Distribution of Chern–Simons invariants

Julien Marche

DISTRIBUTION OF CHERN-SIMONS INVARIANTS

JULIEN MARCHÉ

ABSTRACT. Let M be a 3-manifold with a finite set $X(M)$ of conjugacy classes of representations $\rho : \pi_1(M) \to \mathrm{SU}_2$. We study here the distribution of the values of the Chern-Simons function $\mathrm{CS} : X(M) \to \mathbb{R}/2\pi\mathbb{Z}$. We observe in some examples that it resembles the distribution of quadratic residues. In particular for specific sequences of 3-manifolds, the invariants tends to become equidistributed on the circle with white noise fluctuations of order $|X(M)|^{-1/2}$. We prove that for a manifold with toric boundary the Chern-Simons invariants of the Dehn fillings $M_{p/q}$ have the same behaviour when p and q go to infinity and compute fluctuations at first order.

1. Introduction

1.1. **Distribution of quadratic residues.** Let p be a prime number. We consider the weighted counting measure on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ defined by quadratic residues modulo p, that is:

$$\mu_p = \frac{1}{p}\sum_{k=0}^{p-1} \delta_{\frac{2\pi k^2}{p}}.$$

We investigate the limit of μ_p when p goes to infinity and to that purpose, we consider its ℓ-th momentum i.e $\mu_p^\ell = \int e^{i\ell\theta} d\mu_p(\theta) = \frac{1}{p}\sum_{k=0}^{p-1}\exp(2i\pi\ell k^2/p)$. We have $\mu_p^\ell = 1$ if $p|\ell$, and else by the Gauss sum formula, $\mu_p^\ell = \left(\frac{\ell}{p}\right)\frac{1}{\sqrt{p}}$ where $\left(\frac{\ell}{p}\right)$ is the Legendre symbol.

This shows that μ_p converges to the uniform measure μ_∞ whereas the renormalized measure $\sqrt{p}(\mu_p - \mu_\infty)$ -that we call fluctuation- has l-th momentum ± 1 depending on the residue of l modulo p and hence is a kind of "white noise".

1.2. **Distribution of Chern-Simons invariants.** On the other hand, such Gauss sums appear naturally in the context of Chern-Simons invariants of 3-manifolds. Consider an oriented and compact 3-manifold M and define its character variety as the set $X(M) = \mathrm{Hom}(\pi_1(M), \mathrm{SU}_2)/\mathrm{SU}_2$. In what follows, we will confuse between representations and their conjugacy classes. The Chern-Simons invariant may be viewed as a locally constant map $\mathrm{CS} : X(M) \to \mathbb{T}$. We refer to [3] for background on Chern-Simons invariants and give here a quick definition for the convenience of the reader.

Let ν be the Haar measure of SU_2 normalised by $\nu(\mathrm{SU}_2) = 2\pi$ and let $\pi : \tilde{M} \to M$ be the universal cover of M. There is an equivariant map $F : \tilde{M} \to \mathrm{SU}_2$ in the sense that $F(\gamma x) = \rho(\gamma)F(x)$ for all $\gamma \in \pi_1(M)$ and $x \in M$. The form $F^*\nu$ is invariant hence can be written $F^*\nu = \pi^*\nu_F$. We set $\mathrm{CS}(\rho) = \int_M \nu_F$ and claim that it is independent on the choice of equivariant map F modulo 2π.

Definition 1.1. Let M be a 3-manifold whose character variety is finite. We define its *Chern-Simons measure* as $\mu_M = \frac{1}{|X(M)|} \sum_{\rho \in X(M)} \delta_{\mathrm{CS}(\rho)}$.

1.2.1. *Lens spaces.* For instance, if $M = L(p,q)$ is a lens space, then $\pi_1(M) = \mathbb{Z}/p\mathbb{Z}$ and $X(M) = \{\rho_n, n \in \mathbb{Z}/p\mathbb{Z}\}$ where ρ_n maps the generator of $\mathbb{Z}/p\mathbb{Z}$ to a matrix with eigenvalues $e^{\pm\frac{2i\pi n}{p}}$. We know from [3] that $\mathrm{CS}(\rho_n) = 2\pi\frac{q^* n^2}{p}$ where $qq^* = 1 \bmod p$. Hence, the Chern-Simons invariants of $L(p,q)$ behave exactly like quadratic residues when p goes to infinity.

1.2.2. *Brieskorn spheres.* To give a more complicated but still manageable example, consider the Brieskorn sphere $M = \Sigma(p_1,p_2,p_3)$ where p_1,p_2,p_3 are distinct primes. This is a homology sphere whose irreducible representations in SU_2 have the form ρ_{n_1,n_2,n_3} where $0 < n_1 < p_1, 0 < n_2 < p_2, 0 < n_3 < p_3$. From [3] we have

$$\mathrm{CS}(\rho_{n_1,n_2,n_3}) = 2\pi\frac{(n_1p_2p_3 + p_1n_2p_3 + p_1p_2n_3)^2}{4p_1p_2p_3}$$

Setting $n = n_1p_2p_3 + p_1n_2p_3 + p_1p_2n_3$, we observe that -due to Chinese remainder theorem- n describes $(\mathbb{Z}/p_1p_2p_3\mathbb{Z})^\times$ when n_i describes $(\mathbb{Z}/p_i\mathbb{Z})^\times$ for $i = 1,2,3$. Hence, we compute that the following ℓ-th momentum:

$$\mu^\ell_{p_1p_2p_3} = \frac{1}{|X(\Sigma(p_1,p_2,p_3))|}\sum_{\rho\in X(M)} \exp(i\ell\,\mathrm{CS}(\rho)) \sim \frac{1}{p_1p_2p_3}\sum_{n=0}^{p_1p_2p_3-1} e^{\frac{i\pi\ell n^2}{2p_1p_2p_3}}.$$

Assuming ℓ is coprime with $p = p_1p_2p_3$ we get from [1] the following estimates where $\epsilon_n = 1$ is $n = 1 \bmod 4$ and $\epsilon_n = i$ if $n = 3 \bmod 4$:

$$\mu^\ell_p \sim \begin{cases} \frac{\epsilon_p}{\sqrt{p}}\left(\frac{\ell/4}{p}\right) & \text{if } \ell = 0 \bmod 4 \\ 0 & \text{if } \ell = 2 \bmod 4 \\ \frac{1+i}{2\sqrt{p}\epsilon_l}\left(\frac{p}{\ell}\right) & \text{else.} \end{cases}$$

Again we obtain that μ_p converges to the uniform measure when p goes to infinity. The renormalised measure $\sqrt{p}(\mu_p - \mu_\infty)$ have ℓ-th momentum with modulus equal to $1, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$ depending on ℓ mod 4.

1.3. **Dehn Fillings.** The main question we address in this article is the following: fix a manifold M with boundary $\partial M = \mathbb{T}\times\mathbb{T}$. For any $\frac{p}{q} \in \mathbb{P}^1(\mathbb{Q})$, we denote by $\mathbb{T}_{p/q}$ the curve on $\mathbb{T}^2$ parametrised by (pt, qt) for t in $\mathbb{T}$. We define the manifold $M_{p/q}$ by Dehn filling i.e the result of gluing M with a solid torus such that $\mathbb{T}_{p/q}$ bounds a disc.

We recall from [3] that in the case where M has boundary, there is a principal $\mathbb{T}$-bundle with connection $L \to X(\partial M)$ such that the Chern-Simons invariant is a flat section of $\mathrm{Res}^* L$

$$\begin{tikzcd} & L \arrow[d] \\ X(M) \arrow[ur, "\mathrm{CS}"] \arrow[r, "\mathrm{Res}"] & X(\partial M) \end{tikzcd}$$

where $\mathrm{Res}(\rho) = \rho \circ i_*$ and $i : \partial M \to M$ is the inclusion.

We will denote by $|d\theta|$ the natural density on $X(\mathbb{T}) = \mathbb{T}/(\theta \sim -\theta)$.

We also have $X(\mathbb{T}^2) = \mathbb{T}^2/(x,y) \sim (-x,-y)$ and for any p,q the map $\mathrm{Res}_{p/q} : X(\mathbb{T}^2) \to X(\mathbb{T}_{p/q})$ is given by $(x,y) \mapsto px + qy$.

Moreover, for any $\frac{p}{q}$, $\ell > 0$ and $0 \le k \le \ell$, there are natural flat sections $\mathrm{CS}^{k/\ell}_{p/q}$ of L^ℓ over the preimage $\mathrm{Res}^{-1}_{p/q}(\frac{\pi k}{\ell})$. These sections are called Bohr-Sommerfeld sections and they coincide for $k = 0$ with CS^ℓ. See [3] or [2] for a detailed description.

Theorem 1.2. *Let M be a 3-manifold with $\partial M = \mathbb{T}^2$ satisfying the hypothesis of Section 2.2. Let p,q,r,s be integers satisfying $ps - qr = 1$ and for any integer n, set $p_n = pn - r$ and $q_n = qn - s$. Then setting*

$$\mu_n^\ell = \frac{1}{n} \sum_{\rho \in X(M_{p_n/q_n})} e^{i\ell \,\mathrm{CS}(\rho)}$$

we get first

$$\mu_n^0 = \int_{X(M)} \mathrm{Res}^*_{r/s} |d\theta| + O\Big(\frac{1}{n}\Big)$$

and for $\ell > 0$

$$\mu_n^\ell = \frac{1}{\sqrt{2n}} \sum_{k=0}^{l} \sum_{\rho, k / \mathrm{Res}_{r/s}(\rho) = \pi \frac{k}{l}} \exp(-2i\pi n \frac{k^2}{4\ell} + i\ell \,\mathrm{CS}(\rho) - i\,\mathrm{CS}^{k/l}_{r/s}(\rho)) + O(\frac{1}{n})$$

Hence, we recover the behaviour that we observed for Lens spaces and Brieskorn spheres. The measure converges to a uniform measure μ_∞ and the renormalised measure $\sqrt{n}(\mu_n - \mu_\infty)$ has an oscillating behaviour controlled by representations in $X(M)$ with rational angle along $\mathbb{T}_{r/s}$.

1.4. **Intersection of Legendrian subvarieties.** We will prove Theorem 1.2 in the more general situation of curves immersed in a torus. Indeed, the problem makes sense in an even more general setting that we present here.

1.4.1. *Prequantum bundles.*

Definition 1.3. Let (M,ω) be a symplectic manifold. A prequantum bundle is a principal $\mathbb{T}$-bundle with connection whose curvature is ω.

It is well-known that the set of isomorphism classes of prequantum bundles is homogeneous under $H^1(M,\mathbb{T})$ and non-empty if and only if ω vanishes in $H^2(M,\mathbb{T})$. Let us give three examples:

Example 1.4. (i) Take $\mathbb{R}^2\times\mathbb{T}$ with $\lambda = d\theta + \frac{1}{2\pi}(xdy - ydx)$. This gives a prequantum bundle on $\mathbb{R}^2$. Dividing by the action of $\mathbb{Z}^2$ given by

$$(m,n)\cdot(x,y,\theta) = (x+2\pi m, y+2\pi n, \theta + my - nx) \tag{1}$$

gives a prequantum bundle $\pi : L \to \mathbb{T}^2$.

(ii) Any complex projective manifold $M \subset \mathbb{P}^n(\mathbb{C})$ has such a structure by restricting the tautological bundle whose curvature is the restriction of the Fubini-Study metric.

(iii) The Chern-Simons bundle over the character variety of a surface.

In all these cases, there is a natural subgroup of the group of symplectomorphisms of (M,ω) which acts on the prequantum bundle. The group $\mathrm{SL}_2(\mathbb{Z})$ acts in the first case and the mapping class group in the third case. In the second case, a group acting linearly on $\mathbb{C}^{n+1}$ and preserving M will give an example.

1.4.2. *Legendrian submanifolds and their pairing.* Consider a prequantum bundle $\pi : L \to M$ where M has dimension $2n$ and denote by $\lambda \in \Omega^1(L)$ the connection 1-form. By Legendrian immersion we will mean an immersion $i : N \to L$ where N is a manifold of dimension $n+1$ such that $i^*\lambda = 0$. This condition implies that i is transverse to the fibres of π and hence $\pi\circ i : N \to M$ is a Lagrangian immersion.

Definition 1.5. (1) Given $i_1 : N_1 \to L$ and $i_2 : N_2 \to L$ two Legrendrian immersions, we will say that they are transverse if it is the case of $\pi\circ i_1$ and $\pi\circ i_2$.

(2) Given such transverse Legendrian immersions and an intersection point, i.e. $x_1 \in N_1$ and $x_2 \in N_2$ such that $\pi(i_1(x_1)) = \pi(i_2(x_2))$ we define their phase $\phi(i_1(x_1), i_2(x_2))$ as the element $\theta \in \mathbb{T}$ such that $i_2(x_2) = i_1(x_1) + \theta$.

(3) The phase measure $\phi(i_1, i_2)$ is the measure on the circle defined by

$$\phi(i_1,i_2) = \sum_{\pi(i_1(x_1))=\pi(i_2(x_2))} \delta_{\phi(i_1(x_1),i_2(x_2))}.$$

If M is a 3-manifold obtained as $M = M_1 \cup M_2$ then, assuming transversality, the Chern-Simons measure of M is given by $\mu_M = \phi(\mathrm{CS}_1, \mathrm{CS}_2)$ where $\mathrm{CS}_i : X(M_i) \to L$ is the Chern-Simons invariant with values in the Chern-Simons bundle.

2. The torus case

2.1. **Immersed curves in the torus.** Consider the pre quantum bundle $\pi : L \to \mathbb{T}^2$ given in the first item of Example 1.4. We consider a fixed

Legendrian immersion $i : [a, b] \to L$ and for any coprime integers p, q the Legendrian immersion

$$i_{p/q} : \mathbb{T} \to L, i_{p/q}(t) = (pt, qt, 0).$$

Our aim here is to study the behaviour of $\phi(i, i_{p/q})$ when $(p, q) \to \infty$.

We first lift i to an immersion $I : [a, b] \to \mathbb{R}^2 \times \mathbb{R}$ of the form $I(t) = (x(t), y(t), \theta(t))$. By assumption we have $\dot{\theta} = -\frac{1}{2\pi}(x\dot{y} - y\dot{x})$. For instance, lifting $i_{p/q}$ we get simply the map $I_{p/q} : t \mapsto (pt, qt, 0)$.

Let r, s be integers such that $A = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ has determinant 1. Take $F_A : \mathbb{R}^2 \to \mathbb{R}$ the function

$$F_A(x, y) = \frac{1}{2\pi}(sx - ry)(qx - py)$$

A direct computation shows that this function satisfies $(m, n).I_{p/q}(t) = (pt + 2\pi m, qt + 2\pi n, F(pt + 2\pi m, qt + 2\pi n))$. We obtain from it the following formula:

$$\phi(i, i_{p/q}) = \sum_{a \le t \le b, qx(t) - py(t) \in 2\pi\mathbb{Z}} \delta_{\theta(t) - F(x(t), y(t))}. \tag{2}$$

If we put $i = i_{0/1}$ this formula becomes $\phi(i_{0/1}, i_{p/q}) = \sum_{k=0}^{p-1} \delta_{2\pi \frac{rk^2}{p}}$. This measure is related to the usual Gauss sum in the sense that denoting by q^* an inverse of q mod p we have:

$$\int e^{i\theta} d\phi(i_{0/1}, i_{p/q})(\theta) = \sum_{k \in \mathbb{Z}/q\mathbb{Z}} \exp(2i\pi \frac{q^* k^2}{p}).$$

Suppose that $p_n = pn - r$ and $q_n = qn - s$. A Bézout matrix is given by $A_n = \begin{pmatrix} pn - r & p \\ qn - s & q \end{pmatrix}$. Up to the action of $\mathrm{SL}_2(\mathbb{Z})$, we can suppose that $p = s = 1$ and $q = r = 0$ in which case $F_{A_n}(x, y) = -\frac{y}{2\pi}(x + ny)$. We get from Equation (2) the following formula for $\mu_n^\ell = \frac{1}{n} \int e^{i\ell\theta} d\phi(i, i_{pn/-1})(\theta)$:

$$\mu_n^\ell = \frac{1}{n} \sum_{\substack{x(t) + ny(t) \in 2\pi\mathbb{Z} \\ a \le t \le b}} \exp\left(i\ell(\theta(t) + \frac{y(t)}{2\pi}(x(t) + ny(t))) \right). \tag{3}$$

Taking $\ell = 0$, we are simply counting the number of solutions of $x(t) + ny(t) \in 2\pi\mathbb{Z}$ for $t \in [a, b]$. Assuming that y is monotonic, the number of solutions for $t \in [a, b]$ is asymptotic to $|y(b) - y(a)|$. Hence the asymptotic density of intersection points is $i^*|dy|$ and we get

$$\lim_{n \to \infty} \mu_n^0 = \int_a^b i^*|dy|.$$

To treat the case $\ell > 0$, we need the following version of the Poisson formula:

Lemma 2.1. *If $f, g : [a,b] \to \mathbb{R}$ are respectively C^1 and continuous and f is piecewise monotonic, then if further $f(a), f(b) \notin 2\pi\mathbb{Z}$ we have*

$$\sum_{a\le t\le b, f(t)\in 2\pi\mathbb{Z}} g(t) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_a^b e^{-ikf(t)}|f'(t)|g(t)dt$$

Applying it here, we get

$$\mu_n^\ell = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_a^b e^{-ik(x+ny)+i\ell(\theta+\frac{y}{2\pi}(x+ny))}|\frac{\dot{x}}{n}+\dot{y}|dt$$

We apply a stationary phase expansion in this integral, the phase being $\Phi = -ky + ly^2/2\pi$ and its derivative being $\dot{\Phi} = (-k+ly/\pi)\dot{y}$. We find two types of critical points: the horizontal tangents $\dot{y}=0$ and the points of rational height $y = \pi\frac{k}{l}$. We observe that when $\dot{y}=0$ the amplitude is $O(\frac{1}{n})$ and hence these contributions can be neglected compared with the other ones, where $y = \pi\frac{k}{l}$.

We compute $\ddot{\Phi} = \frac{l}{\pi}\dot{y}^2 + (-k+ly/\pi)\ddot{y} = \frac{l}{\pi}\dot{y}^2$ and $\Phi = -\frac{\pi k^2}{2l}$. As $\ddot{\Phi} > 0$, the stationary phase approximation gives

$$\mu_n^l = \frac{1}{\sqrt{2n}}\sum_{y=\frac{\pi k}{l}} e^{-in\frac{k^2\pi}{2l}-i\frac{kx}{2}+il\theta} + O(\frac{1}{n})$$

In order to give the final result, observe that the map $t \mapsto (t, \pi\frac{k}{l}, \frac{kt}{2})$ defines a flat section of L^ℓ that we denote by $i_{1/0}^{k/\ell}$.

We can sum up the discussion by stating the following proposition.

Proposition 2.2. *Let $i : \mathbb{T} \to L$ be a Legendrian immersion and suppose that $\pi \circ i$ is transverse to $i_{pn/-1}$ for n large enough and to the circles of equation $y = \pi\xi$ for $\xi \in \mathbb{Q}$.*

Then writing $i(t) = (x(t), y(t), \theta(t))$ and $\mu_n^\ell = \frac{1}{n}\int e^{i\ell\theta}d\phi(i, i_{pn/-1})(\theta)$ we have for all $\ell > 0$:

$$\mu_n^\ell = \frac{1}{\sqrt{2n}}\sum_{k\in\mathbb{Z}/2\ell\mathbb{Z}}\sum_{t\in\mathbb{T}, y(t)=\pi k/\ell} e^{-in\pi\frac{k^2}{2\ell}+i\phi\left(i(t), i_{1/0}^{k/l}(x(t))\right)} + O\left(\frac{1}{n}\right)$$

2.2. **Application to Chern-Simons invariants.** Let M be a 3-manifold with $\partial M = \mathbb{T}\times\mathbb{T}$. We assume that $X(M)$ is at most 1-dimensional and that the restriction map $\mathrm{Res} : X(M) \to X(\partial M)$ is an immersion on the smooth part and map the singular points to non-torsion points. Then we know that $\mathrm{Res}(X(M))$ is transverse to $\mathbb{T}_{p/q}$ for all but a finite number of p/q, see [4].

Consider the projection map $\pi : \mathbb{T}^2 \to X(\partial M)$ which is a 2-fold ramified covering. We may decompose $X(M)$ as a union of segments $[a_i, b_i]$ whose extremities contain all singular points. The restriction map Res can be lifted to $\mathbb{T}^2$ and the Chern-Simons invariant may be viewed as a map CS :

$[a_i, b_i] \to L$. Hence, we may apply it the results of Proposition 2.2 and obtain Theorem 1.2.

We may comment that the flat sections $i_{1/0}^{k/\ell}$ of L^ℓ over the line $y = \frac{\pi k}{\ell}$ induces through the quotient $(x, y, \theta) \sim (-x, -y, -\theta)$ a flat section of L^ℓ that we denoted $\mathrm{CS}_{0/1}^{k/l}$ over the subvariety $\mathrm{Res}_{0/1}^{-1}(\frac{\pi k}{\ell})$.

3. Chern-Simons invariants of coverings

3.1. **General setting.** Beyond Dehn fillings, we can ask for the limit of the Chern-Simons measure of any sequence of 3-manifolds. A natural class to look at is the case of coverings of a same manifold M. Among that category, one can restrict to the family of cyclic coverings. One can even specify the problem to the following case.

Question: Let $p : M \to \mathbb{T}$ be a fibration over the circle and M_n be the pull-back of the self-covering of $\mathbb{T}$ given by $z \mapsto z^n$. What is the asymptotic behaviour of μ_{M_n}?

This problem can be formulated in the following way. Let Σ be the fiber of M and $f \in \mathrm{Mod}(\Sigma)$ be its monodromy. Any representation $\rho \in X(M)$ restricts to a representation $\mathrm{Res}(\rho) \in X(\Sigma)$ invariant by the action f_* of f on $X(\Sigma)$. Reciprocally, any irreducible representation $\rho \in X(\Sigma)$ fixed by f_* correspond to two irreducible representations in $X(M)$.

The Chern-Simons invariant corresponding to a fixed point may be computed in the following way: pick a path $\gamma : [0, 1] \to X(\Sigma)$ joining the trivial representation to ρ and consider the closed path obtained by composing γ with $f(\gamma)$ in the opposite direction. Then its holonomy along L is the Chern-Simons invariant of the corresponding representation.

Understanding the asymptotic behaviour of μ_{M_n} consists in understanding the fixed points of f_*^n on $X(\Sigma)$ and the distribution of Chern-Simons invariants of these fixed points, a problem which seems to be out of reach for the moment.

3.2. **Torus bundles over the circle.** In this elementary case, the computation can be done. Let $A \in \mathrm{SL}_2(\mathbb{Z})$ act on $\mathbb{R}^2/\mathbb{Z}^2$. Its fixed points form a group $G_A = \{v \in \mathbb{Q}^2, Av = v \bmod \mathbb{Z}^2\}/\mathbb{Z}^2$. If $\mathrm{Tr}(A) \neq 2$, which we suppose from now, G_A is isomorphic to $\mathrm{Coker}(A - \mathrm{Id})$ and has cardinality $|\det(A - \mathrm{Id})|$.

Following the construction explained above, the phase is a map $f : G_A \to \mathbb{Q}/\mathbb{Z}$ given by $f([v]) = \det(v, Av) \bmod \mathbb{Z}$. Hence, the measure we are trying to understand is the following:

$$\mu_A = \frac{1}{|\det(A - \mathrm{Id})|} \sum_{v \in G_A} \delta_{2\pi \det(v, Av)}.$$

Consider the ℓ-th moment μ_A^ℓ of μ_A. It is a kind of Gauss sum that can be computed explicitly. The map f is a quadratic form on G_A with values in $\mathbb{Q}/\mathbb{Z}$. Its associated bilinear form is $b(v, w) = \det(v, Aw) + \det(w, Av) =$

$\det(v,(A-A^{-1})w)$. As $A+A^{-1}=\mathrm{Tr}(A)\,\mathrm{Id}$ and $\det(A-\mathrm{Id})=2-\mathrm{Tr}(A)$ we get $b(v,w)=2\det(v,(A-\mathrm{Id})w) \mod \mathbb{Z}$. Hence, if 2ℓ is invertible in G_A, then ℓb is non-degenerate and standard arguments (see [5] for instance) show that $|\mu_A^\ell|=|\det(A-\mathrm{Id})|^{-1/2}$. Hence we still get the same kind of asymptotic behaviour for the Chern-Simons measure of the torus bundles over the circle.

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Institut de Mathématiques de Jussieu - Paris Rive Gauche, Université Pierre et Marie Curie, 75252 Paris cédex 05, France

E-mail address: julien.marche@imj-prg.fr