Logarithmic Hennings invariants for restricted quantum sl(2)

Anna Beliakova, Christian Blanchet, Nathan Geer

LOGARITHMIC HENNINGS INVARIANTS FOR RESTRICTED QUANTUM $\mathfrak{sl}(2)$

ANNA BELIAKOVA, CHRISTIAN BLANCHET, AND NATHAN GEER

Abstract. We construct a Hennings type logarithmic invariant for restricted quantum $\mathfrak{sl}(2)$ at a $2\mathsf{p}$-th root of unity. This quantum group U is not braided, but factorizable. The invariant is defined for a pair: a 3-manifold M and a colored link L inside M. The link L is split into two parts colored by central elements and by trace classes, or elements in the 0^{th} Hochschild homology of U, respectively. The two main ingredients of our construction are the universal invariant of a string link with values in tensor powers of U, and the *modified trace* introduced by the third author with his collaborators and computed on tensor powers of the regular representation. Our invariant is a colored extension of the logarithmic invariant constructed by Jun Murakami.

1. Introduction

In the 90th M. Hennings came up with a construction of 3-manifold invariants out of a factorizable ribbon Hopf algebra H [13]. In his construction the right integral $\mu \in H^*$ satisfying

$$(\mu \otimes \mathrm{id})\Delta(x) = \mu(x)\mathbf{1} \quad \text{for all} \quad x \in H$$

plays the role of the Kirby color. If the category of H-modules is semi-simple, Hennings recovers the Reshetikhin–Turaev invariant. However in the non semi-simple case, his invariant vanishes for manifolds with positive first Betti number (see [15]). A TQFT based on the Hennings invariant was constructed by Lyubashenko and Kerler [19, 14]. It satisfies the full TQFT axioms for lagrangian cobordisms between connected surfaces with one boundary component. In the general case, it has weak functoriality and monoidality properties.

More recently, a completely different non-semisimple TQFT based on the unrolled quantum $\mathfrak{sl}(2)$ was defined by Blanchet, Costantino, Geer and Patureau [3]. This construction uses the logarithmic 3-manifold invariant constructed previously by Costantino, Geer and Patureau (CGP) in [5] by generalizing the Kashaev invariant. More precisely, the CGP invariant is defined for an admissible pair: a 3-manifold together with a $\mathbb{C}/2\mathbb{Z}$ valued cohomology class. One of the most innovative ingredients of the CGP construction is the so-called *modified trace* which contrary to the usual quantum trace does not vanish on the projective modules.

In this paper we construct a family of invariants of 3-manifolds with colored links inside by combining Hennings approach with the modified trace methods of [10, 11]. We develop structural properties of restricted quantum $\mathfrak{sl}(2)$, by working in the Hopf algebra itself rather than in its category of modules. Our invariants generalize the logarithmic invariants of knots in 3-manifolds recently introduced by J. Murakami [21].

Main results. Let us denote by U the restricted quantum $\mathfrak{sl}(2)$ at $2\mathsf{p}^{\text{th}}$ root of unity $q = e^{\frac{i\pi}{\mathsf{p}}}$, explicitely defined in the next section. The Hopf algebra U does not contain an R-matrix, but only monodromy or double braiding. However, U is a subalgebra of a ribbon Hopf algebra D obtained by adjoining a square root of the generator K. Since D is not factorizable and U is not braided, neither U, nor D supports the Hennings–Kerler–Lyubashenko TQFT construction.

Let U-mod be the category of finite dimensional U-modules. This is a finite pivotal tensor category. Hence, for any morphism f in U-mod there is a notion of a categorical (or quantum) left and right traces, denoted by $\mathrm{tr}_l(f)$ and $\mathrm{tr}_r(f)$, respectively.

Let U-pmod be its full subcategory of projective U-modules. An explicit description of U-pmod is given in [9]. Let us denote by $\mathcal{P}_j^{\pm}$ with $j = 1, ..., \mathsf{p}$ the indecomposable projective modules. Here $\mathcal{P}_{\mathsf{p}}^{\pm}$ is a simple module with highest weight $\pm q^{\mathsf{p}-1}$. The module $\mathcal{P}_1^+$ is the projective cover of the trivial module. The space of endomorphisms $\mathrm{End}_U(\mathcal{P}_j^{\pm})$ $(1 \leq j \leq \mathsf{p}-1)$ is two dimensional with basis given by the identity $\mathrm{id}_{\mathcal{P}_j^{\pm}}$ and a nilpotent endomorphism $x_j^{\pm}$, defined in Section 2.

The subcategory U-pmod is an ideal of U-mod in the sense of [10, 11]. A *modified trace* on U-pmod, is a family of linear functions

$$\{\mathsf{t}_V : \mathrm{End}_U V \to \mathbb{C}\}_{V \in U\text{-pmod}}$$

such that the following two conditions hold:

(1) **Cyclicity.** If $X, V \in U$-pmod, then for any morphisms $f : V \to X$ and $g : X \to V$ in U-mod we have

$$\mathsf{t}_V(gf) = \mathsf{t}_X(fg).$$

(2) **Partial trace properties.** If $X \in U$-pmod and $W \in U$-mod then for any $f \in \mathrm{End}_U(X \otimes W)$ and $g \in \mathrm{End}_U(W \otimes X)$ we have
$$\mathsf{t}_{X\otimes W}(f) = \mathsf{t}_X\left(\mathrm{tr}_r^W(f)\right),$$
$$\mathsf{t}_{W\otimes X}(g) = \mathsf{t}_X\left(\mathrm{tr}_l^W(g)\right),$$
where tr_r^W and tr_l^W are the right and left partial categorical traces along W defined using the pivotal structure in Eqs. (5).

Our first main result is the following.

Theorem 1. *There exists a unique family of linear functions*
$$\{\mathsf{t}_V : \mathrm{End}_U(V) \to \mathbb{C}\}_{V\in U\text{-pmod}},$$
satisfying cyclicity and the partial trace properties, normalized by
$$\mathsf{t}_{\mathcal{P}^+_{\mathsf{p}}}(\mathrm{id}_{\mathcal{P}^+_{\mathsf{p}}}) = (-1)^{\mathsf{p}-1}.$$
Moreover, $\mathsf{t}_{\mathcal{P}^-_{\mathsf{p}}}(\mathrm{id}_{\mathcal{P}^-_{\mathsf{p}}}) = 1$ *and for* $1 \le j \le \mathsf{p}-1$ *we have*
$$\mathsf{t}_{\mathcal{P}^\pm_j}(\mathrm{id}_{\mathcal{P}^\pm_j}) = (\pm 1)^{\mathsf{p}-1}(-1)^j(q^j+q^{-j}) \quad \text{and} \quad \mathsf{t}_{\mathcal{P}^\pm_j}(x^\pm_j) = (\pm 1)^{\mathsf{p}}(-1)^j[j]^2.$$

This family is called the modified trace on U-pmod. The proof uses the fact that U-mod is unimodular (i.e. projective cover of the trivial module is self-dual) and it has a simple projective object. In this case, there exist unique left and right modified traces on U-pmod by [10, Cor. 3.2.1]. We actually compute these traces on the algebra of endomorphisms of indecomposable projectives and show that they are equal.

Observe that Theorem 1 applies to U, considered as a free left module over itself, called the regular representation, and its tensor powers. Recall from [9] that the regular representation decomposes as
$$U \cong \bigoplus_{j=1}^{\mathsf{p}} j\mathcal{P}^+_j \oplus \bigoplus_{j=1}^{\mathsf{p}} j\mathcal{P}^-_j.$$
The algebra $\mathrm{End}_U(U)$ of the U-endomorphisms of U can be identified with U^{op} (i.e. U with the opposite multiplication). The isomorphism is given by sending an element x of U^{op} to the operator r_x of the right multiplication by x on the regular representation. By definition r_x commutes with the left action. More generally, $\mathfrak{C}_m = \mathrm{End}_U(U^{\otimes m})$, $m \ge 1$, are known as centralizer algebras.

The space of characters (or symmetric functions, see [1]) on U is defined as
$$\mathrm{Char}(U) := \{\phi \in U^* \mid \phi(xy) = \phi(yx) \quad \text{for any} \quad x, y \in U\}.$$

This space is dual to the 0^{th}-Hochschild homology $\mathrm{HH}_0(U)$, which is

$$\mathrm{HH}_0(U) := \frac{U}{[U,U]} \quad \text{where} \quad [U,U] = \mathrm{Span}\{xy - yx \mid x, y \in U\}.$$

There is an obvious action of the center $Z(U)$ on $\mathrm{Char}(U)$ by setting $z\phi(x) := \phi(zx)$ for any $z \in Z(U)$ and $x \in U$.

Let us define the linear map

$$\mathrm{Tr}'_m : \mathfrak{C}_m \to \mathbb{C} \quad \text{by} \quad x \mapsto t_{U^{\otimes m}}(x)$$

and in particular, $\mathrm{Tr}' : U^{op} \simeq \mathfrak{C}_1 \to \mathbb{C}$ is the linear map $\mathrm{Tr}'_1 = t_U$. Due to cyclicity, we have $\mathrm{Tr}' \in \mathrm{Char}(U)$.

Our next theorem states properties of the special symmetric function on U given by the modified trace.

Theorem 2. *The modified trace* $\mathrm{Tr}' \in \mathrm{Char}(U)$ *satisfies the following properties.*

- *(Partial trace property) For any* $f \in \mathrm{End}_U(U^{\otimes 2})$,
$$\mathrm{Tr}'_2(f) = \mathrm{Tr}'\left(\mathrm{tr}^U_r(f)\right) = \mathrm{Tr}'\left(\mathrm{tr}^U_l(f)\right)$$
where tr^U_l *and* tr^U_r *are the right and left partial traces, defined in Equation* (5)*;*
- *(Non-degeneracy) The pairing* $\langle\ ,\ \rangle : Z(U) \times \mathrm{HH}_0(U) \to \mathbb{C}$ *defined by* $\langle z, u\rangle = \mathrm{Tr}'(zu)$ *is non-degenerate.*

The proof is by direct computation of the pairing in the basis of the center and $\mathrm{HH}_0(U)$, defined in Sections 2 and 4, respectively.

For any ribbon Hopf algebra there exists a *universal* invariant associated to an oriented framed tangle T. This invariant is obtained by assigning the R-matrix to crossings and evaluation and coevaluation maps to maxima and minima (see Section 4). Although our restricted quantum group U is not ribbon, it has a ribbon extension D, which produces a universal invariant $J_T \in D^{\otimes m}$ for any tangle T with m components. If T is a string link, we argue that J_T actually belongs to the subspace $(U^{\otimes m})^U \subset U^{\otimes m}$ of invariants under left action.

We are now ready to state our main result. Let $\mu \in U^*$ be the right integral of U. It is unique up to a normalisation which we fix in Section 2. For $m \geq 1$, the function $\mathrm{Tr}'_m = t_{U^{\otimes m}}$ defines a bilinear pairing $\langle\ ,\rangle : (U^{\otimes m})^U \times U^{\otimes m} \to \mathbb{C}$ as follows

$$\langle z, x\rangle = \mathrm{Tr}'_m(l_z r_x)\ .$$

Here l_z, r_x are the left and right multiplications, respectively. This bilinear pairing factorises on the right through $\mathrm{HH}_0(U)^{\otimes m}$.

Assume that a closed 3-manifold M with a (m_+, m_-) component framed link (L^+, L^-) inside is represented by surgery in S^3 along the

m_0 component link L^0. Let us color the components of L^+ (resp. L^-) by central elements $z_j \in Z(U)$, $1 \leq j \leq m_+$ (resp. by trace classes $h_k \in \mathrm{HH}_0(U)$, $1 \leq k \leq m_-$). Let $T = T^+ \cup T_0 \cup T^-$ be a string link in S^3 obtained by opening all $m = m_+ + m_0 + m_-$ components. Let s be the signature of the linking matrix for L^0 and $\delta := \frac{1-i}{\sqrt{2}} q^{\frac{3-\mathsf{p}^2}{2}}$. We set $z^+ = \otimes_j z_j$ (resp. $h^- = \otimes_k h_k$), and denote by L the colored link $((L^+, z^+), (L^-, h^-))$. Note that to define z^+, h^- as well as J_T, we need to fix an order on the components of L, change of this order will result in an obvious permutation of the entries.

We define a number associated to the pair (M, L) as follows.

Theorem 3. *With the notation as above,*

$$\mathrm{H}^{\log}(M, L) := \delta^s \langle \left(z^+ \mu^{\otimes m_+} \otimes \mu^{\otimes m_0} \otimes \mathrm{id} \right) (J_T) \, , \, h^- \rangle$$

is a topological invariant of the pair (M, L).

When L^- is empty, the colored Hennings invariants [13] are recovered. When L^- is a knot, then our invariants are equivalent to the Murakami center valued logarithmic invariants [21]. Thus $\mathrm{H}^{\log}$ can be understood as a colored extension of the Murakami invariants. An action of the modular group $SL(2, \mathbb{Z})$ on the center $Z(U)$ was studied in [8]. We expect that $\mathrm{H}^{\log}$ can be used to extend these mapping class group representation in genus one to a refined TQFT with full functorial and monoidal properties.

The paper is organized as follows. In Section 2 we define the restricted quantum group U and its braided extension D. In Section 3 we discuss their categories of finite dimensional modules. The universal tangle invariant is constructed in Section 4, where we also compute a basis for the space of trace classes $\mathrm{HH}_0(U)$. Our main theorems are proved in the two last sections.

Acknowledgments. The authors are grateful to Azat Gainutdinov and Thomas Kerler for helpful discussions. The work was supported by NCCR SwissMAP founded by the Swiss National Science Foundation. NG was partially supported by NSF grants DMS-1308196 and MS-1452093. He also would like to thank CNRS for its support.

2. Restricted quantum $\mathfrak{sl}(2)$ and its braided extension

Definition of U. Fix an integer $\mathsf{p} \geq 2$ and let $q = e^{\frac{\pi\sqrt{-1}}{\mathsf{p}}}$ be a $2\mathsf{p}^{th}$-root of unity. Let $U = \overline{U}_q(\mathfrak{sl}(2))$ be the $\mathbb{C}$-algebra given by generators

E, F, K, K^{-1} and relations:

$$E^{\mathsf{p}} = F^{\mathsf{p}} = 0, \qquad K^{2\mathsf{p}} = 1, \qquad KK^{-1} = K^{-1}K = 1,$$
$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

The algebra U is a Hopf algebra where the coproduct, counit and antipode are defined by

$$\begin{aligned}
\Delta(E) &= 1 \otimes E + E \otimes K, & \varepsilon(E) &= 0, & S(E) &= -EK^{-1},\\
\Delta(F) &= K^{-1} \otimes F + F \otimes 1, & \varepsilon(F) &= 0, & S(F) &= -KF,\\
\Delta(K) &= K \otimes K & \varepsilon(K) &= 1, & S(K) &= K^{-1},\\
\Delta(K^{-1}) &= K^{-1} \otimes K^{-1} & \varepsilon(K^{-1}) &= 1, & S(K^{-1}) &= K.
\end{aligned}$$

In what follows we will use Sweedler notation. For $x \in U$ we write

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}, \quad \Delta^{[n]}(x) = \sum x_{(1)} \otimes x_{(2)} \otimes ... \otimes x_{(n)} \quad \text{for} \quad n \geq 1.$$

The center of U. The dimension of the center $Z(U)$ is $3\mathsf{p} - 1$. A basis consists of $\mathsf{p} + 1$ central idempotents $\mathbf{e}_j$ $(0 \leq j \leq \mathsf{p})$ and $2\mathsf{p} - 2$ elements $\mathbf{w}_j^{\pm}$ $(1 \leq j \leq \mathsf{p} - 1)$ in the radical [8]. These elements satisfy the following relations:

$$\begin{aligned}
&\mathbf{e}_s\mathbf{e}_t = \delta_{s,t}\mathbf{e}_s & & 0 \leq s, t \leq \mathsf{p}\\
&\mathbf{e}_s\mathbf{w}_t^{\pm} = \delta_{s,t}\mathbf{w}_t^{\pm} & & 0 \leq s \leq \mathsf{p},\ 1 \leq t \leq \mathsf{p} - 1\\
&\mathbf{w}_s^{\pm}\mathbf{w}_t^{\pm} = \mathbf{w}_s^{\pm}\mathbf{w}_t^{\mp} = 0 & & 1 \leq s, t \leq \mathsf{p} - 1.
\end{aligned}$$

Braided extension. The Hopf algebra U is not braided, see [16]. However, it can be realized as a Hopf subalgebra of the following braided Hopf algebra. Let D be the Hopf algebra generated by e, ϕ, k and k^{-1} with the relations:

$$e^{\mathsf{p}} = \phi^{\mathsf{p}} = 0, \qquad k^{4\mathsf{p}} = 1,$$
$$kk^{-1} = k^{-1}k = 1, \quad kek^{-1} = qe, \quad k\phi k^{-1} = q^{-1}\phi, \quad [e, \phi] = \frac{k^2 - k^{-2}}{q - q^{-1}},$$

and Hopf algebra structure:

$$\begin{aligned}
\Delta(e) &= 1 \otimes e + e \otimes k^2, & \varepsilon(e) &= 0, & S(e) &= -ek^{-2},\\
\Delta(\phi) &= k^{-2} \otimes \phi + \phi \otimes 1, & \varepsilon(\phi) &= 0, & S(\phi) &= -k^2\phi,\\
\Delta(k) &= k \otimes k & \varepsilon(k) &= 1, & S(k) &= k^{-1},\\
\Delta(k^{-1}) &= k^{-1} \otimes k^{-1} & \varepsilon(k^{-1}) &= 1, & S(k^{-1}) &= k.
\end{aligned}$$

The Hopf algebra D has two special invertible elements: the R-matrix

$$R = \frac{1}{4\mathsf{p}} \sum_{m=0}^{\mathsf{p}-1} \sum_{n,j=0}^{4\mathsf{p}-1} \frac{(q-q^{-1})^m}{[m]!} q^{m(m-1)/2+m(n-j)-nj/2} e^m k^n \otimes \phi^m k^j$$

and the ribbon element

$$r = \frac{1-i}{2\sqrt{\mathsf{p}}} \sum_{m=0}^{\mathsf{p}-1} \sum_{j=0}^{2\mathsf{p}-1} \frac{(q-q^{-1})^m}{[m]!} q^{-m/2+mj+(j+\mathsf{p}+1)^2/2} \phi^m e^m k^{2j}$$

where $q^{\frac{1}{2}} = e^{\frac{i\pi}{2\mathsf{p}}}$, $[m] = \frac{q^m - q^{-m}}{q-q^{-1}}$ and $[m]! = [m][m-1]...[1]$. The following theorem is well known, see [8].

Theorem 4. *The triple (D, R, r) is a ribbon Hopf algebra.*

Let us call $M = R_{21}R$ the double braiding or *monodromy*, where $R_{21} = \sum_i \beta_i \otimes \alpha_i$ with $R = \sum_i \alpha_i \otimes \beta_i$. A Hopf algebra A is called *factorisable* if its monodromy matrix can be written as

$$M = \sum_i m_i \otimes n_i$$

where m_i and n_i are two bases of A. The Hopf algebra D is not factorisable. There is a Hopf algebra embedding $U \to D$ given by

$$E \mapsto e, \ F \mapsto \phi, \ \text{ and } K \mapsto k^2.$$

It is easy to check that $r \in U$, and the monodromy $M = R_{21}R \in U \otimes U$. Moreover, U is factorisable.

Ribbon and balancing elements of U. Let $u = \sum_i S(\beta_i)\alpha_i$ be the canonical element implementing the inner-automorphism S^2, i.e. $S^2(x) = uxu^{-1}$ for any $x \in D$, and satisfying

$$\Delta(u) = M^{-1}(u \otimes u).$$

Using the formula for the R-matrix, it is easy to check that $u \in U$. The ribbon element $r \in U$ is central and invertible, such that

$$r^2 = uS(u), \quad S(r) = r, \quad \varepsilon(r) = 1 \quad \Delta(r) = M^{-1}(r \otimes r). \tag{1}$$

Using them we define

$$g := r^{-1}u = k^{2\mathsf{p}+2} = K^{\mathsf{p}+1} \in U$$

the *balancing* element. This element is grouplike, i.e.

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad \text{ and } \quad gxg^{-1} = S^2(x) \tag{2}$$

for any $x \in U$. The balancing element will be used to define the pivotal structure.

Remark. As it was shown by Drinfeld, equations (2) determine g^2 only. Hence $g' = K$ is another balancing element, which will not be considered in this paper.

Right integral. Recall that a *right integral* $\mu \in U^*$ is defined by the following system of equations

$$(\mu \otimes \mathrm{id})\Delta(x) = \mu(x)\mathbf{1} \quad \text{for all} \quad x \in U.$$

For any finite dimensional Hopf algebra over a field of zero characteristics, there is a unique solution to these equations up to a scalar. In our case in the PBW basis, it is given by the formula

$$\mu(E^m F^n K^l) = \zeta \delta_{m,\mathsf{p}-1} \delta_{n,\mathsf{p}-1} \delta_{l,\mathsf{p}+1} \ .$$

We fix normalisation as in [21] by setting

$$\zeta = -\sqrt{\frac{2}{\mathsf{p}}}([\mathsf{p}-1]!)^2.$$

The evaluation on the ribbon element and its inverse are given by

$$\mu(r) = \frac{1-i}{\sqrt{2}} q^{\frac{3-\mathsf{p}^2}{2}} = \frac{1}{\mu(r^{-1})} = \delta \ .$$

One can show that μ belongs to the space of quantum characters

$$\mathrm{qChar}(U) := \{\phi \in U^* \mid \phi(xy) = \phi(S^2(y)x) \quad \text{for any} \quad x, y \in U\}.$$

The center $Z(U)$ acts on $\mathrm{qChar}(U)$ by $z\phi(x) := \phi(zx)$ for any $z \in Z(U)$ and $x \in U$. Under this action $\mathrm{qChar}(U)$ is a free module of dimension one with basis given by the right integral μ. Hence, as a vector space $\mathrm{qChar}(U)$ has dimension $3p - 1$.

The space of quantum characters $\mathrm{qChar}(U)$ is naturally isomorphic to the space of characters. Indeed, we can define the map

$$Q : \mathrm{qChar}(U) \to \mathrm{Char}(U) \quad \text{by sending} \quad \phi \mapsto \phi_g \tag{3}$$

where $\phi_g(x) := \phi(gx)$ and g is the balancing element. Cyclicity can be verified as follows:

$$\phi_g(xy) = \phi(gxy) = \phi(S^2(y)gx) = \phi(gyx) = \phi_g(yx)$$

The inverse map is given by sending $\psi \in \mathrm{Char}(A)$ to $\psi_{g^{-1}} \in \mathrm{qChar}(A)$. Hence Q is an isomorphism.

We get that the dimension of the dual vector spaces $\mathrm{Char}(U)$ and $\mathrm{HH}_0(U)$ is also $3p - 1$.

3. Categories of modules

In this section we will study the $\mathbb{C}$-linear categories of finite dimensional modules over D and U, which we denote by D-mod and U-mod, respectively.

Category D-mod**.** The category D-mod is ribbon, with the usual braiding

$$c_{V,W} : V \otimes W \to W \otimes V, \quad \text{given by} \quad u \otimes v \mapsto \tau R(u \otimes v),$$

where $\tau(x \otimes y) = y \otimes x$, twist

$$\theta_V : V \to V, \quad \text{given by} \quad v \mapsto r^{-1} v$$

and compatible duality

$$\begin{aligned} \mathrm{coev}_V : \ & \mathbb{C} \to V \otimes V^*, \ \text{given by } 1 \mapsto \sum_i v_i \otimes v_i^*, \\ \mathrm{ev}_V : \ & V^* \otimes V \to \mathbb{C}, \ \text{given by } f \otimes v \mapsto f(v), \\ \widetilde{\mathrm{coev}}_V : \ & \mathbb{C} \to V^* \otimes V \quad \text{given by } 1 \mapsto \sum_i v_i^* \otimes g^{-1} v_i, \\ \widetilde{\mathrm{ev}}_V : \ & V \otimes V^* \to \mathbb{C} \text{ given by } v \otimes f \mapsto f(gv) \end{aligned} \tag{4}$$

where g is the balancing element. Using properties of r, one can check that the twist is *self-dual*, i.e. $\theta_{V^*} = \theta_V^*$.

The duality morphisms (4) define pivotal structure on D-mod (see e.g. [11]). In particular, in the pivotal setting, one can define left and right (categorical) traces of any endomorphism $f : V \to V$ as

$$\mathrm{tr}_l(f) = \mathrm{ev}_V(\mathrm{id}_{V^*} \otimes f)\widetilde{\mathrm{coev}}_V \quad \mathrm{tr}_r(f) = \widetilde{\mathrm{ev}}_V(f \otimes \mathrm{id}_{V^*})\,\mathrm{coev}_V$$

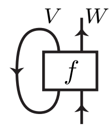
and dimensions of objects.

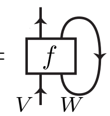
A *spherical* category is a pivotal category whose left and right traces are equal, i.e. $\mathrm{tr}_l(f) = \mathrm{tr}_r(f)$ for any endomorphism f. It is easy to see that any ribbon category is spherical. We call $\mathrm{tr}_l(\mathrm{id}_V) = \mathrm{tr}_r(\mathrm{id}_V)$ the *quantum* dimension of V.

We will use a standard graphical calculus to represent morphisms in D-mod by diagrams in the plane which are read from the bottom to the top.

In what follows, we will need *partial* categorical traces of endomorphisms. Given $V, W \in D$-mod and $f : V \otimes W \to V \otimes W$ let tr_l^V and

tr_r^W be the left and right *partial* traces defined as follows

$$\mathrm{tr}_l^V(f) = (\mathrm{ev}_V \otimes \mathrm{id}_W)(\mathrm{id}_{V^*} \otimes f)(\widetilde{\mathrm{coev}}_V \otimes \mathrm{id}_W) =$$ 

$$\mathrm{tr}_r^W(f) = (\mathrm{id}_V \otimes \widetilde{\mathrm{ev}}_W)(f \otimes \mathrm{id}_{W^*})(\mathrm{id}_V \otimes \mathrm{coev}_W) = \tag{5}$$  .

Category U-mod**.** Let us call a module *simple*, if its endomorphism ring is one dimensional. A module is *projective* if it is a direct summand of a free module.

The category U-mod includes the s-dimensional simple modules $\mathscr{X}_s^\pm$, and their projective covers $\mathcal{P}_s^\pm$ for $1 \leq s \leq \mathsf{p}$, which are $2\mathsf{p}$ dimensional for $1 \leq s < \mathsf{p}$. The simple module $\mathscr{X}_s^\pm$ is determined by its highest weight vector v with the action $Ev = 0$ and $Kv = \pm q^{s-1}v$. It is projective if and only if $s = p$.

A category is called *unimodular*, if the projective cover of the trivial module is self-dual. Since $\mathcal{P}_1^+$ is self-dual (as well as all other modules defined above), U-mod is unimodular.

The category U-mod inherits the pivotal structure, twist and double braiding from D-mod. The double braiding is

$$M_{V,W} : V \otimes W \to V \otimes W, \quad \text{given by} \quad x \otimes y \mapsto M(x \otimes y),$$

where M is the monodromy matrix; the self-dual twist and duality are given by (4). It can be checked that U-mod is *twisted* category with duality in the sense of Bruguières [2].

Let U-pmod be the full subcategory of U-mod consisting of projective modules. This category is non-abelian. To compute the modified trace on U-pmod, we will need an explicit structure of this category.

Structure of U-pmod**.** A module is *indecomposable* if it does not decompose as a direct sum of two modules. The indecomposable projective U-modules are classified up to isomorphism in [9]: they are precisely the projective covers $\mathcal{P}_j^\pm$ of the simple modules where $j = 1, ..., \mathsf{p}$. In particular, $\mathcal{P}_\mathsf{p}^\pm$ is a simple module with highest weight $\pm q^{\mathsf{p}-1}$. The module $\mathcal{P}_1^+$ is the projective cover of the trivial one.

We will recall some facts about these projective modules. For $1 \leq j \leq \mathsf{p} - 1$ let

$$\{\mathbf{x}_k^\pm, \mathbf{y}_k^\pm\}_{0 \leq k \leq \mathsf{p}-j-1} \cup \{\mathbf{a}_n^\pm, \mathbf{b}_n^\pm\}_{0 \leq n \leq j-1} \tag{6}$$

be the basis of $\mathcal{P}_j^\pm$ given in [8] (see Section C.2 of [8] for the defining relations).

Following [6] we call a weight vector v *dominant* if $(FE)^2v = 0$. The vector $\mathbf{b}_0^+$ (resp. $\mathbf{y}_0^-$) is a dominant vector of $\mathcal{P}_j^+$ (resp. $\mathcal{P}_j^-$) with weight $\pm q^{j-1}$. Let x_j^+ (resp. x_j^-) be the nilpotent endomorphism of $\mathcal{P}_j^\pm$ determined by $\mathbf{b}_0^+ \mapsto \mathbf{a}_0^+$ (resp. $\mathbf{y}_0^- \mapsto \mathbf{x}_0^-$), and let $a_j^+, b_j^+ : \mathcal{P}_j^+ \to \mathcal{P}_{\mathsf{p}-j}^-$ and $a_j^-, b_j^- : \mathcal{P}_j^- \to \mathcal{P}_{\mathsf{p}-j}^+$ be the morphisms defined by

$$a_j^+(\mathbf{b}_0^+) = \mathbf{a}_0^-, \;\; b_j^+(\mathbf{b}_0^+) = \mathbf{b}_0^-, \;\; a_j^-(\mathbf{y}_0^-) = \mathbf{x}_0^+ \text{ and } b_j^-(\mathbf{y}_0^-) = \mathbf{y}_0^+,$$

respectively. Analysing the images of the dominant weight vector of $\mathcal{P}_s^\epsilon$, we can completely determine the Hom-spaces between indecomposable projective modules. Here is the list of the non-trivial ones:

- the endomorphism ring $\mathrm{End}_U(\mathcal{P}_j^\pm)$ is one dimensional for $j = p$ and two dimensional with basis $\{\mathrm{id}_{\mathcal{P}_j^\pm}, x_j^\pm\}$ for $1 \le j < \mathsf{p}$,
- the Hom-spaces $\mathrm{Hom}_U(\mathcal{P}_j^+, \mathcal{P}_{\mathsf{p}-j}^-)$ and $\mathrm{Hom}_U(\mathcal{P}_j^-, \mathcal{P}_{\mathsf{p}-j}^+))$ are two dimensional with respective basis $\{a_j^+, b_j^+\}$ and $\{a_j^-, b_j^-\}$, for $1 \le j < \mathsf{p}$.

Proposition 5 (Proposition 4.4.4 of [8])**.** *The action of the center on the indecomposable projective modules is as follows.*

	$\mathcal{P}_p^-$	$\mathcal{P}_p^+$	$\mathcal{P}_j^+,\ 1 \le j < p$	$\mathcal{P}_{p-j}^-,\ 1 \le j < p$
$\boldsymbol{e}_0$	$\mathrm{id}_{\mathcal{P}_p^-}$	0	0	0
$\boldsymbol{e}_p$	0	$\mathrm{id}_{\mathcal{P}_p^+}$	0	0
$\boldsymbol{e}_j,\ 1 \le j < p$	0	0	$\mathrm{id}_{\mathcal{P}_j^+}$	$\mathrm{id}_{\mathcal{P}_{p-j}^-}$
$\boldsymbol{w}_j^+,\ 1 \le j < p$	0	0	x_j^+	0
$\boldsymbol{w}_j^-,\ 1 \le j < p$	0	0	0	x_{p-j}^-

4. Tangle invariants and the trace of U

Our links and tangles are always assumed to be framed and oriented. The diagrams are going from bottom to top. A *string link* is a tangle without closed component whose arcs end at the same order as they start, with upwards orientation. A pure braid is an example.

Reshetikhin–Turaev invariant. Given any ribbon category $\mathcal{C}$, Turaev showed in [24], that there exists a canonical ribbon functor

$$F_\mathcal{C} : \mathrm{Rib}_\mathcal{C} \to \mathcal{C},$$

where $\mathrm{Rib}_\mathcal{C}$ is the category of $\mathcal{C}$-colored ribbon graphs. Applying this construction to D-mod, for an m-component string link T, colored with the regular representation D, we obtain $F_D(T) \in \mathrm{End}_D(D^{\otimes m})$. Here we use shorthand F_D for $F_{D\text{-mod}}$.

The Reshetikhin–Turaev invariant of the colored link $(L, V_1, \dots, V_m)$ is obtained by evaluating the categorical right traces on $F_D(T)$, i.e.

$$\begin{aligned} J_L(V_1, \dots, V_m) :&= (\mathrm{tr}_r^{V_1} \otimes \dots \mathrm{tr}_r^{V_m}) F_D(T) \\ &= (\mathrm{trace}^{V_1} \otimes \dots \otimes \mathrm{trace}^{V_m})(g \otimes \dots \otimes g) F_D(T) \end{aligned}$$

where T is a string link with braid closure isotopic to L. Recall also that $\mathrm{tr}_r^V = \mathrm{tr}_l^V$ since both closures are isotopic. Hence, we can replace g by g^{-1} in the last line. If one of the V_i's is projective, this invariant vanishes.

Universal invariant. Associated with a ribbon Hopf algebra there is another powerful invariant – the *universal* invariant of links and tangles introduced by Lawrence [17] for some quantum groups and defined by Hennings [13] in the general case. For a string link T with m components, its universal invariant J_T is obtained by pasting together pieces shown in Figure 1. Here we write $R = \sum \alpha \otimes \beta$, $R^{-1} = \sum \overline{\alpha} \otimes \overline{\beta}$.

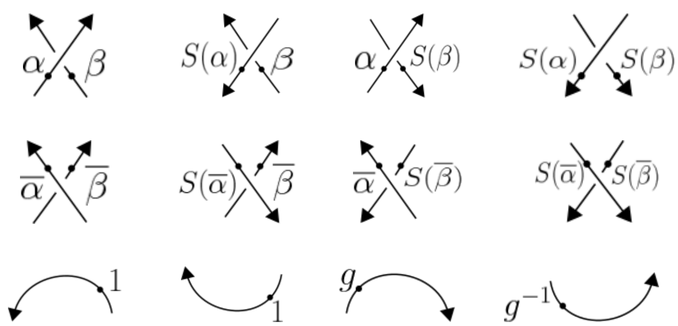


FIGURE 1. Local formulas for the universal invariant

More precisely, for each arc of T we obtain an element of D by writing a word from right to left with labels read using the order following the orientation. Thus we get $J_T \in D^{\otimes m}$. This element does not change by Reidemeister moves. The original proof of the invariance was stated for a link. The argument was extended to tangles in [12, 7.3].

Note that in [12] Habiro uses different conventions. His tangles are depicted from top to bottom and orientations are reversed. Hence, our model can be recovered from his one after reflecting over a horizontal axis. The universal invariants coincide. In [22], Ohtsuki defines the universal invariant using the opposite to our orientation convention, but the word is written there from left to right when following the orientation. Again the universal invariant is finally the same as ours.

Relation between them. The universal invariant is known to dominate Reshetikhin–Turaev invariants in the following sense:

$$J_L(V_1, \dots, V_m) = (\mathrm{tr}_r^{V_1} \otimes \cdots \otimes \mathrm{tr}_r^{V_m}) J_T$$

The proof is given in [22, Theorem 4.9].

Let us denote by $(U^{\otimes m})^U \subset U^{\otimes m}$ the submodule centralising the left action, i.e.

$$x \in (U^{\otimes m})^U \quad \Leftrightarrow \quad \Delta^{[m]}(h)x = x\Delta^{[m]}(h) \quad \text{for all} \quad h \in U\ .$$

The following lemma is folklore, but we are adding the proof for completeness.

Lemma 6. *The Reshetikhin–Turaev intertwinner $F_D(T)$ is equal to left multiplication by J_T. In addition, for an m-component string link T, J_T belongs to $(U^{\otimes m})^U$.*

Proof. The fact that $F_D(T)$ is the left multiplication by J_T follows directly by comparing the definitions of these two invariants. More details are given in the proof of [22, Theorem 4.9]. Hence, multiplication by J_T has to commute with left action, we conclude $J_T \in (D^{\otimes m})^D$.

Let us show that for a string link the universal invariant J_T actually belongs to $U^{\otimes m}$. This implies the claim, since $(U^{\otimes m})^D \subset (U^{\otimes m})^U$.

The linking matrix of a string link is diagonal mod 2. In [2, Section 1.3], Bruguières shows that *any* tangle with this property can be obtained as compositions and tensor products of evaluations, coevaluations and twists. Thus the universal invariant J_T is build up from the ribbon element r, the balancing element g, and their inverses by applying the Hopf algebra operations. We obtain $J_T \in U^{\otimes m}$. $\square$

Evaluations of the universal invariant. Assume for simplicity that T is a $(1,1)$ tangle, whose closure is the knot K. Then for any $\phi \in \mathrm{qChar}(U)$, the evaluation $\phi(J_T) \in \mathbb{C}$ is a knot invariant. To prove this fact, we need to show that this evaluation does not change by cyclic permutations of the word $J_K = g^{-1}J_T$ obtained by applying the algorithm described above to the left closure of T. Using (3), we get

$$\phi(J_T) = \phi_g(g^{-1}J_T) = \phi_g(J_K).$$

The last expression does not change by cyclic permutations since $\phi_g \in \mathrm{Char}(U)$. Applying this argument to the k leftmost components of a string link with m strands, we will get the following.

Lemma 7. *Let T be an m-component string link and $1 \le k \le m$. Let $\Phi = \otimes_{j=1}^k \phi_j$ with $\phi_j \in \mathrm{qChar}(U)$ be a sequence of quantum characters, then*

$$(\Phi \otimes \mathrm{id}_{U^{\otimes(m-k)}})(J_T) \in (U^{\otimes(m-k)})^U$$

is invariant of the tangle obtained from T by closing the first k components.

Further observe that given $\phi \in \mathrm{qChar}(U)$ we can obtain another quantum character ϕ_z by twisting ϕ with a central element $z \in Z(U)$, where $\phi_z(x) := \phi(zx)$ for any $x \in U$. In the logaritmic invariant we will use quantum characters μ and μ_z to evaluate components of T^0 and T^+, and the *modified trace pairing* constructed below for T^-.

In the next section we will define a family of linear functions

$$\{\mathsf{t}_V : \mathrm{End}_{\mathscr{C}}(V) \to \mathbb{C}\}_{V \in U\text{-pmod}}$$

satisfying cyclicity and the partial trace property. For any $m > 0$, we can use $\mathsf{t}_{U^{\otimes m}}$ to define a bilinear pairing

$$\langle\ ,\ \rangle : \left(U^{\otimes m}\right)^U \otimes U^{\otimes m} \to \mathbb{C}\ ,$$

by the formula

$$\langle x, y\rangle = \mathsf{t}_{U^{\otimes m}}(l_x \circ r_y)\ . \tag{7}$$

Here l_x, r_x are the operators of the left and right multiplication by x.

From cyclicity of t, we obtain an induced pairing

$$\langle\ ,\ \rangle : \left(U^{\otimes m}\right)^U \otimes \mathrm{HH}_0(U^{\otimes m}) \to \mathbb{C}\ ,$$

which we call the modified trace pairing. To achieve the full evaluation, besides of the basis for the center, we will need a basis for $\mathrm{HH}_0(U)$.

The trace $\mathrm{HH}_0(U)$. Let us construct a basis of the trace of U.

Recall that 0^{th}-Hochschild homology or trace of a linear category $\mathcal{C}$ is defined by

$$\mathrm{HH}_0(\mathcal{C}) := \frac{\oplus_{x \in \mathcal{C}}\ \mathcal{C}(x,x)}{fg - gf} \quad \text{for any} \quad f : x \to y, g : y \to x.$$

The image of $x \in \mathcal{C}$ in $\mathrm{HH}_0(\mathcal{C})$ will be called its trace class and denoted by $[x]$. For an algebra A (i.e. a category with one object) this reduces to

$$\mathrm{HH}_0(A) := \frac{A}{[A,A]} \quad \text{with} \quad [A,A] = \mathrm{Span}\{xy - yx \mid x, y \in A\}.$$

This space supports a natural action of the center defined by $z[x] = [zx]$ for any $z \in Z(U)$. By definition $\mathrm{HH}_0(U)$ is dual to $\mathrm{Char}(U)$.

We will use the following well-known fact.

Proposition 8. *For any finite dimensional algebra U,*

$$\mathrm{HH}_0(U\text{-pmod}) \simeq \mathrm{HH}_0(U)$$

Let us give a proof for completeness.

Proof. Assume M is projective, then there exists another projective module N such that $M \oplus N = U^{\otimes n}$. Hence in U-pmod there are morphisms

$$i : M \to U^{\otimes n} \quad \text{and} \quad p : U^{\otimes n} \to M$$

with $p \circ i = \mathrm{id}_M$ and $i \circ p$ an idempotent. They can be used to define a map $\mathrm{HH}_0(U\text{-pmod}) \to \mathrm{HH}_0(U)$ as follows: Given $f \in \mathrm{End}(M)$, then

$$[f] = [fpi] = [ifp] \in \mathrm{HH}_0(U^{\otimes n}) \simeq \mathrm{HH}_0(U).$$

The last equivalence in proven in [18]. The inverse of this map sends $x \in \mathrm{HH}_0(U)$ to the right multiplication r_x in $\mathrm{End}_U(U)$. Hence, we have an isomorphism. □

Let us recall that U as a free left U-module decomposes into a direct sum of indecomposable projectives as follows

$$U \cong \bigoplus_{j=1}^{\mathsf{p}} j\mathcal{P}_j^+ \oplus \bigoplus_{j=1}^{\mathsf{p}} j\mathcal{P}_j^- .$$

The module $\mathbb{P} = \oplus_{j,\pm} \mathcal{P}_j^{\pm}$ is called projective generator of U and $B = \mathrm{End}(\mathbb{P})$ the *basic* algebra. By definition,

$$\mathrm{HH}_0(U\text{-pmod}) = \mathrm{HH}_0(B).$$

Remark. By [7, Th. 8.4.5], any finite dimensional algebra A and its basic algebra B are Morita equivalent. The Morita equivalence between A and B also implies

$$\mathrm{HH}_0(A\text{-mod}) \simeq \mathrm{HH}_0(B\text{-mod})$$

but this group could be different from $\mathrm{HH}_0(A)$.

Let us use the notation $\mathrm{id}_j^{\pm} = \mathrm{id}_{\mathcal{P}_j^{\pm}}$ for simplicity.

Lemma 9. *A basis for* $\mathrm{HH}_0(B)$ *is represented by* $[\mathrm{id}_k^{\pm}]$, $1 \leq k \leq \mathsf{p}$, *and* $[x_j^+] = [x_{\mathsf{p}-j}^-]$, $1 \leq j < \mathsf{p}$.

Proof. Recall that

$$B = \bigoplus_{j,\epsilon,j',\epsilon'} \mathrm{Hom}(\mathcal{P}_j^{\epsilon}, \mathcal{P}_{j'}^{\epsilon'}) .$$

A linear basis for B defined in Section 3 consist of

$$\mathrm{id}_k^{\pm}, \quad x_j^{\epsilon}, \quad a_j^{\epsilon} : \mathcal{P}_j^{\epsilon} \to \mathcal{P}_{\mathsf{p}-j}^{-\epsilon}, \quad \text{and} \quad b_j^{\epsilon} : \mathcal{P}_j^{\epsilon} \to \mathcal{P}_{\mathsf{p}-j}^{-\epsilon}$$

where $1 \leq k \leq \mathsf{p}$, $1 \leq j < \mathsf{p}$ and $\epsilon \in \{-,+\}$. For $1 \leq j < \mathsf{p}$, we have $a_j^{\epsilon}\, \mathrm{id}_j^{\epsilon} = a_j^{\epsilon}$, while $\mathrm{id}_j^{\epsilon}\, a_j^{\epsilon} = 0$, so that a_j^{ϵ} and similarly b_j^{ϵ} vanish in $\mathrm{HH}_0(B)$. We also have $b_{\mathsf{p}-j}^{-\epsilon} a_j^{\epsilon} = x_j^{\epsilon}$, while $a_j^{\epsilon} b_{\mathsf{p}-j}^{-\epsilon} = x_{\mathsf{p}-j}^{-\epsilon}$. We get the relation $[x_j^\epsilon] = [x_{\mathsf{p}-j}^{-\epsilon}] \in \mathrm{HH}_0(B)$. Since the resulting set of generators has expected cardinality, this completes the proof of the lemma. $\square$

Combining Proposition 8 with Lemma 9, we conclude that $\mathrm{HH}_0(U)$ has dimension $3p-1$, with basis consisting of

- $\mathbf{h}_k^\pm$, for $1 \le k \le \mathsf{p}$, represented by the minimal (non central) idempotent projecting onto a copy of the module $\mathcal{P}_k^\pm$, and
- $\mathbf{h}_j = \mathbf{w}_j^+ \mathbf{h}_j^+ = \mathbf{w}_j^- \mathbf{h}_{p-j}^-$, for $1 \le j \le \mathsf{p}-1$.

On the regular representation in U-pmod these elements act by the right multiplication.

5. Proofs of Theorems 1 and 2

In this section we construct the modified trace on U-pmod and compute it for the regular representation U and its tensor powers. This will provide the main tool to prove Theorems 1 and 2.

Modified trace. The subcategory U-pmod is an ideal of U-mod in the sense of [10, 11], which means the following:

a) If $V \in U$-pmod and $W \in U$-mod, then $V \otimes W \in U$-pmod and $V \otimes W \in U$-pmod.
b) If $V \in U$-pmod, $W \in U$-mod, and there exists morphisms $f : W \to V$, $g : V \to W$ such that $gf = \mathrm{id}_W$, then $W \in U$-pmod.

Let us recall that a *modified trace* on U-pmod is a family of linear functions
$$\{\mathsf{t}_V : \mathrm{End}_{\mathscr{C}}(V) \to \mathbb{C}\}_{V \in U\text{-pmod}}$$
such that the following two conditions hold:

(1) **Cyclicity.** If $X, V \in U$-pmod, then for any morphisms $f : V \to X$ and $g : X \to V$ in U-mod we have
$$\mathsf{t}_V(gf) = \mathsf{t}_X(fg).$$
(2) **Partial trace properties.** If $X \in U$-pmod and $W \in U$-mod then for any $f \in \mathrm{End}_U(X \otimes W)$ and $g \in \mathrm{End}_U(W \otimes X)$ we have
$$\mathsf{t}_{X\otimes W}(f) = \mathsf{t}_X\left(\mathrm{tr}_r^W(f)\right),$$
$$\mathsf{t}_{W\otimes X}(g) = \mathsf{t}_X\left(\mathrm{tr}_l^W(g)\right)$$
where tr_r^W and tr_l^W are the right and left partial categorical traces along W defined by (5).

If only the first (resp. the second) of the two partial trace properties is satisfied, we call the modified trace right (resp. left).

Proof of Theorem 1. Corollary 3.2.1 of [10] implies the existence of a unique (up to global scalar) right modified trace in any unimodular pivotal category with enough projectives and a simple projective object L, such that $\widetilde{\mathrm{ev}}_L$ is surjective. The category U-pmod does satisfy all these assumptions. Hence there exists a unique right modified trace

$$\{\mathsf{t}^R_V : \mathrm{End}(V) \to \mathbb{C}\}_{V\in U\text{-pmod}}$$

normalized by

$$\mathsf{t}^R_{\mathcal{P}^+_{\mathsf{p}}}(\mathrm{id}_{\mathcal{P}^+_{\mathsf{p}}}) = (-1)^{\mathsf{p}-1}.$$

Analogous arguments imply the existence of the unique left trace

$$\{\mathsf{t}^L_V : \mathrm{End}(V) \to \mathbb{C}\}_{V\in U\text{-pmod}}$$

normalized by

$$\mathsf{t}^L_{\mathcal{P}^+_{\mathsf{p}}}(\mathrm{id}_{\mathcal{P}^+_{\mathsf{p}}}) = (-1)^{\mathsf{p}-1}.$$

We will compute both of them and show that they coincide. For this, we will use additivity of trace functions for direct sums, which follows from the cyclicity. Hence, it is enough to compute modified traces on the endomorphisms of the indecomposable projectives.

Identity endomorphisms. Recall $\mathscr{X}_1^+$ is the one dimensional trivial U-module whose action on any vector v is given by $Ev = Fv = 0$ and $K^{\pm 1}v = v$. There is another one dimensional module $\mathscr{X}_1^-$ whose action on any vector v is given by $Ev = Fv = 0$ and $K^{\pm 1}v = -v$. Hence, $\mathrm{tr}_r(\mathrm{id}_{\mathscr{X}_1^-}) = \mathrm{trace}^{\mathscr{X}_1^-}(K^{p+1}) = (-1)^{p+1}$. Using $\mathcal{P}^-_j \cong \mathcal{P}^+_j \otimes \mathscr{X}_1^-$, we can compute the trace on $\mathcal{P}^-_{\mathsf{p}}$:

$$\mathsf{t}^R_{\mathcal{P}^-_{\mathsf{p}}}(\mathrm{id}_{\mathcal{P}^-_{\mathsf{p}}}) = \mathsf{t}^R_{\mathcal{P}^+_{\mathsf{p}}\otimes\mathscr{X}_1^-}(\mathrm{id}_{\mathcal{P}^+_{\mathsf{p}}\otimes\mathscr{X}_1^-}) = \mathsf{t}^R_{\mathcal{P}^+_{\mathsf{p}}}\left(\mathrm{tr}_r^{\mathscr{X}_1^-}(\mathrm{id}_{\mathcal{P}^+_{\mathsf{p}}\otimes\mathscr{X}_1^-})\right) = 1.$$

Similarly, $\mathcal{P}^-_{\mathsf{p}} \cong \mathscr{X}_1^- \otimes \mathcal{P}^+_p$ implies that $\mathsf{t}^L_{\mathcal{P}^-_{\mathsf{p}}}(\mathrm{id}_{\mathcal{P}^-_{\mathsf{p}}}) = 1$.

Next we compute some partial traces involving $\mathscr{X}_2^+$. The module $\mathscr{X}_2^+$ has a basis $\{w_0, w_1\}$ with action

$$\begin{aligned} &Ew_0 = 0 && Fw_0 = w_1 && Kw_0 = qw_0\\ &Ew_0 = w_1 && Fw_1 = w_0 && Kw_1 = q^{-1}w_1. \end{aligned}$$

Recall that an endomorphism of $\mathcal{P}^+_j$ is determined by the image of the dominant vector $\mathbf{b}^+_0$. We can use this fact to compute the following partial trace:

$$\left(\mathrm{tr}_l^{\mathscr{X}_2^+}(\mathrm{id}_{\mathscr{X}_2^+\otimes\mathcal{P}^+_j})\right)(\mathbf{b}^+_0) = \\ = w_0^*(K^{\mathsf{p}-1}w_0)\mathbf{b}^+_0 + w_1^*(K^{\mathsf{p}-1}w_1)\mathbf{b}^+_0 = (q^{\mathsf{p}-1} + q^{-\mathsf{p}+1})\mathbf{b}^+_0\ .$$

Thus, $\mathrm{tr}_l^{\mathscr{X}_2^+}(\mathrm{id}_{\mathscr{X}_2^+\otimes\mathcal{P}_j^+}) = -(q+q^{-1})\,\mathrm{id}_{\mathcal{P}_j^+}$ for $j=1,...,\mathsf{p}-1$. Similarly, for the right partial trace of the identity of $\mathcal{P}_j^+\otimes\mathscr{X}_2^+$ we get

$$\mathrm{tr}_r^{\mathscr{X}_2^+}(\mathrm{id}_{\mathcal{P}_j^+\otimes\mathscr{X}_2^+}) = (q^{-\mathsf{p}+1}+q^{\mathsf{p}-1})\,\mathrm{id}_{\mathcal{P}_j^+} = (-q-q^{-1})\,\mathrm{id}_{\mathcal{P}_j^+}\,.$$

The decomposition of tensor products of a simple module with a projective indecomposable module is given in [23, proposition 4.1] (also see [16, Theorems 3.1.5, 3.2.1]). In particular,

$$\mathcal{P}_{\mathsf{p}}^{\pm}\otimes\mathscr{X}_2^+ \cong \mathcal{P}_{p-1}^{\pm} \cong \mathscr{X}_2^+\otimes\mathcal{P}_{\mathsf{p}}^{\pm}\,, \tag{8}$$

$$\mathcal{P}_{\mathsf{p}-1}^{\pm}\otimes\mathscr{X}_2^+ \cong \mathcal{P}_{\mathsf{p}-2}^{\pm}\oplus 2\mathcal{P}_{\mathsf{p}}^{\pm} \cong \mathscr{X}_2^+\otimes\mathcal{P}_{\mathsf{p}-1}^{\pm}\,, \tag{9}$$

and

$$\mathcal{P}_{j}^{\pm}\otimes\mathscr{X}_2^+ \cong \mathcal{P}_{j-1}^{\pm}\oplus\mathcal{P}_{j+1}^{\pm} \cong \mathscr{X}_2^+\otimes\mathcal{P}_{j}^{\pm} \tag{10}$$

for $j\in\{2,...,\mathsf{p}-2\}$.

Combining these formulas with properties of the modified trace we have:

$$\mathsf{t}^R_{\mathcal{P}_j^{\pm}\otimes\mathscr{X}_2^+}\left(\mathrm{id}_{\mathcal{P}_j^+\otimes\mathscr{X}_2^+}\right) = \mathsf{t}^R_{\mathcal{P}_j^{\pm}}\left(\mathrm{tr}_r^{\mathscr{X}_2^+}(\mathrm{id}_{\mathcal{P}_j^+\otimes\mathscr{X}_2^+})\right) = (-q-q^{-1})\mathsf{t}^R_{\mathcal{P}_j^{\pm}}\left(\mathrm{id}_{\mathcal{P}_j^+}\right)$$

where $j\in\{2,...,\mathsf{p}\}$. This equality together with the isomorphism on the left hand side of Equation (8) for $j=\mathsf{p}$ gives

$$\mathsf{t}^R_{\mathcal{P}^{\pm}_{\mathsf{p}-1}}(\mathrm{id}_{\mathcal{P}^{\pm}_{\mathsf{p}-1}}) = (\mp 1)^{\mathsf{p}-1}(-q-q^{-1}).$$

Then the isomorphism of the left hand side of (9) implies

$$\mathsf{t}^R_{\mathcal{P}^{\pm}_{\mathsf{p}-2}}(\mathrm{id}_{\mathcal{P}^{\pm}_{\mathsf{p}-2}}) = (\mp 1)^{\mathsf{p}-1}((-q-q^{-1})^2-2) = (\mp 1)^{\mathsf{p}-1}(q^2+q^{-2}).$$

Finally, for $j\in\{2,...,\mathsf{p}-2\}$, the isomorphism of the left hand side of Equation (10) implies

$$-(q+q^{-1})\mathsf{t}^R_{\mathcal{P}^{\pm}_{j}}(\mathrm{id}_{\mathcal{P}^{\pm}_{j}}) = \mathsf{t}^R_{\mathcal{P}^{\pm}_{j-1}}(\mathrm{id}_{\mathcal{P}^{\pm}_{j-1}}) + \mathsf{t}^R_{\mathcal{P}^{\pm}_{j+1}}(\mathrm{id}_{\mathcal{P}^{\pm}_{j+1}})\,. \tag{11}$$

Recursively the last equality implies

$$\mathsf{t}^R_{\mathcal{P}^{\pm}_{j}}(\mathrm{id}_{\mathcal{P}^{\pm}_{j}}) = (\pm 1)^{p-1}(-1)^j(q^j+q^{-j}) \tag{12}$$

for $j\in\{2,...,\mathsf{p}-1\}$.

Using the right hand side of the tensor product in (8),(9),(10) we compute similarly the left trace of identities and get

$$\mathsf{t}^L_{\mathcal{P}^{\pm}_{j}}(\mathrm{id}_{\mathcal{P}^{\pm}_{j}}) = \mathsf{t}^R_{\mathcal{P}^{\pm}_{j}}(\mathrm{id}_{\mathcal{P}^{\pm}_{j}}),\ \text{for } 1\le j\le \mathsf{p}.$$

Endomorphisms $x_j^\pm$. For $x \in U$, let us denote by l_x^V the operator of the left multiplication by x on V. Sometimes, we will omit V for simplicity.

To compute $\mathsf{t}^R_{\mathcal{P}_j^\pm}(x_j^\pm)$ we will use the action of the Casimir element

$$C = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}.$$

Since C is central, $l_C^{\mathcal{P}}$ commutes with the left U-action, and hence defines an endomorphism in U-pmod.

On simple modules, C acts by a scalar, hence

$$\mathsf{t}^R_{\mathcal{P}_\mathsf{p}^\pm}(l_C^{\mathcal{P}_\mathsf{p}^\pm}) = \frac{2(\mp 1)^\mathsf{p}}{(q-q^{-1})^2}. \tag{13}$$

For $j \in \{1, \dots, \mathsf{p}-1\}$, the dominant vector $\mathbf{b}_0^+$ (resp. $\mathbf{y}_0^-$) of $\mathcal{P}_j^+$ (resp. $\mathcal{P}_j^-$) has weight $\pm q^{j-1}$. The action of C on this vector is

$$C\mathbf{b}_0^+ = \mathbf{a}_0^+ + \frac{q^j + q^{-j}}{(q-q^{-1})^2}\mathbf{b}_0^+$$

(resp. $C\mathbf{y}_0^- = \mathbf{x}_0^- - \frac{q^j+q^{-j}}{(q-q^{-1})^2}\mathbf{y}_0^-$). Thus, for $j \in \{1, \dots, \mathsf{p}-1\}$ we have

$$l_C^{\mathcal{P}_j^\pm} = x_j^\pm \pm \frac{q^j + q^{-j}}{(q-q^{-1})^2}\,\mathrm{id}_{\mathcal{P}_j^\pm}\ . \tag{14}$$

To compute the action of C on tensor products we need the following formula.

$$\begin{aligned}\Delta(C) = {} & K^{-1} \otimes FE + K^{-1}E \otimes FK \\ & + F \otimes E + FE \otimes K + \frac{qK \otimes K + q^{-1}K^{-1} \otimes K^{-1}}{(q-q^{-1})^2}\ .\end{aligned}$$

The second and third terms of $\Delta(C)$ have no diagonal contibution, and hence vanish when computing the partial trace $\mathrm{tr}_r^{\mathscr{X}_2^+}\left(l_{\Delta(C)}^{\mathcal{P}_i^\pm \otimes \mathscr{X}_2^+}\right)$. In particular, for $v \in \mathcal{P}_i^\pm$ we have

$$\begin{aligned}&\left[\mathrm{tr}_r^{\mathscr{X}_2^+}\left(l_{\Delta(C)}^{\mathcal{P}_i^\pm \otimes \mathscr{X}_2^+}\right)\right](v) \\ &= -q^{-1}K^{-1}v - (q^2+q^{-2})FEv - \frac{(q^2+q^{-2})q}{(q-q^{-1})^2}Kv + \frac{-2q^{-1}}{(q-q^{-1})^2}K^{-1}v.\end{aligned}$$

When $i \in \{1, ..., \mathsf{p}-1\}$ and v is the generating vector of $\mathcal{P}_i^\pm$ this equality implies

$$\mathrm{tr}_r^{\mathscr{X}_2^+}\left(l_{\Delta(C)}^{\mathcal{P}_i^\pm \otimes \mathscr{X}_2^+}\right) = -(q^2+q^{-2})x_i^\pm \mp \frac{(q^2+q^{-2})}{(q-q^{-1})^2}(q^i + q^{-i})\,\mathrm{id}_{\mathcal{P}_i^\pm}\ .$$

Similarly, if v is the highest weight of $\mathcal{P}^\pm_{\mathsf{p}}$, we obtain

$$\operatorname{tr}_r^{\mathscr{X}_2^+}\left(l_{\Delta(C)}^{\mathcal{P}^\pm_{\mathsf{p}}\otimes\mathscr{X}_2^+}\right) = \pm 2\frac{(q^2+q^{-2})}{(q-q^{-1})^2}\operatorname{id}_{\mathcal{P}^\pm_{\mathsf{p}}}. \tag{15}$$

Now we can compute $\mathsf{t}^R_{\mathcal{P}^\pm_{\mathsf{p}-1}}(x^\pm_{\mathsf{p}-1})$. From the isomorphism in (8), we have

$$\mathsf{t}^R_{\mathcal{P}^\pm_{\mathsf{p}}\otimes\mathscr{X}_2^+}\left(l_{\Delta(C)}^{\mathcal{P}^\pm_{\mathsf{p}}\otimes\mathscr{X}_2^+}\right) = \mathsf{t}^R_{\mathcal{P}^\pm_{\mathsf{p}-1}}\left(l_C^{\mathcal{P}^\pm_{\mathsf{p}-1}}\right).$$

Using Equations (14) and (15) we can simplify the last equality as follows

$$\pm 2\frac{q^2+q^{-2}}{(q-q^{-1})^2}\mathsf{t}^R_{\mathcal{P}^\pm_{\mathsf{p}}}(\operatorname{id}^\pm_{\mathcal{P}^\pm_{\mathsf{p}}}) = \mathsf{t}^R_{\mathcal{P}^\pm_{\mathsf{p}-1}}(x^\pm_{\mathsf{p}-1}) \pm \frac{q^{\mathsf{p}-1}+q^{-\mathsf{p}+1}}{(q-q^{-1})^2}\mathsf{t}^R_{\mathcal{P}^\pm_{\mathsf{p}-1}}(\operatorname{id}^\pm_{\mathcal{P}^\pm_{\mathsf{p}-1}}),$$

or

$$\mathsf{t}^R_{\mathcal{P}^\pm_{\mathsf{p}-1}}(x^\pm_{\mathsf{p}-1}) = \pm(\mp)^{\mathsf{p}-1}\frac{(q^{\mathsf{p}-1}-q^{-\mathsf{p}+1})^2}{(q-q^{-1})^2}.$$

Using the isomorphism in Equation (10), for $j\in\{2,\dots,\mathsf{p}-2\}$ we obtain the following recursive relation

$$\mathsf{t}^R_{\mathcal{P}^\pm_{j-1}}(x^\pm_{j-1}) + \mathsf{t}^R_{\mathcal{P}^\pm_{j+1}}(x^\pm_{j+1}) + (q^2+q^{-2})\mathsf{t}^R_{\mathcal{P}^\pm_{j}}(x^\pm_{j}) = -2(\pm 1)^{\mathsf{p}}(-1)^j. \tag{16}$$

Using Equation (9) we can show that this formula also holds for $j=\mathsf{p}-1$ by setting $x^\pm_{\mathsf{p}}=0$. We deduce the general formula for $j\in\{1,\dots,\mathsf{p}-1\}$

$$\mathsf{t}^R_{\mathcal{P}^\pm_{j}}(x^\pm_{j}) = (\pm 1)^{\mathsf{p}}(-1)^j[j]^2,$$

which is compatible with the computation at $j=\mathsf{p}-2$ or $j=\mathsf{p}-1$ and satisfies the recursive relation for $j\in\{2,\dots,\mathsf{p}-2\}$.

With similar computation we get the same value for $\mathsf{t}^L_{\mathcal{P}^\pm_{j}}(x^\pm_{j})$. Thus, we have proved that the left and right modified traces are equal on U-pmod. Let us summarize our computations of the modified trace ($0<j<\mathsf{p}$):

$\operatorname{id}_{\mathcal{P}^-_{\mathsf{p}}}$	$\operatorname{id}_{\mathcal{P}^+_{\mathsf{p}}}$	$\operatorname{id}_{\mathcal{P}^-_{j}}$	$\operatorname{id}_{\mathcal{P}^+_{j}}$	x^-_j	x^+_j
1	$(-1)^{\mathsf{p}-1}$	$(-1)^{\mathsf{p}+j-1}(q^j+q^{-j})$	$(-1)^j(q^j+q^{-j})$	$(-1)^{\mathsf{p}+j}[j]^2$	$(-1)^j[j]^2$

.

Proof of Theorem 2. Let us apply the modified trace construction to the regular representation $U\in U$-pmod. Using the isomorphism of algebras

$$r: U^{op}\cong \operatorname{End}_U(U),\quad x\mapsto r_x$$

where $r_x(y)=yx$ is the right multiplication, we define

$$\operatorname{Tr}': U\to\mathbb{C} \quad\text{by}\quad \operatorname{Tr}'(x)=\mathsf{t}_U(r_x).$$

By Theorem 1, the linear map Tr' is a character and satisfies the partial trace property.

The fact that the pairing between

$$Z(U) \times \mathrm{HH}_0(U) \to \mathbb{C} \quad \text{given by} \quad (z, x) \mapsto \mathrm{Tr}'(zx)$$

is non-degenerate can now be shown by a direct computation. Using Proposition 5 and Theorem 1 we can explicitly compute this pairing in the base of the center and the trace. For example

$$\mathrm{Tr}'\left(\mathbf{w}_j^+ \mathbf{h}_j^+\right) = \mathsf{t}^R_{\mathcal{P}_j^+}\left(x_j^+\right) = (-1)^j [j]^2$$

or

$$\mathrm{Tr}'(\mathbf{e}_0 \mathbf{h}_{\mathsf{p}}^-) = \mathsf{t}^R_{\mathcal{P}_{\mathsf{p}}^-}(\mathrm{id}_{\mathcal{P}_{\mathsf{p}}^-}) = 1.$$

Completing the computation, we obtain the pairing shown in Table 1. From the table it is easy to see it is non-degenerate.

Table 1. Values of the pairing on a basis, here $1 \leq j < \mathsf{p}$.

	$\mathbf{h}_{\mathsf{p}}^+$	$\mathbf{h}_{\mathsf{p}}^-$	$\mathbf{h}_s$	$\mathbf{h}_s^+$	$\mathbf{h}_{\mathsf{p}-s}^-$
$\mathbf{e}_{\mathsf{p}}$	$(-1)^{\mathsf{p}-1}$	0	0	0	0
$\mathbf{e}_0$	0	1	0	0	0
$\mathbf{e}_j$	0	0	$(-1)^j[j]^2$	$(-1)^j(q^j + q^{-j})$	$(-1)^j(q^j + q^{-j})$
$\mathbf{w}_j^+$	0	0	0	$(-1)^j[j]^2$	0
$\mathbf{w}_j^-$	0	0	0	0	$(-1)^j[j]^2$

6. Proof of Theorem 3

In this section we will define our logarithmic 3-manifold invariant $\mathrm{H}^{\log}(M, L)$, prove Theorem 3 and compare $\mathrm{H}^{\log}(M, L)$ with the invariant defined by Jun Murakami in [21].

Logarithmic invariant. Assume we are given a link (L^+, L^-) with (m_+, m_-) components inside a 3-manifold $M = S^3(L^0)$, where $L^0 \subset S^3$ is a surgery link for M with m_0 components. We suppose that (L^+, L^-) is in $S^3 \setminus L^0$, and choose a string link $T = (T^+, T^0, T^-)$ whose closure is (L^+, L^0, L^-). By Lemma 6, the universal invariant

$$J_T \in \left(U^{\otimes(m_+ + m_0 + m_-)}\right)^U.$$

Let us color the components of L^+ and L^- by central elements $z_j \in Z(U)$, $1 \leq j \leq m_+$ and by trace classes $h_k \in \mathrm{HH}_0(U)$, $1 \leq k \leq m_-$, respectively, and write $z^+ = \otimes_j z_j$, $h^- = \otimes_k h_k$. Let us denote by L the resulting colored link $((L^+, z^+), (L^-, h^-))$. We obtain

$$\mathrm{H}^{\log}(M, L) := \delta^s \langle \left(z^+ \mu^{\otimes m_+} \otimes \mu^{\otimes m_0} \otimes \mathrm{id}\right)(J_T)\,,\, h^- \,\rangle$$

by evaluating components of J_T corresponding to (L^+, L^0) with the right integral μ twisted with central elements and by applying to the result the modified trace pairing. Here s is the signature of the linking matrix for L^0. Theorem 3 claims that $\mathrm{H}^{\log}(M, L)$ is a topological invariant of the pair (M, L).

Proof of Theorem 3. We first show that $\mathrm{H}^{\log}(M, L)$ is an invariant of the colored link L, i.e. it does not depend on the choice of T. By applying Lemma 7 we see that $\mathrm{H}^{\log}(M, L)$ is invariant of the tangle T_1 obtained by closing the first $m_+ + m_0$ components of T.

Using the partial trace property of the modified trace, recursively, we can safely close further $m_- - 1$ components of T_1 with quantum characters. Under this operation, the colors $\mathbf{h}_j^\pm$ and $\mathbf{h}_j$ correspond to the quantum character $\mathrm{tr}_l^{\mathcal{P}_j^\pm}$ and this character pre-composed with x_j^+, respectively, for any $1 \leq j < \mathsf{p}$. The main point is that it does not matter which $m_- - 1$ components we choose!

Hence $\mathrm{H}^{\log}(M, L)$ is invariant of the $(1,1)$-tangle K obtained from T_1 by closing all but one trace class colored components.

But the resulting central element J_K does not depend on the point where we cut L into K. The proof mimics the argument showing that long knots and knots in S^3 are equivalent.

Indeed, let us think about our $(1,1)$-tangle K as being a long knot. We need to show that our central element does not change if we move an arc through "infinity" (or the cutting disc). This move can be alternatively realized by moving the arc in the opposite way through the long knot, which is just a sequence of Reidemeister moves, under which we know J_K to be stable.

It remains to show invariance under Kirby moves: sliding along a component of L^0 and stabilisation with ± 1 framed unknot. The defining property of the right integral ensure the sliding invariance (see e.g. [15]). Note that if we change the orientation on one of the T^0 components, this will change J_T by applying S at the corresponding position, but now $\mu \circ S$ is a left integral, hence the sliding property holds after rearranging the components.

Adding to L^0 a ± 1 framed unknot multiplies $\mathrm{H}^{\log}(M, L)$ by $\mu(v^\mp) = \delta^\mp$, and changes the signature s by ± 1, so $\mathrm{H}^{\log}(M, L)$ remains the same. □

Relation with other invariants. Here we show that the logarithmic invariant of Jun Murakami [21] is a special case of $\mathrm{H}^{\log}(M, L)$ where L^- has precisely one component.

Murakami's invariant is defined for a knot and a colored link in a 3-manifold. We will adapt our notation to his setting. Let us consider a link (L^+, L^-) in a 3-manifold $M = S^3(L_0)$ as before, but with $L_- = K$ a knot and $m_- = 1$. The link L^+ is colored by z^+ as before. The logarithmic knot invariant defined by Murakami is then

$$\mathrm{J}^{\log}(M, (L^+, z^+), K) = \delta^s(z^+\mu^{\otimes m_+} \otimes \mu^{\otimes m_0} \otimes \mathrm{id})(J_T) \in Z(U)\ .$$

Due to different conventions in the definition of J_T, Murakami's original invariant rather corresponds to the opposite link in our notation.

Murakami further expands his invariant in the basis of the center as follows

$$\mathrm{J}^{\log}(M, (L^+, z^+), K) = \sum_{j=0}^{p} a_j \mathbf{e}_j + \sum_{j=1}^{p-1} b_j^+ \mathbf{w}_j^+ + \sum_{j=1}^{p-1} b_j^- \mathbf{w}_j^-\,.$$

The coefficients are clearly topological invariants of the triple $(M, (L^+, z^+), K)$.

Proposition 10. *With the above notation, we have* $(1 \leq j < \mathsf{p})$

$$\begin{aligned}
a_0 &= \mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_p^-)) \\
a_p &= (-1)^{\mathsf{p}-1}\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_p^+)) \\
a_j &= \frac{(-1)^j}{[j]^2}\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_j)) \\
b_j^+ &= \frac{(-1)^j}{[j]^4}\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, [j]^2\boldsymbol{h}_j^+ - (q^j + q^{-j})\boldsymbol{h}_j)) \\
b_j^- &= \frac{(-1)^j}{[j]^4}\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, [j]^2\boldsymbol{h}_{p-j}^- - (q^j + q^{-j})\boldsymbol{h}_j))
\end{aligned}$$

Proof. For any trace class $h \in \mathrm{HH}_0(U)$ we have

$$\mathrm{H}^{\log}((M, (L^+, z^+), (L^-, h)) = \langle \mathrm{J}^{\log}(M, (L^+, z^+), L^-), h\rangle\ .$$

From Table 1 we get

$$\begin{aligned}
\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \mathbf{h}_p^-) &= a_0 \\
\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \mathbf{h}_p^+) &= (-1)^{\mathsf{p}-1}a_p \\
\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \mathbf{h}_j) &= (-1)^j[j]^2 a_j \\
\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \mathbf{h}_j^+) &= (-1)^j(q^j + q^{-j})a_j + (-1)^j[j]^2 b_j^+ \\
\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \mathbf{h}_{p-j}^-) &= (-1)^j(q^j + q^{-j})a_j + (-1)^j[j]^2 b_j^-\ .
\end{aligned}$$

The claim follows. □

References

1. Y. Arike, *A construction of symmetric linear functions on the restricted quantum group $\overline{U}_q(sl_2)$*, Osaka J. Math. 47 (2010), 535–557.
2. A. Bruguières, *Double Braidings, Twists and Tangle Invariants,* Journal of Pure and Applied Algebra, Volume 204, Issue 1 (2006), 170–194.
3. C. Blanchet, F. Costantino, N. Geer, B. Patureau-Mirand, *Non-semi-simple TQFTs, Reidemeister torsion and Kashaev's invariants,* Advances in Mathematics 301 (2016), 1–78.
4. V. Chari, A. Pressley, *A Guide to Quantum Groups,* Cambridge University Press 1994.
5. F. Costantino, N. Geer, B. Patureau-Mirand - *Quantum invariants of 3-manifolds via link surgery presentations and non-semi-simple categories,* Journal of Topology (2014) 7 (4) 1005–1053.
6. F. Costantino, N. Geer, B. Patureau-Mirand - *Some remarks on the unrolled quantum group of $sl(2)$*, J. Pure Appl. Algebra 219 (2015), no. 8, 3238–3262.
7. Yu. A. Drozd, V. V. Kirichenko, *Finite Dimensional Algebras*, Kiev 1980 (Russian), Springer-Verlag 1994.
8. B.L. Feĭgin, A.M.Gaĭnutdinov, A.M. Semikhatov, I.Yu. Tipunin - *Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center.* Comm. Math. Phys. 06 (2006); 265 (1), 47-93.
9. B.L. Feĭgin, A.M.Gaĭnutdinov, A.M. Semikhatov, I.Yu. Tipunin - *Kazhdan-Lusztig correspondence for the representation category of the triplet W-algebra in logarithmic CFT*, Theor.Math.Phys. 148 (2006) 1210-1235.
10. Geer, N., Kujawa, J. and Patureau-Mirand, B., - *Ambidextrous objects and trace functions for nonsemisimple categories,* Proceedings of the American Mathematical Society, 141 (2013), no. 9, 2963–2978.
11. N. Geer, B. Patureau-Mirand, A. Virelizier, *Traces on ideals in pivotal categories,* Quantum Topology, 4 (2013), no. 1, 91–124.
12. K. Habiro, *Bottom tangles and universal invariants*, Algebraic and Geometric Topology 6 (2006) 1113–1214.
13. M. Hennings, *Invariants of links and 3-manifolds obtained from Hopf algebras*, J. London Math. Soc. (2) 54 (1996), no. 3, 594-624.
14. T. Kerler, *Mapping class group actions on quantum doubles*, Comm. Math. Phys., 168, (1995)
15. T. Kerler, *Genealogy of non-perturbative quantum invariants of 3-manifolds: the surgical family*, from "Geometry and physics (Aarhus, 1995)", Lecture Notes in Pure and Appl. Math. 184, Dekker (1997) 503–547.
16. H. Kondo, Y. Saito - *Indecomposable decomposition of tensor products of modules over the restricted quantum universal enveloping algebra associated to $sl(2)$*, J. Algebra 330 (2011), 103129.
17. R. Lawrence, *A universal link invariant using quantum groups*, Differential geometric methods in theoretical physics (Chester, 1988), 5563, World Sci. Publ., Teaneck, NJ, 1989.
18. J.L. Loday, *Cyclic Homology*, Spriger-Verlag 1992.
19. V.V. Lyubashenko, *Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity*, Commun. Math. Phys. 172 (1995) 467–516

20. J. Murakami, *From colored Jones Invariants to Logarithmic Invariants*, arXiv:1406.1287.
21. J. Murakami, *Generalized Kashaev invariants for knots in three manifolds,* Quantum Topology, Vol. 8 (2017) 3573.
22. T. Ohtsuki, *Quantum Invariants,* Series on Knots and Everything Vol. 29.
23. R. Suter - *Modules over* $\mathcal{U}_q(\mathfrak{sl}(2))$*,* Commun. Math. Phys. 163 (1994) 359–393.
24. V.G. Turaev - *Quantum invariants of knots and 3-manifolds,* de Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, 1994.

(anna@math.uzh.ch) Anna Beliakova, Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich.

(christian.blanchet@imj-prg.fr) Christian Blanchet, Univ Paris Diderot, Sorbonne Paris Cité, IMJ-PRG, UMR 7586 CNRS, F-75013, Paris.

(nathan.geer@gmail.com) Nathan Geer, Mathematics & Statistics, Utah State University, Logan, Utah 84322, USA