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COMPARING DESCENT OBSTRUCTION AND BRAUER-MANIN OBSTRUCTION FOR OPEN VARIETIES

YANG CAO, CYRIL DEMARCHE, AND FEI XU

Abstract. We provide a relation between Brauer-Manin obstruction and descent obstruction for torsors over not necessarily proper varieties under a connected linear algebraic group or a group of multiplicative type. Such a relation is also refined for torsors under a torus. The equivalence between descent obstruction and étale Brauer-Manin obstruction for smooth projective varieties is extended to smooth quasi-projective varieties, which provides the perspective to study integral points.

1. Introduction

The descent theory for tori was first established by Colliot-Thélène and Sansuc in [8] and was extended by Skorobogatov to groups of multiplicative type in [34]. In a series of papers [21], [23], [25], Harari and Skorobogatov introduced descent obstruction for a general algebraic group and compared the descent obstruction with the Brauer-Manin obstruction. By various works of Poonen [31], the second named author [14], Stoll [38] and Skorobogatov [35], it was proved that the descent obstruction is equivalent to the étale Brauer-Manin obstruction for smooth projective geometrically integral varieties. In this paper, we study the relation between the descent obstruction and the Brauer-Manin obstruction for open varieties by using new arithmetic tools developed in [2], [6], [9], [17], [22] and [27], and we extend the equivalence between the descent obstruction and the étale Brauer-Manin obstruction to smooth quasi-projective varieties.

Let $k$ be a number field, $\Omega_k$ the set of all primes of $k$ and $A_k$ the adelic ring of $k$. A variety over $k$ is defined to be a separated scheme $X$ of finite type over $k$. Fix an algebraic closure $\bar{k}$ of $k$. We denote by $X_\bar{k}$ the fibre product $X \times_k \bar{k}$. Let

$$Br(X) = H^2_{\text{ét}}(X, G_m), \quad Br_1(X) = \ker(\text{Br}(X) \to \text{Br}(X_\bar{k})) \quad \text{and} \quad Br_0(X) = \text{Im}(\text{Br}(k) \xrightarrow{\pi^*} \text{Br}(X))$$

where $X \xrightarrow{\pi} \text{Spec}(k)$ is the structure morphism, and $Br_a(X) = Br_1(X)/Br_0(X)$. For any subgroup $B$ of $Br(X)$, one can define the Brauer-Manin set

$$X(A_k)^B = \{(x_v)_{v \in \Omega_k} \in X(A_k) : \sum_{v \in \Omega_k} \text{inv}_v(\xi(x_v)) = 0 \text{ for all } \xi \in B\}$$

with respect to $B$. When $B = Br(X)$, we simply write this Brauer-Manin set as $X(A_k)^{Br}$.

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Suppose $Y \xrightarrow{f} X$ is a left torsor under a linear algebraic group $G$ over $k$. The descent obstruction (see [21], [23] and [25]) given by $f$ is defined by the following set

$$X(A_k)^f = \{ (x_v) \in X(A_k) : ([Y](x_v)) \in \text{Im}(H^1(k, G) \to \prod_{v \in \Omega_k} H^1(k_v, G)) \} = \bigcup_{\sigma \in H^1(k,G)} f_{\sigma}(Y^{\sigma}(A_k))$$

where $Y^{\sigma} \xrightarrow{f_{\sigma}} X$ is the twist of $Y \xrightarrow{f} X$ by a 1-cocycle representing $\sigma \in H^1(k, G)$. Moreover, one can define

$$X(A_k)^{desc} = \bigcap_{Y \xrightarrow{f} X} X(A_k)^f$$

following [31], where $Y \xrightarrow{f} X$ runs through all torsors under all linear algebraic groups over $k$.

The main results in this paper are the following theorems.

**Theorem 1.1.** (Theorem 3.5) Let $k$ be a number field, $G$ a connected linear algebraic group or a group of multiplicative type over $k$, and $X$ a smooth and geometrically integral variety over $k$. Suppose $Y \xrightarrow{f} X$ is a left torsor under $G$. For any subgroup $A \subseteq \text{Br}(X)$ which contains the kernel of the natural map $f^* : \text{Br}(X) \to \text{Br}(Y)$ we have

$$X(A_k)^A = \bigcup_{\sigma \in H^1(k,G)} f_{\sigma}(Y^{\sigma}(A_k)^{f_{\sigma}(A)})$$

where $Y^{\sigma} \xrightarrow{f_{\sigma}} X$ is the twist of $Y \xrightarrow{f} X$ by $\sigma$ and $\text{Br}(X) \xrightarrow{f_{\sigma}} \text{Br}(Y^{\sigma})$ is the associated pull-back map, for each $\sigma \in H^1(k, G)$.

When $G$ is a torus, this theorem can be refined in order to get Theorem 1.1 in §4. In particular, we prove:

**Theorem 1.2.** (Corollary 4.3) Under the same assumptions as in Theorem 1.1, if $G$ is assumed to be a torus, then

$$X(A_k)^{\text{Br}1(X)} = \bigcup_{\sigma \in H^1(k,G)} f_{\sigma}(Y^{\sigma}(A_k)^{\text{Br}1(Y^{\sigma})})$$

and

$$X(A_k)^{\text{Br}1} = \bigcup_{\sigma \in H^1(k,G)} f_{\sigma}(Y^{\sigma}(A_k)^{\text{Br}1(Y^{\sigma})} + f^*_{\sigma}(\text{Br}(X))).$$

This result is inspired by some lectures by Yonatan Harpaz. It should be pointed out that the first part in Theorem 1.2 was first obtained by Dasheng Wei in [39]: his proof uses an argument of Harari and Skorobogatov in [20] together with an exact sequence due to Sansuc (see [2], Theorem 2.8). Theorem 1.2 can be applied to study strong approximation, as in [39]. It should be noted that in general, the image of $\text{Br}(X)$ in $\text{Br}(Y^{\sigma})$ in Theorem 1.1 and Theorem 1.2 is not easy to describe, even under the assumption $\kappa[X]^x = \kappa^x$ (see [24, Theorem 1.7(b)]).

**Definition 1.3.** Let $X$ be a variety over a number field $k$ and let $B$ be a subgroup of $\text{Br}(X)$. For a finite subset $S$ of $\Omega_k$, we denote by $p^S : X(A_k) \to X(A_k^S)$ the projection map, where $A_k^S$ is the set of adeles of $k$ without $S$-components.
We say that $X$ satisfies strong approximation off $S$ if $X(\mathbb{A}_k) \neq \emptyset$ and the diagonal image of $X(k)$ is dense in $\text{pr}^S(X(\mathbb{A}_k))$.

We say that $X$ satisfies strong approximation with respect to $B$ off $S$ if $X(\mathbb{A}_k)^B \neq \emptyset$ and the diagonal image of $X(k)$ is dense in $\text{pr}^S(X(\mathbb{A}_k)^B)$.

Corollary 3.20 in [17] provides a sufficient condition for strong approximation with Brauer-Manin obstruction to hold for a connected linear algebraic group. As an application of Theorem 1.2, we prove that this sufficient condition is also a necessary condition:

Theorem 1.4. (Corollary 5.3) Let $G$ be a connected linear algebraic group over a number field $k$ and let $S$ be a finite subset of $\Omega_k$ containing $\infty_k$. Then $G$ satisfies strong approximation with respect to $\text{Br}_1(G)$ off $S$ if and only if $\prod_{v \in S} G'(k_v)$ is not compact for any non-trivial simple factor $G'$ of the semi-simple part $G^{ss}$ of $G$.

For any variety $X$ over a number field $k$, one can define, following [31]:

$$X(\mathbb{A}_k)^{\text{et,Br}} = \bigcap_{Y \rightarrow X} \bigcup_{\sigma \in H^1(k,F)} f_{\sigma}(Y^\sigma(\mathbb{A}_k)^{\text{Br}}),$$

where $Y \rightarrow X$ runs through all torsors under all finite group schemes $F$ over $k$. The last two sections of the paper are devoted to the proof of the following generalization of [14] and [35]:

Theorem 1.5. (Corollary 6.7 and Theorem 7.6) If $X$ is a smooth quasi-projective and geometrically integral variety over a number field $k$, then

$$X(\mathbb{A}_k)^{\text{desc}} = X(\mathbb{A}_k)^{\text{et,Br}}.$$
2. Brauer groups of torsors

In this section, we assume that $k$ is an arbitrary field of characteristic 0.

**Lemma 2.1.** Let $H$ be a semi-simple simply connected group or a unipotent group over $k$. Suppose $X$ is a smooth and geometrically integral variety over $k$. If $Z \xrightarrow{\rho} X$ is a torsor under $H$, then the induced map $\text{Br}(X) \xrightarrow{\rho^*} \text{Br}(Z)$ is an isomorphism.

**Proof.** We first show that $\text{Br}(X) \xrightarrow{\rho^*} \text{Br}(X \times_k H)$, where the map is induced by the natural projection $X \times_k H \rightarrow X$. Using the spectral sequence

$$H^n(k, H^q(X, \mathbb{G}_m)) \Rightarrow H^{n+q}(X, \mathbb{G}_m),$$

one only needs to show that

$$\bar{k}[X_k]^\times / \bar{k}^\times \xrightarrow{\sim} \bar{k}[X_k \times_k H_k]^\times / \bar{k}^\times,$$

Pic($X_k$) $\xrightarrow{\sim}$ Pic($X_k \times_k H_k$) and $\text{Br}(X_k) \xrightarrow{\sim} \text{Br}(X_k \times_k H_k)$.

Since $\bar{k}[H]^\times = \bar{k}^\times$ and Pic($H_k$) = $\text{Br}(H_k) = 0$ by [32] Proposition 6.10]. To prove the last part, Kummer exact sequence ensures that one only needs to prove that

$$H^2_{\acute{e}t}(X_k, \mathbb{Z}/n) \xrightarrow{\sim} H^2_{\acute{e}t}(X_k \times_k H_k, \mathbb{Z}/n)$$

for all $n \geq 1$. This last isomorphism follows from [37] Proposition 2.2] and [13] Exposé XI, Théorème 4.4] with $H^i_{\acute{e}t}(H_k, \mathbb{Z}/n) = 0$ for $i = 1, 2$. So we proved the required isomorphism $\text{Br}(X) \xrightarrow{\sim} \text{Br}(X \times_k H)$.

Let us now deduce Lemma 2.1: since Pic($H$) = 0, [2] Proposition 2.4] gives the following short exact sequence

$$0 \rightarrow \text{Br}(X) \rightarrow \text{Br}(Z) \xrightarrow{m^*-p_Z^*} \text{Br}(H \times_k Z),$$

where $m^*$ and $p_Z^*$ are induced by the multiplication map $H \times_k Z \xrightarrow{m} Z$ and the projection map $H \times_k Z \xrightarrow{p_Z} Z$ respectively. Since $m \circ (1_H \times \text{id}) = p_Z \circ (1_H \times \text{id}) = \text{id}$, one concludes that $m^* = p_Z^*$ by the above argument. Therefore $\text{Br}(X) \xrightarrow{\sim} \text{Br}(Z)$. \qed

Let $H$ be a closed subgroup of an algebraic group $G$ over $k$, and $Y \xrightarrow{1} X$ be a left torsor under $H$. Let $Z \xrightarrow{\rho} X$ be the left torsor under $G$ defined by the contracted product $Z = G \times^H Y$ (see [36] Example 3 in p.21]): the torsor $Z$ is the push-forward of $Y$ by the homomorphism $H \rightarrow G$. The projection map $G \times_k Y \xrightarrow{pr_G} G$ induces the following commutative diagram

$$
\begin{array}{ccc}
G \times_k Y & \longrightarrow & Z = G \times^H Y \\
pr_G \downarrow & & \downarrow \theta \\
G & \xrightarrow{\pi} & G/H,
\end{array}
$$

where $\theta$ is induced by $pr_G$ via the quotient by $H$.

**Lemma 2.4.** With the above notations, for any $\gamma \in (G/H)(k)$, the composite map $\theta^{-1}(\gamma) \rightarrow Z \xrightarrow{\rho} X$ is naturally a left torsor under $H^\sigma$, which is canonically isomorphic to the twist of $Y \xrightarrow{\rho} X$ by the $k$-torsor $\pi^{-1}(\gamma)$ under $H$. 


Proof. It follows from diagram \(2.3\) and \[36\] Example 2 in p.20.

Let \(G\) be a connected linear algebraic group over \(k\), and \(Y\) be a smooth variety over \(k\). Since \(G_k\) is rational over \(\bar{k}\) by Bruhat decomposition, the projections \(G \times_k Y \to G\) and \(G \times_k Y \to Y\) induce an isomorphism

\[
\text{Br}_a(G) \oplus \text{Br}_a(Y) \xrightarrow{\sim} \text{Br}_a(G \times_k Y)
\]

by \[32\] Lemma 6.6. If \(P\) is a (left) torsor under \(G\) over \(k\) and \(H^3(k, \bar{k}^\times) = 0\), the previous result generalizes to an isomorphism

\[
\text{Br}_a(P) \oplus \text{Br}_a(Y) \xrightarrow{\sim} \text{Br}_a(P \times Y)
\]

(2.5)

by \[3\] Lemma 5.1.

Let \(G\) be a connected linear algebraic group over \(k\) and let \(X\) be a smooth variety over \(k\) with \(H^3(k, \bar{k}^\times) = 0\). Suppose that \(Y \xrightarrow{f} X\) is a left torsor under \(G\) and \(P\) is a left \(k\)-torsor under \(G\), associated to a cocycle \(\sigma \in Z^1(k, G)\). One can consider \(P\) as a right torsor under \(G\) by defining a right action \(x \circ g := g^{-1}x\) (see \[36\] Example 2 in p.20). This right torsor is called the inverse right torsor of \(P\) under \(G\), and is denoted by \(P^\sigma\). One can now consider the map given by the quotient of \(P \times_k Y\) by the diagonal action of \(G\) given by \(g \cdot (p, y) := (p \circ g^{-1}, g \cdot y) = (g \cdot p, g \cdot y)\):

\[
\chi_P : P \times_k Y \to Y^\sigma := P^\sigma \times^G Y.
\]

Definition 2.6. With the above notation, assuming that \(H^3(k, \bar{k}^\times) = 0\), consider the map

\[
\psi_\sigma = \psi_P : \text{Br}_a(Y^\sigma) \xrightarrow{\chi_P^*} \text{Br}_a(P \times_k Y) \xleftarrow{\sim} \text{Br}_a(P) \oplus \text{Br}_a(Y) \to \text{Br}_a(Y).
\]

The following lemma, which compares the algebraic Brauer groups of twists of a given torsor, can be regarded as an extension of \[39\] Lemma 1.3] to torsors under connected linear algebraic groups.

Lemma 2.7. The morphism \(\psi_\sigma\) in Definition 2.6 is an isomorphism.

Proof. The natural morphism \((\text{pr}_P, \chi_P) : P \times_k Y \to P \times_k Y^\sigma\) is an isomorphism, and we have a commutative diagram:

\[
\begin{array}{ccc}
P \times_k Y & \xrightarrow{(\text{pr}_P, \chi_P)} & P \times_k Y^\sigma \\
\text{pr}_P \downarrow & & \downarrow \text{pr}_P \\
P & \xrightarrow{\text{id}} & P.
\end{array}
\]

Therefore \((\text{pr}_P, \chi_P)^* : \text{Br}_a(Y^\sigma \times_k P) \to \text{Br}_a(Y \times P)\) induces the identity map on the subgroups \(\text{Br}_a(P) \subset \text{Br}_1(Y^\sigma \times_k P)\) and \(\text{Br}_a(P) \subset \text{Br}_1(Y \times_k P)\), hence

\[
\psi_\sigma : \text{Br}_a(Y^\sigma) \to \text{Br}_a(Y^\sigma \times_k P) \xrightarrow{(\text{pr}_P, \chi_P)^*} \text{Br}_a(Y \times P) \to \text{Br}_a(Y)
\]

is an isomorphism (using the isomorphism (2.5)).

Let \(f : Y \to X\) be a torsor under a connected linear algebraic group \(G\) over \(k\) and let

\[
a_Y : G \times_k Y \to Y
\]
be the action of $G$. There is a canonical map $\lambda : \text{Br}_1(Y) \to \text{Br}_a(G)$ by [32] Lemma 6.4. Let $e : \text{Br}_a(G) \to \text{Br}_1(G)$ be the section of $\text{Br}_1(G) \to \text{Br}_a(G)$ such that $1_G \circ e = 0$. If $X$ is smooth and geometrically integral, then the following diagram

\begin{equation}
\begin{array}{ccc}
\text{Br}_1(Y) & \xrightarrow{\lambda} & \text{Br}_a(G) \\
\downarrow & & \downarrow p^*_G \circ e \\
\text{Br}(Y) & \xrightarrow{a^*_Y - p^*_Y} & \text{Br}(G \times_k Y)
\end{array}
\end{equation}

commutes by [2] Theorem 2.8, where $G \times_k Y \xrightarrow{p^*_G} G$ and $G \times_k Y \xrightarrow{p^*_Y} Y$ are the projections. One can reformulate the commutative diagram (2.8) in the following proposition:

**Proposition 2.9.** With the above notation, one has

$$b(t \cdot x) = \lambda(b)(t) + b(x)$$

for any $x \in Y(k)$, $t \in G(k)$ and $b \in \text{Br}_1(Y)$.

**Proof.** The commutativity of diagram (2.8) implies that

$$a^*_Y - p^*_Y = p^*_G \circ e \circ \lambda : \text{Br}_1(Y) \to \text{Br}_1(G \times Y),$$

therefore one has

$$b(t \cdot x) = a^*_Y(b)(t, x) = p^*_Y(b)(t, x) + p^*_G \circ e \circ \lambda(b)(t, x) = b(x) + \lambda(b)(t)$$

as required. \qed

3. CONNECTED LINEAR ALGEBRAIC GROUPS OR GROUPS OF MULTIPLICATIVE TYPE

In this section, we study the relation between the descent obstruction and the Brauer-Manin obstruction for a general connected linear group or a group of multiplicative type.

First we need the following fact concerning topological groups:

**Lemma 3.1.** Let $f : M \to N$ be an open homomorphism of topological groups. If $K$ is a closed subgroup of $M$ containing $\ker(f)$, then $f(K)$ is a closed subgroup of $N$.

**Proof.** Since $K$ is a closed subgroup containing $\ker(f)$, one has

$$f(K) = f(M) \setminus f(M \setminus K).$$

Since $f$ is an open homomorphism, $f(M)$ is an open subgroup of $N$. This implies that $f(M)$ is closed in $N$. Since $f(M \setminus K)$ is open in $N$, one concludes that $f(K)$ is closed in $N$. \qed

**Remark 3.2.** The assumption $K \supset \ker(f)$ in Lemma 3.1 can not be removed. For example, the projection map $p^S : \mathbf{A}_k^S \to \mathbf{A}_k^S$ is open where $\mathbf{A}_k^S$ is the set of adeles of $k$ without $S$-component. It is clear that $k$ is a discrete subgroup of $\mathbf{A}_k$ by the product formula. However $k$ is dense in $\mathbf{A}_k^S$ by strong approximation for $\mathbb{G}_a$, when $S$ is not empty.

For a short exact sequence of connected linear algebraic groups, one has the following result.
Proposition 3.3. Let

\[ 1 \to G_1 \xrightarrow{\psi} G_2 \xrightarrow{\phi} G_3 \to 1 \]

be a short exact sequence of connected linear algebraic groups over a number field \( k \). Then

1. \( \phi \left( G_2(A_k)^{Br_1(G_2)} \right) \) is a closed subgroup of \( G_3(A_k) \).
2. If \( G'(k_\infty) \) is not compact for each simple factor \( G' \) of the semi-simple part of \( G_3 \), then one has

\[ G_3(A_k)^{Br_1(G_3)} = G_3(k) \cdot \phi \left( G_2(A_k)^{Br_1(G_2)} \right) \]

Proof. Let \( S \) be a sufficiently large finite set of primes of \( \Omega_k \) containing \( \infty_k \) and let \( G_1 \) (resp. \( G_2 \), resp. \( G_3 \)) be a smooth group scheme model of \( G_1 \) (resp. \( G_2 \), resp. \( G_3 \)) over \( O_S \) with connected fibres, such that the short exact sequence of smooth group schemes

\[ 1 \to G_1 \xrightarrow{\psi} G_2 \xrightarrow{\phi} G_3 \to 1 \]

extends the given short exact sequence of their generic fibres. The set \( H^1(\text{et}, G_1) \) is trivial by Hensel’s lemma together with Lang’s theorem, and the following diagram

\[ \begin{array}{cccccccc}
G_3(O_v) & \xrightarrow{\partial_v} & H^1(\text{et}, G_3) \\
\downarrow & & \downarrow \\
G_3(k_v) & \xrightarrow{\partial_v} & H^1(k_v, G_3)
\end{array} \]

commutes, hence we deduce the following commutative diagram of exact sequences in Galois cohomology:

\[ \begin{array}{cccccccc}
G_1(k) & \xrightarrow{\psi} & G_2(k) & \xrightarrow{\phi} & G_3(k) & \xrightarrow{\partial} & H^1(k, G_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G_1(A_k) & \xrightarrow{\psi} & G_2(A_k) & \xrightarrow{\phi} & G_3(A_k) & \xrightarrow{\partial} & \bigoplus_{v \in \Omega_k} H^1(k_v, G_1).
\end{array} \]

In addition, [17, Theorem 5.1] and [32, Corollary 6.11] gives the following commutative diagram of exact sequences of topological groups and pointed topological spaces:

\[ \begin{array}{cccccccc}
G_1(A_k) & \xrightarrow{\theta_1} & \text{Br}_a(G_1)^D & \to & \text{III}^1(k, G_1) \\
\downarrow^{(\phi_1)} & & \downarrow^{(\psi^*)^D} \\
G_2(A_k) & \xrightarrow{\theta_2} & \text{Br}_a(G_2)^D \\
\downarrow^{(\phi_2)} & & \downarrow^{(\psi^*)^D} \\
G_3(A_k) & \xrightarrow{\theta_3} & \text{Br}_a(G_3)^D \\
\downarrow^{(\phi_3)} & & \downarrow^{(\psi^*)^D} \\
\bigoplus_{v \in \Omega_k} H^1(k_v, G_1).
\end{array} \]
where \( \text{Br}_a(G_i)^D \) is the topological dual of the discrete group \( \text{Br}_a(G_i) \), for \( 1 \leq i \leq 3 \). Since \( \theta_1(G_1(A_k)) \) is the kernel of the continuous map \( \text{Br}_a(G_1)^D \to \text{III}^1(k, G_1) \), it is a closed subgroup of \( \text{Br}_1(G)^D \). Since \( (\psi^*)^D \) is a closed map, one obtains that \( (\psi^*)^D(\theta_1(G_1(A_k))) \) is a closed subgroup of \( \text{Br}_1(G_2)^D \). It implies that

\[
\ker(\theta_2) \cdot \psi(G_1(A_k)) = \theta_2^{-1} [(\psi^*)^D(\theta_1(G_1(A_k)))]
\]

is a closed subgroup of \( G_2(A_k) \) by diagram (3.4). Proposition 6.5 in Chapter 6 of [30] ensures that \( \phi : G_2(A_k) \to G_3(A_k) \) is an open homomorphism of topological groups. Then \( \phi(\ker(\theta_2)) = \phi(G_2(A_k)^{\text{Br}_1(G_2)}) \) is closed by Lemma 3.11, and property (1) follows.

Let us now prove statement (2): Corollary 3.20 in [17] (see also the proof of Proposition 4.5 in [5]) implies that

\[
\ker(\theta_3) = G_3(A_k)^{\text{Br}_1(G_3)} = G_3(k) \cdot \phi(G_2(A_k)^{\text{Br}_1(G_2)}),
\]

where \( G_3(k_{\infty})^0 \) is the connected component of identity with respect to the topology of \( k_{\infty} \). One only needs to show that

\[
G_3(A_k)^{\text{Br}_1(G_3)} \subseteq G_3(k) \cdot \phi(G_2(A_k)^{\text{Br}_1(G_2)}).
\]

For any \( (x_v) \in G_3(k) \cdot G_3(k_{\infty})^0 \), there is \( h \in G_3(k) \) and \( h_{\infty} \in G_3(k_{\infty}) \) such that

\[
(\partial_v)(h \cdot h_{\infty}) = (\partial_v)(x_v),
\]

because \( (\partial_v) \) is a continuous map with respect to the discrete topology of \( \bigoplus_{v \in \Omega_k} H^1(k_v, G_1) \). Since \( \phi_{\infty}(G_2(k_{\infty})^0) \) is open and connected, the finiteness of \( H^1(k_{\infty}, G_1) \) gives

\[
G_3(k_{\infty})^0 = \phi_{\infty}(G_2(k_{\infty})^0).
\]

Therefore

\[
(h \cdot h_{\infty}) \in G_3(k) \cdot \phi(G_2(A_k)^{\text{Br}_1(G_2)})
\]

and one can replace \( (x_v) \) by \( (h \cdot h_{\infty})^{-1} \cdot (x_v) \). Without loss of generality, one can therefore assume \( (\partial_v)(x_v) \) is the trivial element in \( \bigoplus_{v \in \Omega_k} H^1(k_v, G_1) \).

Since \( \text{III}^1(k, G_1) \) is finite, one can fix \( \xi_1, \cdots, \xi_n \in G_3(k) \) such that each element of \( \text{III}^1(k, G_1) \cap \partial(G_3(k)) \) is represented by one of the \( \xi_i \)'s. As \( \partial_{\infty}(h_{\infty}) \) is trivial for any \( h_{\infty} \in G_3(k_{\infty})^0 \), one concludes that

\[
(x_v) \in \bigcup_{i=1}^n \xi_i \phi(\ker(\theta_2)) = \bigcup_{i=1}^n \xi_i \cdot \phi(\ker(\theta_2)) \subseteq G_3(k) \cdot \phi(G_2(A_k)^{\text{Br}_1(G_2)})
\]

by Corollary 1 in Page 50 of [53] and assertion (1).

The main result of this section is the following theorem:

**Theorem 3.5.** Let \( X \) be a smooth and geometrically integral variety and let \( G \) be a connected linear algebraic group or a group of multiplicative type over a number field \( k \). Suppose that \( f : Y \to X \) is a left torsor under \( G \). If \( A \) is a subgroup of \( \text{Br}(X) \) which contains the kernel of the natural map \( f^* : \text{Br}(X) \to \text{Br}(Y) \), then

\[
X(A_k)^4 = \bigcup_{\sigma \in H^1(k, G)} f_\sigma(\text{III}^\sigma(A_k)^{f_\sigma(A)})
\]
where $Y^\sigma \xrightarrow{f_\sigma} X$ is the twist of $f$ by $\sigma$ and $Br(X) \xrightarrow{f_*} Br(Y^\sigma)$ is the associated pull-back morphism, for each $\sigma \in H^1(k,G)$.

**Proof.** By the functoriality of Brauer-Manin pairing, one only needs to show that

$$X(A_k)^A \subseteq \bigcup_{\sigma \in H^1(k,G)} f_\sigma (Y^\sigma(A_k))^{f_\sigma(A)} .$$

It is clear that

$$(x_v) \in \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(A_k)) \Leftrightarrow ([Y](x_v)) \in \text{Im} \left[ H^1(k,G) \to \prod_{v \in \Omega_k} H^1(k_v,G) \right] . \quad (3.6)$$

(1) Assume that $G$ is connected.

Recall first that Hensel’s lemma together with Lang’s theorem ensures that $H^1(k,G)$ maps to $\bigoplus_{v \in \Omega_k} H^1(k_v,G)$. Since any element $P \in \text{Pic}(G)$ can be given the structure of a central extension of algebraic groups

$$1 \to \mathbb{G}_m \to P \to G \to 1 \quad (3.7)$$

by [6 Corollary 5.7], one obtains a coboundary map

$$\partial_P : H^1(X,G) \to H^2(X,\mathbb{G}_m) = Br(X)$$

associated to $P$ (see [19 IV.4.4.2]). Then the map defined by

$$\Delta_{Y/X} : \text{Pic}(G) \to Br(X), \quad P \mapsto \partial_P([Y])$$

appears in the following short exact sequence (see [2 Theorem 2.8])

$$\text{Pic}(G) \xrightarrow{\Delta_{X/Y}} Br(X) \xrightarrow{f_*} Br(Y) . \quad (3.8)$$

For any $v \in \Omega_k$, the exact sequence [3.7] defines a coboundary map

$$\partial^k_v : H^1(k_v,G) \to H^2(k_v,\mathbb{G}_m) = Br(k_v) .$$

One can therefore define a pairing

$$\delta_v : H^1(k_v,G) \times \text{Pic}(G) \to Br(k_v) \subseteq \mathbb{Q}/\mathbb{Z}, \quad (\sigma_v, P) \mapsto \partial^k_v(\sigma_v)$$

such that the following diagram

$$\begin{array}{ccc}
X(k_v) & \times & \text{Br}(X) & \xrightarrow{e_v} & \text{Br}(k_v) \\
\downarrow{[Y]} & & \downarrow{\Delta_{X/Y}} & & \downarrow{\text{id}} \\
H^1(k_v,G) & \times & \text{Pic}(G) & \xrightarrow{\delta_v} & \text{Br}(k_v)
\end{array} \quad (3.9)$$

commutes (see Proposition 2.9 in [9]). These pairings induce a pairing

$$(\delta_v)_{v \in \Omega_k} : \bigoplus_{v \in \Omega_k} H^1(k_v,G) \times \text{Pic}(G) \to \mathbb{Q}/\mathbb{Z}, \quad ((\sigma_v)_{v \in \Omega_k}, P) \mapsto \sum_{v \in \Omega_k} \delta_v(\sigma_v, P) \in \mathbb{Q}/\mathbb{Z}$$
and a natural exact sequence of pointed sets
\[ H^1(k,G) \to \bigoplus_{v \in \Omega_k} H^1(k_v,G) \to \text{Hom}(\text{Pic}(G),\mathbb{Q}/\mathbb{Z}) \]
by [9, Theorem 3.1]. Therefore (3.6) is equivalent to the fact that \([Y](x_v) \in \bigoplus_{v \in \Omega_k} H^1(k_v,G)\) is orthogonal to \(\text{Pic}(G)\) for the pairing \((\delta_v)_{v \in \Omega_k}\). The commutative diagram (3.9), together with (3.8), gives
\[ X(\mathbb{A}_k) \ker(f^*) = \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbb{A}_k)). \]
Since \(\ker(f^*) \subseteq A\), one has
\[ X(\mathbb{A}_k)^A \subseteq X(\mathbb{A}_k) \ker(f^*) = \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbb{A}_k)). \]
Then the functoriality of the Brauer-Manin pairing implies that
\[ X(\mathbb{A}_k)^A \subseteq \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbb{A}_k))^{G(A)}. \]

(2) When \(G\) is a group of multiplicative type, one obtains that (3.6) is equivalent to
\[ \sum_{v \in \Omega_k} \text{inv}_v(\chi \cup [Y])(x_v) = 0 \]
for all \(\chi \in H^1(k,\hat{G})\) by [16, Theorem 6.3]. Let
\[ \mathcal{K}_f = \langle \{\chi \cup [Y] : \chi \in H^1(k,\hat{G})\} \rangle \]
be the subgroup of \(\text{Br}(X)\) generated by elements \(\chi \cup [Y]\), where \(\cup\) is the cup product
\[ \cup : H^1(k,\hat{G}) \times H^1(X,G) \to H^2(X,\mathbb{G}_m) = \text{Br}(X). \]
Then
\[ X(\mathbb{A}_k)^{\mathcal{K}_f} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbb{A}_k)) \]
by [26, Proposition 3.1]. Functoriality of the cup product proves that the following diagram
\[ H^1(k,\hat{G}) \times H^1(X,G) \xrightarrow{\cup} H^2(X,\mathbb{G}_m) = \text{Br}(X) \]
\[ \xrightarrow{id \times f^*} \]
\[ H^1(k,\hat{G}) \times H^1(Y,G) \xrightarrow{\cup} H^2(Y,\mathbb{G}_m) = \text{Br}(Y) \]
is commutative. Since \(Y \xrightarrow{f} X\) becomes a trivial torsor over \(Y\), the above diagram gives \(\mathcal{K}_f \subseteq \ker(f^*)\). Since \(\mathcal{K}_f \subseteq \ker(f^*) \subseteq A\), one has
\[ X(\mathbb{A}_k)^A \subseteq X(\mathbb{A}_k)^{\mathcal{K}_f} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y^\sigma(\mathbb{A}_k)). \]
Then the functoriality of the Brauer-Manin pairing implies that
\[ X(\mathbb{A}_k)^A \subseteq \bigcup_{\sigma \in H^1(k,G)} f_\sigma \left( Y^\sigma(\mathbb{A}_k)^{G(A)} \right). \]

\[ \square \]

4. Refinement in the toric case

In this section, we will refine Theorem 3.5 for torsors under tori.

**Theorem 4.1.** Let \( f : Y \to X \) be a torsor under a torus \( G \) over a number field \( k \). Assume that \( X \) is smooth and geometrically integral. Let \( \ker(f^*) \subseteq A \subseteq \text{Br}(X) \) be a subgroup, and for all \( \sigma \in H^1(k,G) \), let \( B_\sigma \subseteq \text{Br}_1(Y^\sigma) \) be a subgroup such that
\[
 f^*\left( \sum_{\sigma \in H^1(k,G)} \psi_\sigma(\widehat{B}_\sigma) \right) \subseteq A,
\]
where \( \text{Br}_a(Y^\sigma) \xrightarrow{\psi_\sigma} \text{Br}_a(Y) \) is the morphism of Definition 2.6 and \( \widehat{B}_\sigma \) is the image of \( B_\sigma \) in \( \text{Br}_a(Y^\sigma) \).

Then one has
\[ X(\mathbb{A}_k)^A = \bigcup_{\sigma \in H^1(k,G)} f_\sigma \left( Y^\sigma(\mathbb{A}_k)^{B_\sigma+f^*_{\sigma}(A)} \right) \]
where \( Y^\sigma f_\sigma X \) is the twist of \( Y f X \) by \( \sigma \).

**Proof.** Since
\[
 \bigcup_{\sigma \in H^1(k,G)} f_\sigma \left( Y^\sigma(\mathbb{A}_k)^{B_\sigma+f^*_{\sigma}(A)} \right) \subseteq \bigcup_{\sigma \in H^1(k,G)} f_\sigma \left( Y^\sigma(\mathbb{A}_k)^{f^*_{\sigma}(A)} \right) \subseteq X(\mathbb{A}_k)^A
\]
by the functoriality of Brauer-Manin pairing, one only needs to prove the converse inclusion.

Step 1. We first prove the result when \( \widehat{G} \) is a permutation Galois module. In this case, Shapiro Lemma and Hilbert 90 gives \( H^1(K,G) = \{1\} \) for any field extension \( K/k \). This implies that
\[
 X(\mathbb{A}_k)^A = f \left( Y(\mathbb{A}_k)^{f^*_{\sigma}(A)} \right)
\]
by the functoriality of Brauer-Manin pairing.

Let \( (x_v) \in X(\mathbb{A}_k)^A \). Then there is \( (y_v) \in Y(\mathbb{A}_k)^{f^*_{\sigma}(A)} \) such that \( (x_v) = f((y_v)) \).

By Proposition 6.10 (6.10.3) in [22], the natural sequence
\[
 \text{Br}_1(X) \xrightarrow{f^*} \text{Br}_1(Y) \xrightarrow{\lambda} \text{Br}_a(G)
\]
is exact, and it induces the exact sequence
\[
 (f^*)^{-1}(B) \xrightarrow{f^*} B \xrightarrow{\lambda} \text{Br}_a(G)
\]
for any subgroup \( B \subseteq \text{Br}_1(Y) \). Therefore the following sequence
\[
 \text{Br}_a(G)^D \xrightarrow{\lambda^D} B^D \xrightarrow{(f^*)^D} ((f^*)^{-1}(B))^D
\]
is exact. Assuming \((f^*)^{-1}(B) \subseteq A\), one has \((f^*)^D([y_v]) = 0\), where we (abusively) identify \((y_v)\) with its image in \(B^D\) via the Brauer-Manin pairing. By the aforementioned exactness, there is \(\xi \in Br_a(G)^D\) such that \(\lambda^D(\xi) = (y_v)\). Since \(\text{III}^1(k, G) = \{1\}\), Theorem 2 in [22] implies that every element in \(Br_a(G)^D\) is given by an element in \(G(A_k)\) via the Brauer-Manin pairing. Namely, there is \((y_v) \in G(A_k)\) such that

\[b(y_v) = \lambda(b)(y_v)\]

for all \(b \in B\). Then \((y_v)^{-1} \cdot (y_v) \in Y(A_k)^{B + f^*(A)}\) by Proposition 6.9 in [27], and \((x_v) = f((y_v)^{-1} \cdot (y_v))\).

Step 2. We now prove the case of an arbitrary torus \(G\). By Proposition-Definition 3.1 in [6], there is a short exact sequence of tori

\[1 \rightarrow G \rightarrow T_0 \xrightarrow{\xi} T_1 \rightarrow 1\]

such that \(\tilde{T}_0\) is a permutation Galois module and \(\tilde{T}_1\) is a coflasque Galois module. Since

\[H^3(k, \tilde{T}_1) \cong \prod_{v \in \infty_k} H^3(k_v, \tilde{T}_1) \cong \prod_{v \in \infty_k} H^1(k_v, \tilde{T}_1) = \{1\}\]

(see for instance Proposition 5.9 in [27]), the map \(Br_1(T_0) \rightarrow Br_1(G)\) is surjective.

Let \(Z \xrightarrow{\xi} X\) be the torsor under \(T_0\) defined by \(Z := T_0 \times^G Y\). We have a morphism of torsors under \(G\):

\[
\begin{array}{ccc}
Y & \xrightarrow{\epsilon_0 \times \text{id}_Y} & T_0 \times_k Y \\
\downarrow p_0 & & \downarrow \theta \\
T_0 & \xrightarrow{q} & T_1
\end{array}
\]

where \(\epsilon_0 \in T_0(k)\) is the unit element, \(p_0\) is the projection map and \(\theta\) is given as in (2.3). For simplicity, denote by \(i := \chi \circ (\epsilon_0 \times \text{id}_Y) : Y \rightarrow Z\) the composite morphism defined in the previous diagram.

Then Proposition 6.10 (6.10.3) in [32] gives the following commutative diagram of exact sequences:

\[
\begin{array}{ccc}
\text{Br}_1(T_1) & \xrightarrow{q^*} & \text{Br}_1(T_0) & \longrightarrow & \text{Br}_a(G) \\
\downarrow \theta^{-1} & & \downarrow p_0^{-1} & & \downarrow \text{id} \\
\text{Br}_1(Z) & \xrightarrow{\chi^*} & \text{Br}_1(T_0 \times_k Y) & \longrightarrow & \text{Br}_a(G)
\end{array}
\]

Since the following sequence

\[
\text{Br}_1(T_0) \xrightarrow{p_0^{-1}} \text{Br}_1(T_0 \times_k Y) \xrightarrow{(\epsilon_0 \times \text{id}_Y)^*} \text{Br}_a(Y) \rightarrow 1
\]

is exact by Lemma 6.6 in [32], the surjectivity of the map \(\text{Br}_1(T_0) \rightarrow \text{Br}_1(G)\) implies that the morphism

\[i^* : \text{Br}_1(Z) \rightarrow \text{Br}_1(Y)\]

is surjective, by a simple diagram chase.
Lemma 2.4 implies that for any \( t \in T_1(k) \), the composite morphism \( \theta^{-1}(t) \to Z \xrightarrow{\rho} X \) is canonically isomorphic to the twist \( f_t : Y^{q^{-1}(t)} \to X \) of \( f : Y \to X \) by the Spec(\( k \))-torsor \( q^{-1}(t) \) under \( G \).

Denote by \( i_t : \theta^{-1}(t) \to Z \) the closed immersion. Then \( f_t = \rho \circ i_t \) for any \( t \in T_1(k) \).

Let \( \chi_t \) be the restriction of \( \chi \) to \( q^{-1}(t) \times_k Y \) for any \( t \in T_1(k) \). Then the following diagram

\[
\begin{array}{ccc}
q^{-1}(t) \times_k Y & \xrightarrow{\chi_t} & Y^{q^{-1}(t)} \\
\downarrow j_t \times \text{id}_Y & & \downarrow i_t \\
Y & \xrightarrow{e_0 \times \text{id}_Y} & T_0 \times_k Y \\
\downarrow p_0 & & \downarrow \theta \\
G & \longrightarrow & T_0 \\
\end{array}
\]

is commutative, where \( j_t : q^{-1}(t) \to T_0 \) is the closed immersion of the fiber of \( q \) at \( t \). Therefore Definition 2.6 implies that we have a commutative triangle:

\[
\begin{array}{ccc}
\text{Br}_a(Z) & \xrightarrow{i_t^*} & \text{Br}_a(Y^{q^{-1}(t)}) \\
\downarrow i_t^* & & \downarrow \sim \psi_{q^{-1}(t)} \\
& \text{Br}_a(Y), &
\end{array}
\]

i.e. that \( \psi_{q^{-1}(t)} \circ i_t^* = i^* \).

Let

\[
B = i^*^{-1} \left( \sum_{t \in T_1(k)} \psi_{q^{-1}(t)} \left( \widetilde{B_{q^{-1}(t)}} \right) \right) \subseteq \text{Br}_a(Y)
\]

where \( \widetilde{B_{q^{-1}(t)}} \) is the image of \( B_{q^{-1}(t)} \) in \( \text{Br}_a(Y^{q^{-1}(t)}) \) and \( \psi_{q^{-1}(t)} \) is given by Definition 2.6 for all \( t \in T_1(k) \).

Since \( i^* \circ \rho^* = f^* \), we have

\[
\rho^{q^{-1}}(B) = f^* \left( \sum_{t \in T_1(k)} \psi_{q^{-1}(t)} \left( \widetilde{B_{q^{-1}(t)}} \right) \right) \subseteq A,
\]

hence step 1 applied to the torsor \( Z \xrightarrow{\rho} X \) under \( T_0 \) implies that

\[
X(A_k)^A = \rho \left( Z(A_k)^{B+\rho^*(A)} \right). \tag{4.2}
\]

Let \((x_v) \in X(A_k)^A\). By (4.2), there is \((z_v) \in Z(A_k)^{B+\rho^*(A)}\) such that \((x_v) = \rho((z_v))\). Since

\[
i^* \circ \theta^*(\text{Br}_1(T_1)) = (e_0 \times \text{id}_Y)^* \circ p_0^* \circ q^*(\text{Br}_1(T_1)) = \text{Br}_0(Y)
\]

and \( i^*(\text{Br}_0(Z)) = \text{Br}_0(Y) \), one gets \( \theta^*(\text{Br}_1(T_1)) \subseteq \text{Br}_0(Z) + B \) (by construction, \( B \) contains \( \ker(i^* : \text{Br}_1(Z) \to \text{Br}_1(Y)) \)). Functoriality of the Brauer-Manin pairing now gives

\[
\theta((z_v)) \in T_1(A_k)^{\text{Br}_1(T_1)}.
\]
By Proposition 3.3, there are $\alpha \in T_1(k)$ and $(\beta_v) \in T_0(\mathbb{A}_k)^{Br_1(T_0)}$ such that $\theta((z_v)) = \alpha \cdot q(\beta_v)$. Therefore $(\beta_v)^{-1} \cdot (z_v) \in \theta^{-1}(\alpha)$, hence $(\beta_v)^{-1} \cdot (z_v) \in Z(\mathbb{A}_k)^{Br_1(\mathbb{A}_k)}$.

Since $i^* : Br_1(Z) \to Br_1(Y)$ is surjective, one has

$$\psi_{q^{-1}(a)} \circ i^*_{\alpha}(B) = i^*(B) = \sum_{t \in T_1(k)} \psi_{q^{-1}(t)}(\widetilde{B}_{q^{-1}}) \supseteq \psi_{q^{-1}(a)}(\widetilde{B}_{q^{-1}(a)}),$$

where $\widetilde{B}$ is the image of $B$ in $Br_a(Z)$. It implies that $i^*(B) + Br_0(\theta^{-1}(\alpha)) \supseteq B_{q^{-1}(a)}$ by Lemma 2.7 and

$$(\beta_v)^{-1} \cdot (z_v) \in \left[\theta^{-1}(\alpha)(\mathbb{A}_k)\right]^{i^*(B) + (i^* \circ \eta^*) \circ (\mathcal{A})} \subseteq \left[\theta^{-1}(\alpha)(\mathbb{A}_k)\right]^{B_{q^{-1}(a)} \circ (i^* \circ \eta^*) \circ (\mathcal{A})}$$

as desired. \hfill \Box

The first part of the following result is also proved in Theorem 1.7 of [39].

Corollary 4.3. Let $X$ be a smooth and geometrically integral variety. If $f : Y \to X$ is a torsor under a torus $G$ over a number field $k$, then

$$X(\mathbb{A}_k)^{Br_1(X)} = \bigcup_{\alpha \in H^1(k, G)} f_\alpha \left(Y^{\sigma}(\mathbb{A}_k)^{Br_1(Y^{\sigma})}\right)$$

and

$$X(\mathbb{A}_k)^{Br} = \bigcup_{\alpha \in H^1(k, G)} f_\alpha \left(Y^{\sigma}(\mathbb{A}_k)^{Br_1(Y^{\sigma})} + f^*_\alpha(\text{Br}(X))\right).$$

Proof. To get the first equality, apply Theorem 4.2 to $A = Br_1(X)$ and $B_\sigma = Br_1(Y^{\sigma})$ for each $\sigma \in H^1(k, G)$. Since $\text{Pic}(G_k) = 0$, Proposition 6.10 in [32] gives

$$f^{*-1} \left(\sum_{\sigma \in H^1(k, G)} \psi_\sigma(\widetilde{B}_\sigma)\right) \subseteq f^{*-1}(Br_a(Y)) \subseteq Br_1(X) = A,$$

as required.

The second equality follows from Theorem 4.1 by taking $A = Br(X)$ and $B_\sigma = Br_1(Y^{\sigma})$ for each $\sigma \in H^1(k, G)$. \hfill \Box

5. An Application

In this section, we apply the previous results to study the necessary conditions for a connected linear algebraic group to satisfy strong approximation with Brauer-Manin obstruction.

When $X$ is affine, the set $X(k)$ is discrete in $X(\mathbb{A}_k)$ by the product formula. Therefore if such an $X$ satisfies strong approximation off $S$, then $\prod_{v \in S} X(k_v)$ is not compact. However this necessary condition for strong approximation is no longer true for strong approximation with Brauer-Manin obstruction if $Br(X)/Br(k)$ is not finite. For example, a torus $X$ always satisfies strong approximation with Brauer-Manin obstruction off $\infty_k$, $X$ being anisotropic over $k_\infty$ or not: see [22] Theorem 2. When $X$ is a semi-simple linear algebraic group, the necessary and sufficient condition for $X$ to satisfy strong approximation with Brauer-Manin obstruction is
given by Proposition 6.1 in [5]. In this section, we extend this result to a general connected linear algebraic group.

The following lemma explains that strong approximation with Brauer-Manin obstruction for a general connected linear algebraic group can be reduced to the reductive case.

**Lemma 5.1.** Let $G$ be a connected linear algebraic group over a number field $k$.

If $\pi : G \to G^\text{red}$ is the quotient map, then $G^\text{red}(A_k)^{\text{Br}_1(G^\text{red})} = \pi(G(A_k)^{\text{Br}_1(G)})$.

In particular, for any finite subset $S$ of $\Omega_k$, $G$ satisfies strong approximation with respect to $\text{Br}_1(G)$ off $S$ if and only if $G^\text{red}$ satisfies strong approximation with respect to $\text{Br}_1(G^\text{red})$ off $S$.

**Proof.** By applying Lemma 2.1 for $k$ and $\widetilde{k}$, one obtains that $\pi^*(\text{Br}_1(G^\text{red})) = \text{Br}_1(G)$. The first part follows from Theorem 3.5 and Proposition 6 of §2.1 of Chapter III in [33].

Suppose $G$ satisfies strong approximation with respect to $\text{Br}_1(G)$ off $S$. For any open subset $M = \prod_{v \in S} G^\text{red}(k_v) \times \prod_{v \notin S} M_v$ of $G^\text{red}(A_k)$ such that $M \cap \left[G^\text{red}(A_k)^{\text{Br}_1(G^\text{red})}\right] \neq \emptyset$, one has that

$$\pi^{-1}(M) = \prod_{v \in S} G(k_v) \times \prod_{v \notin S} \pi^{-1}(M_v)$$

with $\pi^{-1}(M) \cap G(A_k)^{\text{Br}_1(G)} \neq \emptyset$ by the first part. Then by assumption there is $x \in G(k) \cap \pi^{-1}(M)$. It implies that $\pi(x) \in M \cap G^\text{red}(k)$, as required.

Conversely, suppose $G^\text{red}$ satisfies strong approximation with respect to $\text{Br}_1(G^\text{red})$ off $S$. For any open subset $N = \prod_{v \in S} G(k_v) \times \prod_{v \notin S} N_v$ of $G(A_k)$ such that $N \cap G(A_k)^{\text{Br}_1(G)} \neq \emptyset$, we have

$$\pi(N) = \prod_{v \in S} G^\text{red}(k_v) \times \prod_{v \notin S} \pi(N_v)$$

and this set is an open subset of $G^\text{red}(A_k)$, with $\pi(M) \cap \left[G^\text{red}(A_k)^{\text{Br}_1(G^\text{red})}\right] \neq \emptyset$: here we use Proposition 6 of §2.1 of Chapter III in [33], Proposition 6.5 in Chapter 6 of [30] and the functoriality of Brauer-Manin pairing. Then by assumption there is $y \in G^\text{red}(k) \cap \pi(N)$. Using Proposition 6 of §2.1 of Chapter III in [33] one more time, one concludes that $\pi^{-1}(y)$ is isomorphic to $R_u(G)$ as an algebraic variety, hence it satisfies strong approximation off $S$. Since

$$\pi^{-1}(y) \cap N = \prod_{v \in S} \pi^{-1}(y)(k_v) \times \prod_{v \notin S} (\pi^{-1}(y)(k_v) \cap N) \neq \emptyset,$$

there is $z \in \pi^{-1}(y)(k) \cap N \subset G(k) \cap N$, as desired. □

The main result of this section is the following statement:
Theorem 5.2. Let $G$ be a connected linear algebraic group over a number field $k$ and let $G^{qs} := G/R(G)$, where $R(G)$ is the solvable radical of $G$. If $\pi : G \to G^{qs}$ is the quotient map, then

$$G^{qs}(A_k)^{Br_1(G^{qs})} = \pi \left( G(A_k)^{Br_1(G)} \right) \cdot G^{qs}(k).$$

In particular, if $G$ satisfies strong approximation with respect to $Br_1(G)$ off a finite subset $S$ of $\Omega_k$, then $G^{qs}$ satisfies strong approximation with respect to $Br_1(G^{qs})$ off $S$.

Proof. For the first part, by functoriality of the Brauer-Manin pairing, one only needs to prove

$$\pi \left( G(A_k)^{Br_1(G)} \right) \cdot G^{qs}(k).$$

By Lemma 5.1 we can assume that $G$ is reductive. Then $R(G)$ is a torus contained in the center of $G$ (see Theorem 2.4 in Chapter 2 of [30]) and $\pi : G \to G^{qs}$ is a torsor under $R(G)$. By Corollary 4.3, for any $(x_v) \in G^{qs}(A_k)^{Br_1(G^{qs})}$, there are $\sigma \in H^1(k, R(G))$ and $(y_v) \in G^\sigma(A_k)^{Br_1(G^\sigma)}$ such that $(x_v) = \pi_\sigma((y_v))$. Since $G^\sigma(k) \neq \emptyset$ by Corollary 8.7 in [32] (see also Theorem 5.2.1 in [36]), there is $\gamma \in G^{qs}(k)$ such that $\partial(\gamma) = \sigma$, where $\partial$ is the coboundary map in the following exact sequence in Galois cohomology:

$$1 \to R(G)(k) \to G(k) \to G^{qs}(k) \xrightarrow{\partial} H^1(k, R(G)) \to H^1(k, G).$$

In addition, the choice of an element $\bar{\gamma} \in G(k)$ such that $\pi(\bar{\gamma}) = \gamma$ defines a commutative diagram defined over $k$:

$$\begin{array}{ccc}
G^\sigma & \xrightarrow{\bar{\gamma} \cdot} & G \\
\pi_\sigma \downarrow & & \downarrow \pi \\
G^{qs} & \xrightarrow{\gamma \cdot} & G^{qs}
\end{array}$$

(see for instance Example 2 of p.20 in [36]). This implies that

$$\pi_\sigma \left( G^\sigma(A_k)^{Br_1(G^\sigma)} \right) = \pi \left( G(A_k)^{Br_1(G)} \right) \cdot \gamma,$$

as desired.

Suppose now that $G$ satisfies strong approximation with respect to $Br_1(G)$ off $S$. For any open subset

$$M = \prod_{v \in S} G^{qs}(k_v) \times \prod_{v \notin S} M_v$$

of $G^{qs}(A_k)$ such that $M \cap G^{qs}(A_k)^{Br_1(G^{qs})} \neq \emptyset$, the first part implies that there is $g \in G^{qs}(k)$ such that

$$\pi^{-1}(M \cdot g) = \prod_{v \in S} G(k_v) \times \prod_{v \notin S} \pi^{-1}(M_v \cdot g),$$

with $\pi^{-1}(M \cdot g) \cap G(A_k)^{Br_1(G)} \neq \emptyset$. Since $G$ satisfies strong approximation with algebraic Brauer-Manin obstruction off $S$, there exists $x \in G(k) \cap \pi^{-1}(M \cdot g)$. This implies that $\pi(x) \cdot g^{-1} \in M \cap G^{qs}(k)$ as required. \qed
Corollary 5.3. Let $G$ be a connected linear algebraic group over a number field $k$ and let $S$ a finite subset of $\Omega_k$ containing $\infty_k$. Then $G$ satisfies strong approximation with respect to $\text{Br}_1(G)$ off $S$ if and only if $\prod_{v \in S} G'(k_v)$ is not compact for any non-trivial simple factor $G'$ of the semi-simple part $G^{ss}$ of $G$.

Proof. By Theorem 2.3 and Theorem 2.4 of Chapter 2 in [30], the quotient map
$$G^{\text{red}} \rightarrow G/R(G) = G^{\text{qs}}$$
induces an isogeny $G^{ss} \rightarrow G^{qs}$. One side follows from Corollary 3.20 in [17]. The other side follows from Theorem 5.2 and Proposition 6.1 in [5].

Remark 5.4. All the results in this section involve the group $\text{Br}_1(G)$, and they remain true with $\text{Br}_1(G)$ replaced by $\text{Br}(G)$. Indeed, there is a sufficiently large subset $S$ of $\Omega_k$ containing $\infty_k$ such that $\prod_{v \in S} G'(k_v)$ is not compact for any non-trivial simple factor $G'$ of $G^{ss}$, therefore Corollary 3.20 in [17], Proposition 2.6 in [9] and the functoriality of Brauer-Manin pairing gives the following inclusions:
$$G(A_k)^{\text{Br}_1(G)} = G(k) \cdot \rho(\prod_{v \in S} G^{\text{sc}}(k_v)) \subseteq G(A_k)^{\text{Br}(G)} \subseteq G(A_k)^{\text{Br}_1(G)}$$
where $G^{\text{sc}} = G^{\text{sc}} \times G^{\text{red}}$ G with the projection map $G^{\text{sc}} \rightarrow G$ and $G^{\text{sc}}$ is the simply connected covering of $G^{ss}$. In particular, we have $G(A_k)^{\text{Br}(G)} = G(A_k)^{\text{Br}_1(G)}$.

6. Comparison I, $X(A_k)^{\text{desc}} \subseteq X(A_k)^{\text{ét,Br}}$

Let $Y \rightarrow X$ be a left torsor under a linear algebraic group $G$ over a number field $k$. The fundamental problem to define the descent obstruction for strong approximation with respect to $Y \rightarrow X$ is to decide whether the set
$$X(A_k)^{\text{f}} = \left\{ (x_v) \in X(A_k) : ([Y](x_v)) \in \text{Im} \left( H^1(k, G) \rightarrow \prod_v H^1(k_v, G) \right) \right\} = \bigcup_{\sigma \in H^1(k, G)} f_\sigma(Y^\sigma(A_k))$$
is closed or not in $X(A_k)$. We already know that this is true when $G$ is either connected or a group of multiplicative type, by Theorem [3,5]. For a general linear algebraic group $G$, this result is proved by Skorobogatov in Corollary 2.7 of [35], when $X$ is assumed to be proper over $k$. The proof depends on Proposition 5.3.2 in [36] or Proposition 4.4 in [23], which are not true for open varieties, as explained in the following example.

Example 6.1. The short exact sequence of linear algebraic groups
$$1 \rightarrow \mu_2 \rightarrow \mathbb{G}_m \xrightarrow{f} \mathbb{G}_m \rightarrow 1,$$
where $f(x) = x^2$, can be viewed as torsor over $\mathbb{G}_m$ under $\mu_2$. For any $\sigma \in H^1(k, \mu_2) \cong k^\times/(k^\times)^2$, the twist $\mathbb{G}_m^\sigma$ of $\mathbb{G}_m$ by $\sigma$ is given by the equation $x = a_\sigma y^2$ in $\mathbb{G}_m \times_k \mathbb{G}_m$, where $a_\sigma$ is an element in $k^\times$ representing the class $\sigma$ by the above isomorphism. It is clear that $\mathbb{G}_m^\sigma \cong \mathbb{G}_m$ as varieties over $k$, hence it always contains adelic points.

We use the same definition of an integral model as in [28].
**Definition 6.2.** Let $X$ be a variety over a number field $k$ and let $S$ be a finite subset of $\Omega_k$ containing $\infty_k$. An integral model of $X$ over $O_S$ is a faithfully flat separated $O_S$-scheme $X_S$ of finite type such that $X_S \times_{O_S} k \cong X$.

The replacement for Proposition 5.3.2 in [36] or Proposition 4.4 in [23] is the following proposition:

**Proposition 6.3.** Let $X$ be a variety over a number field $k$ and let $S$ be a finite subset of $\Omega_k$ containing $\infty_k$. Fix an integral model $X_S$ of $X$ over $O_S$. If $Y \xrightarrow{f} X$ is a left torsor under a linear algebraic group $G$ over $k$, then the set

$$\{ [\sigma] \in H^1(k, G) : f_\sigma(Y^\sigma(A_k)) \cap \left( \prod_{v \in S} X(k_v) \times \prod_{v \in S \setminus \Omega_k} X_S(O_v) \right) \neq \emptyset \}$$

is finite.

**Proof.** It follows from the same argument as the proof of Proposition 4.4 in [23].

One can now extend Corollary 2.7 in [35] to open varieties by using the above replacement for Proposition 4.4 in [23].

**Proposition 6.4.** Let $X$ be a (not necessarily proper) variety over a number field $k$. If $Y \xrightarrow{f} X$ is a left torsor under a linear algebraic group $G$ over $k$, then the set $X(A_k)^f$ is closed in $X(A_k)$.

**Proof.** Take an integral model $X_{S_0}$ of $X$ over $O_{S_0}$, where $S_0$ is a finite subset of $\Omega_k$ containing $\infty_k$. Then

$$\left\{ \prod_{v \in S} X(k_v) \times \prod_{v \in \Omega_k \setminus S} X_{S_0}(O_v) \right\}_{|S}$$

is an open covering of $X(A_k)$ (see Theorem 3.6 in [11]), where $S$ runs through all finite subsets of $\Omega_k$ containing $S_0$. By Proposition 6.3 and Corollary 2.5 in [35], the set

$$X(A_k)^f \cap \left( \prod_{v \in S} X(k_v) \times \prod_{v \in \Omega_k \setminus S} X_{S_0}(O_v) \right)$$

is closed in $\prod_{v \in S} X(k_v) \times \prod_{v \in \Omega_k \setminus S} X_{S_0}(O_v)$, therefore the set $X(A_k)^f$ is closed in $X(A_k)$.

Applying Proposition 6.3 one can also extend Lemma 2.2 and Theorem 1.1 in [35] to open varieties. For any variety over a number field $k$, and following [35], we write

$$X(A_k)^{\text{des}} = \bigcap_{Y \xrightarrow{f} X} X(A_k)^f,$$

where $Y \xrightarrow{f} X$ runs through all torsors under all linear algebraic groups over $k$ (see also [11]).
Lemma 6.5. Let $X$ be a (not necessarily proper) variety and let $Y \rightarrow X$ be a torsor over a number field $k$. For any $(P_v) \in X(A_k)^{\text{desc}}$, there is a twist $Y' \rightarrow X$ of $Y \rightarrow X$ such that the following property holds:

For any surjective $X$-torsor morphism $Z \rightarrow Y'$ (see Definition 2.1 in [35]), there is a twist $Z' \rightarrow Y'$ of $Z \rightarrow Y'$ such that $(P_v)$ lies in the image of $Z'(A_k)$.

Proof. There are a finite subset $S_0$ of $\Omega_k$ containing $\infty_k$ and an integral model $X_{S_0}$ over $O_{S_0}$ such that

$$(P_v) \in \prod_{v \in S_0} X(k_v) \times \prod_{v \in \Omega_k \setminus S_0} X_{S_0}(O_v)$$

(see for instance Theorem 3.6 in [11]), hence Proposition 6.3 implies that there are only finitely many twists of a given torsor over $X$ such that $(P_v)$ lifts as an adelic point of this torsor. As pointed out in the proof of Lemma 2.2 in [35], the finite combinatorics in the first part of the proof of Proposition 5.17 in [38] are still valid. It concludes the proof. □

Proposition 6.6. Let $X$ be a (not necessarily proper) variety over a number field $k$. If $Y \xrightarrow{f} X$ is a left torsor under a finite group scheme $F$ over $k$, then

$$X(A_k)^{\text{desc}} = \bigcup_{\sigma \in H^1(k,F)} f_\sigma (Y^\sigma(A_k)^{\text{desc}}).$$

Proof. One only needs to modify the proof of Theorem 1.1 in [35] by replacing Lemma 2.2 in [35] with Lemma 6.5 Corollary 2.7 in [35] with Proposition 6.4. Moreover, since $f$ is finite, the induced map $Y(A_k) \xrightarrow{f} X(A_k)$ is topologically proper by Proposition 4.4 in [11]. This implies that $f^{-1}((P_v))$ is compact. □

Recall that, following [31], one can define for any variety $X$ over a number field $k$, the set

$$X(A_k)^{\text{et,Br}} = \bigcap_{Y \xrightarrow{f} X} \bigcup_{\sigma \in H^1(k,F)} f_\sigma (Y^\sigma(A_k)^{\text{Br}}),$$

where $Y \xrightarrow{f} X$ runs over all torsors under all finite groups $F$ over $k$ (see [11]). Since the induced map $Y(A_k) \xrightarrow{f} X(A_k)$ is topologically closed for any finite morphism $Y \xrightarrow{f} X$ by Proposition 4.4 in [11], one concludes that $X(A_k)^{\text{et,Br}}$ is closed in $X(A_k)$ by the same argument as in Proposition 6.4.

Corollary 6.7. If $X$ is a smooth quasi-projective variety over a number field $k$, then

$$X(A_k)^{\text{desc}} \subseteq X(A_k)^{\text{et,Br}} \subseteq X(A_k)^{\text{Br}}.$$ 

Proof. One only needs to show that $X(A_k)^{\text{desc}} \subseteq X(A_k)^{\text{et,Br}}$. For any torsor $Y \xrightarrow{f} X$ under a finite group scheme $F$, Proposition 6.6 gives the equality

$$X(A_k)^{\text{desc}} = \bigcup_{\sigma \in H^1(k,F)} f_\sigma (Y^\sigma(A_k)^{\text{desc}}).$$
Since $X$ is quasi-projective, $Y^\sigma$ is quasi-projective as well. By a theorem of Gabber (see [12]), one has

$$Y^\sigma(A_k)^{\text{desc}} \subseteq Y^\sigma(A_k)^{\text{Br}}$$

(see the proof of Lemma 2.8 in [35]) and the result follows. □

7. Comparison II, $X(A_k)^{\text{et}, \text{Br}} \subseteq X(A_k)^{\text{desc}}$

In this section, we prove the inclusion $X(A_k)^{\text{et}, \text{Br}} \subseteq X(A_k)^{\text{desc}}$ for open varieties, which implies Theorem 1.5. The strategy of proof is the same as in [14].

The second named author would like to thank Laurent Moret-Bailly warmly for finding a mistake and for suggesting the following alternative proof of Lemma 4 in [14] (which already appeared in [15]). The statement of this lemma is correct, but the proof in [14] uses a result of Stoll (see [38]) that is not. Note that in contrast with [14], all torsors (unless explicitly mentioned) are assumed to be left torsors.

Lemma 7.1. Let $X$ be a smooth geometrically connected $k$-variety. Let $(P_v) \in X(A_k)^{\text{et}, \text{Br}}$ and let $Z \to X$ be a torsor under a finite $k$-group $F$.

Then there are a cocycle $\sigma \in Z^1(k, F)$ and a connected component $X'$ of $Z^\sigma$ over $k$ such that the restriction of $g_\sigma$ to $X'$ is a torsor $X' \to X$ under the stabilizer $F'$ of $X'$ for the action of $F^\sigma$, and the point $(P_v)$ lifts to a point $(Q_v) \in X'(A_k)^{\text{Br}}$.

In particular, $X'$ is geometrically integral.

Proof. By assumption, the point $(P_v)$ lifts to some point $(Q_v) \in Z^\sigma(A_k)^{\text{Br}}$ for some cocycle $\sigma$ with values in $F$. Since $Z^\sigma$ is smooth, $Z^\sigma$ is a disjoint union of connected components over $k$. By Proposition 3.3 in [28], there is a $k$-connected component $X'$ of $Z^\sigma$ such that $(Q_v)_{v \in \Xi} \in P_\Xi(X'(A_k)^{\text{Br}})$, where $\Xi$ is the set of all complex places of $k$, $A_k^\Xi$ is the ring of adeles without $\Xi$-components and $P_\Xi$ is the projection from $X'(A_k)$ to $X'(A_k^\Xi)$. Since for $v \in \Xi$, $Z^\sigma \times_k k_v$ is a trivial torsor under the finite constant group scheme $F^\sigma \times_k k_v$, we have $g_\sigma(X'(k_v)) = X(k_v)$ for all $v \in \Xi$. Hence one can assume that $Q_v \in X'(k_v)$ for $v \in \Xi$, so that we have $(Q_v) \in X'(A_k)^{\text{Br}}$.

Since $X'$ is connected and $X'(A_k) \neq \emptyset$, the proof of Lemma 5.5 in [38] implies that $X'$ is geometrically connected. Eventually, $X'$ being geometrically connected guarantees that the variety $X'$ is an $X$-torsor under the stabilizer $F'$ of $X'$ in $F^\sigma$. □

Let us continue the proof of the aforementioned inclusion. Let $X$ be a smooth and geometrically integral $k$-variety, and $(P_v) \in X(A_k)^{\text{et}, \text{Br}}$. We need to prove that $(P_v) \in X(A_k)^{\text{desc}}$.

For a linear algebraic group $G$ over $k$, one has the following short exact sequence of algebraic groups over $k$:

$$1 \to H \to G \to F \to 1,$$

where $H$ is the connected component of $G$ and $F$ is finite over $k$. This induces the following diagram of short exact sequences

$$
\begin{array}{cccccc}
1 & \to & H & \to & G & \to & F & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & T & \to & G' & \to & F & \to & 1
\end{array}
$$
where $T$ denotes the maximal toric quotient of $H$ and $G'$ is the quotient of $G$ by the kernel of $H \rightarrow T$.

Let $Y \rightarrow X$ be a torsor under $G$ and let $Z \rightarrow X$ be the push-forward of $Y \rightarrow X$ by the morphism $G \rightarrow F$, which is a torsor under $F$. If $\sigma \in Z^1(k, F)$ is a 1-cocycle given by Lemma 7.1 applied to the torsor $Z \rightarrow X$ and to the point $(P_0)$, we want to show that the cocycle $\sigma \in Z^1(k, F)$ lifts to a cocycle $\tau \in Z^1(k, G)$, as in Proposition 5 in [14]. The obstruction to lift $\sigma$ to a cocycle in $Z^1(k, G)$ gives a natural cohomology class $\eta_\sigma \in H^2(k, \kappa_\sigma)$ by (5.1) in [18] (see also (7.7) in [1]). The obstruction to lift $\sigma$ to a cocycle in $Z^1(k, G)$ gives a natural cohomology class $\eta_\sigma \in H^2(k, \kappa_\sigma)$ by (5.1) in [18] (see also (7.7) in [1]).

Let $X$ be a torsor under $Z$ and let $\sigma \in Z^1(k, F)$ be any point lifting $f$. We denote by $\tilde{T}^\sigma$ the quotient of $T$ such that $\sigma$ is neutral. Lemma 6 in [14] implies that there is a canonical map $H^2(k, \kappa_\sigma) \rightarrow H^2(k, T^\sigma)$ such that the class $\eta_\sigma$ is neutral if and only if its image $\eta'_\sigma \in H^2(k, T^\sigma)$ is zero.

We now apply the open descent theory and the extended type developed by Harari and Skorobogatov in [26] to establish the analogue of Lemma 7 in [14] for open varieties. As in the proof of [14], the torsor $Y \rightarrow Z$ under $H$ induces a torsor $W \rightarrow Z$ under $T$ by the natural map $H^1(Z, H) \rightarrow H^1(Z, T)$. Instead of using the type of the torsor $\varpi$ that was used in [14], we consider the so-called “extended type” of the torsor $\varpi$ that was introduced by Harari and Skorobogatov (see Definition 8.2 in [26]).

For a variety $Z$ over $k$, let $KD'(Z)$ denote the complex of Galois modules $\mathbb{Z}(Z)^*/\mathbb{Z} \rightarrow \text{Div}(Z_{\overline{k}})$ in the derived category $D^b_{\text{et}}(k)$ of bounded complexes of étale sheaves over $\text{Spec}(k)$. One can associate to the torsor $W \rightarrow Z$ under $T$ a canonical morphism in this derived category

$$\lambda_W : \tilde{T} \rightarrow KD'(Z),$$

called the extended type of $\varpi$. This induces a morphism in the derived category of bounded complexes of abelian groups

$$\lambda_W^\sigma : \tilde{T}^\sigma \rightarrow KD'(Z^\sigma)$$

for the above $\sigma \in Z^1(k, F)$.

**Lemma 7.2.** The morphism $\lambda_W^\sigma : \tilde{T}^\sigma \rightarrow KD'(Z^\sigma)$ is a morphism in the derived category of bounded complexes of étale sheaves over $\text{Spec}(k)$.

**Proof.** The natural left actions of $F$ on both $T$ and $Z$ induces right actions of $F$ on $\tilde{T}$ and on $KD'(Z)$.

We first prove that the morphism $\lambda_W$ is $F$-equivariant for those actions.

Let $f \in F(\overline{k})$. We denote by $f_z : Z_{\overline{k}} \rightarrow Z_{\overline{k}}$ the morphism of $\overline{k}$-varieties defined by $z \mapsto f \cdot z$. This morphism induces a natural morphism in the derived category $f^*_{\overline{k}} : KD'(Z_{\overline{k}}) \rightarrow KD'(Z_{\overline{k}})$. Similarly, the element $f$ defines a natural morphism of $\overline{k}$-tori $f_T : T_{\overline{k}} \rightarrow T_{\overline{k}}$ such that $f_T(t) := gtg^{-1}$, where $g \in G'(\overline{k})$ is any point lifting $f \in F(\overline{k})$. This morphism $f_T$ induces a morphism of abelian groups $f_T : \tilde{T} \rightarrow \tilde{T}$ such that $f_T(\chi) := \chi \circ f_T$. 

One needs to prove that the following diagram

\[
\begin{array}{ccc}
\hat{T} & \xrightarrow{\lambda_{W_k}} & KD'(Z_k) \\
\bar{f}_T & \downarrow & \downarrow f_Z \\
\hat{T} & \xrightarrow{\lambda_{W_k}} & KD'(Z_k)
\end{array}
\]

is commutative.

Let \( f_{T,*} W_k \) be the push-forward of the torsor \( W_k \to Z_k \) under \( T_k \) by the \( \bar{k} \)-morphism \( T_k \xrightarrow{\bar{f}_T} T_k \) and let \( f_Z^* W_k \) be the pullback of the torsor \( W_k \to Z_k \) under \( T_k \) by the \( k \)-morphism \( f_Z : Z_k \to Z_k \). Then functoriality of the extended type gives:

\[
f_Z^* \circ \lambda_{W_k} = \lambda_{f_Z^* W_k} \quad \text{and} \quad \lambda_{f_{T,*} W_k} = \lambda_{W_k} \circ \bar{f}_T.
\]

To prove the required commutativity \( f_Z^* \circ \lambda_{W_k} = \lambda_{f_Z^* W_k} \), it is enough to show that the torsors \( f_Z^* W_k \to Z_k \) and \( f_{T,*} W_k \to Z_k \) under \( T_k \) are isomorphic. Indeed, we have the following commutative diagram

\[
\begin{array}{ccc}
T_k \times W_k & \xrightarrow{g} & W_k \\
\varepsilon \circ p_W & \downarrow & \downarrow \psi \\
Z_k & \xrightarrow{f_Z} & Z_k
\end{array}
\]

where \( p_W \) denotes the projection on \( W_k \) and the morphism \( g \) is defined by \( (t, w) \mapsto (tg) \cdot w \). This diagram induces a natural \( Z_k \)-morphism \( \phi : T_k \times W_k \to f_Z^* W_k \). Consider now the right action of \( T_k \) on \( T_k \times W_k \) defined by \( (s, w) \cdot t := (sf_T(t), t^{-1} \cdot w) = (sgt^{-1}, t^{-1} \cdot w) \). Then the morphism \( \phi \) is \( T_k \)-invariant under this action, hence it induces a \( Z_k \)-morphism \( \psi : f_{T,*} W_k \to f_Z^* W_k \). One can check by a simple computation that \( \psi \) is \( T_k \)-equivariant, i.e. that \( \psi \) is a morphism of (left) torsors over \( Z_k \) under \( T_k \). It concludes the proof of the required commutativity, hence the morphism \( \lambda_W \) is \( F \)-equivariant.

By definition of the twists \( T^\sigma \) and \( Z^\sigma \), the fact that \( \lambda_W \) is \( F \)-equivariant implies that the morphism \( \lambda_{W_k}^\sigma \) is Galois equivariant, i.e. that \( \lambda_{W_k}^\sigma \) is a morphism in the derived category of \( \bar{k} \)-invariant under this action, hence it induces a \( Z_k \)-equivariant \( \psi^\sigma : f_{T,*} W_k \to f_Z^* W_k \). One can check by a simple computation that \( \psi^\sigma \) is \( F \)-equivariant, i.e. that \( \psi^\sigma \) is a morphism of (left) torsors over \( Z_k \) under \( T_k \). It concludes the proof of the required commutativity, hence the morphism \( \lambda_W \) is \( F \)-equivariant.

\[
f_Z^* \circ \lambda_{W_k} = \lambda_{f_Z^* W_k} \quad \text{and} \quad \lambda_{f_{T,*} W_k} = \lambda_{W_k} \circ \bar{f}_T.
\]

By Proposition 8.1 in [26], there is a natural exact sequence of abelian groups

\[
H^1(k, T^\sigma) \to H^1(X', T^\sigma) \xrightarrow{\lambda} \text{Hom}_k(T^\sigma, KD'(X')) \xrightarrow{\partial} H^2(k, T^\sigma)
\]

where the map \( \lambda \) is the extended type. Let \( \lambda_{W_k}^\sigma = \psi^\sigma \circ \lambda_{W_k}^\sigma \), where \( \psi : X' \to W \) is the inclusion of the \( \bar{k} \)-connected component given by Lemma 7.1 and \( KD'(Z^\sigma) \xrightarrow{\psi^\sigma} KD'(X') \) is the map induced by \( \psi \).

The following lemma, which is an analogue of Lemma 8 in [14], is a crucial step for proving the main result of this section. We give here a more conceptual proof than that in [14], where a similar statement was proven by cocycle computations under the assumption that \( \bar{k}[X]^\times = \bar{k}^\times \).
Lemma 7.3. With the above notation, one has
\[ \partial(\lambda'_\sigma) = 0 \] if and only if \( \eta'_\sigma = 0 \).

Proof. In the following proof, we work over the small étale site of \( \text{Spec}(k) \).

Recall that we are given a cocycle \( \sigma \in Z^1(k,F) \) as in Lemma 7.1; one can associate to \( \sigma \) a \( \text{Spec}(k) \)-torsor \( U \) under \( F \) with a point \( u_0 \in U(\bar{k}) \). This torsor \( U \) is naturally a homogeneous space of the group \( G' \) with geometric stabilizer isomorphic to \( T_k \). Section IV.5.1 in [19] implies that the element \( \eta'_\sigma \in H^2(k,T^\sigma) \) is the class of the \( \text{Spec}(k) \)-gerbe \( \mathcal{E}_\sigma \) banded by \( T^\sigma \) such that for all étale schemes \( S \) over \( \text{Spec}(k) \), the category \( \mathcal{E}_\sigma(S) \) is defined as follows: the objects of \( \mathcal{E}_\sigma(S) \) are triples \((P,p,\alpha)\) where \( P \to S \) is a torsor under \( G' \), \( p \in P(S_\sigma) \) and \( \alpha : P \to U_S \) is a \( G' \)-equivariant \( S \)-morphism. The morphisms of \( \mathcal{E}_\sigma(S) \) between triples \((P,p,\alpha)\) and \((P',p',\alpha')\) are given by morphisms of torsors \( P \to P' \) over \( S \) under \( G' \) that commute with \( \alpha \) and \( \alpha' \).

Similarly, one can associate to the morphism \( \lambda'_\sigma \) a \( \text{Spec}(k) \)-gerbe banded by \( T^\sigma \) that will be the obstruction for the morphism \( \lambda'_\sigma \) to be the extended type of a torsor over \( X' \) under \( T^\sigma \). The morphism \( \lambda'_\sigma \) induces a morphism \( \mathcal{N}_\sigma : \tilde{T}^\sigma_k \to K\mathcal{D}'(X'_k) \) in \( \mathcal{D}_{\text{et}}(\bar{k}) \). By construction, \( \mathcal{N}_\sigma \) is the extended type of the torsor \( Y_0 := W_k \times_{Z_k} X'_k \) over \( X'_k \) under \( T^\sigma_k = T_k \).

We now define \( \mathcal{L}_\sigma \) to be the fibered category defined as follows: for all étale schemes \( S \) over \( \text{Spec}(k) \), the objects of the category \( \mathcal{L}_\sigma(S) \) are pairs \((V,\varphi)\), where \( V \to X'_k \) is a torsor under \( T^\sigma_k \) of extended type \( \lambda^\nu \) compatible with \( \lambda'_\sigma \) and \( \varphi : V_k \to Y_0 \times_{Z_k} S_k \) is an isomorphism of torsors over \( X' \times_k S_k \) under \( T^\sigma_k \). Given two such objects \((V,\varphi)\) and \((V',\varphi')\), a morphism between \((V,\varphi)\) and \((V',\varphi')\) in the category \( \mathcal{L}_\sigma(S) \) is a pair \((\alpha,t)\), where \( \alpha : V \to V' \) is a morphism of torsors over \( X'_k \) under \( T^\sigma_k \) and \( t \in T^\sigma(S_k) \) such that the diagram
\[
\begin{array}{ccc}
V_k & \xrightarrow{\pi} & V'_k \\
\varphi \downarrow & & \varphi' \downarrow \\
Y_0 \times_{Z_k} S_k & \xrightarrow{\eta} & Y_0 \times_{Z_k} S_k \\
\end{array}
\]
commutes.

One can check that \( \mathcal{L}_\sigma \) is a stack for the étale topology over \( \text{Spec}(k) \), and the fact that this is a gerbe is a consequence of the exact sequence of Proposition 8.1 in [26]
\[
H^1(S,T^\sigma) \to H^1(X'_S,T^\sigma) \xrightarrow{\partial} \text{Hom}_S(T^\sigma,K\mathcal{D}'(X'_S)) \xrightarrow{\partial} H^2(S,T^\sigma)
\]
(which holds provided that \( S \) is integral, regular and noetherian).

The band of this gerbe is the abelian band represented by \( T^\sigma \).

In addition, it is clear that \( \mathcal{L}_\sigma \) is neutral if and only if \( \mathcal{L}_\sigma(k) \neq \emptyset \) if and only if there exists a torsor over \( X' \) under \( T^\sigma \) of type \( \lambda'_\sigma \) if and only if \( \partial(\lambda'_\sigma) = 0 \).

Let us now construct an equivalence of gerbes between \( \mathcal{E}_\sigma \) and \( \mathcal{L}_\sigma \).

For all étale \( \text{Spec}(k) \)-schemes \( S \), consider the functor
\[
m_S : \mathcal{E}_\sigma(S) \to \mathcal{L}_\sigma(S)
\]
that maps an object \((P,p,\alpha)\) to the object \((V,\varphi)\), where \( V \) is defined to be the contracted product \( V := (P \times^G_S W_S) \times_{Z_S} X'_S \) and \( \varphi : V_k \to Y_0 \times_{Z_k} S_k = (W_k \times_{Z_k} X'_k) \times_{Z_k} S_k \) is induced by the
point \( p \in P(S_k) \). Indeed, by construction, we have a natural map \( P \times_S \bar{W} \to U_S \times_S \bar{Z} = Z_\bar{S} \), and a simple computation proves that this map is a torsor under \( T^\sigma \) of extended type compatible with \( \lambda_W^\sigma \).

By definition, the functor \( m_S \) sends a morphism \( \varphi : (P, p, \alpha) \to (P', p', \alpha') \) to the morphism \((\varphi, t_0)\) such that \( \tilde{\varphi} : (P \times_S \bar{W}) \times_{\bar{Z}^\varepsilon} X_S \to (P' \times_S \bar{W}) \times_{\bar{Z}^\varepsilon} X'_S \) is the morphism induced by the morphism of torsors \( \varphi : P \to P' \), and \( t_0 \in T^\sigma(S_k) \) is the element such that \( p' = t_0 \cdot \varphi(p) \) as \( S_k \)-points in \( (P' \times_S \bar{W}) \times_{\bar{Z}^\varepsilon} X'_S \).

Finally, one checks that the collection of functors \( m_S \) defines a morphism of gerbes \( m : \mathcal{E}_\sigma \to \mathcal{L}_\sigma \) banded by the identity of \( T^\sigma \), which implies that \( \eta'_\sigma := [\mathcal{E}_\sigma] = [\mathcal{L}_\sigma] \in H^2(k, T^\sigma) \).

Therefore, \( \eta'_\sigma = 0 \) if and only if \( \mathcal{E}_\sigma(k) \neq \emptyset \) if and only if \( \mathcal{L}_\sigma(k) \neq \emptyset \) if and only if \( \partial(\lambda'_W) = 0 \). □

The immediate consequence of Lemma [7.3] is the following result which extends Proposition 5 in [14] to open varieties.

**Proposition 7.4.** Let \( X \) be a smooth geometrically integral \( k \)-variety. Let \((P_v) \in X(A_k)^{\text{ét}, Br} \) and let \( Y \to X \) be a torsor under a linear \( k \)-group \( G \). Let \[
1 \to H \to G \to F \to 1
\]
be an exact sequence of linear \( k \)-groups, where \( H \) is connected and \( F \) finite. Let \( Z \to X \) be the push-forward of \( Y \to X \) by the morphism \( G \to F \), which is a torsor under \( F \). Let \( \sigma \in Z^1(k, F) \) be a 1-cocycle given by Lemma [7.7] applied to the torsor \( Z \to X \) and the point \((P_v)\).

Then the cocycle \( \sigma \in Z^1(k, F) \) lifts to a cocycle \( \tau \in Z^1(k, G) \).

**Proof.** As mentioned above, Construction (5.1) in [18] (see also (7.7) in [1]) gives a class \( \eta_\sigma \) of \( H^2(k, \kappa_\sigma) \) such that \( \sigma \) can be lifted to \( Z^1(k, G) \) if and only if \( \eta_\sigma \) is neutral, where \( \kappa_\sigma \) is a \( k \)-kernel on \( H_k \). By (6.1.2) of [1] and Lemma 6 in [14], there is a canonical map \( H^2(k, \kappa_\sigma) \to H^2(k, T^\sigma) \) such that the class \( \eta_\sigma \) is neutral if and only if its image \( \eta'_\sigma \in H^2(k, T^\sigma) \) is zero. By Lemma [7.3] one only needs to show that \( \partial(\lambda'_W) = 0 \) where \( \lambda'_W = \psi^* \circ \lambda_W^\sigma \), with \( KD'(Z^\sigma) \xrightarrow{\psi'} KD'(X') \) given by Lemma [7.1] and \( \lambda'_W \) defined by Lemma [7.2].

By Lemma [7.1] we know that \( X'(^{\text{Br}}A_k) \neq \emptyset \). Therefore the map \( \lambda \) in the exact sequence (see Proposition 8.1 in [26])
\[
H^1(X', T^\sigma) \xrightarrow{\lambda} \text{Hom}_k(T^\sigma, KD'(X')) \xrightarrow{\partial} H^2(k, T^\sigma)
\]
is surjective by Corollary 8.17 in [26]. Hence the map \( \partial \) is the zero map and \( \partial(\lambda'_W) = 0 \), which concludes the proof. □

**Remark 7.5.** The proof of Proposition 7.4 also gives the following result:

Let \( X \) be a smooth geometrically integral \( k \)-variety and let \( Y \to X \) be a torsor under a linear algebraic \( k \)-group \( G \). Let \[
1 \to H \to G \to F \to 1
\]
be an exact sequence of linear \( k \)-groups, where \( H \) is connected and \( F \) finite. Let \( Z \to X \) be the push-forward of \( Y \to X \) by the morphism \( G \to F \).

If \( \sigma \in H^1(k, F) \) satisfies \( Z^\sigma(\text{A}_k)^{\text{Br}}(Z^\sigma) \neq \emptyset \), then \( \sigma \) can be lifted to \( H^1(k, G) \).

One can now prove the main result of this section:
Theorem 7.6. If $X$ is a smooth and geometrically integral variety over a number field $k$, then $X(\mathbb{A}_k)^{\alpha, Br} \subseteq X(\mathbb{A}_k)^{\text{desc}}$.

Proof. Since the statement 2 of Theorem 2 in [21] (which we apply to $X'$) holds for any geometrically integral variety (without any assumption on $k[X']^\times$), the proof of this theorem using Proposition 7.4 is exactly the same as the proof of Theorem 1 using Proposition 5 in [14] (see in particular [14], p. 244-245). □

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