.8) and (2.9) follow from the classical variational characterization of the two smallest eigenvalues, applied in this case to the operator $A^{-1}B^2$ in the space $H_0^1((0,1))$.

A   Appendix

In the second paragraph of section 2 we described in finite dimension the role of the eigenvalues of $A^{-1}B^2$ in the study of the negative inertia index of $B^2 - \lambda A$ as a function of λ. In the rest of the paper we developed our theory in the infinite dimensional case without mentioning $A^{-1}B^2$ explicitly.

In this appendix we present a possible functional setting in which the spectral theory can be applied to the operator $A^{-1}B^2$, both in the general and in the concrete case.

A.1   The correct framework for $A^{-1}B^2$ in general

Let H be a Hilbert space, and let A and B be two coercive self-adjoint unbounded operators on H with dense domains $D(B) \subseteq D(A)$. Then a reasonable definition of $A^{-1}B^2$ seems to be the following.

Let us consider the Hilbert spaces $V := D(A^{1/2})$ and $W := D(B)$. If we identify H with its dual space H', then we have the inclusions

$$W \subseteq V \subseteq H = H' \subseteq V' \subseteq W'.$$

With these notations we can consider A as a bounded operator $\widehat{A} : V \to V'$, and represent the scalar product in V in terms of the duality pairing as

$$\langle u, v \rangle_V = \langle \widehat{A}u, v \rangle_{V',V} \qquad \forall (u, v) \in V^2.$$

Similarly, we con consider B as a bounded operator $\widehat{B} : W \to H$, whose adjoint is a bounded operator $\widehat{B}^* : H' \to W'$ with $\widehat{B}^* u = Bu$ if $u \in W \subseteq H'$.

Now we can consider the unbounded operator C in V with domain

$$D(C) := \left\{ u \in W : \widehat{B}^* B u \in V' \right\},$$

and defined by

$$Cu := \widehat{A}^{-1} \widehat{B}^* B u \qquad \forall u \in D(C).$$

We claim that C is an extension of $A^{-1}B^2$ that is symmetric and maximal monotone as an unbounded operator in V.

To begin with, for every u and v in $D(C)$ it turns out that

$$\langle v, Cu \rangle_V = \langle v, \widehat{A}Cu \rangle_{V',V} = \langle v, \widehat{B}^* B u \rangle_{V',V} = \langle \widehat{B}v, Bu \rangle_H = \langle Bv, Bu \rangle_H,$$

which proves that C is symmetric and monotone.

It remain to show that C is maximal, namely that, for every $f \in V$, the equation $u + Cu = f$ has a (unique) solution $u \in D(C)$. Applying $\widehat{A}$ to both sides, this equation becomes

$$\widehat{A}u + \widehat{B}^* B u = \widehat{A}f \in V'.$$

Now the operator $\widehat{A} + \widehat{B}^* B$ is coercive from W to W', and hence surjective (see for example [2, Corollary 14]). Since $\widehat{A}f \in V'$, the solution belongs to $D(C)$.

A.2 The operator $A^{-1}B^2$ in the concrete case

Instead of fussing with generalities, we give an explicit description of the operator $A^{-1}B^2$ in the case where A and B are as in Proposition 2.7.

Let us consider the Hilbert space $V = H_0^1((0,1))$ with scalar product

$$\langle u, v \rangle_V := \int_0^1 u_x(x) v_x(x) \, dx \qquad \forall (u, v) \in H_0^1((0,1))^2. \tag{A.1}$$

Let us consider the unbounded linear operator C in V with domain

$$D(C) := H^3((0,1)) \cap H_0^2((0,1)),$$

and defined by

$$[Cu](x) := -u_{xx}(x) + u_{xx}(0) + [u_{xx}(1) - u_{xx}(0)]x \qquad \forall u \in D(C). \tag{A.2}$$

We observe that

$$Cu = A^{-1}B^2 u \qquad \forall u \in D(B^2) = H^4((0,1)) \cap H_0^2((0,1)),$$

and therefore C is a natural extension of $A^{-1}B^2$. Indeed $B^2u = u_{xxxx} \in L^2((0,1))$, and hence $A^{-1}B^2u$ is the solution $v(x)$ to equation $-v_{xx} = u_{xxxx}$ in $(0,1)$, with Dirichlet boundary conditions in $x = 0$ and $x = 1$. The solution is exactly the function Cu defined in (A.2).

We claim that C is a symmetric positive operator in V with domain $D(C)$, and compact inverse. If we prove this claim, then the eigenfunctions of C are a basis of $H_0^1((0,1))$, and hence also a basis of $L^2((0,1))$, but orthogonal with respect to the scalar product (A.1). These eigenfunctions are exactly the solutions to (7.1) that we characterized in section 7.

To begin with, for every u and v in $D(C)$ it turns out that

$$
\begin{aligned}
\langle v, Cu \rangle_V &= \int_0^1 v_x(x) \cdot [Cu]_x(x)\, dx \\
&= \int_0^1 v_x(x) \cdot \{-u_{xxx}(x) + u_{xx}(1) - u_{xx}(0)\}\, dx \\
&= -\int_0^1 v_x(x) \cdot u_{xxx}(x)\, dx \\
&= \int_0^1 v_{xx}(x) \cdot u_{xx}(x)\, dx,
\end{aligned}
$$

which is enough to conclude that C is both symmetric and positive.

It remains to show that the inverse is compact, namely that for every $f \in H_0^1((0,1))$ the equation

$$
-u_{xx}(x) + u_{xx}(0) + [u_{xx}(1) - u_{xx}(0)]x = f(x) \qquad \forall x \in (0,1) \tag{A.3}
$$

has a unique solution $Tf \in D(C)$ (note that lying in this domain entails four boundary conditions), and T is compact as an operator $T : H_0^1((0,1)) \to H_0^1((0,1))$.

Uniqueness follows from the positivity of C. Existence follows from the explicit formula for the solution. Indeed, if we set

$$
F(x) := \int_0^x f(t)\, dt, \qquad\qquad \widehat{F}(x) := \int_0^x F(t)\, dt,
$$

then a standard computation shows that the solution to (A.3) is

$$
u(x) := -\widehat{F}(x) + \left(3\widehat{F}(1) - F(1)\right)x^2 + \left(F(1) - 2\widehat{F}(1)\right)x^3.
$$

The same formula reveals that if a sequence $\{f_n\}$ is bounded in $H_0^1((0,1))$, then the sequence of corresponding solutions is bounded in $H^3((0,1))$, and therefore relatively compact in $H_0^1((0,1))$.

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References

[1] F. ALOUI, A. HARAUX. Sharp ultimate bounds of solutions to a class of second order linear evolution equations with bounded forcing term. *J. Funct. Anal.* **265** (2013), no. 10, 2204–2225.

[2] H. BREZIS. Équations et inéquations non linéaires dans les espaces vectoriels en dualité. *Ann. Inst. Fourier (Grenoble)* **18** (1968), no. fasc. 1, 115–175.

[3] H. BRÉZIS. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).

[4] C. FITOURI, A. HARAUX. Sharp estimates of bounded solutions to some semilinear second order dissipative equations. *J. Math. Pures Appl. (9)* **92** (2009), no. 3, 313–321.

[5] C. FITOURI, A. HARAUX. Boundedness and stability for the damped and forced single well Duffing equation. *Discrete Contin. Dyn. Syst.* **33** (2013), no. 1, 211–223.

[6] M. GHISI, M. GOBBINO, A. HARAUX. An infinite dimensional Duffing-like evolution equation with linear dissipation and an asymptotically small source term. *Nonlinear Anal. Real World Appl.* **43** (2018), 167–191.

[7] P. HOLMES, J. MARSDEN. A partial differential equation with infinitely many periodic orbits: chaotic oscillations of a forced beam. *Arch. Rational Mech. Anal.* **76** (1981), no. 2, 135–165.

[8] W. S. LOUD. Boundedness and convergence of solutions of $x'' + cx' + g(x) = e(t)$. *Duke Math. J.* **24** (1957), 63–72.

[9] F. MOON, P. HOLMES. A magnetoelastic strange attractor. *Journal of Sound and Vibration* **65** (1979), no. 2, 275 – 296.

[10] M. REED, B. SIMON. *Methods of modern mathematical physics. I.* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, second edition, 1980. Functional analysis.