Dilation surfaces and their Veech groups

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EDUARD DURYEV, CHARLES FOUGERON, AND SELIM GHAZOUANI

To the memory of William Veech

ABSTRACT. We introduce a class of objects which we call *'dilation surfaces'*. These provide families of foliations on surfaces whose dynamics we are interested in. We present and analyze a couple of examples, and we define concepts related to these in order to motivate several questions and open problems. In particular we generalize the notion of Veech group to dilation surfaces, and we prove a structure result about these Veech groups.

NOTATIONS

- $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ is the open unit disk;
- $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ is the upper-half plane;
- $\mathrm{GL}_2^+(\mathbb{R})$ is a group of 2 by 2 matrices with positive determinant;
- $\mathrm{SL}_2(\mathbb{R})$ is a group of 2 by 2 matrices with determinant 1;
- $\mathrm{Aff}(\mathbb{C})$ is a group of complex affine transformations of a plane: $\{z \mapsto az + b \mid a \in \mathbb{C}^*, b \in \mathbb{C}\}$;
- $\mathrm{Aff}_{\mathbb{R}_+}(\mathbb{C}) \subset \mathrm{Aff}(\mathbb{C})$ is a group of dilations: $\{z \mapsto az + b \mid a \in \mathbb{R}_+, b \in \mathbb{C}\}$.

1. INTRODUCTION.

A *translation* structure on a surface is a geometric structure modelled on the complex plane $\mathbb{C}$ with structural group the set of translations. A large part of the interest that these structures have drawn lies in the directional foliations inherited from the standard directional foliations of $\mathbb{C}$ (the latter being invariant under the action of translations). Examples of such structures are polygons whose sides are glued along parallel sides *of same length*, see Figure 1 below.

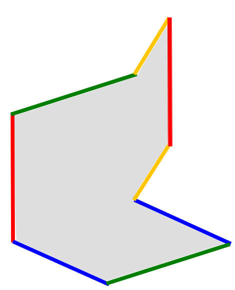


FIGURE 1. A translation surface of genus 2.

The directional foliation, say in the horizontal direction, can be drawn very explicitly: a leaf is a horizontal line until it meets a side, and continues as the horizontal line starting from the point on the other side to which it is identified. These foliations have been more than extensively studied over the past forty years. They are very closely linked to one dimensional dynamical systems called *interval exchange transformations*, and most of the basic features of these objects (as well as less basic ones!) have long been well understood, see [Zor06] for a broad and clear survey on the subject.

The starting point of this article is the following remark: to define the horizontal foliation discussed in the example above, there is no need to ask the identified sides to be of the same length, but only to be parallel, in which case we can glue using certain affine identifications. In terms of geometric structures, it means that we extend the structural group to all the transformations of the form $z \mapsto az + b$ with $a \in \mathbb{R}_+$ and $b \in \mathbb{C}$. Formally, these corresponds to (branched) complex affine structures whose holonomy group lies in the subgroup of complex affine transformations $\mathrm{Aff}(\mathbb{C})$, whose linear parts are real positive. A simple example of such a *'dilation surface'* is given by the gluing below:

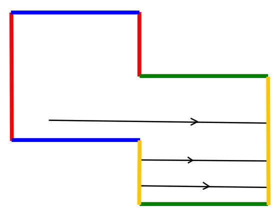


Figure 2. A *'dilation surface'* of genus 2 and a leaf of its horizontal foliation.

A notable feature of these dilation surfaces is that they present dynamical behaviors of *hyperbolic* type: the directional foliations sometimes have a closed leaf which 'attracts' all the nearby leaves. This is the case of the closed leave drawn in black on Figure 3 below. This situation is in sharp contrast with the case of translation surfaces and promises a very different picture in the affine case.

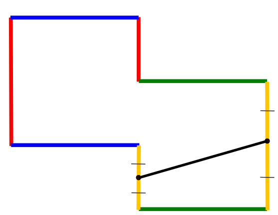


Figure 3. A *'hyperbolic'* closed leaf.

It is somewhat surprising to find no systematic study of these *'dilation surfaces'* in the literature. However, related objects and concepts have kept popping up every now and then, both of geometric and dynamical nature. To our knowledge, their earliest appearance is in the work of Prym on holomorphic 1-forms with values in a flat bundle, see [Pry69]. These provide an algebro-geometric interpretation of the dilation surfaces, see also Mandelbaum ([Man72, Man73]) and Gunning ([Gun81]). We would also like to mention Veech's remarkable papers [Vee93] and [Vee97] where he investigates the moduli spaces of complex affine surfaces with singularities as well as the Delaunay partitions for such surfaces. On the dynamical side, the first reference to related questions can be found in Levitt's paper on foliations on surfaces [Lev82], where he builds an affine interval exchange (AIET) with a wandering interval (these AIETs must be thought of as the one-dimensional reduction of the foliations we are going to consider). It is followed by a series of works initiated by Camelier and Gutierrez [CG97] and pursued by Bressaud, Hubert and Maass [BHM10], and Marmi, Moussa, and Yoccoz [MMY10]. They generalize a well known construction of Denjoy to build out of a standard IET an AIET having a wandering interval, behavior which is (conjecturally) highly non-generic. Very striking is that the question of the behavior of a typical AIET has been very little investigated. In this direction, we mention the nice article of Liousse [Lio95] where the author deals with the topological generic behavior of transversely affine foliations on surfaces.

Contents of the paper and results. After introducing formal definitions as well as a couple of interesting examples of *'dilation surfaces'*, we prove a structure result about Veech groups of dilation surfaces.

The Veech group of a dilation surface Σ is the straightforward generalization of its translation analogue: it is the subgroup of $\mathrm{SL}_2(\mathbb{R})$ made of linear parts of locally affine transformations of Σ. It is a well-known fact that the Veech group of a translation surface is always discrete. This fails to be true in the more general case of dilation surface, although the examples of surfaces whose Veech group is not discrete are fairly distinguishable. We completely describe the class of surfaces whose Veech group is not discrete. Roughly, those are the surface obtained starting from a ribbon graph and gluing to its edges a finite number of *'dilation cylinders'*. We call such surfaces **Hopf surfaces**, because they must be thought of as higher genus analogues of Hopf tori, that are quotients $\mathbb{C}^*/_{(z\sim\lambda z)}$ with λ a positive constant different from 1. Precisely we prove

Theorem 1. Let Σ be a dilation surface of genus ≥ 2. There are two possible cases :

(1) $\mathbf{V}(\Sigma)$ is the subgroup of upper triangular elements of $\mathrm{SL}_2(\mathbb{R})$ and Σ is a Hopf surface.
(2) $\mathbf{V}(\Sigma)$ is discrete.

We also prove the following theorem on the existence of closed geodesics in genus 2 :

Theorem 3. Any dilation surface of genus 2 has a closed regular geodesic.

The proof is elementary and relies on combinatoric arguments. Nonetheless it is a good motivation for a list of open problems we address in Section 7. We end the article with a short appendix reviewing Veech's results on the geometry of affine surfaces contained in the article [Vee97] and the unpublished material [Vee08] that W. Veech kindly shared.

About Bill Veech's contribution. Bill Veech's sudden passing away encouraged us to account for his important contribution to the genesis of the present article. About twenty years ago, he published a very nice paper called *Delaunay partitions* in the journal *Topology* (see [Vee97]), in which he investigated the geometry of complex affine surfaces (of which our 'dilation surfaces' are particular cases). A remarkable result contained in it is that dilation surfaces all have geodesic triangulations in the same way flat surfaces have. We used it extensively when we first started working on dilation surfaces, overlooking the details of [Vee97]. But at some point, we discovered a family of dilation surfaces that seemed to be a counter-example to Veech's result and which provides an obstruction for dilation surfaces to have a geodesic triangulation. We then decided to contact Bill Veech, who replied to us almost instantly with the most certain kindness. He told us that he realized the existence of the mistake long ago, but since the journal *Topology* no longer existed and that the paper did not draw a lot of attention, he did not bother to write an erratum. However, he shared with us courses notes from 2008 in which he *'fixed the mistake'*. It was a pleasure for us to discover that in these long notes (more than 100p) he completely characterizes the obstruction for the slightly flawed theorem of *Delaunay partitions* to be valid, overcoming serious technical difficulties. We extracted from the notes the Proposition 3 which is somewhat the technical cornerstone of this paper.

A few weeks before his passing away, Bill Veech allowed us to reproduce some of the content of his notes in an appendix to this article. It is a pity he did not live to give his opinion and modify accordingly to his wishes this part of the paper.

Acknowledgements. We are very grateful to Vincent Delecroix, Bertrand Deroin, Pascal Hubert, Erwan Lanneau, Leonid Monin and William Veech for interesting discussions. The third author is grateful to Luc Pirio for introducing him to the paper [Vee97]. The third author acknowledges partial support of ANR Lambda (ANR-13-BS01-0002).

2. Dilation surfaces.

We give in this section formal definitions of *dilation surfaces* and several concepts linked to both their geometry and dynamics.

2.1. **Basics.** An affine structure on a complex manifold M of dimension n is an atlas of charts (U_i, φ_i) with values in $\mathbb{C}^n$ such that the transition maps belong to a group of complex affine transformations $\mathrm{Aff}_n(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C}) \ltimes \mathbb{C}^n$, which we will shortly call *affine maps*. It is well known that the only compact surface (thought of as a 2-dimensional real manifold) carrying an affine structure is torus.

In order to include interesting examples mentioned in the introduction, we make the definition of an affine structure less rigid by allowing a finite number of points where the structure is singular:

Definition 1. Let M be a closed orientable topological surface.

(1) An **affine surface** Σ on M is an affine structure on $M \setminus \mathcal{S}$, where $\mathcal{S}$ is a finite set $\{s_1, \dots, s_n\} \subset M$, that extends to an Euclidean cone structure of angle a multiple of 2π at the s_i's. Affine surfaces Σ_1 and Σ_2 are *isomorphic* if there exist a homeomorphism $h : M \to M$, whose induced action on the charts of the atlases defining Σ_1 and Σ_2 is a complex affine map.

(2) A **dilation surface** Σ is an affine surface whose transition maps belong to the group of dilations $\mathrm{Aff}_{\mathbb{R}_+}(\mathbb{C}) = \{z \mapsto az + b \mid a \in \mathbb{R}_+, b \in \mathbb{C}\}$. Dilation surfaces Σ_1 and Σ_2 are *isomorphic* if there exist a homeomorphism $h : M \to M$, whose induced action on the charts of the atlases defining Σ_1 and Σ_2 is a dilation.

We have to mention that in general one can allow singular points to look like affine cones of arbitrary angles. However, we prefer to restrict to the flat singularities of angles, which are multiples of 2π. A first important remark is that an affine surface satisfy a *discrete Gauss-Bonnet* equality. If $\mathcal{S} = \{s_1, \dots, s_n\}$ is the set of singular points of an affine surface Σ, where s_i has cone angle $2k_i\pi$ for some integer $k_i \geq 2$, then:

$$\sum_{i=1}^{n} (1 - k_i) = \chi(\Sigma) = 2 - 2g.$$

Geometric structures. We will use extensively the language and formalism of geometric structure. The reader can refer to [Gol88] or [Thu97] for an introduction to the topic. For self-completeness, we give here a definition of the *holonomy* and *developing* maps.

Let $\gamma : [0,1] \to \Sigma$ be a path in Σ, covered by a finite number of charts $\phi_i : U_i \to \mathbb{C}$, $i = 1, \dots, k$ such that $U_i \cap U_{i+1} \neq \varnothing$ and the transition functions $\psi_{i+1,i} : U_{i+1} \to U_i \in \mathrm{Aff}_{\mathbb{R}_+}(\mathbb{C})$. Assume $\psi_{i+1,i} = a_i z + b_i$, the *holonomy* of γ is a composition:

$$\mathrm{hol}(\gamma) = \psi_{2,1} \circ \ldots \circ \psi_{k,k-1},$$

and the *linear holonomy* is:

$$\mathrm{lhol}(\gamma) = a_1 \cdot \ldots \cdot a_{k-1}.$$

Observe that by definition, an affine surface has trivial holonomy around its singularities, since they are modeled out of an Euclidian cone.

The *developing map* can be thought of as the result of analytical continuation of a some chart along all possible paths in Σ. Let $\widetilde{\Sigma}$ be the universal cover of Σ, *i.e.* the set of homotopy classes of the paths in Σ starting at a base point x_0. Fix a chart ϕ_1 at x_0. The *developing map* $\mathrm{dev} : \widetilde{\Sigma} \to \mathbb{C}$ is defined by sending a class of paths $\gamma \in \widetilde{\Sigma}$ to $\mathrm{hol}(\gamma) \circ \phi_k(\gamma(1))$, where ϕ_i are as above. The holonomy and the developing map only depend on the choice of the charts and transitions up to a map in $\mathrm{Aff}_{\mathbb{R}_+}(\mathbb{C})$. Changing ϕ_1 post-composes the developing map dev with a corresponding element of $\mathrm{Aff}_{\mathbb{R}_+}(\mathbb{C})$.

A general principle with geometric structures is that any object that is defined on the model and is invariant under the transformation group is well defined on the manifolds carrying such a structure. In our case the model is $\mathbb{C}$ with structure group $\mathrm{Aff}_{\mathbb{R}_+}(\mathbb{C})$. Among others, angles and straight lines are well defined on dilation surfaces. In fact, the direction of a line is well defined, and to each angle $\theta \in S^1$ we can associate a *foliation* in direction θ denoted by $\mathcal{F}_\theta$, whose leaves are exactly the directed lines with direction θ. In particular, $\mathcal{F}_{\theta+\pi}$ and $\mathcal{F}_\theta$ define foliations in opposite directions.

Finally note that although the speed of a path is only defined up to a fixed constant, it makes sense to say that a path has a constant speed (speed which is not itself well defined), as well as to say that a path has finite or infinite length.

Vocabulary. We will use the following terms:

- a **geodesic** is an affine immersion of a segment $]a, b[$ (a or b can be $\pm\infty$);
- a **saddle connection** is a geodesic joining two singular points;
- a **leaf** of a directional foliation is a maximal geodesic in the direction of the foliation;
- a **closed geodesic** (or **closed leaf** if the direction of the foliation is unambiguous) is an affine embedding of $\mathbb{R}/\mathbb{Z}$;
- the first return on a little segment $(-\varepsilon, \varepsilon)$ around a point x on a closed geodesic orthogonal to it is a map of the form $x \mapsto \lambda x$ with $\lambda \in \mathbb{R}_+$. We say it is **flat** if $\lambda = 1$ and that it is **hyperbolic** otherwise.

2.2. **Hopf torus.** These definitions being set, we introduce the first fundamental example, a *Hopf torus*. Consider a real number $\lambda \neq 1$ and identify every two points on $\mathbb{C}^*$ which differ by scalar multiplication by λ. The quotient surface $\mathbb{C}^*/(z \sim \lambda z)$ is a called a *Hopf torus* and we call λ its *dilation factor*.

This provides a real 1-parameter family of dilation structures on the torus. These surfaces have a very specific kind of dynamics. For any $\theta \in S^1$ the directional foliation $\mathcal{F}_\theta$ has two closed leaves. They can be obtained the intersections of the annulus (a fundamental domain for $z \mapsto \lambda z$ action) and a ray starting at 0 with angle θ (the *attractive leaf*) and $\theta + \pi$ (the *repulsive leaf*). One can check that any other leaf in a direction θ accumulates to the attractive leaf in the forward direction and to the repulsive leaf in the backward direction.

Based on Hopf torus, we introduce a *slit construction.* It is a set of higher genus examples obtained by gluing two of these tori along a slit in the same direction (see Figure 5). Consider an embedded segment along a directional foliation in a fixed direction on one Hopf torus, and another one in the same direction on the second Hopf torus. We cut the two surfaces along these segments and identify

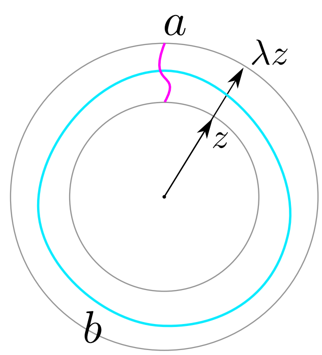


FIGURE 4. A Hopf torus and the basis of its homology.

the upper part of one with the lower part of the other and vice versa by the corresponding dilation transformation.

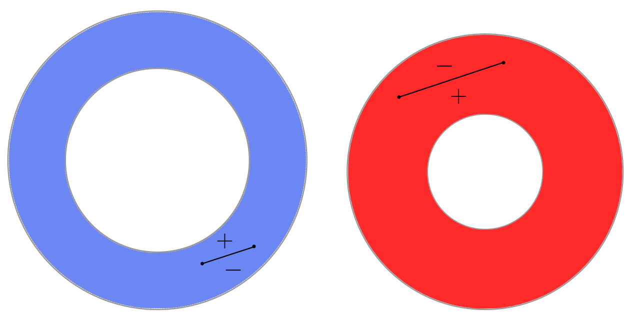


FIGURE 5. The franco-russian slit construction

Another construction based on Hopf torus is given by considering a finite cover. Denote by a the closed curve of the Hopf torus in direction of the dilation and b the closed curve turning around zero once in the complex plane as in Figure 4. Take the k index subgroup of $\pi_1(T^2)$ generated by a and b^k and consider the associated cover with the induced affine structure. It is also a torus, which *makes k turns around zero*. We call it a k-Hopf torus. Similarly, an ∞-Hopf cylinder is given by the cover associated to the subgroup generated by a.

Remark. *We can also construct these dilation structures as slit constructions along horizontal closed leaves of $k \leq \infty$ different Hopf tori.*

2.3. **Dilation cylinders.** A k-Hopf torus has a remarkable property of being a disjoint union of uncountably many closed geodesics in different directions. Each of these geodesics is an attractive leaf of the foliation in a suitable direction. Note that the angular sectors of these tori embedded in a dilation surface enjoy the same property. This is the motivation for the following definition:

Definition 2. Consider a dilation surface Σ. Let C_{θ_1,θ_2} be an open angular domain of a ∞-Hopf cylinder between *angles* $\theta_1 \in \mathbb{R}_+$ and $\theta_1 < \theta_2$ such that

$\theta_2 - \theta_1 = \theta$. A *dilation cylinder* of angle θ is the image of a maximal affine embedding of some C_{θ_1,θ_2} in Σ. Two dilation cylinders are called isomorphic if there is a homeomorphism between them which acts by an affine map in the defining charts.
We call λ the *dilation factor* of the dilation cylinder and θ its *angle*.

Note that the isomorphism class of a dilation cylinder is determined by the two numbers θ and λ.

On a translation surface the boundary of any maximal flat cylinder is a union of saddle connections. We show that the same holds for dilation surfaces and dilation (or flat) cylinders:

Proposition 1. *Let Σ be a dilation surface of genus ≥ 2. Then the boundary of a maximal cylinder embedded in Σ is a union of saddle connections.*

Proof. Assume $C_{0,\theta}$ is affinely embedded in a dilation surface Σ. If $\theta = \infty$ we would have a half-infinite cylinder in the surface. But this half-cylinder would have an accumulation point at ∞ in Σ which contradicts its being embedded.

When $\theta < \infty$, there are two reasons why $C_{0,\theta'}$ cannot be embedded for $\theta' > \theta$:

(1) The embedding $C_{0,\theta} \to \Sigma$ extends continuously to the boundary of the cylinder in direction θ. If the surface is not a k-Hopf torus the image contains a singular point. The image of the boundary is closed, and it is an union of saddle connections.
(2) The embedding does not extend to the boundary at a point z_0 on the θ boundary of $C_{0,\theta}$. Pick a small geodesic $\gamma : [0, \epsilon) \to C_{0,\theta}$ starting close to the boundary and ending at z_0 orthogonally to the boundary of $C_{0,\theta}$. Then the embedding of γ to Σ has no limit in Σ as it approaches ϵ. Consider an open disk in $C_{0,\theta}$ tangent to the boundary at z_0 and centered at a point c_0. Then Corollary 3 from the Appendix implies that the embedding of $\gamma([c_0, z_0))$ in Σ is a closed hyperbolic geodesic in Σ. This cannot happen, since γ is embedded in Σ.

□

Again a cylinder will be a union of closed leaves. The dynamics of a geodesic entering such a cylinder is clear. If the cylinder is of angle less than π and the direction of the flow is not between θ_1 and θ_2 modulo 2π, then it will leave the cylinder in finite time. Otherwise it will be attracted to a closed leaf corresponding to its direction, and be trapped in the cylinder.

As to enter a cylinder we have to cross its border, we see that for cylinder of angle larger than π every geodesic entering the cylinder is also trapped. These *'trap'* cylinders can be ignored when studying dynamics. We can study instead the surface with boundaries where we remove all these cylinders. We will see in the following section that these cylinders are also responsible for degenerate behavior when trying to triangulate the surface.

Remark. *A degenerate case of a dilation cylinder is a* flat cylinder*. It is an embedding of the dilation surface* $C_{\frac{a}{b}} = \{z \in \mathbb{C} \mid 0 < \mathrm{Im}(z) < a\}/(z \sim z + b)$*, for any* $a, b \in \mathbb{R}_+$*. The ratio* $\frac{a}{b}$ *is called the modulus of the cylinder.*

2.4. **Triangulations.** An efficient way to build dilation surfaces is to glue the parallel sides of a (pseudo-)polygon. A surface obtained this way enjoys the property to have a *geodesic triangulation*. It is a triangulation whose edges are geodesic segments and whose set of vertices is exactly the set of singular points. It is natural to wonder if any dilation surface has such triangulation from which we could easily deduce a polygonal presentation. The question only makes sense for surfaces of genus $g \geq 2$, for in genus 1 there are no singular points.

Unfortunately, a simple example shows that it is not to be expected in general. The double Hopf torus constructed above cannot have a geodesic triangulation: any geodesic issued from the singular point accumulates on a closed regular geodesic, except for those coming from the slit. This obstruction can be extended to any dilation surface containing a dilation cylinder of angle $\geq \pi$: any geodesic entering such a cylinder never exits it which is incompatible with the fact that a triangulated surface deprived of its 1-skeleton is a union of triangles. A remarkable theorem of Veech proves that this obstruction is the only one:

Theorem (Veech, [Vee97, Vee08])**.** *Let* Σ *be a dilation surface which does not contain any dilation cylinder of angle at least* π*. Then* Σ *admits a geodesic triangulation.*

The exact theorem generalizes a classical construction known as *Delaunay partitions* to a more general class of affine surfaces. We review this construction and more of the material contained in [Vee97, Vee08] in Appendix A.

3. Examples.

3.1. **The double-chamber surface.** By gluing the sides of the same color of Figure 6 below, we get a genus 2 dilation surface with a unique singular point of angle 6π. Note that every leaf that goes from one chamber to another stays in the latter chamber. We call it a *double-chamber* surface and denote it with 2CH.

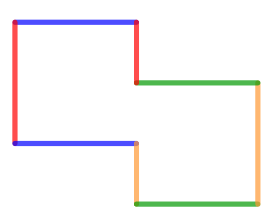


Figure 6. The double-chamber surface.

Let $\mathcal{F}_\theta$ be the directional foliation in direction θ on the double-chamber surface. Below we give some examples of directions with various dynamical behaviours.

If $\theta = \pm\frac{\pi}{2}$, then $\mathcal{F}_\theta$ is *completely periodic*: the surface decomposes into two flat cylinders, each of which is a disjoint union of closed leaves of $\mathcal{F}_\theta$.

If $\theta = \arctan(n)$ or $\arctan(n+\frac{1}{2})$ for $n \in \mathbb{Z}$, the leaves of $\mathcal{F}_\theta$ accumulate on a closed saddle connection.

If $0 < \theta < \arctan(\frac{1}{2})$, the surface has two dilation cylinders (see Figure 7) and every leaf of $\mathcal{F}_\theta$ and $\mathcal{F}_{-\theta}$ accumulates to a closed curve inside one of the cylinders. We call the two latter behaviors *hyperbolic*.

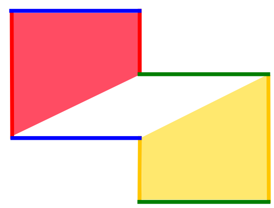
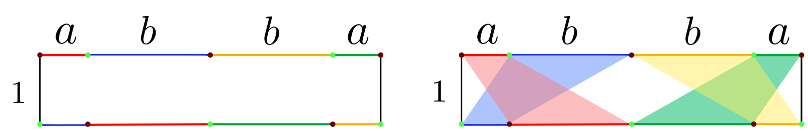


FIGURE 7. Dilation cylinders of the double-chamber surface.

It turns out that the behaviors above are not the only ones that can be observed. A complete classification of dynamical behaviors of directional foliations of the double-chamber surface is given in [BS18]. In particular, it is shown that there exist *"Cantor-like"* directions, in which the leaves accumulate to a set that is transversally a Cantor set.

3.2. **The disco surface.** Choose two positive real numbers a, b. Consider an $2(a+b) \times 1$ rectangle, and consider the identifications as on Figure 8a. This defines a dilation surface with two singularities of angles 4π and genus 2. We call it the *disco surface* and denote it by $\mathrm{D}_{a,b}$. Notice that the surface contains several dilation cylinders. We represent some of them on Figure 8b. These cylinders can overlap, and some zones can a priori be without cylinder coverage.



(A) Suspension construction (B) Cylinders

FIGURE 8. The disco surface $\mathrm{D}_{a,b}$.

We give an alternative representation of the surface which makes a vertical flat cylinder decomposition appear. To do so we cut out the left part of the surface of width a along a vertical line. We now rescale it by a factor $\frac{b}{a}$ and reglue it on the top b interval. Reproduce the same surgery with the right part of the surface and the new surface is the one drawn on Figure 9.

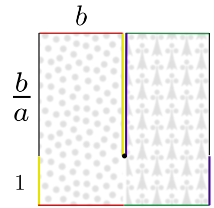


FIGURE 9. An alternative representation of the disco surface

These two representations will be useful later to describe elements of the surface Veech group.

3.3. **Affine interval exchange transformations.** The disco surface is a special example of *suspension of the affine interval exchange transformation.* In this subsection we describe this construction of Camelier and Gutierrez ([CG97]), improved by Bressaud, Hubert and Maas ([BHM10]), and generalized by Marmi, Moussa and Yoccoz ([MMY10]).

An *affine interval exchange transformation* (AIET) is a piecewise affine bijective map from $[0,1]$ to itself. It can be thought of as a generalization of either interval exchange transformation or of piecewise affine homeomorphism of the circle. To any AIET one can associate a dilation surface obtained as a *suspension*: $[0,1] \times [0,\frac{1}{2}]$ rectangle with two vertical parallel sides identified by a parallel translation, and two horizontal sides identified according to the AIET. Note that the disco surface is a suspension of the AIET associated to the permutation $(1,2)(3,4)$ with the intervals of length a, b, b, a (see Figure 8a).

The dynamics of the vertical foliation of such a dilation surface is exactly the same as the dynamics of the affine interval transformation we started from. More precisely, the orbits of the latter are in correspondence with the non-singular leaves of the vertical foliation, while the singular leaves correspond to the orbits of discontinuity points (endpoints of the intervals) of the AIET.

The result of Marmi, Moussa and Yoccoz ([MMY10]) about AIET implies an existence of a surprising dynamical behavior of a directional flow on a dilation surface:

Theorem (Marmi-Moussa-Yoccoz, [MMY10])**.** *For all combinatorics of genus at least* 2*, there exists a uniquely ergodic affine interval exchange whose invariant measure is supported by a Cantor set in* $[0,1]$*.*

Corollary 1. *For $g \geq 2$ there exists a dilation surface of genus g whose leaves of the vertical foliation accumulate to a union of leaves, which intersects every transverse curve along a Cantor set.*

We will call such behavior *Cantor-like.* It is in sharp contrast with both standard interval exchanges and piecewise affine homeomorphisms of the circle. The construction is quite involved and we will not give details here. Nonetheless, the

Cantor-like behavior is analyzed in [BFG18] on the Disco surface $D_{1,2}$ and in [BS18] on the double-chamber surface 2CH.

4. Veech groups: definitions and examples.

Given a matrix $M \in \mathrm{GL}_2^+(\mathbb{R})$ and an dilation structure $\mathcal{A}$ on Σ, there is a way to create a new dilation structure by replacing the atlas $(U_i, \varphi_i)_{i \in I}$ by $(U_i, M.\varphi_i)_{i \in I}$. This new dilation structure is denoted by $M \cdot \mathcal{A}$. A way to put our hands on this operation is to describe it when $\mathcal{A}$ is given by gluing sides of a polygon p. If one sees P embedded in the complex plane $\mathbb{C} \simeq \mathbb{R}^2$, $M \cdot \mathcal{A}$ is the structure one gets after gluing sides of the polygon $M \cdot P$ along the same pattern.

We have defined this way an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the set of dilation surfaces which factors through $\mathrm{SL}_2(\mathbb{R})$, since the action of the diagonal subgroup is obviously trivial. If $\mathcal{A}$ is a dilation structure on Σ, we introduce its *Veech group* $\mathbf{V}(\mathcal{A})$, which is the stabilizer in $\mathrm{SL}_2(\mathbb{R})$ of $\mathcal{A}$, namely

$$\mathbf{V}(\mathcal{A}) = \{M \in \mathrm{SL}_2(\mathbb{R}) \mid M \cdot \mathcal{A} = \mathcal{A}\}$$

The Veech group is the set of real affine symmetries of the considered dilation surface. It is the direct generalization of the Veech group in the case of translation surfaces (see [HS06] for a nice introduction to the subject). For example, if $\mathcal{T}$ is a Hopf torus, $\mathbf{V}(\mathcal{T}) = \mathrm{SL}_2(\mathbb{R})$. It is a consequence of the fact that $\mathcal{T} = \mathbb{C}^*/(z \sim \lambda z)$ for a certain $\lambda > 1$, and that the action of $\mathrm{SL}_2(\mathbb{R})$ commutes with $z \mapsto \lambda z$. This fact is in sharp contrast with the case of translation surfaces where the Veech group is known to always be discrete.
Precisely, let $\mathrm{Aff}^+(\Sigma)$ denote the group of orientation-preserving affine automorphisms, i.e. homeomorphism $f : \Sigma \to \Sigma$ whose action on charts of Σ is affine. The derivative of f is a matrix in $\mathrm{GL}_2^+(\mathbb{R})$ defined up to a scalar multiplication. Its projection to $\mathrm{SL}_2(\mathbb{R})$ gives an element of the Veech group $\mathbf{V}(\Sigma) \subset \mathrm{SL}_2(\mathbb{R})$. More precisely, there is an exact sequence:

$$0 \to \mathrm{Aut}(\Sigma) \to \mathrm{Aff}^+(\Sigma) \to \mathbf{V}(\Sigma) \to 0, \tag{1}$$

where $\mathrm{Aut}(\Sigma)$ is the group of dilation automorphisms; in particular it is a subgroup of automorphisms of the underlying complex structure and is therefore finite.

In the remaining part of this section we go over several examples of the Veech groups of dilation surfaces.

4.1. **The double-chamber surface.**

We will use different types of the dynamical behavior of directional foliations of the double-chamber surface to describe its Veech group completely:

Proposition 2. *The Veech group of the double-chamber surface is the group generated by the two following matrices:*

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Proof. First notice that for any non-vertical direction there is a leaf that leaves one chamber and stays in the other. Therefore there is no completely periodic directions except for the vertical directions. Any element of the Veech group must preserve the set of completely periodic directions. Hence for the double-chamber surface it must lie in the set of lower triangular matrices.

The moduli of the flat cylinders get multiplied by λ^2 under the action of the matrix

$$\begin{pmatrix} \lambda & 0 \\ * & \lambda^{-1} \end{pmatrix}.$$

Since there are only two identical flat cylinders their moduli must be preserved and therefore $\lambda = 1$.

Clearly the rotation of angle π belongs to the Veech group since both the polygon that defines the double-chamber surface and the identifications are invariant under this rotation.

The matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is also in the Veech group. A simple cut-and-paste operation proves this fact, see Figure 10 below.

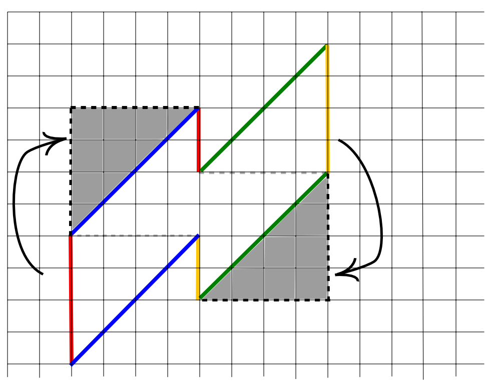


FIGURE 10. Cut-and-paste operation applied to the image of the double-chamber surface under the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

As the flat cylinders are sent to themselves, the only matrices of the form $\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$ that belong to the Veech group of Σ are for $t \in \mathbb{Z}$. $\square$

4.2. **The disco surface.** Let us describe some elements of the Veech group of the disco surface. First note that when we act by the matrix:

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

on a vertical cylinder of height 1 and width t, we can cut-and-paste the surface and end up back with the same cylinder as we saw previously in Figure 10. It is exactly a Dehn twist on its core curve. This works also for any cylinder of modulus t. Hence if we have a surface which we can decompose into horizontal cylinders of the same modulus t, the matrix above is in its Veech group.

As we noted when introducing the disco surface $\mathrm{D}_{a,b}$, they decompose into one cylinder of modulus $2(a+b)$ in horizontal direction (Figure 8b) and into two cylinders of modulus $\frac{1+b/a}{b} = \frac{1}{a} + \frac{1}{b}$ in vertical direction (Figure 9).

As a consequence,

$$\left\langle \begin{pmatrix} 1 & 2(a+b) \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \frac{1}{a} + \frac{1}{b} & 1 \end{pmatrix} \right\rangle \subset \mathbf{V}(\mathrm{D}_{a,b})$$

Note that these two matrices never generate a lattice in $\mathrm{SL}_2(\mathbb{R})$, since it would imply that $2(a+b)(\frac{1}{a}+\frac{1}{b}) \leq 4$, which never happens.

Remark. *These elements of the Veech group enable us to find a lot of dilation cylinders, since starting with one, we can construct new ones by taking its orbit.*

4.3. **Hopf surfaces.** We present in this subsection a general construction of dilation surfaces whose Veech groups are conjugate to:

$$\left\{ \begin{pmatrix} \lambda & t \\ 0 & \lambda^{-1} \end{pmatrix} \mid \ t \in \mathbb{R} \text{ and } \lambda \in \mathbb{R}_+^* \right\},$$

and we prove that these are the only surfaces whose Veech groups are not discrete. In particular this construction includes the Hopf torus and their derivatives introduced in Section 2.2.

Definition 3 (Hopf surface)**.** A *Hopf surface* is a dilation surface for which there exists a finite union of closed geodesics and saddle connections such that the complement is a disjoint union of dilation cylinders of angle π.

Note that all closed geodesics and saddle connections in such union are parallel and that all saddle connection must be in this one direction. Also remark that the franco-russian slit construction from Section 2.2 is a Hopf surface if and only if the slits are radial. Hopf surface give examples of dilation surface with non-discrete Veech groups, more precisely:

Proposition 3. *For every Hopf surface of genus larger or equal to* 2*, the Veech group is conjugate to:*

$$\{ \begin{pmatrix} \lambda & t \\ 0 & \lambda^{-1} \end{pmatrix} \mid t \in \mathbb{R} \text{ and } \lambda \in \mathbb{R}_+^* \}$$

Proof. As the genus is larger or equal to 2 there exists at least one saddle connection in the surface, and all saddle connections are in the same direction. Elements of $\mathrm{SL}_2(\mathbb{R})$ that fix this direction are in the Veech group, since they do not change the slits and preserve each of the cylinders. On the other hand any element of the Veech group has to fix this distinguished direction. □

We will see in the next subsection that Hopf surfaces of genus larger or equal to 2 are exactly the dilation surfaces whose Veech group is not discrete and is not $\mathrm{SL}_2(\mathbb{R})$. Before we explain this property, we would like to present a general construction of Hopf surfaces.

Definition 4. A *ribbon graph* is a finite graph with a cyclic ordering of its semi-edges at its vertices.

We can think of a ribbon graph as an embedding of a given graph in a surface, the manifold structure giving the ordering at the vertices. The structure of a tubular neighbourhood of the embedding of the graph completely determines the ribbon graph.

Given a ribbon graph, we can make the following construction: along the boundary components of the infinitesimal thickening of the ribbon graph, we can glue cylinders of angle $k\pi$ respecting the orientation of the foliation to get a dilation surface. We have to make sure that the factors of the cylinders produce Euclidean singular points; we give an example from which it will be easy to deduce the general pattern.

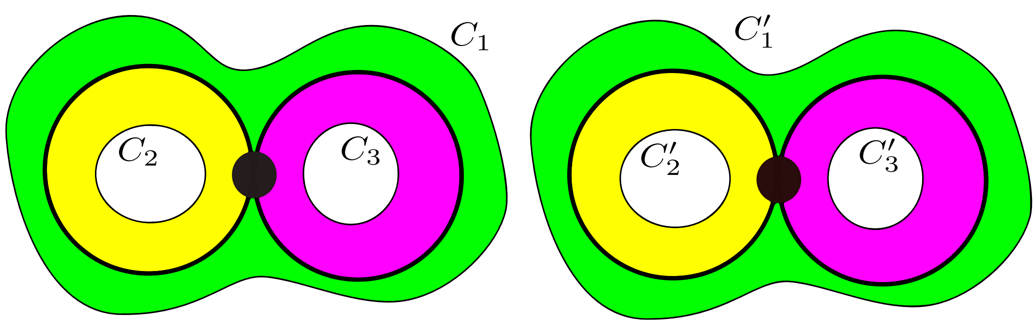


FIGURE 11. A ribbon graph with two vertices.

Consider a ribbon graph as on Figure 11 with two vertices and four edges. We turn it into a genus two surface by gluing three angle $k\pi$ dilation cylinders D_i to the boundary components each joining C_i to C'_i for $i = 1, 2, 3$ as in Figure 12.

Let λ_i be the dilation factor of D_i. For the angular points to be Euclidean, it is necessary that the product of the factors of the cylinders adjacent at a singular point is trivial. In our case, for an appropriate choice of orientation, we obtain:

$$\lambda_1 = \lambda_2\lambda_3.$$

This constraint is the only obstruction to complete the general construction of Hopf surfaces.

4.4. **Dilation torus.** But first we will deal with the case of genus 1 showing the following proposition:

Proposition 4. *A non-flat dilation torus is the exponential of some flat torus*

$$\mathbb{C}/\alpha\mathbb{Z} \oplus (\beta + 2ik\pi)\,\mathbb{Z},$$

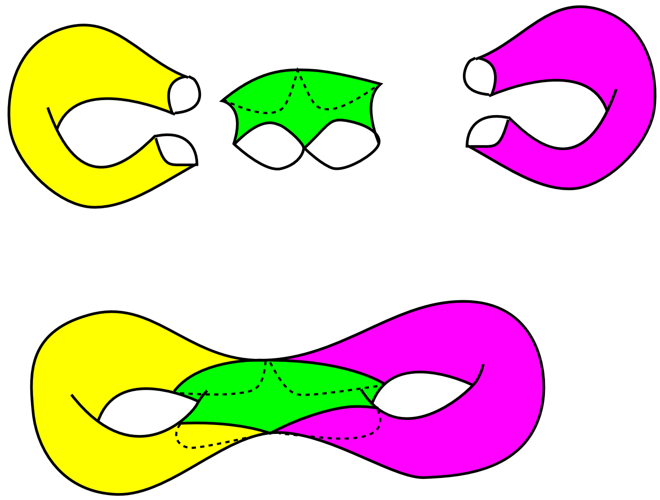


FIGURE 12. A cylinder decomposition of the surface of genus 2.

where $\alpha, \beta \in \mathbb{R}$ *and* $k \in \mathbb{N}$. *Moreover its Veech group is* $\mathrm{SL}_2(\mathbb{R})$.

Proof. Consider a dilation torus. Its developing map (see section 2.1 for definitions) goes from $\mathbb{C}$ to $\mathbb{C}$ and its holonomy is commutative. Hence its holonomy is generated either by two translations or two affine transformations with the same fixed point (which we assume to be zero). The former case is to be excluded since we are only considering a non-flat surface.

In the second case, we can choose a developing map f which avoids zero, and associate to it the 1-form $\mathrm{d}\log f = \frac{df}{f}$. As the surface has no singularity, the derivative of f is never zero, and the 1-form is invariant with respect to the holonomy. Thus $d\log f$ is an *a priori* meromorphic form on the torus. It has no zeroes and by the residue formula no poles, therefore it is holomorphic.

In conclusion, $d\log f$ gives a flat structure on the torus that is isomorphic to $\mathbb{C}/\alpha\mathbb{Z}\oplus\tau\mathbb{Z}$ with $\alpha, \tau \in \mathbb{C}^*$. The exponential of this flat structure $d\exp(\log f) = df$ is the initial dilation structure. This implies $e^\alpha, e^\tau \in \mathbb{R}_+$ and hence $\mathrm{Im}(\alpha)$ and $\mathrm{Im}(\tau)$ are integer multiples of 2π. Therefore the basis of the lattice generated by α and τ can be chosen to be of the form $\tau = \beta + 2ik\pi, \alpha, \beta \in \mathbb{R}$ and $k \in \mathbb{N}$.

Any matrix of $\mathrm{SL}_2(\mathbb{R})$ commutes with the scalar multiplications, thus the Veech group of such a surface is the whole $\mathrm{SL}_2(\mathbb{R})$. ☐

One can think geometrically of such dilation torus in the following way. For $k = 1$, consider a region of $\mathbb{C}$ bounded by the intervals $I_1 = [1, e^\alpha]$, $I_2 = [e^\beta, e^{\alpha+\beta}]$ and the spirals S_1 starting at 1 and ending at e^β and S_2 starting at e^α and ending at $e^{\alpha+\beta}$, that both wind around 0. Identifying I_1 to I_2 by $z \mapsto e^\beta z$ and S_1 to S_2 by $z \mapsto e^\alpha z$ we obtain the required dilation torus. The case of general k is obtained by slitting along $I_1 \cup I_2$ and gluing a $(k-1)$-Hopf torus of the dilation factor e^α

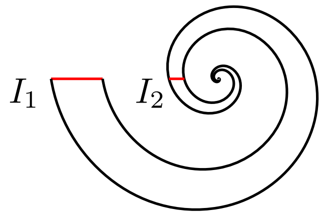


FIGURE 13. A dilation torus, which is not a Hopf torus.

via identity on one boundary and $z \mapsto e^{\beta} z$ on the other. This construction is represented in Figure 13.

5. Veech group dichotomy.

In this section we will show that for dilation surfaces of genus $g \geq 2$ there are only two types of Veech groups. More precisely, we will show that dilation surface is either a Hopf surface (see Section 4.3) or its Veech group is discrete:

Theorem 1. *Let Σ be a dilation surface of genus ≥ 2. There are two possible cases:*

(1) Σ is a Hopf surface and $\mathbf{V}(\Sigma)$ is conjugate to the subgroup of upper triangular matrices of $\mathrm{SL}_2(\mathbb{R})$: $\left\{ \begin{pmatrix} \lambda & * \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{R}^* \right\}$; or

(2) $\mathbf{V}(\Sigma)$ is discrete.

We will also show that the Veech group of a dilation surface is 'probably' not a lattice:

Theorem 2. *If Σ is a dilation surface with a dilation cylinder then $\mathbf{V}(\Sigma)$ is not a lattice.*

We say 'probably', because it seems highly plausible that every dilation surface has a dilation cylinder, however we do not have a proof of this fact.

To prove Theorem 1 we will distinguish between dilation surfaces having saddle connections in at least two non-parallel directions and those which do not. We will show that the Veech group of the former is discrete generalizing a classical argument in the case of translation surfaces (see Section 3.1 of [HS06] and [Vor96]). The latter will turn out to be a Hopf surfaces introduced in Section 4.3. We begin with the following observation:

Lemma 1. *For every singular point of a dilation surface and angle θ there exists at least one saddle connection in a direction in the interval $[\theta, \theta + \pi]$.*

Proof. Without loss of generality we assume that $\theta = 0$. By the assumption singularities of dilation surfaces are conical points modeled on Euclidian cones of angles multiple of 2π. Thus there exists affine charts around these points in a finite cover of the pointed plane $\mathbb{R}^2 \setminus \{0\}$.

In such an affine chart around the singular point p, consider the exponential map centered at 0 and let Δ_r be an open half-disk of radius r and center at 0 bounded by two horizontal half-lines emanating from 0. Let $r > 0$ be the largest radius such that Δ_r immerses in Σ by means of the exponential map.

If $r = \infty$, we would have a maximal affine immersion of $\mathbb{H}$. By Lemma 4 this immersion factors through an embedding of a dilation cylinder of angle π whose boundary would project to two closed leaves. Note that one of them contains p, since this immersion extends to 0. Therefore there is a saddle connection passing through p.

If $r < \infty$, we distinguish two obstructions of why $\Delta_{r'}$ does not immerse for $r' > r$:

(1) the immersion $\varphi : \Delta_r \longrightarrow \Sigma$ extends continuously to the semi-circle on the boundary of Δ_r and the image of this extension contains a singular point p';
(2) the immersion $\varphi : \Delta_r \longrightarrow \Sigma$ does not extend continuously to the semi-circle on the boundary of Δ_r.

In the first case, there is a saddle connection that connects p to p'. Assume we are in the second case, then similarly to Lemma 5, there exist a point z_0 on the semi-circle of the boundary of Δ_r such that $\lim_{z\to z_0} \varphi(z)$ does not exists. Consider a small disk $\mathbb{D}_\varepsilon \subset \Delta_r$ that is tangent to Δ_r at z_0. The segment starting at p and ending at z_0 is eventually contained in $\mathbb{D}_\varepsilon$. According to Corollary 3 this last part of the segment, and hence the whole segment, projects to a closed geodesic in Σ containing p. Therefore there is a closed saddle connection through p. □

Using Lemma 1 together with Gauss-Bonnet formula we obtain a useful criterion of when a dilation surface is a Hopf surface:

Proposition 5. *A dilation surface Σ is a Hopf surface if and only if all saddle connections of Σ are in the same direction.*

Proof. Let Σ be a Hopf surface. A Hopf surface by definition is a dilation surface that decomposes into dilation cylinders of angle π with boundaries being closed curves and saddle connections. All saddle connections on these boundaries are parallel. Moreover, there are no other saddle connections of Σ. Indeed, any separatrix that does not follow a boundary enters a dilation cylinder of angle π and never leaves it. Therefore all saddle connections of Σ are in the same direction.

For the converse, assume without loss of generality that all saddle connections of Σ are horizontal. Consider a union of all horizontal closed curves and saddle connections. A complement of such union is a disjoint union of subsurfaces with boundaries. Due to Lemma 1 each such subsurface enjoys two properties: there are no singularities in the interior and each singularity on the boundary has total angle π around it. Therefore, by Gauss-Bonnet such subsurface is topologically a cylinder. It cannot be a flat cylinder, otherwise we would have a non-horizontal saddle connection. Then the cylinder is affine and has angle π since its boundaries are parallel and we also cut along all horizontal closed curves. This implies that Σ is a Hopf surface. □

Next we will show that if not all saddle connections are in the same direction, the Veech group is discrete. The classical proof of the discreteness of the Veech group for translation surfaces (Section 3.1 of [HS06] and [Vor96]) relies on the fact that the set of holonomy vectors that encodes the direction and length of saddle connections in a translation surface is discrete in $\mathbb{R}^2$.

In the case of dilation surface, we can define the image through the developing map of saddle connections emanating from a singularity with a given image by the developing map. This set is defined as a subset of $\mathbb{R}^2$ up to scalar, therefore it still makes sense to talk about its discreteness. However, the next proposition shows that it is not necessarily discrete in the case of dilation surfaces:

Proposition 6. *There exists a dilation surface, whose images of saddle connections starting from a fixed singularity through the developing map is not discrete.*

Proof. Consider a slit sum of a Hopf torus and a flat square torus as on Figure 14 so that the ration of the lengths of the saddle connection b and c is equal to the ration of the length of b and l. These ratios are well defined since the saddle connections intersect in a vertex, so we can compute their lengths in a common chart. We fix a developing map and consider the image in the plane of the saddle connections emanating from the singularity s to the right of b. Note that the image of the set of saddle connections emanating from this singularity to one of the singularities of the slit a accumulates to the image of s translated by the vector l, *i.e.* the limit image of the center of the Hopf torus. Moreover the image of the saddle connection c is precisely l in the plane, by assumption on the ratio of lengths. Therefore the set of holonomy vectors of saddle connections starting from s is not discrete. □

We deal with this obstruction by introducing $\mathbb{R}^2_k$, a cyclic k-cover of $\mathbb{R}^2$ fully ramified at 0. Let p be a conical singularity of total angle $2\pi k$. By definition of the cone singularity, there exists a Euclidean chart $\phi : U \to \mathbb{R}^2_k$ on an open neighborhood U around p, which is an affine embedding on $U \setminus p$ and sends p to 0. We define S a subset of $\mathbb{R}^2_k$ of the endpoints of the paths obtained by developing all saddle connections that start at p into $\mathbb{R}^2_k$ using this map. Note that as the map ϕ changes the set S changes by a multiplication by scalar, therefore discreteness of S does not depend on the choice of ϕ.

Proposition 7. *Let Σ be a dilation surface and p a singularity of angle $2\pi k$. The set S, obtained by developing all saddle connections starting at p into $\mathbb{R}^2_k$, is discrete.*

Proof. Let U be an open neighborhood of p and let $\phi : U \to \mathbb{R}^2_k$ be a Euclidean chart. Consider a saddle connection s that starts at p and let s_ϕ be a path obtained by developing s into $\mathbb{R}^2_k$ using ϕ. We will show that the endpoint $v_s \in S$ of s_ϕ is an isolated point of S.

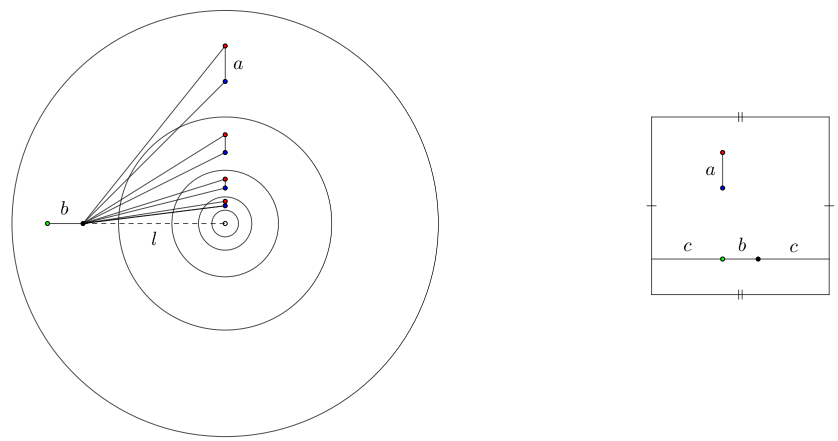


FIGURE 14. A dilation surface with a non-discrete set of holonomy vectors of saddle connections starting at the black point.

By compactness of s there exists an open neighborhood $V \subset \Sigma$ of the path s that contains no other singularities apart from the two endpoints of s. Then ϕ gives rise to a continuous map of V into $\mathbb{R}^2_k$. Denote the image of this map by V_ϕ. Note that V_ϕ is an open subset of $\mathbb{R}^2_k$ that contains the endpoint v_s of s_ϕ. It suffices to show that V_ϕ contains no other elements of S apart from the endpoint of s_ϕ. There are two cases how a straight line segment γ starting at 0 and contained in V_ϕ can be obtained:

- by developing a unique separatrix contained in V that starts at p, if $s_\phi \not\subset \gamma$; or
- by developing a union of the saddle connection s and a separatrix starting at the second endpoint of s that is contained in V, if $s_\phi \subset \gamma$.

Both of the above are contained in V, therefore they do not end in a singularity of Σ. This implies that $\gamma \cap S = \varnothing$ and hence v_s is an isolated point of S. Therefore S is discrete. $\square$

We now have, using notation from Section **??**,

Proposition 8. *Let Σ be a dilation surface with two saddle connections in different directions. Then $\mathrm{Aff}^+(\Sigma)$ is a discrete group, and the Veech group $\mathbf{V}(\Sigma)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$.*

Proof. Consider $\mathrm{Aff}_0^+(\Sigma)$ the subgroup of $\mathrm{Aff}^+(\Sigma)$ fixing pointwise the set of singular points of Σ. The subgroup $\mathrm{Aff}_0^+(\Sigma)$ has finite index in $\mathrm{Aff}^+(\Sigma)$ and its discreteness implies the discreteness of $\mathrm{Aff}^+(\Sigma)$. Since $\mathbf{V}(\Sigma)$ is a quotient of $\mathrm{Aff}^+(\Sigma)$ by the finite group $\mathrm{Aut}(\Sigma)$, its discreteness is also implied by the discreteness of $\mathrm{Aff}_0^+(\Sigma)$. We will consider two cases:

(1) First, assume there are two saddle connections s_1 and s_2 in different directions from the same singularity p of conical angle $2\pi k$. Consider the set S, obtained by developing all saddle connections starting at p into $\mathbb{R}^2_k$. This set is discrete due to Proposition 7 and $\mathrm{Aff}_0^+(\Sigma)$ acts on it. Denote by $v_1, v_2 \in S \subset \mathbb{R}^2_k$ the elements obtained by developing the saddle connections s_1, s_2 respectively.

Suppose that $\mathrm{Aff}_0^+(\Sigma)$ is not discrete, then there exist a sequence of distinct elements $f_n \in \mathrm{Aff}_0^+(\Sigma)$ converging to identity. Since S is discrete one obtains:

$$f_n(v_1) = v_1, f_n(v_2) = v_2 \text{ for } n >> 1.$$

The derivatives of f_n are matrices in $\mathrm{GL}_2^+(\mathbb{R})$ defined up to scalar. There is a natural projection $\mathbb{R}^2_k \to \mathbb{R}^2$ and the images of v_1 and v_2 under this projection are non-collinear vectors. Therefore the derivatives of f_n project to identity in $\mathbf{V}(\Sigma) \subset \mathrm{SL}_2(\mathbb{R})$ for $n >> 1$. Since an affine automorphism is defined up to a finite choice by its derivative (see (1)), this implies that $\mathrm{Aff}_0^+(\Sigma)$ and therefore $\mathrm{Aff}^+(\Sigma)$ and $\mathbf{V}(\Sigma)$ are discrete.

(2) Second, assume there are no saddle connections in different directions starting at the same point. Then there are at most finitely many saddle connections. Each saddle connection defines a direction $\theta \in [0, 2\pi)$. The set of all such directions is finite, hence discrete, and contains at least two elements θ_1 and θ_2. The group $\mathrm{Aff}^+(\Sigma)$ acts on this set, because parallel saddle connections remain parallel under the action of $\mathrm{Aff}^+(\Sigma)$. Therefore $\mathrm{Aff}^+(\Sigma)$ and hence $\mathbf{V}(\Sigma)$ are discrete due to an arguement used in the first case. □

This criterion can be effectively used to check discreteness of a given dilation surface:

Corollary 2. *The double-chamber surface and the disco surfaces have discrete Veech groups.*

We are now ready to proof Theorem 1:

Proof of Theorem 1. By Lemma 1 the set of saddle connections is not empty. We distinguish two cases: when all saddle connections are parallel and when there are two saddle connections in different directions. In the first case, Proposition 5 implies that Σ is a Hopf surface and from Proposition 3 we obtain that $\mathbf{V}(\Sigma)$ is conjugate to:

$$\{ \begin{pmatrix} \lambda & t \\ 0 & \lambda^{-1} \end{pmatrix} \mid t \in \mathbb{R} \textit{ and } \lambda \in \mathbb{R}_+^* \}.$$

In the second case, Proposition 8 implies that $\mathbf{V}(\Sigma)$ is discrete. □

5.1. **The Veech group of a dilation surface is probably not a lattice.** As we saw in Section 2.2 dilation cylinders trap the linear flow in a corresponding angular sector. We will show that this behavior restricts the potential directions for saddle connections around a singularity in the boundary of cylinders and prevents Veech groups from being lattices.

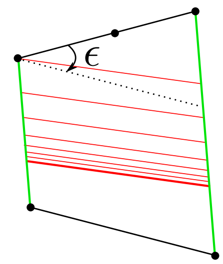


FIGURE 15. An angular section in which all leaves are hyperbolic.

Consider Σ a dilation surface endowed with a dilation cylinder, take any singular point at the boundary of this cylinder and consider a straight line heading inside the cylinder whose angular direction falls just in between the two extreme angles of the cylinders. Such line will be trapped inside the cylinder and will accumulate to a closed geodesic (Figure 15). As a consequence none of such straight lines will meet a singular point. Therefore there are no saddle connections starting from the chosen singular point in the angular sector defined by the cylinder.

Proof of Theorem 2. By the assumption there is a dilation cylinder in Σ and according to Proposition 1 there is a singularity in the boundary of the cylinder. From the above observation it follows that there is a small angular sector in which any separatrix starting from this singularity will accumulate to a closed leaf (see Figure 15).

Assume that the Veech group of Σ is a lattice. We will show that the set of directions in which every separatrix is a saddle connection is dense, which will end the proof since it contradicts the above remark.

Recall that $\mathrm{Aff}_0^+(\Sigma)$ is the subgroup of $\mathrm{Aff}^+(\Sigma)$ fixing pointwise the set of singular points of Σ. The image of $\mathrm{Aff}_0^+(\Sigma)$ in $\mathrm{SL}_2(\mathbb{R})$ is a finite index subgroup of $\mathbf{V}(\Sigma)$ and therefore is also a lattice. Hence it contains a parabolic element and its limit set is all of the boundary of $\mathbb{H}$. By conjugating a parabolic element we construct a dense set of directions in which there is a parabolic element. It remains to show that in a parabolic direction any separatrix meets eventually a singularity and thus every separatrix in a parabolic direction is a saddle connection. Indeed, if this did not hold for a separatrix s, then s would accumulate to a point x and a neighborhood of x would be crossed infinitely many times by s. But the parabolic element fixes s pointwise, hence it fixes x and the whole neighborhood of x, which brings us to a contradiction since parabolic elements do not act by identity. $\square$

6. Cylinders on genus 2 surfaces.

The purpose of this section is to show the following result:

Theorem 3. *Any dilation surface of genus* 2 *has a cylinder.*

First, note that Veech's theorem on Delaunay triangulation (see Veech's theorem in 2.4) tells us that if a dilation surface does not have a triangulation, it must contain a dilation cylinder (of angle at least π). We can therefore forget about this case and assume that all the surfaces we are considering have a geodesic triangulation. As the surface has genus 2, there are two cases concerning the angles of its singular points: there can be two singularities of angle 4π each or one of angle 6π. We will show that the existence of a geodesic triangulation implies the existence of a cylinder in both cases using some combinatorial and topological restrictions.

We start by dealing with the first case. If the surface has two singular points of angle 4π, a triangle of a geodesic triangulation of the surface has at least two of its vertices that are equal to the same singular point. The side corresponding to these two vertices is a simple closed curve, it must cut the angle of the associated singular point into two angular sectors of respective angle 3π and π. Then this closed curve bounds a cylinder on the side of angle π.

We now discuss the second case where Σ has a unique singular point of angle 6π denoted by p. A geodesic triangulation of Σ must have exactly 9 edges and 6 triangles due to Euler characteristic. Here the triangulation has a unique vertex and each edge defines a saddle connection cutting each conical singularity into two angular sectors. Since the foliation is oriented, such an edge must cut the angle 6π into two angles of respective values either 5π and π or 3π and 3π. The lemma below proves that any such geodesic triangulation has at least one edge cutting the singular point into two angles 5π and π. As in the previous case, this implies existence of a cylinder.

Lemma 2. *A geodesic triangulation of Σ cannot have its 9 edges cutting p into two sectors of angles 3π.*

Proof. Assume that every edge of the triangulation cuts the neighborhood U of the singularity p into two angular sectors of angle 3π each. Note that each edge starts and ends at the same singularity p, therefore locally around p we see each edge twice. Label the edges with elements $\{a, b, c, d, e, f, g, h, i\}$. Reading the labels of the edges in U that we meet turning around p in a clockwise direction, we first encounter each letter once and then all the letters appear second time in the same order (see Figure 16). Indeed, due to the assumption that each edge cuts U into two angular sectors of angle 3π, the neighborhood of the triangulation around p has a 3π rotation symmetry, which shifts all the labels by 9 steps. In other words, there is a well-defined cyclic order on those 9 labels, in which they appear in the neighborhood of p.

Now consider a triangle of the triangulation. Without loss of generality assume that its edges are a, b, c as on Figure 16. Moving clockwise around the vertices of the triangle we see that in the order defined above b follows a, c follows b and a

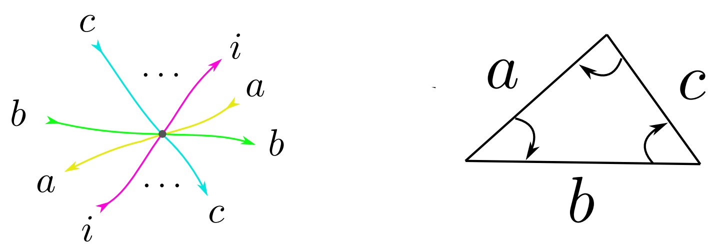


Figure 16. Topological setting of the triangulation

follows c. However this cannot happen since the cyclic order above is defined on 9 elements. This leads us to a contradiction. □

7. Open problems.

In this last section we would like to present several possible directions for further research on dilation surfaces. We will also formulate some open problems.

Dynamics of vertical foliation. The first direction has to do with better understanding the classification of the dynamical behavior of the vertical flow on dilation surfaces. We have already observed several dynamical behaviors:

- **minimal**, when every non-singular leaf is dense on a surface;
- **completely periodic**, when every non-singular leaf is closed;
- **hyperbolic**, when every non-singular leaf is attracted to a finite union of *hyperbolic closed leaves* and saddle connections;
- **Cantor-like**, when every non-singular leaf accumulates to a union of leaves which is transversely a Cantor set. The result of Camelier-Gutierrez discussed in Section 3.3 proves the existence of such directional foliations.

It is possible that a dilation surface decomposes into subsurfaces with different dynamical behaviors of the vertical flow listed above. The first question is whether there are any other possibilities for it:

(1) Does every dilation surface decompose into subsurfaces on each of which the vertical flow is either minimal, periodic, hyperbolic or Cantor-like?

It would be interesting to investigate the behavior of the vertical flow on the specific examples of the Disco surfaces $\mathrm{D}_{a,b}$ (see Section 3.2) and the double-chamber surface DC (see Section 3.1). These examples are simple and explicit, however even in these cases the dynamical questions are not straightforward:

(2) Does there exist a minimal direction on $\mathrm{D}_{a,b}$ or DC?
(3) Does there exist a Cantor-like direction on $\mathrm{D}_{a,b}$ or DC?
(4) Is the set of hyperbolic directions on $\mathrm{D}_{a,b}$ or DC dense? Does it have a full Lebesgue measure?

These questions have been recently answered for $\mathrm{D}_{1,2}$ in [BFG18]. The hyperbolic directions on DC are investigated in [BS18].

Some numerical experiments indicate that the hyperbolic dynamical behaviour is generic and that brings us to the following questions:

(5) Does every dilation surface have a hyperbolic closed leaf in some direction?
(6) Is the set of hyperbolic directions dense for every dilation surface? Does it have a full Lebesgue measure?
(7) Does there exists a dilation surface with infinitely many Cantor-like directions?

We conjecture that the answers to these four questions are positive, with a greater reserve about the full measure one.

Geometry of dilation surfaces. Another natural direction is to investigate the geometrical properties of dilation surfaces:

(8) Does every dilation surface have a regular closed geodesic?
(9) Which dilation surfaces have only finitely many saddle connections?
Conjecture: A dilation surface has finitely many saddle connection if and only if it is a Hopf surface.
(10) Does every point on a dilation surface admit a closed geodesic or a saddle connection passing through it?
Remark: This is the case for the double-chamber surface.
(11) What does the set of holonomy vectors of saddle connections on a given dilation surface look like? When is it non-discrete (see Proposition 6)?

Veech groups. Note that the positive answer to the question (8) together with Theorem 2 would imply that the Veech group of a dilation surface is never a lattice. Nonetheless, it seems to be an interesting invariant of these dilation surfaces.

(12) What kind of Fuchsian groups can appear as Veech groups?

Moduli spaces. Similar questions could be asked about a "generic" dilation surface. It is possible that in this setting some of the above questions are easier to answer and that specific surfaces have a very different behavior from the "generic" ones.

To make meaning of the word "generic" one needs to define the moduli spaces of dilation surfaces. This brings us to a new series of questions about the moduli spaces of dilation surfaces:

(13) What is a natural definition of the moduli space of dilation surfaces? How one defines its topology, analytic structure and measure on it?
(14) Are there analogues of period coordinates of dilation surfaces?
(15) Is the moduli space of dilation surfaces connected? Irreducible?
(16) What are the connected components of the strata of the moduli space of dilation surfaces?

A construction of a moduli space of dilation surfaces of genus g was recently given in [DM18]. It was also shown that it is irreducible. However the question of its naturalness is still open. One way to verify it would be to construct a "period map" and show that it is a homeomorphism in a topology defined by the moduli space.

Finally, we present several other questions related to the loci of surfaces with interesting properties in the moduli spaces of dilation surfaces:

(17) Is the set of dilation surfaces with a minimal direction dense? Does it have a full measure?

(18) Is the set of dilation surfaces having *no* minimal direction dense? Does it have a full measure?

Appendix A. Veech's results on the geometry of dilation surfaces.

We review in this appendix the results of Veech on the geometry of affine surfaces appearing in [Vee97] and [Vee08]. Note that Veech works in a more general context of affine (not necessarily dilation) surfaces with (not necessarily Euclidean) singularities.

For the sake of clarity, we will restrict his results to the case under scrutiny in this paper, namely branched affine structures with real positive linear holonomy. The notes [Vee08] remain unpublished and Veech kindly allowed us to reproduce here the proofs that are contained in these notes.

A.1. **The property $\mathcal{V}$.** In this section we will give a proof of the following theorem:

Theorem 4 (Veech, [Vee08])**.** *Let Σ be a dilation surface. The following three properties are equivalent:*

(1) There is no affine immersion of $\mathbb{H}$ in Σ.

(2) Every affine immersion of $\mathbb{D}$ in Σ extends continuously to a map $\overline{\mathbb{D}} \to \Sigma$.

(3) The dilation surface Σ has no dilation cylinder of angle larger than π.

We say such dilation surface Σ has property $\mathcal{V}$.

We begin with the proof of a few lemmas.

Lemma 3. *Let U be an open subset of $\mathbb{C}$ and $\Gamma \subset \mathrm{Aff}_{\mathbb{R}_+}(\mathbb{C})$ that acts on U. If an affine immersion $\Phi : U \longrightarrow \Sigma$ factors through $U/\Gamma \to \Sigma$ then Γ is either generated by $z \mapsto \lambda z + c$ for some $\lambda \neq 1 \in \mathbb{R}_+$ and $c \in \mathbb{C}$ or by translations.*

Proof. Note that for any $z \in U$ the orbit $\Gamma z \subset U$ has to be discrete and closed since Φ is an immersion. One can check that two affine maps generate a group which acts properly discontinuously on U if and only if they are both translations or they both belong to a cyclic subgroup generated by a dilation. This finishes the proof. □

Lemma 4. *Any affine immersion $\varphi : \mathbb{H} \longrightarrow \Sigma$ can be extended to an affine immersion $\varphi' : \mathbb{H}' \longrightarrow \Sigma$ that factors through an affine embedding of a dilation cylinder of angle π in Σ. Where $\mathbb{H}' \subset \mathbb{C}$ is a half-plane containing $\mathbb{H}$.*

Proof. We first show that the immersion φ cannot be one-to-one. Note that the image of φ does not contain singular points, as it is not possible to extend an immersion beyond singularities. Consider the geodesic

$$\gamma : t \in (0, i\infty) \mapsto \varphi(t) \in \Sigma.$$

The geodesic $\gamma(t)$ cannot have a limit point in Σ as t goes to infinity. Indeed, the length of a geodesic in a dilation surface is defined up to a choice of an affine chart, hence the finiteness of the length is a well-defined property of a geodesic. The immersion φ produces charts, in which γ has an infinite length. If there was a limit point then in a chart around that limit point, the length of γ would be finite.

Therefore γ is a non-compact subset of Σ, hence it has an accumulation point $x \in \Sigma$. Fix an affine chart $f : U \subset \Sigma \mapsto \mathbb{C}$ around x such that $f(U)$ is a disk. Then γ enters and leave U infinitely many times. In particular it must cross U twice along parallel segments. Now we can find a horizontal path η in U that connects those two parallel segments at points $x_1, x_2 \in \Sigma$. Take $t_1 \neq t_2 \in (0, i\infty)$ such that $\gamma(t_i) = x_i$. Lift η starting at t_1 and let $v \in \mathbb{H}$ be the second endpoint of this lift. Then $\varphi(v) = \varphi(t_2)$ and $v \notin (0, i\infty)$ hence $v \neq t_2$. This shows that φ is not one-to-one.

For any $v \neq w \in \mathbb{H}$ such that $\varphi(v) = \varphi(w)$. As the immersion φ is affine, there exists a dilation map $\gamma_{vw} : \mathbb{C} \to \mathbb{C}$, such that in the neighborhoods of v, $\varphi(z) = \varphi(\gamma_{vw}(z))$. Two affine maps that coincide on the neighborhood coincide on the whole domains of definition, therefore the same holds for any $z \in \mathbb{H}$ as long as $\gamma_{vw}(z) \in \mathbb{H}$. Define a group:

$$\Gamma = \{\gamma_{vw} \mid v, w \in \mathbb{H} \text{ such that } \varphi(v) = \varphi(w)\} \subset \mathrm{Aff}_{\mathbb{R}_+}(\mathbb{C}).$$

and an open subset:

$$U = \bigcup_{g \in \Gamma} g\mathbb{H} \subset \mathbb{C}.$$

Γ acts on U and, since φ is not one-to-one, Γ is not trivial. The immersion φ extends to an affine immersion $\Phi : U \to \Sigma$ invariant under Γ. Then by Lemma 3 Γ is either generated by translations or a single dilation and U/Γ embeds into Σ. Note that since $\mathbb{H} \subset U$, if Γ is generated by translations, one obtains an embedding of a half-infinite cylinder or a torus into Σ. An embedding of a half-infinite cylinder contradicts the non-injectivity argument above and an embedding of a torus contradicts Riemann-Hurwitz theorem. Therefore Γ has to be generated by a dilation. Then $U/\Gamma \to \Sigma$ is an embedding of a dilation cylinder of angle π as needed. □

Lemma 5. *Let $\varphi : \mathbb{D} \to \Sigma$ be an affine immersion of the open unit disk $\mathbb{D} \subset \mathbb{C}$ into Σ that does not extend continuously to $\partial\mathbb{D}$. Then:*

- *(i) there exists a point $z_0 \in \partial\mathbb{D}$ such that the limit $\lim_{z \to z_0} \varphi(z)$ does not exists;*
- *(ii) the immersion φ extends to the half-plane that contains $\mathbb{D}$ and is tangent to it at z_0;*
- *(iii) this extension is invariant under a dilation centered at z_0.*

Proof. (i) Assume the limits exist in every direction, consider a lift of the map $\varphi : \mathbb{D} \to \Sigma$ to the universal cover of Σ and compose it with the developing map to obtain an affine map $\tilde{\varphi} : \mathbb{D} \to \mathbb{C}$. Then this map can be continuously extended to $\overline{\mathbb{D}}$. Since the developing map is locally injective and the limits exist this extension pulls back to Σ, which contradicts the hypothesis. Therefore there exists $z_0 \in \partial\mathbb{D}$ such that $\lim_{z\to z_0} \varphi(z)$ does not exists. Without loss of generality, we will assume that $z_0 = 1$.

(ii) Let $\gamma : [0, 1) \to \Sigma$ be a path $t \mapsto \varphi(t)$. Then according to the assumption above the limit at $t = 1$ does not exists. By compactness there exists x an accumulation point of γ. Pick a small neighborhood of x, then γ has to enter and exit this neighborhood infinitely many times, otherwise it would have a limit. This implies that accumulation set contains a straight line segment passing through x, which contains a non-singular point.

Now assume x is non-singular. We will use the notation $B(z, r) \subset \mathbb{D}$ and $\Delta(z, r) \subset \mathbb{C}$ for open disks of radius r centered at z. Choose an affine chart $f : U \to \mathbb{C}$ around x such that $f(x) = 0$ and $f(U) = \Delta(0, 1)$.

We will show that there exists an open neighborhood V of x in Σ and a sequence $t_k \in [0, 1)$ converging to 1, such that $\varphi(B(t_k, 1 - t_k))$ contains V for any k. Note that we have chosen the disk $B(t_k, 1 - t_k)$ centered at t_k and tangent to the boundary of $\mathbb{D}$ at 1. Let us set the following notation :

- $V \subset \Sigma$ is the preimage of $\Delta(0, 1/4)$ under f;
- t_k is an increasing sequence in $[0, 1)$ such that $\varphi(t_k)$ converges to x and is contained in V;
- $V_k \subset U$ is the preimage of $\Delta\left((f \circ \varphi)(t_k), 1/2\right)$ under f;
- $B(t_k, r_k)$ is the connected component of $\varphi^{-1}(V_k)$ containing t_k.

First note that V_k contains V. Secondly, $r_k \leq 1 - t_k$, since φ does not extend to 1, hence $B(t_k, r_k) \subset B(t_k, 1-t_k)$. Therefore $V \subset V_k = \varphi(B(t_k, r_k)) \subset \varphi(B(t_k, 1-t_k))$.

In the following we will show that $f^{-1} : \Delta(0, 1/4) \to V$ extends to affine immersions on nested disks Δ_k of arbitrary large radius to Σ. Let $\psi_k : B(t_k, r_k) \to \Delta((f \circ \varphi)(t_k), 1/2)$ be the restriction of $f \circ \varphi$. This map is an affine bijection and it admits a unique affine extension $\Psi_k : \mathbb{D} \to \Delta_k$ for some $\Delta_k \subset \mathbb{C}$. We have the following diagram,

$$\begin{tikzcd} B(t_k, r_k) \arrow[r, "\psi_k"] \arrow[d, "i_1"] & \Delta\left((f \circ \varphi)(t_k), 1/2\right) \arrow[d, "i_2"] \\ \mathbb{D} \arrow[r, "\Psi_k"] & \Delta_k. \end{tikzcd}$$

The map Ψ_k preserves the ratio of the radii of disks, therefore

$$\frac{r(\Delta_k)}{1/2} = \frac{r(\mathbb{D})}{r_k} \geq \frac{1}{1 - t_k} \to \infty.$$

This implies that $r(\Delta_k) \to 0$.

We consider a subsequence k_i such that the radii satisfy $r(\Delta_{k_{i+1}}) > 2r(\Delta_{k_i})$. Since the centers of all balls belong to $\Delta(0,1/4)$ this condition guarantees that the balls Δ_{k_i} are nested. Taking the inverses of theses extensions of Ψ_{k_i} and post-composing them with φ we obtain affine immersions of nested disks $\Delta_{k_i} \to \Sigma$ that coincide with f^{-1} on $\Delta(0,1/4)$.

Now notice that all Δ_{k_i} are tangent at $l_0 = \lim_{t\to 1}\Psi_1(t)$. Indeed, there is no extension of $f^{-1} : \Delta(0,1/4) \to V$ to l_0, since φ does not extend to z_0. This implies that $l_0 \notin \Delta_k$ for all k. Since Δ_k are nested, they must be all tangent at l_0. The union of these nested balls with radius going to infinity is the half-plane tangent to Δ_{k_i} at l_0. Therefore we obtain an immersion of the half-plane into Σ, which extends $f^{-1} : \Delta(0,1/4) \to V$. Composing it with the affine transformation, which extends Ψ_{k_1} and sends Δ_{k_1} to $\mathbb{D}$, we obtain an affine extension of $\varphi : \mathbb{D} \to \Sigma$ to the half-plane tangent to $\mathbb{D}$ at z_0.

(iii) By Lemma 3 the extension can only be invariant under translations or a single dilation. Similarly to the argument in the proof of Lemma 4 we exclude translations and therefore our extension is invariant under a dilation. It remains to show that this dilation is centered at $z_0 = 1$. Note that the only accumulation point of an orbit of a dilation is the center of the dilation. Now recall that we constructed a sequence $t_k \in [0,1)$ converging to 1, such that $\varphi(B(t_k, 1-t_k))$ contains V, a neighborhood of x, for any k. Taking preimages of x inside $B(t_k, 1-t_k)$ we obtain a sequence of points in $\mathbb{D}$ accumulating to $z_0 = 1$ and invariant under dilation. Therefore z_0 is the center of the dilation. □

Proof of Theorem 4.
$\neg(2) \implies \neg(1)$: follows from Lemma 5.
$\neg(1) \implies \neg(3)$: follows from Lemma 4.
$\neg(3) \implies \neg(2)$: there exists an affine immersion of upper-half plane $\mathbb{H}$ to Σ equivariant with respect to multiplication by a real positive scalar. The embedding of the open disk of radius 1 centered at i does not extend to zero. □

We conclude by presenting a corollary of Theorem 4 that relates closed hyperbolic geodesics to affine immersions of disks which do not extend to the boundary.

Corollary 3 (Veech, [Vee08])**.** *Let $\varphi : \mathbb{D} \longrightarrow \Sigma$ be an affine immersion of the unit disk such that the path $\gamma : t \mapsto \varphi(te^{i\theta})$ does not have a limit in Σ as $t \to 1$ for some $\theta \in \partial\mathbb{D}$. Then γ is a hyperbolic closed geodesic in Σ.*

A.2. **Delaunay decompositions and triangulations.** The main results of [Vee97] and [Vee08] establish the existence of geodesic triangulations for dilation surfaces with property $\mathcal{V}$. These results together with Theorem 4 can be summarized by the following theorem:

Theorem (Veech)**.** *A dilation surface admits a geodesic triangulation if and only if it satisfies the property $\mathcal{V}$.*

This theorem is a corollary of the existence of *Delaunay polygonization* of dilation surfaces that Veech deals with. We give some details on this construction below. Let Σ be a dilation surface satisfying the property $\mathcal{V}$ and let $x \in \Sigma$ be a non-singular point. We are going to distinguish points depending on the number of singular points on the boundary of the maximal immersed disk centered at x. We denote this number by $\nu(x)$. The set $\{x \in \Sigma \mid \nu(x) = 1\}$ is an open dense set in the surface. The special points are those for which $\nu(x) \geq 3$. They form a discrete and therefore finite set on the surface. For each such x, consider the convex hull of the singular points on the boundary of the largest immersed disk centered at x. The following properties are satisfied :

- the image of such convex hull under the immersion is an **embedded** convex polygon in the surface;
- the union of such polygons covers the whole surface;
- such polygons only intersects at their boundary;
- the set of vertices of such polygons coincides with the set of singular points;
- the union of the interior of these polygons coincides with the set $\{x \in \Sigma \mid \nu(x) = 1\}$;
- the union of the interior of their sides coincides with the set $\{x \in \Sigma \mid \nu(x) = 2\}$.

This decomposition of the dilation surface Σ into a union of convex polygons is unique and it is called its **Delaunay polygonizations**. A further triangulation of each polygon leads to a *geodesic triangulation* of the surface.

The key assumption that one needs to make to carry out the construction above (see [Vee97] for details) is the property $\mathcal{V}$, in particular that maximal affine embeddings of the disk extend to their boundary.

Remark. *The converse of the triangulation theorem is quite easy. An affine cylinder of angle at least π behaves like a 'trap': any geodesic entering it never leaves it. Then no edge of the triangulation intersects the cylinder and therefore the surface does not admit a geodesic triangulation, otherwise the cell containing this cylinder would have a non-contractible loop.*

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