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Quentin Berger

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# Strong renewal theorems and local large deviations for multivariate random walks and renewals 

Quentin Berger* ${ }^{*}$


#### Abstract

We study a random walk $\mathbf{S}_{n}$ on $\mathbb{Z}^{d}(d \geqslant 1)$, in the domain of attraction of an operatorstable distribution with index $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{d}\right) \in(0,2]^{d}$ : in particular, we allow the scalings to be different along the different coordinates. We prove a strong renewal theorem, i.e. a sharp asymptotic of the Green function $G(\mathbf{0}, \mathbf{x})$ as $\|\mathbf{x}\| \rightarrow+\infty$, along the "favorite direction or scaling": (i) if $\sum_{i=1}^{d} \alpha_{i}^{-1}<2$ (reminiscent of Garsia-Lamperti's condition when $d=1$ [17]); (ii) if a certain local condition holds (reminiscent of Doney's [13, Eq. (1.9)] when $d=1$ ). We also provide uniform bounds on the Green function $G(\mathbf{0}, \mathbf{x})$, sharpening estimates when $\mathbf{x}$ is away from this favorite direction or scaling. These results improve significantly the existing literature, which was mostly concerned with the case $\alpha_{i} \equiv \alpha$, in the favorite scaling, and has even left aside the case $\alpha \in[1,2)$ with non-zero mean. Most of our estimates rely on new general (multivariate) local large deviations results, that were missing in the literature and that are of interest on their own.


Keywords: multivariate random walks; strong renewal theorems; local large deviations.
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## 1 Setting of the paper

### 1.1 Multivariate random walks, domains of attraction

We consider a d-dimensional random walk $\mathbf{S}=\left(\mathbf{S}_{n}\right)_{n} \geqslant{ }_{0}: \mathbf{S}_{0}=\mathbf{0}$, and $\mathbf{S}_{n}:=\sum_{j=1}^{n} \mathbf{X}_{j}$, where $\left(\mathbf{X}_{j}\right)_{j \geqslant 0}$ is an i.i.d. sequence of $\mathbb{Z}^{d}$-valued random variables (we treat only the case of a lattice distribution for the simplicity of exposition, but non-lattice counterparts should hold). We assume that $\mathbf{X}_{1}$ is non-defective, i.e. $\mathbf{P}\left(\left\|X_{1}\right\|<+\infty\right)=1$ (let $\|\cdot\|$ denote the $L^{1}$ norm). If $\mathbf{X}_{1} \in \mathbb{N}^{d}$, we then call $\mathbf{S}_{n}$ a multivariate renewal process, and $\mathbf{S}=\left\{\mathbf{S}_{0}, \mathbf{S}_{1}, \mathbf{S}_{2}, \ldots\right\}$ is interpreted as a random subset of $\mathbb{N}^{d}$ (with a slight abuse of notations).

[^0]We assume that $\mathbf{S}$ is aperiodic and in the domain of attraction of a non-degenerate multivariate stable distribution with index $\boldsymbol{\alpha}:=\left(\alpha_{1}, \ldots, \alpha_{d}\right) \in(0,2]^{d}$ : there is a recentering sequence $\mathbf{b}_{n}=\left(b_{n}^{(1)}, \ldots, b_{n}^{(d)}\right)$ and scaling sequences $a_{n}^{(1)}, \ldots, a_{n}^{(d)}$ such that, setting $A_{n}$ the diagonal matrix with $A_{n}(i, i)=a_{n}^{(i)}$, we have as $n \rightarrow+\infty$

$$
\begin{equation*}
A_{n}^{-1}\left(\mathbf{S}_{n}-\mathbf{b}_{n}\right)=\left(\frac{S_{n}^{(1)}-b_{n}^{(1)}}{a_{n}^{(1)}}, \ldots, \frac{S_{n}^{(d)}-b_{n}^{(d)}}{a_{n}^{(d)}}\right) \Rightarrow \mathbf{Z} \quad \text { in distribution. } \tag{1.1}
\end{equation*}
$$

Here, $\mathbf{Z}$ is a multivariate stable law, whose non-degenerate density is denoted $g_{\boldsymbol{\alpha}}(\mathbf{x})$. As in [35, 11, 28], we allow the scaling sequences to be different along different coordinates. The case where $a_{n}^{(i)} \equiv a_{n}$ for all $1 \leqslant i \leqslant d$ (that is $A_{n}=a_{n} \mathrm{I}_{d}$ ) was considered by Lévy [26] and Rvaceva [36], and will be referred to as the balanced case. We refer to Appendix A for further discussion on generalized domains of attractions (here we only consider the case where $A_{n}$ is diagonal), and for a brief description of multivariate regular variation.

### 1.2 First notations

For every $i \in\{1, \ldots, d\}, S_{n}^{(i)}$ has to be in the domain of attraction of a $\alpha_{i}$-stable distribution. Let us set $F_{i}(x):=\mathbf{P}\left(X_{1}^{(i)} \leqslant x\right)$ and $\bar{F}_{i}(x):=\mathbf{P}\left(X_{1}^{(i)}>x\right)$.

When $\alpha_{i} \in(0,2)$, there exist some slowly varying function $L_{i}(\cdot)$, and constants $p_{i}, q_{i} \geqslant 0$ (with $p_{i}+q_{i}=1$ ) such that

$$
\begin{equation*}
\bar{F}_{i}(x) \sim p_{i} L_{i}(x) x^{-\alpha_{i}} \quad \text { and } \quad F_{i}(-x) \sim q_{i} L_{i}(x) x^{-\alpha_{i}} \quad \text { as } x \rightarrow+\infty, \tag{1.2}
\end{equation*}
$$

and when $p_{i}=0$ or $q_{i}=0$, we interpret this as $o\left(L_{i}(x) x^{-\alpha_{i}}\right)$. Note that (1.2) is equivalent to $S_{n}^{(i)}$ being in the domain of attraction of an $\alpha_{i}$-stable law, $\alpha_{i} \in(0,2)$, see [15, IX.8, Eq. (8.14)]. When $\alpha_{i}=2$, then we set

$$
\begin{equation*}
\sigma_{i}(x):=\mathbf{E}\left[\left(X_{1}^{(i)}\right)^{2} \mathbb{1}_{\left\{\left|X_{1}^{(i)}\right| \leqslant x\right\}}\right] . \tag{1.3}
\end{equation*}
$$

By [15, IX.8, Thm. 1], having $\sigma_{i}(x)$ slowly varying is equivalent to $S_{n}^{(i)}$ being in the domain of attraction of the normal distribution.

The scaling sequence $a_{n}^{(i)}$ is then characterized by the following relation

$$
\begin{align*}
L_{i}\left(a_{n}^{(i)}\right)\left(a_{n}^{(i)}\right)^{-\alpha_{i}} \sim 1 / n \quad \text { as } n \rightarrow+\infty, & \text { if } \alpha_{i} \in(0,2) ;  \tag{1.4}\\
\sigma_{i}\left(a_{n}^{(i)}\right)\left(a_{n}^{(i)}\right)^{-2} \sim 1 / n \quad \text { as } n \rightarrow+\infty, & \text { if } \alpha_{i}=2 .
\end{align*}
$$

Note that in any case, $a_{n}^{(i)}$ is regularly varying with exponent $1 / \alpha_{i}$.
Regarding the recentering sequences $b_{n}^{(i)}$, we set (see [15, IX.8, Eq. (8.15)]):

$$
\begin{equation*}
b_{n}^{(i)} \equiv 0 \quad \text { if } \alpha_{i} \in(0,1) ; \quad b_{n}^{(i)}:=n \mu_{i} \quad \text { if } \alpha_{i}>1 ; \quad b_{n}^{(i)}=n \mu_{i}\left(a_{n}^{(i)}\right) \quad \text { if } \alpha_{i}=1 \tag{1.5}
\end{equation*}
$$

We defined $\mu_{i}:=\mathbf{E}\left[X_{1}^{(i)}\right]$ when $X_{1}^{(i)}$ is integrable, and $\mu_{i}(x):=\mathbf{E}\left[X_{1}^{(i)} \mathbb{1}_{\left\{\left|X_{1}^{(i)}\right| \leqslant x\right\}}\right]$.

### 1.3 Overview of the literature and of our results

The main focus of our paper is the behavior of the Green's function $G(\mathbf{0}, \mathbf{x})=G(\mathbf{x}):=$ $\sum_{n=1}^{+\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)$, as $\|\mathbf{x}\| \rightarrow+\infty$. The literature is vast in the case of dimension $d=1$, see e.g. [17, 14, 13] or [7] for some landmarks. It has also been studied in a variety of papers in the case of dimension $d \geqslant 2$, but only in the balanced case ( $\alpha_{i} \equiv \alpha$ ), and in some specific cases. Let us now present an overview of the conditions under which the asymptotic behavior of $G(\mathbf{x})$ is known ( $d \geqslant 2$ ):

* In the case $\alpha=2$ (Normal domain of attraction), with non-zero mean: with some moment conditions and along the correct angle $\mathbf{x}=\left(t,\left\lfloor t \mu_{2} / \mu_{1}\right\rfloor\right)$, see [34] (this has been
improved in [12] and [39]), with an exponential tail condition, in a small cone around the mean vector, see [8]. Some estimates away from the favorite directions are provided in [32, Lem. 5], under a zero mean, finite variance condition.
* For $\alpha \in(0,2)$, in the centered case (i.e. $b_{n}^{(i)} \equiv 0$ ): if $d / 2<\alpha<2$ and along a given angle, see [41]; if $\alpha \in(0,1)$ and along a given angle, with an additional local condition, see [41, Cor. 3.B]. This has also been proven more recently in [9] under an integro-local condition. We also mention [38, Prop. 26.1] and [40] for simple moment conditions to obtain the asymptotic behavior of $G(\mathbf{x})$, in the case $\alpha=2$.

The contribution of the present paper is threefold: (i) we give the sharp behavior of $G(\mathbf{x})$ in the case $\alpha \in[1,2)$ with non-zero mean, in a cone around the mean vector (we call it favorite direction): this was missing in the literature-we also treat the case $\alpha=1$ with infinite mean; (ii) we give uniform bounds on $G(\mathbf{x})$, giving improved estimates when $\mathbf{x}$ is outside the favorite direction; (iii) we extend the results to the case of random walks in the domain of attraction of an operator stable distribution, allowing for different scalings along the different components (and we weaken Williamson's condition [41, Eq. (3.10)] in the case $\alpha \in(0,1)$ ).

As a central tool, we prove some multivariate local large deviations estimates, i.e. we go beyond the local limit theorem in a large deviation regime. This is of its own interest since such estimates were missing in the literature, and appear central in controlling the small- $n$ contribution to $G(\mathbf{x})$. We prove a local large deviation in the general setting, see Theorem 2.1. Then we propose a new (and natural) multivariate Assumption 2.2, which extends Doney's condition [13, Eq. (1.9)] to the multivarate settind, and generalizes Williamson's condition [41, Eq. (3.10)]: we obtain a better local large deviation result under this assumption.

Let us now give a brief overview on how the rest of the paper is organized. First, we present our local large deviations estimates and our Assumption 2.2 (that gives a sharper result), in Section 2. In Section 3, we state our strong renewal theorems (along the favorite direction or scaling), that we divide into three parts: the centered case, i.e. when $\mathbf{b}_{n} \equiv 0$; the non-zero mean case with $\alpha_{i}>1$; the case $\alpha_{i}=1$, that we set aside because it needs additional care. In Section 9, we present the uniform bounds on $G(\mathbf{x})$ (in dimension $d=2$ for simplicity). The rest of the paper, Sections 5 to 9 , is devoted to the proofs: Section 5 for the local large deviations, Sections 6-7-8 for the strong renewal theorems, and Section 9 for the estimates when x is away from the favorite direction or scaling. Finally, we collect in the appendix some useful comments: in Appendix A, we recall some definitions and results about multivariate regular variation and generalized domains of attraction; in Appendix B, we discuss further on our Assumption 2.2.

### 1.4 A general working assumption

We assume in the rest of the paper, mostly for simplicity of notations, that the left and right tail distributions of $\mathbf{X}_{1}, F_{i}(-x)$ and $\bar{F}_{i}(x)$, are dominated by subexponential distributions.
Assumption 1.1. There exists some slowly varying functions $\left(\varphi_{i}\right)_{i \leqslant d}$, and some $\gamma_{i} \geqslant \alpha_{i}$ such that for all $x \in \mathbb{N}$ and $i \in\{1, \ldots, d\}$

$$
\begin{equation*}
F_{i}(-x)+\bar{F}_{i}(x):=\mathbf{P}\left(X_{1}^{(i)} \leqslant-x\right)+\mathbf{P}\left(X_{1}^{(i)}>x\right) \leqslant \varphi_{i}(x) x^{-\gamma_{i}} \tag{1.6}
\end{equation*}
$$

When $\mathbf{E}\left[\left(X_{1}^{(i)}\right)^{2}\right]=+\infty$, we may take $\gamma_{i}=\alpha_{i}$, and $\varphi_{i}(\cdot)$ a constant multiplicative of $L_{i}(\cdot)$. When $\mathbf{E}\left[\left(X_{1}^{(i)}\right)^{2}\right]<+\infty$, we may take $\varphi_{i}(\cdot)$ and $\gamma_{i}$ such that $\sum_{n \geqslant 1} \varphi_{i}(n) n^{1-\gamma_{i}}<+\infty$.

This assumption is essentially used to generalize (1.2) to the case $\alpha_{i}=2$ : the exponent $\gamma_{i}$ gives further information on the left and right tail distribution. It does not appear to be
a real restriction (components are allowed to have a much stronger tail, having formally $\left.\gamma_{i}=\infty\right)$, but is easier for presenting the results. Also, we used the same exponent for the left and right tail distribution for simplicity, but all results can be adapted to the case of different tail behaviors. A typical example we have in mind is when the distribution of $\mathbf{X}_{1}$ is regularly varying in $\mathbb{R}^{d}$ with exponent $-\left(\gamma_{1}, \ldots, \gamma_{d}\right)$. We refer to Appendix $A$ for a definition of multivariate regular variation, see in particular (A.2)-we also present two examples (Examples A.1-A.2) of distribution of $\mathbf{X}_{1}$ we keep in mind.

## 2 Local large deviations

Let us start by stating the local limit theorem obtained by Griffin in [19] in our setting, and disentangled by Doney [11] (it is proven in dimension 2, but as stressed by Doney its proof is valid in any dimension): uniformly for $\mathbf{x}=\left(x_{1}, \ldots, x_{d}\right) \in \mathbb{Z}^{d}$,

$$
\begin{equation*}
a_{n}^{(1)} \cdots a_{n}^{(d)} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)-g_{\boldsymbol{\alpha}}\left(\mathbf{x}_{n}\right) \rightarrow 0 \quad \text { as } n \rightarrow+\infty \tag{2.1}
\end{equation*}
$$

with $\mathbf{x}_{n}:=A_{n}^{-1}\left(\mathbf{x}-\mathbf{b}_{n}\right)=\left(\frac{x_{1}-b_{n}^{(1)}}{a_{n}^{(1)}}, \ldots, \frac{x_{d}-b_{n}^{(d)}}{a_{n}^{(d)}}\right)$.
Our first set of results concerns local large deviation estimates, which improve (2.1) in the case $\left\|\mathrm{x}_{n}\right\| \rightarrow+\infty$. But let us start by reviewing some of the existing literature. A great part of it focuses on the balanced case ( $A_{n}=a_{n} \mathrm{I}_{d}$ ): in [25], large deviations are proven, and in $[42,33]$, some sufficient conditions (that we do not detail here) are given to obtain a local limit theorem of the type $\mathbf{P}\left(\mathbf{S}_{n} \in A\right) \sim n \mathbf{P}\left(\mathbf{X}_{1} \in A\right)$-the case $\alpha=1$ is left aside. As far as the "non-balanced" case is concerned, we refer to [29, Ch. 9] for large deviations estimates, see for example Theorem 9.1.3, where it is shown that $\mathbf{P}\left(\left\langle\mathbf{S}_{n}, \theta\right\rangle>x_{n}\right)$ is of the order of $n \mathbf{P}\left(\left\langle\mathbf{X}_{1}, \theta\right\rangle>x_{n}\right)$ when in the domain of attraction of an operator stable distribution with no normal component.

To summarize, there exists no general result that would treat "mixed" Normal and stable cases, and that would give a good (and general) local large deviation, under a weak assumption. Our aim is therefore to provide simple local large deviation estimates, that will be a crucial tool for our renewal results of Sections 3-4. We also give an improved result below, under some more local assumption on the distribution of $\mathbf{X}_{1}$. The proof of the local large deviation results are presented in Section 5.

### 2.1 A first local limit theorem

Let us denote $\hat{\mathbf{S}}_{n}:=\mathbf{S}_{n}-\left\lfloor\mathbf{b}_{n}\right\rfloor$ the recentered walk (we take the integer part of $\mathbf{b}_{n}$ simply so that $\hat{\mathbf{S}}_{n}$ is still $\mathbb{Z}^{d}$ valued). As far as a large deviation estimate is concerned, univariate large deviation estimates already give (we recall these results in Section 5.1 below) that there is a constant $C_{0}$ such that for any $\mathrm{x} \geqslant 0$,

$$
\begin{equation*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n} \geqslant \mathbf{x}\right) \leqslant C_{0} \min _{i \in\{1, \ldots d\}}\left\{n \varphi\left(x_{i}\right)\left(x_{i}\right)^{-\gamma_{i}}+\exp \left(-\frac{c x_{i}^{2}}{n \sigma\left(x_{i}\right)}\right) \mathbb{1}_{\left\{\alpha_{i}=2\right\}}\right\}, \tag{2.2}
\end{equation*}
$$

where the inequality $\hat{\mathbf{S}}_{n} \geqslant \mathrm{x}$ is componentwise. We now give a local version of it.
Theorem 2.1. Assume that Assumption 1.1 holds. There exist constants $c_{1}, C_{1}$ such that for any fixed $i \in\{1, \ldots, d\}$ and $\mathbf{x} \in \mathbb{Z}$ with $\left|x_{i}\right| \geqslant a_{n}^{(i)}$, we have

$$
\begin{equation*}
a_{n}^{(1)} \cdots a_{n}^{(d)} \times \mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}\right) \leqslant C_{1} n \varphi_{i}\left(\left|x_{i}\right|\right)\left|x_{i}\right|^{-\gamma_{i}}+C_{1} \exp \left(-\frac{c_{1}\left|x_{i}\right|^{2}}{n \sigma_{i}\left(\left|x_{i}\right|\right)}\right) \mathbb{1}_{\left\{\alpha_{i}=2\right\} .} . \tag{2.3}
\end{equation*}
$$

The idea of this result is similar to that of [7, Theorem 1.1] for the univariate case (where only the case $\alpha \in(0,1) \cup(1,2)$ is treated), and we give the details in Section 5.2.

### 2.2 A local multivariate assumption for an improved local limit theorem

In dimension $d=1$, better local large deviations can be obtained under a local assumption on the distribution of $\mathbf{X}_{1}$, see [13] in the case $\alpha \in(0,1)$ and [3, Thm 2.7] in the case $\alpha \in(0,2)$. We present here an assumption which can be thought as the analogous of Doney's condition [13, Eq. (1.9)] to the multivariate setting, and generalizes Williamson's condition [41, Eq. (3.10)]. We comment on that Assumption below.

Assumption 2.2. There exist a constant $\mathrm{C}_{d}$, slowly varying functions $\left(\varphi_{i}\right)_{1 \leqslant i \leqslant d}$ and exponents $\left(\gamma_{i}\right)_{1 \leqslant i \leqslant d}$ (the same as in Assumption 1.1) such that for any fixed $i \in$ $\{1, \ldots, d\}$

$$
\begin{equation*}
\mathbf{P}\left(\mathbf{X}_{1}=\mathbf{x}\right) \leqslant \frac{\mathrm{C}_{d} \varphi_{i}\left(\left|x_{i}\right|\right)\left(1+\left|x_{i}\right|\right)^{-\gamma_{i}}}{\prod_{j=1}^{d}\left(1+\left|x_{j}\right|\right)} \times \prod_{j \neq i} h_{\left|x_{i}\right|}^{(i)}\left(\left|x_{j}\right|\right), \tag{2.4}
\end{equation*}
$$

where the functions $h_{u}^{(i)}(v)(u, v \in \mathbb{N})$ for $i \in\{1, \ldots, d\}$ verify:
(i) $h_{u}^{(i)}(v) \leqslant 1$;
(ii) $\sup _{u \geqslant 0} \sum_{v \geqslant 0} \frac{h_{u}^{(i)}(v)}{1+|v|}<+\infty$;
(iii) $\sup _{u, v \geqslant 0 ; u^{\prime} \in[u, 2 u]} \frac{h_{u^{\prime}}^{(i)}(v)}{h_{u}^{(i)}(v)}<+\infty$.

First of all, we present two important examples that verify Assumption 2.2: they are local versions of Example A. 1 (independent case) and Example A. 2 (dependent case).
Example 2.3. There are positive exponents $\gamma_{i}$ and slowly varying functions $\varphi_{i}(\cdot)(i \in$ $\{1, \ldots, d\})$, such that $\mathbf{P}\left(\mathbf{X}_{1}=\mathbf{x}\right)=\prod_{i=1}^{d} \varphi_{i}\left(x_{i}\right) x_{i}^{-\left(1+\gamma_{i}\right)}$, for $\mathbf{x} \in \mathbb{N}^{d}$.
Example 2.4. There are positive exponents $\beta,\left(\beta_{i}\right)_{1 \leqslant i \leqslant d}$ with $\beta>\sum_{i=1}^{d} \beta_{i}^{-1}$, and $\psi(\cdot)$ a slowly varying function, such that $\mathbf{P}\left(\mathbf{X}_{1}=\mathbf{x}\right)=\psi\left(\sum_{i=1}^{d} x_{i}^{\beta_{i}}\right) \times\left(\sum_{i=1}^{d} x_{i}^{\beta_{i}}\right)^{-\beta}$, for $\mathbf{x} \in \mathbb{N}^{d}$. Assumption 2.2 is verified with $\gamma_{i}:=\beta_{i}\left(\beta-\sum_{i=1}^{d} \beta_{i}^{-1}\right)$, see Appendix B.

We mention that a two-dimensional, balanced, version of Example 2.4 is used in [18] (it comes from the biophysics literature, see [16]): the dimension is $d=2, \beta_{i} \equiv 1$, and $\beta=2+\alpha, \alpha>0$.

Let us now give a general idea behind the choice of Assumption 2.2—assume for simplicity that all $x_{i}$ 's are positive. We start with writing

$$
\begin{aligned}
\mathbf{P}\left(\mathbf{X}_{1}=\mathbf{x}\right)=\mathbf{P}( & \left.\mathbf{X}_{1}=\mathbf{x} \mid X_{1}^{(i)} \in\left[x_{i}, 2 x_{i}\right] \forall i \in\{1, \ldots, d\}\right) \\
& \times \mathbf{P}\left(X_{1}^{(i)} \in\left[x_{i}, 2 x_{i}\right]\right) \times \mathbf{P}\left(X_{1}^{(j)} \in\left[x_{j}, 2 x_{j}\right] \forall j \neq i \mid X_{1}^{(i)} \in\left[x_{i}, 2 x_{i}\right]\right)
\end{aligned}
$$

First, conditioned on the event that $\mathbf{X}_{1}$ is in the rectangle $\left[x_{1}, 2 x_{1}\right] \times \cdots \times\left[x_{d}, 2 x_{d}\right]$, a natural assumption is that the probability of being at one particular site is bounded by $c\left(\prod_{i=1}^{d} x_{i}\right)^{-1}$ (i.e. uniform on the rectangle): this gives the first denominator of (2.4). Then, $\mathbf{P}\left(X_{1}^{(i)} \in\left[x_{i}, 2 x_{i}\right]\right)$ is bounded by a constant times $\varphi\left(x_{i}\right) x_{i}^{-\gamma_{i}}$ by Assumption 1.1: it gives the first numerator in (2.4). The last term is, by Hölder's inequality, bounded by

$$
\prod_{j \neq i} \mathbf{P}\left(X_{1}^{(j)} \in\left[x_{j}, 2 x_{j}\right] \mid X_{1}^{(i)} \in\left[x_{i}, 2 x_{i}\right]\right)^{1 /(d-1)}
$$

which accounts for the product of the $h_{x_{i}}^{(i)}\left(x_{j}\right)$. We keep in mind two cases: (i) when the coordinates are independent (see Example 2.3), we recover $h_{x_{i}}^{(i)}\left(x_{j}\right) \leqslant x_{j}^{-a}$ for some $a>0$; (ii) when the coordinates are dependent (see Example 2.4), there is some threshold $t\left(x_{i}\right)$ such that $h_{x_{i}}^{(i)}\left(x_{j}\right) \leqslant\left(\frac{x_{j}}{t\left(x_{i}\right)} \vee \frac{t\left(x_{i}\right)}{x_{j}}\right)^{-a}$ for some $a>0$, and this satisfies the conditions (2.5) (we refer to Appendix B for more details, see (B.1)-(B.2) and below).

We stress that the term $h_{\left|x_{i}\right|}^{(i)}\left(\left|x_{j}\right|\right)$ in (2.4) is central: in particular, item (ii) in (2.5) insures that there is a constant C such that for any $i$,

$$
\begin{equation*}
\mathbf{P}\left(X_{1}^{(i)}=x_{i}\right) \leqslant \mathrm{C} \varphi_{i}\left(\left|x_{i}\right|\right)\left(1+\left|x_{i}\right|\right)^{-\left(1+\gamma_{i}\right)}, \tag{2.6}
\end{equation*}
$$

which is Doney's condition [13, Eq. (1.9)] for each component (generalized to the case $\alpha_{i} \geqslant 1$ ). Also, we point out that Assumption 2.2 is similar in spirit but weaker than Williamson's condition [41, Eq. (3.10)], which considers the balanced case $\alpha_{i} \equiv \alpha<$ $\min (d, 2)$, and says that there is a constant $K_{0}<+\infty$ such that for any $\mathbf{x} \in \mathbb{Z}^{d}$,

$$
\begin{equation*}
\mathbf{P}\left(\mathbf{X}_{1}=\mathbf{x}\right) \leqslant K_{0}(1+\|x\|)^{-d} \mathbf{P}\left(\left\|\mathbf{X}_{1}\right\|>\|\mathbf{x}\|\right) \tag{2.7}
\end{equation*}
$$

((2.7) does not include the case of independent $X^{(i)}$ 's, whereas our Assumption 2.2 does.)

Under Assumption 2.2, we are able to improve Theorem 2.1.
Theorem 2.5. Suppose that Assumption 2.2 holds. Then there are constants $c_{2}, C_{2}$ such that for any $\mathbf{x} \in \mathbb{Z}^{d}$

$$
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}\right) \leqslant \frac{C_{2}}{\prod_{i=1}^{d} \max \left\{\left|x_{i}\right|, a_{n}^{(i)}\right\}} \times \min _{i \in\{1, \ldots d\}}\left\{n \varphi_{i}\left(\left|x_{i}\right|\right)\left|x_{i}\right|^{-\gamma_{i}}+e^{-c_{2}\left(\left|x_{i}\right| / a_{n}^{(i)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{i}=2\right\}}\right\} .
$$

The case of dimension $d=1$ with $\alpha_{1} \in(0,2)$ is proven in [3, Theorem 2.7]: Theorem 2.5 therefore generalizes it to the case $\alpha_{1}=2$, and to the multivariate, non-balanced case. It is a significant improvement of Theorem 2.1, in particular when (several) $x_{i}$ 's are much larger than $a_{n}^{(i)}$.

### 2.3 About the balanced case, and Williamson's condition

We may obtain another bound if we consider the balanced case, and assume that there is a positive exponent $\gamma$, and some slowly varying $\varphi(\cdot)$ such that

$$
\begin{equation*}
\mathbf{P}\left(\mathbf{X}_{1}=\mathbf{x}\right) \leqslant \varphi(\|\mathbf{x}\|)\|\mathbf{x}\|^{-(d+\gamma)} \tag{2.8}
\end{equation*}
$$

This is a natural extension of Williamson's condition (2.7) to the case $\alpha=2$, and as seen in Appendix B (when treating Example 2.4), it implies Assumption 2.2.
Theorem 2.6. Suppose that $a_{n}^{(i)} \equiv a_{n}$ (balanced case) and that (2.8) holds. Then there are constants $c_{3}, C_{3}$, such that for $\|\mathbf{x}\| \geqslant a_{n}$ we have

$$
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}\right) \leqslant C_{3} n \varphi(\|\mathbf{x}\|)\|\mathbf{x}\|^{-(d+\gamma)}+\frac{1}{\left(a_{n}\right)^{d}} e^{-c_{3}\left(\|\mathbf{x}\| / a_{n}\right)^{2}} \mathbb{1}_{\{\alpha=2\}}
$$

In practice, we will not use assumption (2.8) in the rest of the paper: it requires to work in the balanced case, and would not improve our renewal results. We however include Theorem 2.6 since it is an important improvement of Theorem 2.5, and may reveal useful (in particular in the setting of [18] and [5] where (2.8) is verified).

### 2.4 Some conventions for the rest of the paper

First of all, all regularly varying quantities $\left(a_{n}^{(i)}, b_{n}^{(i)}, L_{i}(\cdot), \mu_{i}(\cdot), \varphi_{i}(\cdot) \ldots\right)$ will be interpreted as functions of positive real numbers, which may be taken infinitely differentiable (see [6, Th. 1.8.2]).

As we may work along subsequences and exchange the role of the $X^{(i)}$ 's, we assume that $a_{n}^{(1)} \leqslant \cdots \leqslant a_{n}^{(d)}$ (insuring in particular that $\alpha_{1} \geqslant \cdots \geqslant \alpha_{d}$ )-the first coordinate is the one with the less fluctuations. Finally, assume that $a_{n}^{(j)} / a_{n}^{(i)} \rightarrow a_{i, j} \in\{0,1\}$ for $j \leqslant i$ (if $a_{n}^{(i)} / a_{n}^{(j)} \rightarrow a \in(0,1)$ then rescale the limiting law by $a$ ). Having $a_{i, 1}=1$ for all
$i$ corresponds to the balanced case. We will also assume that: either $b_{n}^{(i)} \equiv 0$ (as it is the case when $\alpha_{i}<1 ; \alpha_{i}>1$ with $\mu_{i}=0$; in the symmetric case for $\alpha_{i}=1$ ), or that $b_{n}^{(i)} / a_{n}^{(i)} \rightarrow \pm \infty$ (as it is the case when $\alpha_{i} \geqslant 1$ with $\mu_{i} \in \mathbb{R}^{*}$ or $\alpha_{i}=1$ with $p_{i} \neq q_{i}$ )一 the only case where subtleties may arise is when $\alpha_{i}=1$ with $\left|\mu_{i}\right|=0$ or $+\infty$ and $p_{i}=q_{i}$. (If $b_{n}^{(i)} / a_{n}^{(i)} \rightarrow b_{i} \in \mathbb{R}$, then we can reduce to the case $b_{n}^{(i)} \equiv 0$, at the expense of a translation of the limiting law.)

In the rest of the paper, we denote $u \vee v=\max (u, v)$ and $u \wedge v=\min (u, v)$. For two sequences $\left(u_{n}\right)_{n} \geqslant 0,\left(v_{n}\right)_{n} \geqslant 0$, we write $u_{n} \sim v_{n}$ is $u_{n} / v_{n} \rightarrow 1$ as $n \rightarrow+\infty, u_{n}=O\left(v_{n}\right)$ if $u_{n} / v_{n}$ stays bounded, and $u_{n} \asymp v_{n}$ if $u_{n}=O\left(v_{n}\right)$ and $v_{n}=O\left(u_{n}\right)$.

## 3 Strong renewal theorems

We now consider the Green function $G(\mathbf{x}):=\sum_{n=1}^{\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)$, and we study its behavior as $\|\mathbf{x}\| \rightarrow+\infty$. If $\left(\mathbf{S}_{n}\right)_{n} \geqslant 0$ is a (multivariate) renewal process, we interpret $G(\mathbf{x})$ as the renewal mass function, $\mathbf{P}(\mathbf{x} \in \mathbf{S})$.

### 3.1 About the favorite direction or scaling

In the sum $\sum_{n=1}^{\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)$, the main contribution comes from some typical number of jumps: identifying that number allows us to determine a favorite direction or scaling along which we will get sharp asymptotics of $G(\mathbf{x})$. Let us define $n_{i}:=n_{i}(\mathbf{x})$ for $i \in\{1, \ldots, d\}$ by the relation

$$
\begin{array}{ll}
b_{n_{i}}^{(i)}=x_{i} & \text { if }\left|b_{n_{i}}^{(i)}\right| / a_{n}^{(i)} \rightarrow+\infty \quad\left(b_{n} \text { and } x_{i} \text { need to have the same sign }\right),  \tag{3.1}\\
a_{n_{i}}^{(i)}=\left|x_{i}\right| & \text { if } b_{n}^{(i)} \equiv 0
\end{array}
$$

Then $n_{i}$ is the typical number of steps for the $i$-th coordinate to reach $x_{i}$. This definition might not give a unique $n_{i}$, but any choice will work, and $n_{i}$ is unique up to asymptotic equivalence. If $\alpha_{i}>1$ with $\mu_{i} \neq 0$, then we have $n_{i}=\left|x_{i}\right| /\left|\mu_{i}\right|$; if $\alpha_{i}=1$ and $\mu_{i} \in \mathbb{R}^{*}$ or $\alpha_{i}=1$ and $p_{i} \neq q_{i}$ then we have $n_{i} \sim\left|x_{i}\right| /\left|\mu_{i}\left(\left|x_{i}\right|\right)\right|$ (see details below, in Section 8.1); and if $b_{n}^{(i)} \equiv 0$ then $n_{i} \sim x_{i}^{-\alpha_{i}} \phi_{i}\left(x_{i}\right)^{-1}$ with $\phi_{i}=L_{i}$ if $\alpha_{i} \in(0,2)$ and $\phi_{i}=\sigma_{i}$ if $\alpha_{i}=2$, thanks to the definition (1.4) of $a_{n}^{(i)}$.

There are mainly three regimes that we consider,
I. Centered case: $\mathbf{b}_{n} \equiv \mathbf{0}$. The typical number of steps to reach $\mathbf{x}$ is $n_{i_{0}}=\min _{i} n_{i}$; the favorite scaling are the points $\mathbf{x}$ with $x_{i} \asymp a_{n_{i_{0}}}^{(i)}$ for all $i$, see (3.2) below.
II. Non-zero mean case: $\mu_{i} \in \mathbb{R}^{*}$ for some $i$, with $\alpha_{i}>0$. Let $i_{0}=\min \left\{i, \mu_{i} \neq 0\right\}$ : the typical number of steps to reach $\mathbf{x}$ is $n_{i_{0}}+O\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)$; the favorite direction are the points x with $x_{i}=b_{n_{i_{0}}}^{(i)}+O\left(a_{n_{i_{0}}}^{(i)}\right)$ for all $i$, see (3.5) below.
III. Case $\alpha_{i_{0}}=1$, where $i_{0}=\min \left\{i, b_{n}^{(i)} \not \equiv 0\right\}$. Assume that either $\mu_{i_{0}} \in \mathbb{R}^{*}$ or $p_{i_{0}} \neq q_{i_{0}}$. The typical number of steps to reach $\mathbf{x}$ is $n_{i_{0}}+O\left(m_{i_{0}}\right)$ with $m_{i_{0}}:=a_{n_{i_{0}}}^{\left(i_{0}\right)} /\left|\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\right|$ (see Section 8.1); the favorite direction are the points $\mathbf{x}$ with $x_{i}=b_{n_{i_{0}}}^{(i)}+O\left(a_{n_{i_{0}}}^{(i)}\right)$ for all $i$, see (3.9) below. Some more subtleties arise in that case.

We now present strong renewal theorems, i.e. sharp asymptotics of $G(\mathbf{x})$, in cases I-II-III, along the favorite direction or scaling (the proofs are presented in Sections 6-7-8). Recall that $g_{\boldsymbol{\alpha}}(\cdot)$ is the density of the limiting multivariate stable law.

### 3.2 Case I (centered): $\mathbf{b}_{n} \equiv \mathbf{0}$

We assume here that $\mathbf{b}_{n} \equiv \mathbf{0}$, and that $\sum_{i=1}^{d} \alpha_{i}^{-1}>1$, so that $\sum_{n \geqslant 1}\left(a_{n}^{(1)} \cdots a_{n}^{(d)}\right)^{-1}<$ $+\infty$, and $\mathbf{S}_{n}$ is transient. We leave aside for the moment the case $d=1, \alpha_{1}=1$ (considered in [3]), and the case $d=2, \boldsymbol{\alpha}=(2,2)$, which are marginal cases-the transience of the random walk depends on the slowly varying functions $L_{i}(\cdot)$.

Theorem 3.1. Suppose $\mathbf{b}_{n} \equiv \mathbf{0}$ and $\sum_{i=1}^{d} \alpha_{i}^{-1}>1$, and that (i) $\sum_{i=1}^{d} \alpha_{i}^{-1}<2$ or (ii) Assumption 2.2 holds. Recall the definition (3.1) of $n_{i}$. If $\|\mathbf{x}\| \rightarrow+\infty$ such that for all $1 \leqslant i \leqslant d$

$$
\begin{equation*}
x_{i} / a_{n_{1}}^{(i)} \rightarrow t_{i} \in \mathbb{R}^{*} \quad \text { as }\left|x_{1}\right| \rightarrow+\infty \quad\left(t_{1}=\operatorname{sign}\left(x_{1}\right)\right), \tag{3.2}
\end{equation*}
$$

then we have that,

$$
\begin{equation*}
G(\mathbf{x}) \sim \frac{\mathrm{C}_{\boldsymbol{\alpha}} n_{1}}{a_{n_{1}}^{(1)} \cdots a_{n_{1}}^{(d)}}, \quad \text { with } \mathrm{C}_{\boldsymbol{\alpha}}=\int_{0}^{\infty} u^{-2+\sum \alpha_{i}^{-1}} g_{\boldsymbol{\alpha}}\left(t_{1} u^{1 / \alpha_{1}}, \ldots, t_{d} u^{1 / \alpha_{d}}\right) \mathrm{d} u \tag{3.3}
\end{equation*}
$$

Recall $n_{1} \sim\left|x_{1}\right|^{\alpha_{1}} \phi_{1}\left(\left|x_{1}\right|\right)^{-1}$ with $\phi_{1}=L_{1}$ if $\alpha_{1} \in(0,2)$ and $\phi_{1}=\sigma_{1}$ if $\alpha_{1}=2$.
We refer to (3.2) as x going to infinity along the favorite scaling. Note that under (3.2) we have $n_{i} \sim\left|t_{i}\right|^{\alpha_{i}} n_{1}$, so we can exchange the role of the coordinates if needed.

## Comments on the balanced case

In the balanced case, $a_{n_{1}}^{(i)} \equiv\left|x_{1}\right|$ and $\alpha_{i} \equiv \alpha$ : we obtain that if either $\alpha>2 / d$ or Assumption 2.2 holds and if $x_{i} / x_{1} \rightarrow t_{i} \in \mathbb{R}^{*}$ for all $i \in\{1, \ldots, d\}$,

$$
\begin{equation*}
G(\mathbf{x}) \sim \mathrm{C}_{\boldsymbol{\alpha}}\left|x_{1}\right|^{\alpha-d} \phi\left(\left|x_{1}\right|\right)^{-1}, \quad \text { with } \mathrm{C}_{\boldsymbol{\alpha}}=\alpha \int_{0}^{\infty} v^{d-1-\alpha} g_{\boldsymbol{\alpha}}\left(t_{1} v, \ldots, t_{d} v\right) \mathrm{d} u \tag{3.4}
\end{equation*}
$$

with $\phi=L$ if $\alpha \in(0,2)$ and $\phi=\sigma$ if $\alpha=2(d \neq 2)$. This recovers Williamson's result [41] (we used a change of variable for the integral), under weaker conditions if $\alpha \leqslant d / 2$.

The marginal case $d=2, \boldsymbol{\alpha}=(2,2)$
In the same spirit as for the case $d=1, \alpha_{1}=1, b_{n}^{(1)} \equiv 0$ (studied in [3, Sect. 3.2]), we treat here the case $d=2$ with $\boldsymbol{\alpha}=(2,2)$ and $\mathbf{b}_{n} \equiv \mathbf{0}$. We give here a renewal theorem (along the favorite scaling) in the case where $\mathbf{S}_{n}$ is transient, i.e. if $\sum_{n=1}^{+\infty}\left(a_{n}^{(1)} a_{n}^{(2)}\right)^{-1}<+\infty$.
Theorem 3.2. Suppose that $d=2$ with $\boldsymbol{\alpha}=(2,2)$ and $\mu_{1}=\mu_{2}=0\left(\mathbf{b}_{n} \equiv \mathbf{0}\right)$, and assume also $\sum_{n=1}^{+\infty}\left(a_{n}^{(1)} a_{n}^{(2)}\right)^{-1}<+\infty$. Recall the definition (3.1) of $n_{1}, n_{2}$. If $\|\mathbf{x}\| \rightarrow+\infty$ such that $x_{2} / a_{n_{1}}^{(2)}$ (equivalently $n_{1} / n_{2}$ ) stays bounded away from 0 and $+\infty$, we have that

$$
G(\mathbf{x}) \sim g_{\boldsymbol{\alpha}}(0,0) \sum_{n \geqslant n_{1}} \frac{1}{a_{n}^{(1)} a_{n}^{(2)}} .
$$

Note that $n_{1} \mapsto \sum_{n \geqslant n_{1}}\left(a_{n}^{(1)} a_{n}^{(2)}\right)^{-1}$ vanishes as a slowly varying function.
In the balanced case $\left(a_{n}^{(i)} \equiv a_{n}\right)$, then $\mathbf{S}_{n}$ is transient if and only if $\int_{1}^{+\infty} \frac{d u}{u \sigma(u)}<+\infty$ (recall the definition (1.3) of $\sigma(\cdot)$ ), and we can rewrite the above as: if $\|\mathbf{x}\| \rightarrow+\infty$ such that $\left|x_{1}\right| /\left|x_{2}\right|$ stays bounded away from 0 and $+\infty$, then

$$
G(\mathbf{x}) \sim 2 g_{\boldsymbol{\alpha}}(0,0) \int_{\left|x_{1}\right|}^{+\infty} \frac{d u}{u \sigma(u)}, \quad \text { as }\left|x_{1}\right| \rightarrow+\infty .
$$

### 3.3 Case II (non-zero mean): $\mu_{i} \neq 0$ for some $i$ with $\alpha_{i}>1$

Let $i_{0}$ be the first $i$ such that $\mu_{i} \neq 0$, and assume that $x_{i_{0}}$ and $\mu_{i_{0}}$ have the same sign. Theorem 3.3. Assume that $\alpha_{i_{0}}>1, \mu_{i_{0}} \neq 0$, and that $\mu_{i}=0$ for $i<i_{0}$. Assume that one among the following three conditions holds:
(i) $\sum_{i=1}^{d} \alpha_{i}^{-1}<2$;
(ii) $\gamma_{i_{0}}>\sum_{i \neq i_{0}} \alpha_{i}^{-1}$;
(iii) Assumption 2.2 .

Recall that $n_{i_{0}}=\left|x_{i_{0}}\right| /\left|\mu_{i_{0}}\right|$, see (3.1). If $\|\mathbf{x}\| \rightarrow+\infty$ such that for all $1 \leqslant i \leqslant d$

$$
\begin{equation*}
\left(x_{i}-b_{n_{i_{0}}}^{(i)}\right) / a_{n_{i_{0}}}^{(i)} \rightarrow t_{i} \in \mathbb{R} \quad \text { as }\left|x_{i_{0}}\right| \rightarrow+\infty \quad\left(t_{i_{0}}=0\right) \tag{3.5}
\end{equation*}
$$

(if (i) or (ii) does not hold, assume that $t_{i} \neq 0$ for $i$ 's with $b_{n}^{(i)} \equiv 0$ ), then we have that

$$
\begin{equation*}
G(\mathbf{x}) \sim \frac{\mathrm{C}_{\boldsymbol{\alpha}}^{\prime} a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}}, \quad \text { with } \mathrm{C}_{\boldsymbol{\alpha}}^{\prime}=\int_{-\infty}^{\infty} g_{\boldsymbol{\alpha}}\left(t_{1}+\kappa_{1} u, \ldots, t_{d}+\kappa_{d} u\right) \mathrm{d} u \tag{3.6}
\end{equation*}
$$

where we set $\kappa_{i}=\mu_{i} a_{i, i_{0}} \mathbb{1}_{\left\{i \geqslant i_{0}\right\}}$ (recall Section 2.4, $a_{i, i_{0}}=0$ if $\alpha_{i}<\alpha_{i_{0}}$ ).
As for Theorem 3.1, we refer to (3.5) as x going to infinity along the favorite direction.

## Comments on the balanced case

If $a_{n}^{(i)} \equiv a_{n}$ and $\alpha_{i} \equiv \alpha$, case II corresponds to having $\alpha>1$ and one $\boldsymbol{\mu}:=\left(\mu_{1}, \ldots, \mu_{d}\right) \neq \mathbf{0}$. If $\alpha>d / 2$, if $\gamma_{i_{0}}>(d-1) / 2$ (in the case $\gamma_{i_{0}}>\alpha=2$ ) or if Assumption 2.2 holds (put otherwise if (i),(ii) or (iii) in Theorem 3.3 holds), we therefore obtain that for $\mathbf{t}=\left(t_{1}, \ldots, t_{d}\right)$ with $t_{i} \neq 0$ if $\mu_{i}=0$,

$$
\begin{equation*}
G\left(\left\lfloor r \boldsymbol{\mu}+a_{r} \mathbf{t}\right\rfloor\right) \sim \frac{\mathrm{C}_{\mathbf{t}}^{\prime}}{\left(a_{r}\right)^{d-1}} \quad \text { with } \mathrm{C}_{\mathbf{t}}^{\prime}=\int_{-\infty}^{+\infty} g_{\boldsymbol{\alpha}}(\mathbf{t}+u \boldsymbol{\mu}) d u, \quad \text { as } r \rightarrow+\infty \tag{3.7}
\end{equation*}
$$

In the symmetric case where we have $\mu_{i} \equiv \mu \neq 0$, the result simplifies: let us state it along the diagonal $\mathbf{1}=(1, \ldots, 1)$ for simplicity,

$$
G(r \mathbf{1}) \sim \frac{|\mu|^{\frac{d-1}{\alpha}-1}}{\left(a_{r}\right)^{d-1}} \int_{-\infty}^{+\infty} g(v \mathbf{1}) d v \quad \text { as } r \rightarrow+\infty
$$

Indeed, we used that $a_{r /|\mu|} \sim|\mu|^{-1 / \alpha} a_{r}$, and a change of variable for the integral.

### 3.4 Case III: $\alpha_{i_{0}}=1$

Let us define $i_{0}=\min \left\{i, b_{n}^{(i)} \not \equiv 0\right\}$, and assume that $\alpha_{i_{0}}=1$ with either $\mu_{i_{0}} \in \mathbb{R}^{*}$ or $p_{i_{0}} \neq q_{i_{0}}$. For an overview of results and estimates on (univariate) random walks of Cauchy type, we refer to [3]-many of the estimates we use below come from there. Having $\mu_{i_{0}} \in \mathbb{R}^{*}$ or $p_{i_{0}} \neq q_{i_{0}}$ ensures in particular that $\left|b_{n}^{\left(i_{0}\right)}\right| / a_{n}^{\left(i_{0}\right)} \rightarrow+\infty$ :
$*$ If $\mu_{i_{0}} \in \mathbb{R}^{*}$ then $b_{n}^{\left(i_{0}\right)} \sim \mu_{i_{0}} n$ and $a_{n}^{\left(i_{0}\right)}=o(n)\left(\left|\mu_{i_{0}}\right|<+\infty\right.$ implies that $\left.L_{i_{0}}(x)=o(1)\right)$.

* If $\left|\mu_{i_{0}}\right|=+\infty$ then $b_{n}^{\left(i_{0}\right)} \sim\left(p_{i_{0}}-q_{i_{0}}\right) n \ell_{i_{0}}\left(a_{n}^{\left(i_{0}\right)}\right)$ with $\ell_{i_{0}}(x):=\int_{1}^{x} L_{i_{0}}(u) u^{-1} d u$ which verifies $\ell_{i_{0}}(x) / L_{i_{0}}(x) \rightarrow+\infty$ as $x \rightarrow+\infty$, see [6, Prop. 1.5.9.a]. Since on the other hand $a_{n}^{\left(i_{0}\right)} \sim n L_{i_{0}}\left(a_{n}^{\left(i_{0}\right)}\right)$, we get that $a_{n}^{\left(i_{0}\right)}=o\left(\left|b_{n}^{\left(i_{0}\right)}\right|\right)$.
* If $\mu_{i_{0}}=0$, then similarly, $b_{n}^{\left(i_{0}\right)} \sim-\left(p_{i_{0}}-q_{i_{0}}\right) n \ell_{i_{0}}^{\star}\left(a_{n}^{\left(i_{0}\right)}\right)$ with $\ell_{i_{0}}^{\star}(x):=\int_{x}^{\infty} L_{i_{0}}(u) u^{-1}$, which also verifies $\ell_{i_{0}}(x) / L_{i_{0}}(x) \rightarrow+\infty$ as $x \rightarrow+\infty$. We also get that $a_{n}^{\left(i_{0}\right)}=o\left(\left|b_{n}^{\left(i_{0}\right)}\right|\right)$.

Analogously to Section 2.4 , if $\alpha_{i}=1$, we work along a subsequence such that the following limit exists

$$
\begin{equation*}
\tilde{a}_{i, i_{0}}:=\lim _{n \rightarrow \infty} \frac{a_{n}^{\left(i_{0}\right)}}{\mu_{i_{0}}\left(a_{n}^{\left(i_{0}\right)}\right)} \frac{\mu_{i}\left(a_{n}^{(i)}\right)}{a_{n}^{(i)}} \in \mathbb{R} \quad \text { for } i \geqslant i_{0} \quad\left(\tilde{a}_{i_{0}, i_{0}}=1\right) \tag{3.8}
\end{equation*}
$$

If $\alpha_{i}<1$ we set $\tilde{a}_{i, i_{0}}=0$. We stress that it is possible to have $\tilde{a}_{i, i_{0}}>0$ even if $a_{i, i_{0}}=0$. For instance, take $L_{i_{0}}(x)=1$ and $L_{i}(x)=\log x$ : we get that $a_{n}^{\left(i_{0}\right)} \sim n$, and $a_{n}^{(i)} \sim n \log n$ so $a_{i, i_{0}}=0$; but we have that $\mu_{i_{0}}(n) \sim \log n$ and $\mu_{i}(n) \sim \frac{1}{2}(\log n)^{2}$, so $\tilde{a}_{i, i_{0}}=1 / 2$.

Theorem 3.4. Assume that $\alpha_{i_{0}}=1$ with $\mu_{i_{0}} \in \mathbb{R}^{*}$ or $p_{i_{0}} \neq q_{i_{0}}$, and that $b_{n}^{(i)} \equiv 0$ for $i<i_{0}$. Suppose that Assumption 2.2 holds. Define $\tilde{\kappa}_{i}=\tilde{a}_{i, i_{0}} \mathbb{1}_{\left\{i \geqslant i_{0}\right\}}$, and recall the definition (3.1) of $n_{i_{0}}$. If $\|\mathbf{x}\| \rightarrow+\infty$ such that for all $1 \leqslant i \leqslant d$,

$$
\begin{equation*}
\left(x_{i}-b_{n_{i_{0}}}^{(i)}\right) / a_{n_{i_{0}}}^{(i)} \rightarrow t_{i} \in \mathbb{R} \quad \text { as }\left|x_{i_{0}}\right| \rightarrow+\infty \quad\left(t_{i_{0}}=0\right), \tag{3.9}
\end{equation*}
$$

(with $t_{i} \neq 0$ when $b_{n}^{(i)} \equiv 0$ ) then we have

$$
\begin{equation*}
G(\mathbf{x}) \sim \frac{1}{\left|\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\right|} \cdot \frac{\mathrm{C}_{\boldsymbol{\alpha}}^{\prime \prime} a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}}, \quad \text { with } \mathrm{C}_{\boldsymbol{\alpha}}^{\prime \prime}=\int_{-\infty}^{\infty} g_{\boldsymbol{\alpha}}\left(t_{1}+\tilde{\kappa}_{1} u, \ldots, t_{d}+\tilde{\kappa}_{d} u\right) \mathrm{d} u \tag{3.10}
\end{equation*}
$$

Note that $n_{i_{0}} \sim\left|x_{i_{0}}\right| /\left|\mu_{i_{0}}\left(\left|x_{i_{0}}\right|\right)\right|$, and $\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right) \sim \mu_{i_{0}}\left(\left|x_{i_{0}}\right|\right)$ as $\left|x_{i_{0}}\right| \rightarrow+\infty$, see Section 8.1.
Again, we refer to (3.9) as $\mathbf{x}$ going to infinity along the favorite direction.

## Comments on the balanced case

In the balanced and symmetric case, we have $a_{n}^{(i)} \equiv a_{n}$ and $b_{n}^{(i)} \equiv b_{n}\left(\mu_{i}(x) \equiv \mu(x)\right.$, $\tilde{\kappa}_{i} \equiv 1$ ). The favorite direction is the diagonal $\mathbf{1}=(1, \ldots, 1)$, and we can write, for $\mathbf{t}=\left(t_{1}, \ldots, t_{d}\right)$,

$$
\begin{equation*}
G\left(r \mathbf{1}+\left\lfloor a_{r /|\mu(r)|} \mathbf{t}\right\rfloor\right)^{r \rightarrow+\infty} \stackrel{\mathrm{C}_{\mathbf{t}}^{\prime \prime}}{|\mu(r)| \times\left(a_{r /|\mu(r)|}\right)^{d-1}} \quad \text { with } \mathrm{C}_{\mathbf{t}}^{\prime \prime}=\int_{-\infty}^{\infty} g_{\boldsymbol{\alpha}}(\mathbf{t}+u \mathbf{1}) \mathrm{d} u \tag{3.11}
\end{equation*}
$$

Indeed, $n_{i_{0}} \sim r /|\mu(r)|$, and we also used that $\mu\left(a_{n_{i_{0}}}\right) \sim \mu\left(\left|b_{n_{i_{0}}}\right|\right)=\mu(r)$, see [3, Lemma 4.3].

As a simple example, take Example 2.4 with $\beta_{i} \equiv 1, \beta=1+d: \mathbf{P}\left(\mathbf{X}_{1}=\mathbf{x}\right)=$ $c_{d}\|\mathbf{x}\|^{-(1+d)}$ for $x \in \mathbb{N}^{d}$, and $\mathbf{P}\left(X_{1}^{(i)}>n\right) \sim c_{1} / n$. We have $\alpha_{i} \equiv 1$, and $a_{n} \sim n / c_{1}$, $\mu(n) \sim \mathrm{c}_{1} \log n$ : we therefore get that $G(r \mathbf{1}+(r / \log r) \mathbf{t}) \sim \mathrm{c}_{\mathbf{t}}(\log r)^{d-2} r^{-(d-1)}$ as $r \rightarrow+\infty$.

## 4 Renewal estimates away from the favorite direction or scaling

In this section, we provide bounds on $G(\mathbf{x})$ that hold uniformly on $\mathbf{x}$ : in particular, this sharpens our estimates when x goes away from the favorite direction or scaling (one would have $\mathbf{C}_{\boldsymbol{\alpha}}, \mathbf{C}_{\boldsymbol{\alpha}}^{\prime}$ or $\mathbf{C}_{\boldsymbol{\alpha}}^{\prime \prime} \rightarrow 0$ in Theorems 3.1, 3.3 or 3.4). We do not obtain sharp asymptotics for $G(\mathbf{x})$, mostly because the local large deviation estimates of Section 2 are not sharp-first of all because our Assumption 2.2 does not give the precise asymptotic of $\mathbf{P}\left(\mathbf{X}_{1}=\mathbf{x}\right)$. Let us stress that in [4], the authors manage to obtain the sharp asymptotic of $G(\mathbf{x})$ in a specific setting (with application to a DNA model): $\mathbf{X}_{1} \in \mathbb{N}^{2}$, and the local probabilities $\mathbf{P}\left(\mathbf{X}_{1}=\mathbf{x}\right)$ are known asymptotically, one coordinate having a heavy-tail, the second one having an exponential tail. One should also be able to obtain the sharp asymptotics of $G(\mathbf{x})$ for instance in Example 2.4, but we do not pursue it here to avoid additional lengthy and technical calculations.

We also stress that having uniform bounds on $G(\mathbf{x})$ turn out to be useful, for instance when studying the intersection of two independent (multivariate) renewal processes $\mathbf{S}=$ $\left\{\mathbf{S}_{n}\right\}_{n} \geqslant 0, \mathbf{S}^{\prime}=\left\{\mathbf{S}_{n}^{\prime}\right\}_{n} \geqslant 0$ with same distribution. Indeed, $\mathbf{E}\left[\left|\mathbf{S} \cap \mathbf{S}^{\prime}\right|\right]=\sum_{\mathbf{x} \in \mathbb{Z}^{d}} \mathbf{P}(\mathbf{x} \in \mathbf{S})^{2}$, and to known whether $\mathbf{S} \cap \mathbf{S}^{\prime}$ is finite, good bounds on $G(\mathbf{x})=\mathbf{P}(\mathbf{x} \in \mathbf{S})$ are essential. The main contribution to $\mathbf{E}\left[\left|\mathbf{S} \cap \mathbf{S}^{\prime}\right|\right]$ will come from points along the favorite direction, and one needs to know how fast $G(\mathbf{x})$ decreases when $\mathbf{x}$ moves away from it. We refer to [5, App. A.2] for some results on the intersection of two independent renewal processes.

For the simplicity of the exposition, we only present the case of dimension $d=2$. Also, we will work under Assumption 2.2. Often, results will be sharper in the case of renewal processes, as will be outlined in our theorems. We divide our statements into three parts: $\mathbf{b}_{n} \equiv 0$ (centered); $b_{n}^{(1)}, b_{n}^{(2)} \not \equiv 0$ (non-zero mean for both coordinates); $b_{n}^{\left(i_{0}\right)} \not \equiv 0$ and $b_{n}^{\left(i_{1}\right)} \equiv 0$ (mixed case). The proofs are presented in Section 9.

### 4.1 Case I (centered case), $\mathbf{b}_{n} \equiv 0$

Let us leave aside the marginal case $d=2 \boldsymbol{\alpha}=(2,2)$ : we have $\alpha_{1}^{-1}+\alpha_{2}^{-1}>1$. Recall the definition (3.1) of $n_{i}\left(n_{i} \sim\left|x_{i}\right|^{\alpha_{i}} \phi_{i}\left(\left|x_{i}\right|\right)^{-1}\right.$ with $\phi_{i}=L_{i}$ if $\alpha_{i} \in(0,2)$ and $\phi_{i}=\sigma_{i}$ if $\alpha_{i}=2$ ), and let $i_{0}, i_{1}$ be such that $n_{i_{0}}=\min \left\{n_{1}, n_{2}\right\}$ and $n_{i_{1}}=\max \left\{n_{1}, n_{2}\right\}$.
Theorem 4.1. Assume that $\mathbf{b}_{n} \equiv 0$, and that Assumption 2.2 holds. Then for any $\delta>0$, we have a constant $C_{\delta}$ such that for any $\mathrm{x} \in \mathbb{Z}^{2}$,

$$
\begin{equation*}
G(\mathbf{x}) \leqslant \frac{C_{\delta} n_{i_{0}}}{a_{n_{i_{0}}}^{(1)} a_{n_{i_{0}}}^{(2)}}\left(\frac{n_{i_{1}}}{n_{i_{0}}}\right)^{-\nu+\delta}, \quad \text { with } \nu=\left(1+\alpha_{i_{1}}^{-1}\right) \frac{\alpha_{1}^{-1}+\alpha_{2}^{-1}-1}{\alpha_{1}^{-1}+\alpha_{2}^{-1}+1} \tag{4.1}
\end{equation*}
$$

If $\left(\mathbf{S}_{n}\right)_{n \geqslant 0}$ is a renewal process (necessarily $\alpha_{1}, \alpha_{2}<1$ ), we can replace $\nu$ by $1+\alpha_{i_{1}}^{-1}$.
Clearly, Theorem 4.1 improves (3.3) in the regime $n_{i_{1}} / n_{i_{0}} \rightarrow+\infty$.

## About the balanced case

If $\alpha_{i} \equiv \alpha \in(0,2]$ and $a_{n}^{(i)} \equiv a_{n}$, we obtain that under Assumption 2.2, for any $\delta>0$ there exists a constant $C_{\delta}$ such that for any $\mathbf{x} \in \mathbb{Z}^{2}$, setting $x_{i_{0}}=\min \left\{x_{1}, x_{2}\right\}$ and $x_{i_{1}}=\max \left\{x_{1}, x_{2}\right\}$,

$$
\begin{equation*}
G(\mathbf{x}) \leqslant C_{\delta}\left|x_{i_{0}}\right|^{\alpha-2} \phi\left(\left|x_{i_{0}}\right|\right)^{-1}\left(\frac{x_{i_{1}}}{x_{i_{0}}}\right)^{-\theta+\delta} \quad \text { with } \theta:=(1+\alpha) \frac{2-\alpha}{2+\alpha} \tag{4.2}
\end{equation*}
$$

with $\phi=L$ if $\alpha \in(0,2)$ and $\phi=\sigma$ if $\alpha=2$. where (recall $n_{i} \sim x_{i}^{-\alpha} \phi\left(x_{i}\right)^{-1}$ ). If $\left(\mathbf{S}_{n}\right)_{n \geqslant 1}$ is a renewal process (necessarily $\alpha \in(0,1)$ ), then we can replace $\theta$ by $1+\alpha$.

### 4.2 Case II-III (non-zero mean), subcase (a): $b_{n}^{(1)}, b_{n}^{(2)} \neq 0$

Let us consider the case when for both $i=1,2$ we have: either $\alpha_{i} \geqslant 1$ and $\mu_{i} \in \mathbb{R}^{*}$, or $\alpha_{i}=1$ and $p_{i} \neq q_{i}$. This insures that $b_{n}^{(i)} \neq 0$ for $i=1,2$, and places us in the setting of cases II and III of Section 3.

Recall the definition (3.1) of $n_{i}$ : we have $n_{i} \sim\left|x_{i}\right| /\left|\mu_{i}\left(\left|x_{i}\right|\right)\right|$ (both if $\alpha_{i}>1$ or $\alpha_{i}=1$ ). Let us also define $m_{i}:=a_{n_{i}}^{(i)} /\left|\mu_{i}\left(a_{n_{i}}^{(i)}\right)\right|$ : in Section 8.1, we see that $m_{i}=o\left(n_{i}\right)$, and that the typical number of steps for the $i^{\text {th }}$ coordinate to reach $x_{i}$ is $n_{i}+O\left(m_{i}\right)$ (this is trivial if $\left.\alpha_{i}>1\right)$. Let us stress that the favorite direction $\left(\left|x_{2}-b_{n_{1}}^{(2)}\right|=O\left(a_{n_{1}}^{(1)}\right),\left|x_{1}-b_{n_{2}}^{(1)}\right|=O\left(a_{n_{2}}^{(2)}\right)\right.$, see (3.5)-(3.9)) corresponds to having $n_{1}-n_{2}=O\left(m_{i}\right)$ for $i=1,2$. We will state only the case $n_{1} \leqslant n_{2}$, the other case being symmetric.
Theorem 4.2. Suppose that Assumption 2.2 holds, and that for $i=1,2$ : either $\alpha_{i} \geqslant 1$ and $\mu_{i} \in \mathbb{R}^{*}$, or $\alpha_{i}=1$ and $p_{i} \neq q_{i}$. Then for every $\delta>0$ there is a constant $C_{\delta}$ such that, for all $\mathbf{x} \in \mathbb{Z}^{2}$ (recalling the definition (3.1) of $n_{i}$, and of $m_{i}:=a_{n_{i}}^{(i)} /\left|\mu_{i}\left(a_{n_{i}}^{(i)}\right)\right|$ ),
(i) If $n_{1} \leqslant n_{2} \leqslant 2 n_{1}$,

$$
\begin{align*}
G(\mathbf{x}) \leqslant \frac{C_{\delta}}{a_{n_{2}}^{(2)}\left|\mu_{1}\left(a_{n_{1}}^{(1)}\right)\right|} & \times\left(\frac{n_{2}-n_{1}}{m_{2}}\right)^{-1+\delta}\left(\frac{n_{2}-n_{1}}{m_{1}}\right)^{\delta} \\
& \times\left\{\left(\frac{n_{2}-n_{1}}{m_{1}}\right)^{-\alpha_{1}} R^{(1)}\left(n_{2}-n_{1}\right)+\left(\frac{n_{2}-n_{1}}{m_{2}}\right)^{-\alpha_{2}} R^{(2)}\left(n_{2}-n_{1}\right)\right\} \tag{4.3}
\end{align*}
$$

with $R^{(i)}(m):=\mathbb{1}_{\left\{\alpha_{i} \in(0,2)\right\}}+\left(m^{2-\gamma_{i}}+e^{-c m / m_{i}}\right) \mathbb{1}_{\left\{\alpha_{i}=2\right\}}$.
(ii) If $n_{2} \geqslant 2 n_{1}$,

$$
\begin{equation*}
G(\mathbf{x}) \leqslant C_{\delta}\left(n_{1} \vee n_{2}^{1 \wedge\left(\gamma_{2} / \gamma_{1}\right)}\right) \times n_{2}^{-\left(1+\gamma_{2}\right)+\delta} \leqslant C_{\delta} n_{2}^{-\gamma_{2}+\delta} \tag{4.4}
\end{equation*}
$$

If $\mathbf{S}_{n}$ is a renewal process, then $G(\mathbf{x})=0$ as soon as $n_{2} \geqslant\left|x_{1}\right|$, in particular if $n_{2} \geqslant n_{1}^{1+\delta}$.

We stress that in the case $\alpha_{1}, \alpha_{2}>1$, then we can replace $n_{i}$ by $x_{i} / \mu_{i}$ ( $x_{i}$ and $\mu_{i}$ with the same sign) and $m_{i}$ by $a_{\left|x_{i}\right|}^{(i)}$.

## About the balanced case

In the balanced case $\left(a_{n}^{(i)} \equiv a_{n}\right)$, Theorem 4.3 gives the following:

* If $\alpha>1, n_{i}=x_{i} / \mu_{i}$, and $\left|n_{1}-n_{2}\right|=\left|x_{1} / \mu_{1}-x_{2} / \mu_{2}\right|$ : the bound (4.3) (together with (3.7) for the case $|s| \leqslant a_{r}$ ) gives, for any $|s| \leqslant r$

$$
\begin{equation*}
G\left(\left(r,\left\lfloor\frac{\mu_{2}}{\mu_{1}} r\right\rfloor+s\right)\right) \leqslant \frac{C}{a_{r}}\left(1 \wedge\left(\frac{|s|}{a_{r}}\right)^{-(1+\alpha)+\delta} R^{(2)}(|s|)\right) \tag{4.5}
\end{equation*}
$$

$\left(R^{(2)}(|s|)=1\right.$ if $\alpha<2$ and $R^{(2)}(|s|)=|s|^{2-\gamma_{2}}+e^{-c|s| / a_{r}}$ if $\alpha=2$ ). For $|s| \geqslant r$, then (4.4) gives that $G\left(\left(r,\left\lfloor\frac{\mu_{2}}{\mu_{1}} r\right\rfloor+s\right)\right) \leqslant C_{\delta}|s|^{-\gamma_{2}+\delta}$.

* If $\alpha=1$ and $\mu_{1}, \mu_{2} \in \mathbb{R}^{*}$, then we have $\left|\mu_{i}-\mu_{i}\left(a_{n}\right)\right|=O\left(L\left(a_{n}\right)\right) \mid$, so $\left|n_{1}-n_{2}\right|=$ $\left|x_{1} / \mu_{1}\left(a_{n_{1}}\right)-n_{2} / \mu_{2}\left(a_{n_{2}}\right)\right|=\left|x_{1} / \mu_{1}-x_{2} / \mu_{2}\right|+O\left(n_{1} L\left(a_{n_{1}}\right)\right)$, provided that $x_{1} \asymp x_{2}$ (equivalently $n_{1} \asymp n_{2}$ )-note also that $n_{1} L\left(a_{n_{1}}\right)=O\left(a_{n_{1}}\right)$. We therefore get the same conclusion as in (4.5). The case $|s| \geqslant r$ is similar to the case $\alpha>1$ above.
* If $\alpha=1$ with $\left|\mu_{i}\right|=+\infty$ or 0 , we assume additionally that the distribution is symmetric: we have $\mu_{i}(n) \equiv \mu(n)$ (we actually only need this for $n$ large). Then, using Claim 5.3 below, we have $\left|\mu\left(a_{n_{1}}\right)-\mu\left(a_{n_{2}}\right)\right|=O\left(L\left(a_{n_{2}}\right)\right)$ provided that $x_{1} / x_{2}$ (hence $\left.n_{1} / n_{2}\right)$ is bounded away from 0 and $+\infty$ : we get $\left|n_{1} \mu\left(a_{n_{1}}\right)-n_{2} \mu\left(a_{n_{2}}\right)\right|=\left|n_{1}-n_{2}\right| \mu\left(a_{n_{1}}\right)+$ $O\left(n_{2} L\left(a_{n_{2}}\right)\right)$, with $n_{2} L\left(a_{n_{2}}\right)=O\left(a_{n_{2}}\right)=O\left(a_{n_{1}}\right)$. It gives, as long as $x_{1} \asymp x_{2}$, that $\left|n_{1}-n_{2}\right| \leqslant \mu\left(x_{1}\right)\left|x_{1}-x_{2}\right|+O\left(a_{n_{1}}\right)$. Using (4.3) (and (3.11) for the case $s \leqslant a_{r / \mu(r)}$ ), we obtain that for any $|s| \leqslant r$

$$
\begin{equation*}
G((r, r+s)) \leqslant \frac{C}{|\mu(r)| a_{r /|\mu(r)|}}\left(\frac{s}{a_{r /|\mu(r)|}} \vee 1\right)^{-2+\delta} . \tag{4.6}
\end{equation*}
$$

We used that $m_{1}=a_{n_{1}} /\left|\mu\left(a_{n_{1}}\right)\right| \sim a_{r /|\mu(r)|} /|\mu(r)|\left(m_{1}=m_{2}\right)$. In the case $|s| \geqslant r$, then applying (4.4) gives that $G((r, r+s)) \leqslant C_{\delta}|s|^{-1+\delta}$, using also that $n_{2} \geqslant c_{\delta^{\prime}} s^{1-\delta^{\prime}}$.

We mention that assumption (2.8) would not improve much (4.5)-(4.6): the improvement would be only at the level of the slowly varying function, that are absorbed by the exponent $\delta$. We refer to the end of Section 9.2 for a discussion.

### 4.3 Case II-III (non-zero mean, mixed), subcase (b): $b_{n}^{\left(i_{0}\right)} \neq 0, b_{n}^{\left(i_{1}\right)} \equiv 0$

Here, we consider again the setting of cases II and III of Section 3, in the case where the second coordinate is "centered".
Theorem 4.3. Suppose that Assumption 2.2 holds, and that there is $\left\{i_{0}, i_{1}\right\}=\{1,2\}$ such that: $b_{n}^{\left(i_{1}\right)} \equiv 0$ and, either $\alpha_{i_{0}} \geqslant 1$ and $\mu_{i_{0}} \in \mathbb{R}^{*}$, or $\alpha_{i_{0}}=1$ and $p_{i_{0}} \neq q_{i_{0}}$. Recall the definition (3.1): we have $n_{i_{0}} \sim\left|x_{i_{0}}\right| /\left|\mu_{i_{0}}\left(\left|x_{i_{0}}\right|\right)\right|$, and $n_{i_{1}} \sim\left|x_{i_{1}}\right|^{-\alpha_{i_{1}}} \phi_{i_{1}}\left(\left|x_{i_{1}}\right|\right)^{-1}$ with $\phi_{i_{1}}=L_{i_{1}}$ if $\alpha_{i_{1}} \in(0,2)$ and $\phi_{i_{1}}=\sigma_{i_{1}}$ if $\alpha_{i_{1}}=2$. There is a constant $C$ and for any $\delta>0$ there is a constant $C_{\delta}$ such that for any $\mathbf{x} \in \mathbb{Z}^{2}$ :
(i) If $n_{i_{1}} \leqslant n_{i_{0}}$,

$$
G(\mathbf{x}) \leqslant \frac{C}{\left|\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\right| a_{n_{i_{0}}}^{\left(i_{1}\right)}}+ \begin{cases}C_{\delta} n_{i_{1}}^{2-1 / \alpha_{i_{1}}}\left|x_{i_{0}}\right|^{-\left(1+\gamma_{i_{0}}\right)+\delta} & \text { if } \alpha_{i_{1}} \leqslant 1 / 2  \tag{4.7}\\ 0 & \text { if } \alpha_{i_{1}}>1 / 2\end{cases}
$$

If $\mathbf{S}_{n}$ is a renewal process (necessarily $\alpha_{i_{1}} \in(0,1)$ ), then there is an exponent $\zeta_{\delta}>0$ such that $G(\mathbf{x}) \leqslant C_{\delta}\left(n_{i_{1}}\right)^{2-1 / \alpha_{i_{1}}}\left|x_{i_{0}}\right|^{-\left(1+\gamma_{i_{0}}\right)+\delta}+e^{-c\left(n_{i_{0}} / n_{i_{1}}\right)^{\zeta_{\delta}}}$.
(ii) If $n_{i_{1}} \geqslant n_{i_{0}}$, we set $m_{\delta}:=\left(\left|x_{i_{1}}\right|^{\gamma_{i_{1}} / \gamma_{i_{0}}+\delta} \vee\left(n_{i_{0}}\right)^{1+\delta}\right) \wedge\left(n_{i_{1}}\right)^{1-\delta}$ (and $m_{\delta}=+\infty$ if $\left(n_{i_{0}}\right)^{1+\delta}>\left(n_{i_{1}}\right)^{1-\delta}$ ), and we have

$$
\begin{equation*}
G(\mathbf{x}) \leqslant \frac{C_{\delta}}{a_{n_{i_{1}}}^{(1)}}\left(1 \wedge m_{\delta}\left|x_{i_{1}}\right|^{-\gamma_{i_{1}}}\right) \tag{4.8}
\end{equation*}
$$

If $\mathbf{S}_{n}$ is a renewal process, $G(\mathbf{x}) \leqslant \frac{C_{\delta} n_{i_{0}}^{\delta}}{a_{n_{i_{1}}}^{(1)}}\left(n_{i_{0}}\left|x_{i_{1}}\right|^{-\alpha_{i_{1}}+\delta}+e^{-c_{\delta}\left(n_{i_{1}} / n_{i_{0}}\right)^{1-\delta}} \mathbb{1}_{\left\{\alpha_{i_{1}}=2\right\}}\right)$.
Notational warning: In the rest of the paper, we use $c, C, c^{\prime}, C^{\prime}, \ldots$ as generic constants, and we will keep the dependence on parameters when necessary, writing for example $c_{\varepsilon}, C_{\varepsilon}$ for constants depending on a parameter $\varepsilon$.

## 5 Proof of the local large deviations

In this section, we prove the local limit theorems of Section 2: Theorem 2.1 in Section 5.2, Theorem 2.5 in Section 5.3, and Theorem 2.6 in Section 5.4. But first of all, let us recall some univariate large deviation results.

### 5.1 Univariate large deviations: a reminder of Fuk-Nagaev inequalities

We start by giving a brief reminder of useful large deviation results for univariate random walks (i.e. we focus on $S^{(1)}$ ) in the domain of attraction of an $\alpha_{1}$-stable distribution-this will be useful throuhout the section. Most of these estimates can be found in [31], but the case $\alpha_{1}=1$ was improved recently, cf. [3]. This will enable us to obtain local limit theorems for multivariate random walks in the next section.

In the rest of the section, we denote $M_{n}^{(i)}:=\max _{1 \leqslant k \leqslant n} X_{k}^{(i)}$. We refer to Section 5 in [3] for an overview on how to derive the following statement from [31].
Theorem 5.1. Suppose that Assumption 1.1 holds. There are constants $c, c^{\prime}$ such that

* if $\alpha_{1} \in(0,1) \cup(1,2)$, for any $1 \leqslant y \leqslant x$

$$
\mathbf{P}\left(S_{n}^{(1)}-b_{n}^{(1)} \geqslant x ; M_{n}^{(1)} \leqslant y\right) \leqslant\left(c \frac{y}{x} n L_{1}(y) y^{-\alpha_{1}}\right)^{x / y}
$$

* if $\alpha_{1}=1$, for every $\varepsilon>0$, there is some $C_{\varepsilon}>0$ such that, for any $x \geqslant C_{\varepsilon} a_{n}^{(1)}$ and $1 \leqslant y \leqslant x$

$$
\mathbf{P}\left(S_{n}^{(1)}-b_{n}^{(1)} \geqslant x ; M_{n}^{(1)} \leqslant y\right) \leqslant\left(c \frac{y}{x} n L_{1}(y) y^{-1}\right)^{(1-\varepsilon) x / y}+e^{-\left(x / a_{n}^{(1)}\right)^{1 / \varepsilon}}
$$

* if $\alpha_{1}=2$, for any $y \leqslant x$

$$
\mathbf{P}\left(S_{n}^{(1)}-b_{n}^{(1)} \geqslant x ; M_{n}^{(1)} \leqslant y\right) \leqslant\left(c \frac{y}{x} n y^{-\gamma_{1}} \varphi_{1}(y)\right)^{\frac{x}{2 y}}+C e^{-c \frac{x^{2}}{n \sigma_{1}(y)}} .
$$

The case $\alpha_{1} \in(0,1) \cup(1,2)$ is given by Theorems 1.1 and 1.2 in [31] (we also refer to Section 3 of [7], which contains a simpler proof of that fact). The case $\alpha_{1}=1$ is given in [3, Theorem 2.2]. The case $\alpha_{1}=2$ is given by Corollary 1.7 in [31].

As a consequence of Theorem 5.1, there is a constant $c_{0}$ such that, whenever $x \geqslant a_{n}^{(1)}$,

$$
\begin{equation*}
\mathbf{P}\left(S_{n}^{(1)}-b_{n}^{(1)} \geqslant x\right) \leqslant c_{0} n \varphi_{1}(x) x^{-\gamma_{1}}+c_{0} e^{-\frac{x^{2}}{c_{0} n \sigma_{1}(x)}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}} \tag{5.1}
\end{equation*}
$$

Indeed, the left-hand side is bounded by $\mathbf{P}\left(M_{n}^{(1)} \geqslant x / 4\right)+\mathbf{P}\left(S_{n}^{(1)}-b_{n}^{(1)} \geqslant x ; M_{n}^{(1)} \leqslant x / 4\right)$. Using a union bound, and because of Assumption 1.1, the first term is bounded by a constant times $n \varphi_{1}(x) x^{-\gamma_{1}}$. For the second term, we use Theorem 5.1, which gives that

- if $\alpha_{1} \in(0,1) \cup(1,2)$, it is bounded by a constant times $\left(n x^{-\alpha_{1}} L_{1}(x)\right)^{4}$;
- if $\alpha_{1}=1$, is it bounded by a constant times $\left(n L_{1}(x) x^{-1}\right)^{4(1-\varepsilon)}+e^{-c\left(x / a_{n}^{(1)}\right)^{1 / \varepsilon}}$;
- if $\alpha_{1}=2$, it is bounded by $\left(n x^{-\gamma_{1}} \varphi_{1}(x)\right)^{2}+e^{-c x^{2} /\left(n \sigma_{1}(x)\right)}$.

Another useful consequence of Theorem 5.1 is the following: let $C, C^{\prime}$ be two (large) constants, with $C^{\prime}<C / 10$, then there is a constant $c^{\prime \prime}$ such that for any $x \geqslant C a_{n}^{(1)}$, we have

$$
\begin{equation*}
\mathbf{P}\left(S_{n}^{(1)}-b_{n}^{(1)} \geqslant x, M_{n}^{(1)} \leqslant C^{\prime} a_{n}^{(1)}\right) \leqslant\left(n \varphi_{1}(x) x^{-\gamma_{1}}\right)^{2} e^{-c^{\prime \prime} x_{1} / a_{n}^{(1)}}+e^{-c^{\prime \prime}\left(x_{1} / a_{n}^{(1)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}} . \tag{5.2}
\end{equation*}
$$

We used $\left(n \varphi_{1}(x) x^{-\gamma_{1}}\right)^{2}$ for technical purposes (it is needed in the following), but the bound is also valid without the square (or even without this term), bounding $n \varphi_{1}(x) x^{-\gamma_{1}}$ by 1 if $x$ is larger than $C a_{n}^{(1)}$.

Indeed, Theorem 5.1 gives that the left-hand side is bounded by

$$
\left(c n \varphi_{1}\left(a_{n}^{(1)}\right)\left(a_{n}^{(1)}\right)^{-\gamma_{1}} \frac{a_{n}^{(1)}}{x_{1}}\right)^{c^{\prime} x_{1} / a_{n}^{(1)}}+e^{-c^{\prime} x_{1} / a_{n}^{(1)}} \mathbb{1}_{\left\{\alpha_{1}=1\right\}}+e^{-c^{\prime} x_{1}^{2} /\left(n \sigma_{1}\left(a_{n}^{(1)}\right)\right)} \mathbb{1}_{\left\{\alpha_{1}=2\right\}} .
$$

To obtain (5.2) from this, we use the following. (1) If $\alpha_{1} \in(0,2)$ then $L_{1}=\varphi_{1}, \gamma_{1}=$ $\alpha_{1}$ and $n L_{1}\left(a_{n}^{(1)}\right) \sim\left(a_{n}^{(1)}\right)^{\alpha_{1}} L_{1}\left(a_{n}^{(1)}\right)^{-1}$ so the first and second term are smaller than $\exp \left(-c^{\prime} x_{1} / a_{n}^{(1)}\right)$ provided that $x_{1} / a_{n}^{(1)} \geqslant C$. Then we use that $\exp \left(-c^{\prime} x_{1} / a_{n}^{(1)}\right)$ is bounded by a constant times $\left(x_{1} / a_{n}^{(1)}\right)^{-4 \alpha_{1}} \exp \left(-c^{\prime \prime} x_{1} / a_{n}^{(1)}\right)$ with $c^{\prime \prime}>c^{\prime}$ since $x_{1} \geqslant C a_{n}^{(1)}$, and then that $\left(x_{1} / a_{n}^{(1)}\right)^{-4 \alpha_{1}}$ is bounded by a constant times $\left(n L_{1}\left(x_{1}\right) x_{1}^{-\alpha_{1}}\right)^{2}$ thanks to Potter's bound [6, Thm. 1.5.6] (recall the definition (1.4) of $\left.a_{n}^{(1)}\right)$. (2) If $\alpha_{1}=2, \varphi_{1}\left(a_{n}^{(1)}\right)\left(a_{n}^{(1)}\right)^{-\gamma_{1}}$ is bounded above by a constant times $\varphi\left(x_{1}\right) x_{1}^{-\gamma_{1}}\left(a_{n}^{(1)} / x_{1}\right)^{-1}$ (by Potter's bound, since $\gamma_{1}>1$ ). Therefore, the first term is bounded by $\left(n \varphi\left(x_{1}\right) x_{1}^{-\gamma_{1}}\right)^{c x_{1} / a_{n}^{(1)}}$ times $\exp \left(-c^{\prime \prime} x_{1} / a_{n}^{(1)}\right)$ since $x_{1} \geqslant C a_{n}^{(1)}$. We also used that $n \sigma_{1}\left(a_{n}^{(1)}\right) \sim a_{n}^{(1)}$ when $\alpha=2$.

### 5.2 Proof of Theorem 2.1

We fix $i \in\{1, \ldots, d\}$, and consider some $\mathbf{x} \in \mathbb{Z}^{d}$ with $x_{i} \geqslant a_{n}^{(i)}$. Recall that $\hat{\mathbf{S}}_{n}=$ $\mathbf{S}_{n}-\left\lfloor\mathbf{b}_{n}\right\rfloor$. We denote $\mathbf{d}_{n}:=\frac{1}{2}\left\lfloor\mathbf{b}_{n}\right\rfloor-\mathbf{b}_{\lfloor n / 2\rfloor}$, so that $\mathbf{S}_{n}-\frac{1}{2}\left\lfloor\mathbf{b}_{n}\right\rfloor=\hat{\mathbf{S}}_{\lfloor n / 2\rfloor}-\mathbf{d}_{n}$.

We decompose $\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}\right)$ according to whether $S_{\lfloor n / 2\rfloor}^{(i)}-\frac{1}{2}\left\lfloor b_{n}^{(i)}\right\rfloor \geqslant x_{i} / 2$ or not, so that

$$
\begin{align*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}\right. & =\mathbf{x}) \\
& \leqslant \mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x} ; S_{\lfloor n / 2\rfloor}^{(i)}-\frac{1}{2}\left\lfloor b_{n}^{(i)}\right\rfloor \geqslant x_{i} / 2\right)+\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x} ; S_{n}^{(i)}-S_{\lfloor n / 2\rfloor}^{(i)}-\frac{1}{2}\left\lfloor b_{n}^{(i)}\right\rfloor \geqslant x_{i} / 2\right) . \tag{5.3}
\end{align*}
$$

The two terms are treated similarly, so we only focus on the first one. We have

$$
\begin{align*}
& \mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x} ; \hat{S}_{\lfloor n / 2\rfloor}^{(i)} \geqslant x_{i} / 2+d_{n}^{(i)}\right)  \tag{5.4}\\
& =\sum_{\substack{\mathbf{z} \in \mathbb{Z}^{d}}} \mathbf{P}\left(\mathbf{S}_{\lfloor n / 2\rfloor}=\mathbf{z}\right) \mathbf{P}\left(\mathbf{S}_{n}-\mathbf{S}_{\lfloor n / 2\rfloor}=\left\lfloor\mathbf{b}_{n}\right\rfloor+\mathbf{x}-\mathbf{z}\right) \\
& \\
& \leqslant \frac{z_{i} \geqslant \frac{1}{2}\left\lfloor b_{n}^{(i)}\right\rfloor+x_{i} / 2}{a_{n}^{(1)} \cdots a_{n}^{(d)}} \sum_{\substack{\mathbf{z} \in \mathbb{Z}^{d} \\
z_{i} \geqslant \frac{1}{2}\left\lfloor b_{n}^{(i)}\right\rfloor+x_{i} / 2}} \mathbf{P}\left(\mathbf{S}_{\lfloor n / 2\rfloor}=\mathbf{z}\right)=\frac{C}{a_{n}^{(1)} \cdots a_{n}^{(d)}} \mathbf{P}\left(S_{\lfloor n / 2\rfloor}^{(i)}-\frac{1}{2}\left\lfloor b_{n}^{(i)}\right\rfloor \geqslant x_{i} / 2\right),
\end{align*}
$$

where we used the local limit theorem (2.1) to get that there is a constant $C>0$ such that for any $k \geqslant 1$ and $\mathbf{y} \in \mathbb{Z}^{d}$, we have $\mathbf{P}\left(\mathbf{S}_{k}=\mathbf{y}\right) \leqslant C\left(a_{k}^{(1)} \cdots a_{k}^{(d)}\right)^{-1}$.

Then, in order to use (5.1) for the last probability, we need to control $\frac{1}{2}\left\lfloor b_{n}^{(i)}\right\rfloor-b_{\lfloor n / 2\rfloor}^{(i)}$.

Claim 5.2. There exists a constant $c>0$ such that for all $n$

$$
d_{n}^{(i)}:=\frac{1}{2}\left\lfloor b_{n}^{(i)}\right\rfloor-b_{\lfloor n / 2\rfloor}^{(i)} \geqslant-c a_{n}^{(i)} .
$$

Proof. When $\alpha_{i} \in(0,1)$ we have that $b_{n}^{(i)} \equiv 0$ so this quantity is equal to 0 . When $\alpha_{i}>1$ we have $b_{k}^{(i)}=k \mu_{i}$ in which case $\frac{1}{2}\left\lfloor n \mu_{i}\right\rfloor-\lfloor n / 2\rfloor \mu_{i} \geqslant-\mu_{i}$. When $\alpha_{i}=1$, this is more delicate but not too hard:

$$
\begin{aligned}
\frac{n}{2} \mu_{i}\left(a_{n}^{(i)}\right)-\lfloor n / 2\rfloor \mu_{i}\left(a_{\lfloor n / 2\rfloor}^{(i)}\right) & \geqslant \frac{n}{2}\left(\mu_{i}\left(a_{n}^{(i)}\right)-\mu_{i}\left(a_{\lfloor n / 2\rfloor}^{(i)}\right)\right)-\left|\mu_{i}\left(a_{\lfloor n / 2\rfloor}\right)\right| \\
& \geqslant-c\left(n L_{i}\left(a_{n}^{(i)}\right)+\left|\mu_{i}\left(a_{n}^{(i)}\right)\right|\right) \geqslant-c^{\prime} a_{n}^{(i)} .
\end{aligned}
$$

For the second inequality we used [3, Claim 5.3] that we reproduce below (separate the positive and negative part of $X_{1}^{(i)}$ ), using also that $a_{n}^{(i)} / a_{\lfloor n / 2\rfloor}^{(i)}$ is bounded by a constant.

Claim 5.3 (Claim 5.3 in [3]). Assume that $\alpha_{i}=1$. For every $\delta>0$, there is a constant $c_{\delta}$ such that for every $u \geqslant v \geqslant 1$ we have

$$
\frac{1}{L_{i}(v)}\left|\mu_{i}(u)-\mu_{i}(v)\right| \leqslant c_{\delta}(u / v)^{\delta} .
$$

Additionally, if $c^{-1} \leqslant u / v \leqslant c$, we have that $\frac{1}{L_{i}(v)}\left|\mu_{i}(u)-\mu_{i}(v)\right| \leqslant C|\log (u / v)|$.
Therefore, provided that $x_{i} \geqslant C_{4} a_{n}^{(i)}$ with some constant $C_{4}$ large enough, Claim 5.2 gives that $\frac{1}{2}\left\lfloor b_{n}^{(i)}\right\rfloor-b_{\lfloor n / 2\rfloor}^{(i)} \geqslant-x_{i} / 4$, so that

$$
\begin{equation*}
\mathbf{P}\left(S_{\lfloor n / 2\rfloor}^{(i)}-\frac{1}{2}\left\lfloor b_{n}^{(i)}\right\rfloor \geqslant x_{i} / 2\right) \leqslant \mathbf{P}\left(S_{\lfloor n / 2\rfloor}^{(i)}-b_{\lfloor n / 2\rfloor}^{(i)} \geqslant x_{i} / 4\right), \tag{5.5}
\end{equation*}
$$

and then (5.1) provides an upper bound. Plugged in (5.4), this concludes the proof of Theorem 2.1, possibly by changing the constants to cover the range $x \geqslant a_{n}^{(i)}, x<$ $C_{4} a_{n}^{(i)}$.

Note that with the same method, using Theorem 5.1 instead of (5.1), one is able to obtain a local version of Theorems 5.1.
Proposition 5.4. There are some $C_{4}, C_{5}>0$ such that, for any x with $x_{i} \geqslant C_{4} a_{n}^{(i)}$, and $1 \leqslant y \leqslant x_{i}$

$$
\begin{equation*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x} ; M_{n}^{(i)} \leqslant y\right) \leqslant \frac{C_{5}}{a_{n}^{(1)} \cdots a_{n}^{(d)}} \mathbf{P}\left(\hat{S}_{\lfloor n / 2\rfloor}^{(i)} \geqslant x_{i} / 4, M_{\lfloor n / 2\rfloor}^{(i)} \leqslant y\right) \tag{5.6}
\end{equation*}
$$

The proof of this proposition is a straightforward transposition of the proof of Theorem 2.1, we leave the details to the reader (for the univariate setting, we refer to Proposition 6 in [3] and its proof). We also state two other bounds (in dimension $d=2$ for simplicity), that will be useful in the proof of Theorem 2.5.
Claim 5.5. There are constants $C_{6}, C_{7}$ such that, for any $\mathbf{x}=\left(x_{1}, x_{2}\right)$ with $x_{1} \geqslant C_{6} a_{n}^{(1)}$, and any $1 \leqslant y \leqslant x_{1}$

$$
\begin{equation*}
\mathbf{P}\left(\hat{S}_{n}^{(1)} \geqslant x_{1}, \hat{S}_{n}^{(2)}=x_{2}, M_{n}^{(1)} \leqslant y_{1}\right) \leqslant \frac{C_{7}}{a_{n}^{(2)}} \mathbf{P}\left(\hat{S}_{\lfloor n / 2\rfloor}^{(1)} \geqslant x_{1} / 4, M_{\lfloor n / 2\rfloor}^{(1)} \leqslant y_{1}\right) . \tag{5.7}
\end{equation*}
$$

For any $\mathbf{x}=\left(x_{1}, x_{2}\right)$ with $x_{1} \geqslant C_{6} a_{n}^{(1)}, x_{2} \geqslant C_{6} a_{n}^{(2)}$, and any $1 \leqslant y_{1} \leqslant x_{1}, 1 \leqslant y_{2} \leqslant x_{2}$,

$$
\begin{align*}
& \mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \leqslant y_{1}, M_{n}^{(2)} \leqslant y_{2}\right)  \tag{5.8}\\
& \quad \leqslant \frac{C_{7}}{a_{n}^{(1)} a_{n}^{(2)}} \mathbf{P}\left(\hat{S}_{\lfloor n / 4\rfloor}^{(1)} \geqslant x_{1} / 16, M_{\lfloor n / 4\rfloor}^{(1)} \leqslant y_{1}\right)^{1 / 2} \mathbf{P}\left(\hat{S}_{\lfloor n / 4\rfloor}^{(2)} \geqslant x_{2} / 16, M_{\lfloor n / 4\rfloor}^{(2)} \leqslant y_{2}\right)^{1 / 2} .
\end{align*}
$$

Then we can use Theorem 5.1 to control the probabilities in the right-hand sides.
Proof of Claim 5.5. We prove only (5.8), the proof of (5.7) being identical as that of (5.6). We decompose the probability into four parts, according to whether $S_{\lfloor n / 2\rfloor}^{(i)}-\frac{1}{2}\left\lfloor b_{n}^{(i)}\right\rfloor \geqslant x_{i} / 2$ or not, for $i=1,2$ : there are two terms we need to control (the other two being symmetric).
(1) The first term we need to control is

$$
\begin{aligned}
& \mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \leqslant y_{1}, M_{n}^{(2)} \leqslant y_{2}, S_{\lfloor n / 2\rfloor}^{(i)}-\frac{1}{2}\left\lfloor b_{n}^{(i)}\right\rfloor \geqslant \frac{1}{2} x_{i} \text { for } i=1,2\right) \\
& \leqslant \sum_{\substack{z_{1} \geqslant \frac{1}{2} x_{1}+\frac{1}{2}\left\lfloor b_{n}^{(1)}\right\rfloor \\
z_{2} \geqslant \frac{1}{2} x_{2}+\frac{1}{2}\left\lfloor b_{n}^{(2)}\right\rfloor}} \mathbf{P}\left(\mathbf{S}_{\lfloor n / 2\rfloor}=\left(z_{1}, z_{2}\right), M_{\lfloor n / 2\rfloor}^{(1)} \leqslant y_{1}, M_{\lfloor n / 2\rfloor}^{(2)} \leqslant y_{2}\right) \\
& \\
& \leqslant \\
& \quad \times \mathbf{P}\left(\mathbf{S}_{n}-\mathbf{S}_{\lfloor n / 2\rfloor}=\left(x_{1}-z_{1}, x_{2}-z_{2}\right)\right) \\
& a_{n}^{(1)} a_{n}^{(2)} \\
& \mathbf{P}\left(S_{\lfloor n / 2\rfloor}^{(1)} \geqslant \frac{1}{2} x_{1}+\frac{1}{2}\left\lfloor b_{n}^{(1)}\right\rfloor, S_{\lfloor n / 2\rfloor}^{(2)} \geqslant \frac{1}{2} x_{2}+\frac{1}{2}\left\lfloor b_{n}^{(2)}\right\rfloor, M_{\lfloor n / 2\rfloor}^{(1)} \leqslant y_{1}, M_{\lfloor n / 2\rfloor}^{(2)} \leqslant y_{2}\right) .
\end{aligned}
$$

For the last inequality, we used the local limit theorem (2.1) to bound the last probability by $C /\left(a_{n}^{(1)} a_{n}^{(2)}\right)$ uniformly in $x_{1}, x_{2}, z_{1}, z_{2}$, and then summed over $z_{1}, z_{2}$. Then, we use Claim 5.2 to get that, provided $x_{i} \geqslant C_{6} a_{n}^{(i)}$ with $C_{6}$ large enough, the last probability is bounded by

$$
\begin{aligned}
& \mathbf{P}\left(\hat{S}_{\lfloor n / 2\rfloor}^{(1)} \geqslant \frac{1}{4} x_{1}, \hat{S}_{\lfloor n / 2\rfloor}^{(2)} \geqslant \frac{1}{4} x_{2}, M_{\lfloor n / 2\rfloor}^{(1)} \leqslant y_{1}, M_{\lfloor n / 2\rfloor}^{(2)} \leqslant y_{2}\right) \\
& \leqslant \mathbf{P}\left(\hat{S}_{\lfloor n / 2\rfloor}^{(1)} \geqslant \frac{1}{4} x_{1}, M_{\lfloor n / 2\rfloor}^{(1)} \leqslant y_{1}\right)^{1 / 2} \mathbf{P}\left(\hat{S}_{\lfloor n / 2\rfloor}^{(2)} \geqslant \frac{1}{4} x_{2}, M_{\lfloor n / 2\rfloor}^{(2)} \leqslant y_{2}\right)^{1 / 2},
\end{aligned}
$$

where we used Cauchy-Schwarz inequality at last.
(2) The second term we need to control is

$$
\begin{align*}
& \mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \leqslant y_{1}, M_{n}^{(2)} \leqslant y_{2}, S_{\lfloor n / 2\rfloor}^{(1)}-\frac{1}{2}\left\lfloor b_{n}^{(1)}\right\rfloor \geqslant \frac{1}{2} x_{1}, S_{\lfloor n / 2\rfloor}^{(2)}-\frac{1}{2}\left\lfloor b_{n}^{(2)}\right\rfloor<\frac{1}{2} x_{2}\right)  \tag{5.9}\\
& \quad \leqslant \sum_{\substack{z_{1} \geqslant \frac{1}{2} x_{1}+\frac{1}{2}\left\lfloor b_{n}^{(1)}\right\rfloor \\
z_{2}<\frac{1}{2} x_{2}+\frac{1}{2}\left\lfloor b_{n}^{(2)}\right\rfloor}} \mathbf{P}\left(\mathbf{S}_{\lfloor n / 2\rfloor}=\left(z_{1}, z_{2}\right), M_{\lfloor n / 2\rfloor}^{(1)} \leqslant y_{1}\right) \\
& \quad \times \mathbf{P}\left(\mathbf{S}_{n}-\mathbf{S}_{\lfloor n / 2\rfloor}=\left(x_{1}-z_{1}, x_{2}-z_{2}\right), \max _{\lfloor n / 2\rfloor \leqslant i \leqslant n} X_{i}^{(2)} \leqslant y_{2}\right) .
\end{align*}
$$

Then, we can use Proposition 5.4, say for the second probability: indeed, we have that uniformly for the range of $z_{2}$ considered,

$$
\begin{aligned}
\mathbf{P}\left(\mathbf{S}_{n}-\mathbf{S}_{\lfloor n / 2\rfloor}\right. & \left.=\left(x_{1}-z_{1}, x_{2}-z_{2}\right), \max _{\lfloor n / 2\rfloor \leqslant i \leqslant n} X_{i}^{(2)} \leqslant y_{2}\right) \\
& =\mathbf{P}\left(\hat{\mathbf{S}}_{\lfloor n / 2\rfloor}=\left(x_{1}-z_{1}-\left\lfloor b_{n / 2}^{(1)}\right\rfloor, x_{2}-z_{2}-\left\lfloor b_{n / 2}^{(2)}\right\rfloor\right), M_{n-\lfloor n / 2\rfloor}^{(2)} \leqslant y_{2}\right) \\
& \leqslant \frac{C}{a_{n}^{(1)} a_{n}^{(2)}} \mathbf{P}\left(\hat{S}_{\lfloor n / 4\rfloor}^{(2)} \geqslant x_{2} / 16, M_{\lfloor n / 4\rfloor}^{(2)} \leqslant y_{2}\right)
\end{aligned}
$$

where we used that $x_{2}-z_{2}-\left\lfloor b_{n / 2}^{(2)}\right\rfloor \geqslant x_{2} / 4 \geqslant C_{4} a_{n}^{(2)}$ (thanks to Claim 5.2). Using this in (5.9) and summing over $z_{1}$ and $z_{2}$ (and using again Claim 5.2), we finally get that (5.9) is bounded by

$$
\frac{C}{a_{n}^{(1)} a_{n}^{(2)}} \mathbf{P}\left(\hat{S}_{\lfloor n / 2\rfloor}^{(1)} \geqslant x_{1} / 4, M_{\lfloor n / 2\rfloor}^{(1)} \leqslant y_{1}\right) \mathbf{P}\left(\hat{S}_{\lfloor n / 4\rfloor}^{(2)} \geqslant x_{2} / 16, M_{\lfloor n / 4\rfloor}^{(2)} \leqslant y_{2}\right) .
$$

Let us stress that, to obtain the statement of Claim 5.5 , we additionally use that

$$
\mathbf{P}\left(\hat{S}_{\lfloor n / 2\rfloor}^{(1)} \geqslant x_{1} / 4, M_{\lfloor n / 2\rfloor}^{(1)} \leqslant y_{1}\right) \leqslant 2 \mathbf{P}\left(\hat{S}_{\lfloor n / 4\rfloor}^{(1)} \geqslant x_{1} / 16, M_{\lfloor n / 4\rfloor}^{(1)} \leqslant y_{1}\right) .
$$

This comes from splitting the left-hand side according to whether $S_{\lfloor n / 4\rfloor}^{(1)}-\frac{1}{2}\left\lfloor b_{\lfloor n / 2\rfloor}\right\rfloor \geqslant x_{1} / 8$ or not, and using again Claim 5.2 to get that $\left|\left\lfloor b_{n / 4}\right\rfloor-\frac{1}{2}\left\lfloor b_{\lfloor n / 2\rfloor}\right\rfloor\right| \geqslant x_{1} / 16$.

### 5.3 Proof of Theorem 2.5

Let us write the details only in dimension $d=2$ to avoid lengthy notations, the proof works identically when $d \geqslant 3$. Also, we only deal with $\mathrm{x} \geqslant 0$. We fix a constant $C_{8}$ (large). The case $x_{1} \leqslant C_{8} a_{n}^{(1)}, x_{2} \leqslant C_{8} a_{n}^{(2)}$ falls in the range of the local limit theorem (2.1), so we need to consider only two cases: $x_{1}>C_{8} a_{n}^{(1)}, x_{2} \leqslant C_{8} a_{n}^{(2)}$ (the case $x_{1} \leqslant C_{8} a_{n}^{(1)}, x_{2}>C_{8} a_{n}^{(2)}$ is symmetric) and $x_{1}>C_{8} a_{n}^{(1)}, x_{2}>C_{8} a_{n}^{(2)}$.

### 5.3.1 Case $x_{1} \geqslant C_{8} a_{n}^{(1)}, x_{2} \leqslant C_{8} a_{n}^{(2)}$

We will treat three different contributions, by writing, for some $C_{9}>0$

$$
\begin{align*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}\right)= & \mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \geqslant x_{1} / 8\right)  \tag{5.10}\\
& +\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \in\left(C_{9} a_{n}^{(1)}, x_{1} / 8\right)\right)+\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \leqslant C a_{n}^{(1)}\right) .
\end{align*}
$$

For the last term, we use Proposition 5.4, together with Theorem 5.1 (more precisely (5.2)), to get that it is bounded by a constant times

$$
\frac{C}{a_{n}^{(1)} a_{n}^{(2)}}\left(n \varphi_{1}\left(x_{1}\right) x_{1}^{-\gamma_{1}} e^{-c^{\prime \prime} x_{1} / a_{n}^{(1)}}+e^{-c^{\prime \prime}\left(x_{1} / a_{n}^{(1)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}\right)
$$

Then, we can use that $e^{-c^{\prime} x_{1} / a_{n}^{(1)}} \leqslant c\left(a_{n}^{(1)} / x_{1}\right) e^{-c^{\prime \prime} x_{1} / a_{n}^{(1)}}$ provided that $x_{1} / a_{n}^{(1)}$ is large enough (and similarly for the last term), to get that

$$
\begin{equation*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \leqslant C_{9} a_{n}^{(1)}\right) \leqslant \frac{C}{x_{1} a_{n}^{(2)}}\left(n \varphi_{1}\left(x_{1}\right) x_{1}^{-\gamma_{1}}+e^{-c^{\prime}\left(x_{1} / a_{n}^{(1)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}\right) . \tag{5.11}
\end{equation*}
$$

In order to treat the first two terms in (5.10), we control the probability, for $k \in \mathbb{Z}$,

$$
\begin{align*}
& \mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \in\left[2^{k} x_{1}, 2^{k+1} x_{1}\right)\right)  \tag{5.12}\\
& =n \sum_{u=2^{k} x_{1}}^{2^{k+1} x_{1}} \sum_{v \in \mathbb{Z}} \mathbf{P}\left(\mathbf{X}_{1}=(u, v)\right) \mathbf{P}\left(\hat{\mathbf{S}}_{n-1}=\left(x_{1}-u, x_{2}-v\right)+\boldsymbol{\delta}_{n}, M_{n-1}^{(1)} \leqslant 2^{k+1} x_{1}\right),
\end{align*}
$$

where we set $\boldsymbol{\delta}_{n}:=\left\lfloor\mathbf{b}_{n}\right\rfloor-\left\lfloor\mathbf{b}_{n-1}\right\rfloor$, which is uniformly bounded by a constant. By Assumption 2.2, we get that, for any $u \in\left[2^{k} x_{1}, 2^{k+1} x_{1}\right)$ and $v \in \mathbb{Z}$,

$$
\begin{align*}
\mathbf{P}\left(\mathbf{X}_{1}=(u, v)\right) & \leqslant c \varphi_{1}(u) u^{-\left(1+\gamma_{1}\right)} \times \frac{1}{1+|v|} h_{u}^{(1)}(|v|)  \tag{5.13}\\
& \leqslant c^{\prime} 2^{-k\left(1+\gamma_{1}\right)+\eta|k|} \varphi\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} \times \frac{h_{2^{k} x_{1}}(|v|)}{1+|v|} .
\end{align*}
$$

We used Potter's bound [6, Thm. 1.5.6] to get that for $x_{1}$ sufficiently large, for every $\eta>0$ there is a constant $c_{\eta}>0$ such that $\varphi\left(2^{k} x_{1}\right) \leqslant c_{\eta} 2^{\eta|k|} \varphi\left(x_{1}\right)$ for any $k \in \mathbb{Z}$, together with item (iii) in (2.5).

Since this bound is uniform over $u \in\left[2^{k} x_{1}, 2^{k+1} x_{1}\right)$, we may sum over $u$ the last probability in (5.12): note that

$$
\begin{align*}
& \sum_{u=2^{k} x_{1}}^{2^{k+1} x_{1}} \mathbf{P}\left(\hat{\mathbf{S}}_{n-1}=\left(x_{1}-u, x_{2}-v\right)+\boldsymbol{\delta}_{n}, M_{n}^{(1)} \leqslant 2^{k+1} x_{1}\right) \\
& \leqslant \begin{cases}\mathbf{P}\left(\hat{S}_{n-1}^{(2)}=x_{2}-v+\delta_{n}^{(2)}\right) & \text { if } k \geqslant-3 \\
\mathbf{P}\left(\hat{S}_{n-1}^{(1)} \geqslant \frac{1}{2} x_{1}, \hat{S}_{n-1}^{(2)}=x_{2}-v+\delta_{n}^{(2)}, M_{n}^{(1)} \leqslant 2^{k+1} x_{1}\right) & \text { if } k \leqslant-4\end{cases} \tag{5.14}
\end{align*}
$$

- When $k \geqslant-3$, we therefore get from (5.12) that (taking $\eta<\gamma_{1}$ in (5.13))

$$
\begin{align*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)}\right. & \left.\in\left[2^{k} x_{1}, 2^{k+1} x_{1}\right)\right)  \tag{5.15}\\
& \leqslant c^{\prime} n 2^{-k} \varphi\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} \sum_{v \in \mathbb{Z}} \frac{h_{2^{k} x_{1}}(|v|)}{1+|v|} \mathbf{P}\left(\hat{S}_{n-1}^{(2)}=x_{2}-v+\delta_{n}^{(2)}\right) \\
& \leqslant c^{\prime} n 2^{-k} \varphi\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} \times \frac{1}{a_{n}^{(2)}} .
\end{align*}
$$

We used the local limit theorem to get that there is a constant $C$ such that for any $z \in \mathbb{Z}$, $\mathbf{P}\left(\hat{S}_{n-1}^{(2)}=z\right) \leqslant C / a_{n}^{(2)}$, and then that $\sum_{v \in \mathbb{Z}} h_{2^{k} x_{1}}(|v|) /(1+|v|) \leqslant C$ for some constant $C$ not depending on $k$ or $x_{1}$, thanks to item (ii) in (2.5). From this, we obtain that

$$
\begin{align*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \geqslant x_{1} / 8\right) & =\sum_{k \geqslant-3} \mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \in\left[2^{k} x_{1}, 2^{k+1} x_{1}\right)\right)  \tag{5.16}\\
& \leqslant \frac{C^{\prime}}{a_{n}^{(2)}} n \varphi\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)}
\end{align*}
$$

- When $k \leqslant-4$, we use Claim 5.5 in (5.14), so that plugged in (5.12) we obtain that

$$
\begin{aligned}
& \mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \in\left[2^{k} x_{1}, 2^{k+1} x_{1}\right)\right) \\
& \quad \leqslant c n 2^{-k\left(1+\gamma_{1}+\eta\right)} \varphi\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} \sum_{v \in \mathbb{Z}} \frac{h_{2^{k} x_{1}}(v)}{1+|v|} \frac{1}{a_{n}^{(2)}} \mathbf{P}\left(\hat{S}_{\lfloor(n-1) / 2\rfloor} \geqslant \frac{1}{8} x_{1}, M_{n}^{(1)} \leqslant 2^{k+1} x_{1}\right) \\
& \quad \leqslant \frac{c}{a_{n}^{(2)}} n \varphi\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} 2^{-k\left(2+\gamma_{1}\right)} \mathbf{P}\left(\hat{S}_{\lfloor(n-1) / 2\rfloor} \geqslant x_{1} / 8, M_{n}^{(1)} \leqslant 2^{k+1} x_{1}\right),
\end{aligned}
$$

where we used again item (ii) in (2.5) to bound $\sum_{v \in \mathbb{Z}} h_{2^{k} x_{1}}(|v|) /(1+|v|)$ by a (uniform) constant, and took $\eta=1$. Then, we can use Theorem 5.1 to get that there are constants $c, c^{\prime}$ such that uniformly for $k \leqslant-4$ with $2^{k} x_{1} \geqslant C_{9} a_{n}^{(1)}$,

$$
\begin{align*}
& \mathbf{P}\left(\hat{S}_{\lfloor(n-1) / 2\rfloor} \geqslant x_{1} / 8, M_{n}^{(1)} \leqslant 2^{k+1} x_{1}\right)  \tag{5.17}\\
& \quad \leqslant\left(\frac{c}{x_{1}} n \varphi_{1}\left(2^{k} x_{1}\right)\left(2^{k} x_{1}\right)^{1-\gamma_{1}}\right)^{c^{\prime} 2^{-k}}+e^{-c^{\prime} x_{1} / a_{n}^{(1)}} \mathbb{1}_{\left\{\alpha_{1}=1\right\}}+e^{-c^{\prime} x_{1}^{2} / n \sigma_{1}\left(2^{k} x_{1}\right)} \mathbb{1}_{\left\{\alpha_{1}=2\right\}} \\
& \quad \leqslant\left(c^{\prime \prime} 2^{-k}\right)^{-c^{\prime} 2^{-k}}+e^{-c^{\prime} x_{1} / a_{n}^{(1)}} .
\end{align*}
$$

Indeed, we used that since $2^{k} x_{1} \geqslant C_{9} a_{n}^{(1)}$, we have that $n \varphi_{1}\left(2^{k} x_{1}\right)\left(2^{k} x_{1}\right)^{-\gamma_{1}}$ is bounded by a constant. Also, in the case $\alpha_{1}=2$, we used that $\sigma_{1}\left(2^{k} x_{1}\right) \leqslant \sigma_{1}\left(x_{1}\right)$, and that by definition of $a_{n}^{(1)}$ we have $x_{1}^{2} / n \sigma\left(x_{1}\right) \geqslant c\left(x_{1} / a_{n}^{(1)}\right)^{2} \sigma_{1}\left(a_{n}^{(1)}\right) / \sigma_{1}(x) \geqslant c x_{1} / a_{n}^{(1)}$ (the last inequality comes from Potter's bound).

Therefore, summing over $k$ between -4 and $-\left\lfloor\log _{2}\left(x_{1} / C_{9} a_{n}^{(1)}\right)\right\rfloor$, we finally obtain that $\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \in\left(C a_{n}^{(1)}, x_{1} / 8\right)\right)$ is bounded by a constant times $\frac{n}{a_{n}^{(2)}} \varphi\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)}$, times

$$
\begin{equation*}
\sum_{k^{\prime}=4}^{\left\lfloor\log _{2}\left(x_{1} / C_{9} a_{n}^{(1)}\right)\right\rfloor} 2^{k^{\prime}\left(2+\gamma_{1}\right)}\left(\left(c^{\prime} 2^{k^{\prime}}\right)^{-c 2^{k^{\prime}}}+e^{-c x_{1} / a_{n}^{(1)}}\right) \leqslant C+\left(\frac{c x_{1}}{a_{n}^{(1)}}\right)^{3+\gamma_{1}} e^{-c x_{1} / a_{n}^{(1)}} . \tag{5.18}
\end{equation*}
$$

Note that the second term is bounded by a constant, uniformly for $x_{1} / a_{n}^{(1)} \geqslant C$. Therefore, we conclude that

$$
\begin{equation*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \in\left(C a_{n}^{(1)}, x_{1} / 8\right)\right) \leqslant \frac{C}{a_{n}^{(2)}} n \varphi\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} \tag{5.19}
\end{equation*}
$$

## Multivariate strong renewal theorems and local large deviations

As a conclusion, (5.10), combined with (5.11), (5.16) and (5.19), gives that

$$
\begin{align*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}\right) & \leqslant \frac{C}{a_{n}^{(2)}}\left(n \varphi_{1}\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)}+\frac{1}{a_{n}^{(1)}} e^{-c\left(x_{1} / a_{n}^{(1)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}\right) \\
& \leqslant \frac{C}{x_{1} a_{n}^{(2)}}\left(n \varphi_{1}\left(x_{1}\right) x_{1}^{-\gamma_{1}}+e^{-c^{\prime}\left(x_{1} / a_{n}^{(1)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}\right) . \tag{5.20}
\end{align*}
$$

Notice that, in the case $\alpha_{1}<2$, we have $\gamma_{1}=\alpha_{1}$, and $n \sim\left(a_{n}^{(1)}\right)^{\alpha_{1}} \varphi_{1}\left(a_{n}^{(1)}\right)^{-1}$, so that the second term is negligible, since the first term is bounded below by a power of $x_{1} / a_{n}^{(1)}$. We also used that $\left(x_{1} / a_{n}^{(1)}\right) \exp \left(-c x_{1} / a_{n}^{(1)}\right)$ is bounded by a constant times $\exp \left(-c^{\prime} x_{1} / a_{n}^{(1)}\right)$ with $c^{\prime}>c$, provided that $x_{1} \geqslant C_{9} a_{n}^{(1)}$.
5.3.2 Case $x_{1} \geqslant C_{8} a_{n}^{(1)}, x_{2} \geqslant C_{8} a_{n}^{(2)}$

Again, we decompose the probability according to the value of $M_{n}^{(1)}, M_{n}^{(2)}$. As a first step, we write

$$
\begin{align*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}\right)=\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(i)} \leqslant\right. & \left.C_{9} a_{n}^{(i)} i=1,2\right)+\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(i)}>C_{9} a_{n}^{(i)} i=1,2\right) \\
+ & \mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)}>C_{9} a_{n}^{(1)}, M_{n}^{(2)} \leqslant C_{9} a_{n}^{(2)}\right) \\
& \quad+\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \leqslant C_{9} a_{n}^{(1)}, M_{n}^{(2)}>C_{9} a_{n}^{(2)}\right) . \tag{5.21}
\end{align*}
$$

Term 1. Let us bound the first term in (5.21). We use Claim 5.5 (more precisely (5.8)), together with Theorem 5.1 (more precisely (5.2)) to get that

$$
\begin{align*}
& \mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(i)} \leqslant C_{9} a_{n}^{(i)} i=1,2\right)  \tag{5.22}\\
& \quad \leqslant \frac{C}{a_{n}^{(1)} a_{n}^{(2)}}\left(\left(n \varphi_{1}\left(x_{1}\right) x_{1}^{-\gamma_{1}}\right)^{2} e^{-c^{\prime \prime} x_{1} / a_{n}^{(1)}}+e^{-c^{\prime \prime}\left(x_{1} / a_{n}^{(1)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}\right)^{1 / 2}\left(e^{-c x_{2} / a_{n}^{(2)}}\right)^{1 / 2} \\
& \quad \leqslant \frac{C^{\prime}}{x_{1} x_{2}}\left(n \varphi_{1}\left(x_{1}\right) x_{1}^{-\gamma_{1}}+e^{-c^{\prime}\left(x_{1} / a_{n}^{(1)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}\right) .
\end{align*}
$$

Note that we also used that (5.2) is also bounded by $\exp \left(-c x_{2} / a_{n}^{(2)}\right)$ for the first inequality. Then, we used that $(a+b)^{1 / 2} \leqslant a^{1 / 2}+b^{1 / 2}$ for any $a, b \geqslant 0$, and then that $\frac{1}{a_{n}^{(i)}} e^{-c x_{i} / a_{n}^{(i)}} \leqslant \frac{1}{x_{i}} e^{-c^{\prime} x_{i} / a_{n}^{(i)}}$ for $x_{i} / a_{n}^{(i)}$ large.

Term 3. We now bound the third term in (5.21) by a constant times $\left(x_{1} x_{2}\right)^{-1} n \varphi_{1}\left(x_{1}\right) x_{1}^{-\gamma_{1}}$. We proceed as for the previous section (5.12)-(5.19). The analogous of (5.12) is, for $k \in \mathbb{Z}$
$\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \in\left[2^{k} x_{1}, 2^{k+1} x_{1}\right), M_{n}^{(2)} \leqslant C_{9} a_{n}^{(2)}\right)$

$$
\leqslant n \sum_{u=2^{k} x_{1}}^{2^{k+1} x_{1}} \sum_{v \leqslant C_{9} a_{n}^{(2)}} \mathbf{P}\left(\mathbf{X}_{1}=(u, v)\right) . \quad \mathbf{P}\left(\hat{\mathbf{S}}_{n-1}=\left(x_{1}-u, x_{2}-v\right)+\boldsymbol{\delta}_{n}, M_{n-1}^{(1)} \leqslant 2^{k+1} x_{1}, M_{n-1}^{(2)} \leqslant C_{9} a_{n}^{(2)}\right) .
$$

Then, one bounds $\mathbf{P}\left(\mathbf{X}_{1}=(u, v)\right)$ by using Assumption 2.2 (as in (5.13)), and by summing over $u \in\left[2^{k} x_{1}, 2^{k+1} x_{1}\right.$ ) one needs to control (analogously to (5.14))

$$
\begin{align*}
& \text { if } k \geqslant-3, \quad \mathbf{P}\left(\hat{S}_{n-1}^{(2)}=x_{2}-v+\delta_{n}^{(2)}, M_{n-1}^{(2)} \leqslant C_{9} a_{n}^{(2)}\right), \\
& \text { if } k \leqslant-4, \quad \mathbf{P}\left(\hat{S}_{n-1}^{(1)} \geqslant x_{1} / 2, \hat{S}_{n-1}^{(2)}=x_{2}-v+\delta_{n}^{(2)}, M_{n-1}^{(1)} \leqslant 2^{k+1} x_{1}, M_{n-1}^{(2)} \leqslant C_{9} a_{n}^{(2)}\right), \tag{5.23}
\end{align*}
$$

uniformly over $v \leqslant C_{9} a_{n}^{(2)}$.

The first probability in (5.23) is treated by using Proposition 5.4, together with (5.2) (and the remark below): since $x_{2}-v+\delta_{n}^{(2)}$ is bounded below by $x_{2} / 2$ uniformly in the range of $v$ considered (and assuming that $C_{9}<C_{8} / 2$ ), we get that

$$
\begin{equation*}
\mathbf{P}\left(\hat{S}_{n-1}^{(2)}=x_{2}-v+\delta_{n}^{(2)}, M_{n-1}^{(2)} \leqslant C_{9} a_{n}^{(2)}\right) \leqslant \frac{C}{a_{n}^{(2)}} e^{-c x_{2} / a_{n}^{(2)}} \leqslant \frac{C}{x_{2}} \tag{5.24}
\end{equation*}
$$

We used the fact that $x_{2} \geqslant C_{8} a_{n}^{(2)}$ for the last inequality. Hence, we get that for $k \geqslant-3$

$$
\begin{aligned}
& \mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \in\left[2^{k} x_{1}, 2^{k+1} x_{1}\right), M_{n}^{(2)} \leqslant C_{9} a_{n}^{(2)}\right) \\
& \leqslant \frac{C}{x_{2}} 2^{-k\left(1+\gamma_{1}\right)+\eta|k|} n \varphi\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} \times \sum_{v \in \mathbb{Z}^{d}} \frac{h_{2^{k} x_{1}}(|v|)}{1+|v|},
\end{aligned}
$$

and the last sum is bounded by a constant uniform in $k, x_{1}$, thanks to item (ii) in (2.5). Summing over $k \geqslant-3$, we get that, analogously to (5.16),

$$
\begin{equation*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \geqslant x_{1} / 8, M_{n}^{(2)} \leqslant C_{9} a_{n}^{(2)}\right) \leqslant \frac{C}{x_{2}} n \varphi\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} . \tag{5.25}
\end{equation*}
$$

For the second probability in (5.23) (with $k \leqslant-4$ ), we invoke Claim 5.5: one can easily adapt the proof of (5.8), using that $x_{2}-v+\delta_{n}^{(2)} \geqslant x_{2} / 2$ uniformly for the range of $v$ considered, to get that

$$
\begin{align*}
& \mathbf{P}\left(\hat{S}_{n-1}^{(1)} \geqslant x_{1} / 2, \hat{S}_{n-1}^{(2)}=x_{2}-v+\delta_{n}^{(2)}, M_{n-1}^{(1)} \leqslant 2^{k+1} x_{1}, M_{n-1}^{(2)} \leqslant C_{9} a_{n}^{(2)}\right) \\
& \quad \leqslant \frac{C}{a_{n}^{(2)}} \mathbf{P}\left(\hat{S}_{\lfloor n / 4\rfloor}^{(1)} \geqslant x_{1} / 32, M_{\lfloor n / 4\rfloor}^{(1)} \leqslant 2^{k+1} x_{1}\right)^{1 / 2} \mathbf{P}\left(\hat{S}_{\lfloor n / 4\rfloor}^{(2)} \geqslant x_{2} / 32, M_{\lfloor n / 4\rfloor}^{(2)} \leqslant C_{9} a_{n}^{(2)}\right)^{1 / 2} \\
& \quad \leqslant \frac{C}{a_{n}^{(2)}}\left(\left(c 2^{-k}\right)^{c 2^{k}}+e^{-c x_{1} / a_{n}^{(1)}}\right) e^{-c x_{2} / a_{n}^{(2)}} \leqslant \frac{C}{x_{2}}\left(\left(c 2^{-k}\right)^{c^{\prime} 2^{k}}+e^{-c^{\prime} x_{1} / a_{n}^{(1)}}\right) \tag{5.26}
\end{align*}
$$

For the second inequality, we used Theorem 5.1, more precisely (5.17). Therefore, we obtain that for $k \leqslant-4$ with $2^{k} x_{1} \geqslant C_{9} a_{n}^{(1)}$,

$$
\begin{aligned}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}\right. & \left.=\mathbf{x}, M_{n}^{(1)} \in\left[2^{k} x_{1}, 2^{k+1} x_{1}\right), M_{n}^{(2)} \leqslant C_{9} a_{n}^{(2)}\right) \\
& \leqslant \frac{C}{x_{2}} 2^{-k\left(1+\gamma_{1}\right)+\eta|k|} n \varphi\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)}\left(\left(c 2^{-k}\right)^{c^{\prime} 2^{k}}+e^{-c^{\prime} x_{1} / a_{n}^{(1)}}\right) \times \sum_{v \in \mathbb{Z}^{d}} \frac{h_{2^{k} x_{1}}(v)}{1+|v|},
\end{aligned}
$$

with the last sum bounded by a constant uniform in $k, x_{1}$. Summing over $k$ between -4 and $-\left\lfloor\log _{2}\left(x_{1} / C_{9} a_{n}^{(1)}\right)\right\rfloor$ (as done in (5.18)), we get that

$$
\begin{equation*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \in\left(C_{9} a_{n}^{(1)}, x_{1} / 8\right), M_{n}^{(2)} \leqslant C_{9} a_{n}^{(2)}\right) \leqslant \frac{C}{x_{2}} n \varphi_{1}\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} . \tag{5.27}
\end{equation*}
$$

To conclude, we have that

$$
\begin{equation*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \geqslant C_{9} a_{n}^{(1)}, M_{n}^{(2)} \leqslant C_{9} a_{n}^{(2)}\right) \leqslant \frac{C}{x_{2}} n \varphi_{1}\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} \tag{5.28}
\end{equation*}
$$

Term 4. We now bound the fourth term in (5.21). We stress that the treatment is not completely symmetric to that of Term 3 , since we wish to obtain a bound that depends on the tail of the first coordinate (i.e. on $\varphi_{1}(\cdot)$ and $\gamma_{1}$ ), whereas (5.28) above yields the bound $\frac{C}{x_{1}} n \varphi_{2}\left(x_{2}\right) x_{1}^{-\left(1+\gamma_{2}\right)}$. We however proceed analogously: we control

$$
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \leqslant C_{9} a_{n}^{(1)}, M_{n}^{(2)} \in\left[2^{k} x_{1}, 2^{k+1} x_{1}\right]\right)
$$

## Multivariate strong renewal theorems and local large deviations

Then, for $k \geqslant-3$, instead of (5.24), we use Proposition 5.4 together with (5.2) to get that

$$
\begin{equation*}
\mathbf{P}\left(\hat{S}_{n-1}^{(1)}=x_{1}-v+\delta_{n}^{(1)}, M_{n-1}^{(1)} \leqslant C_{9} a_{n}^{(1)}\right) \leqslant \frac{C}{x_{1}}\left(n \varphi\left(x_{1}\right) x_{1}^{-\gamma_{1}}+e^{-c\left(x_{1} / a_{n}^{(1)}\right)^{2}}\right) . \tag{5.29}
\end{equation*}
$$

We end up with, analogously to (5.25),

$$
\begin{aligned}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \leqslant C_{9} a_{n}^{(1)}, M_{n}^{(2)}\right. & \left.\geqslant x_{2} / 8\right) \\
& \leqslant \frac{C}{x_{1}}\left(n \varphi\left(x_{1}\right) x_{1}^{-\gamma_{1}}+e^{-c\left(x_{1} / a_{n}^{(1)}\right)^{2}}\right) n \varphi\left(x_{2}\right) x_{2}^{-\left(1+\gamma_{1}\right)} .
\end{aligned}
$$

Also, for $k \leqslant-4$, instead of (5.26), we get

$$
\begin{aligned}
\mathbf{P}\left(\hat{S}_{n-1}^{(1)}=x_{1}-v+\delta_{n}^{(1)}\right. & \left., \hat{S}_{n-1}^{(2)} \geqslant x_{2} / 2, M_{n-1}^{(1)} \leqslant C_{9} a_{n}^{(2)}, M_{n-1}^{(2)} \leqslant 2^{k+1} x_{1},\right) \\
& \leqslant \frac{C}{a_{n}^{(1)}}\left(n \varphi\left(x_{1}\right) x_{1}^{-\gamma_{1}}+e^{-c\left(x_{1} / a_{n}^{(1)}\right)^{2}}\right)\left(\left(c 2^{-k}\right)^{c^{\prime} 2^{-k}}+e^{-c^{\prime} x_{2} / a_{n}^{(2)}}\right)
\end{aligned}
$$

and, analogously to (5.27), we obtain

$$
\begin{aligned}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \leqslant C_{9} a_{n}^{(1)}, M_{n}^{(2)}\right. & \left.\in\left(C_{9} a_{n}^{(2)}, x_{2} / 8\right)\right) \\
& \leqslant \frac{C}{x_{1}}\left(n \varphi\left(x_{1}\right) x_{1}^{-\gamma_{1}}+e^{-c\left(x_{1} / a_{n}^{(1)}\right)^{2}}\right) n \varphi\left(x_{2}\right) x_{2}^{-\left(1+\gamma_{1}\right)} .
\end{aligned}
$$

All together, and since $n \varphi\left(x_{2}\right) x_{2}^{-\gamma_{1}}$ is bounded by a constant (since $x_{2} \geqslant C_{8} a_{n}^{(2)}$ ), we obtain

$$
\begin{equation*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \leqslant C_{9} a_{n}^{(1)}, M_{n}^{(2)} \geqslant C_{9} a_{n}^{(2)}\right) \leqslant \frac{C}{x_{1} x_{2}}\left(n \varphi\left(x_{1}\right) x_{1}^{-\gamma_{1}}+e^{-c\left(x_{1} / a_{n}^{(1)}\right)^{2}}\right) . \tag{5.30}
\end{equation*}
$$

Term 2. It remains to deal with the second term in (5.21), which is the most technical. We will estimate the probabilities, for $k, j \in \mathbb{Z}$

$$
\begin{equation*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \in\left[2^{k} x_{1}, 2^{k+1} x_{1}\right), M_{n}^{(2)} \in\left[2^{j} x_{2}, 2^{j+1} x_{2}\right)\right)=: P_{1}(k, j)+P_{2}(k, j) . \tag{5.31}
\end{equation*}
$$

Here, we split the probability into two contributions: either the two maxima in $M_{n}^{(1)}, M_{n}^{(2)}$ are attained in one increment (with both coordinates large), see (5.32), or the two maxima are attained by separate increments, see (5.39).

Part 1. The first contribution is, using a union bound and the exchangeability of the $\mathbf{X}_{i}{ }^{\prime} \mathrm{S}$

$$
\begin{align*}
& P_{1}(k, j):=\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, \exists i \in \llbracket 1, n \rrbracket \text { s.t. } \mathbf{X}_{i} \in\left[2^{k} x_{1}, 2^{k+1} x_{1}\right) \times\left[2^{j} x_{2}, 2^{j+1} x_{2}\right),\right.  \tag{5.32}\\
& \leqslant n \sum_{u=2^{k} x_{1}} \sum_{v=2^{j} x_{2}}^{2^{k+1} x_{1}}{ }^{2^{j+1} x_{2}} \leqslant\left(\mathbf{X}_{1}=(u, v)\right) \\
& \quad \times \mathbf{P}\left(\hat{\mathbf{S}}_{n-1}=\left(x_{1}-u, x_{2}-v\right)+\boldsymbol{\delta}_{n}, M_{n-1}^{(1)} \leqslant 2^{k+1} x_{1}, M_{n-1}^{(2)} \leqslant 2^{j+1} x_{2}\right) \\
& \\
& \left.\qquad 2+1 x_{2}\right) .
\end{align*}
$$

Then we use Assumption 2.2 (item (i) in (2.5)) to get that there is a constant $C$ such that for any $j, k$, and any $(u, v) \in\left[2^{k} x_{1}, 2^{k+1} x_{1}\right) \times\left[2^{j} x_{2}, 2^{j+1} x_{2}\right)$, we have

$$
\begin{align*}
\mathbf{P}\left(\mathbf{X}_{1}=(u, v)\right) & \leqslant C \varphi_{1}\left(2^{k} x_{1}\right)\left(2^{k} x_{1}\right)^{-\left(1+\gamma_{1}\right)}\left(2^{j} x_{2}\right)^{-1}  \tag{5.33}\\
& \leqslant c 2^{-k\left(1+\gamma_{1}\right)+\eta|k|} 2^{-j} \frac{\varphi_{1}\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)}}{x_{2}} .
\end{align*}
$$

Therefore, in (5.32), we can sum over $u, v$ the last probability, and we treat it differently according to whether $k \geqslant-3$ or not and $j \geqslant-3$ or not (similarly to (5.14)): after summation over $u, v$, we obtain the following upper bound

$$
\begin{aligned}
& \text { if } k \leqslant-4, j \leqslant-4, \quad \mathbf{P}\left(\hat{S}_{n-1}^{(1)} \geqslant x_{1} / 2, \hat{S}_{n-1}^{(2)} \geqslant x_{2} / 2, M_{n-1}^{(1)} \leqslant 2^{k+1} x_{1}, M_{n-1} \leqslant 2^{j+1} x_{2}\right), \\
& \text { if } k \leqslant-4, j \geqslant-3, \quad \mathbf{P}\left(\hat{S}_{n-1}^{(1)} \geqslant x_{1} / 2, M_{n-1}^{(1)} \leqslant 2^{k+1} x_{1}\right) \\
& \text { if } k \geqslant-3, j \leqslant-4, \quad \mathbf{P}\left(\hat{S}_{n-1}^{(2)} \geqslant x_{2} / 2, M_{n-1} \leqslant 2^{j+1} x_{2}\right) \\
& \text { if } k \geqslant-3, j \geqslant-3, \quad 1 .
\end{aligned}
$$

Then, we can use Theorem 5.1 to get that for $k \leqslant-4$ with $2^{k} x_{1} \geqslant C_{9} a_{n}^{(1)}$ we have, with the same argument as for (5.17),

$$
\begin{equation*}
\mathbf{P}\left(\hat{S}_{n-1}^{(1)} \geqslant x_{1} / 2, M_{n-1}^{(1)} \leqslant 2^{k+1} x_{1}\right) \leqslant\left(c 2^{-k}\right)^{-c^{\prime} 2^{-k}}+e^{-c^{\prime} x_{1} / a_{n}^{(1)}} \tag{5.34}
\end{equation*}
$$

and similarly for the second coordinate. In the case $k \leqslant-4, j \leqslant-4$, Cauchy-Schwarz inequality allows us to to reduce to this estimate.

Going back to (5.32), and using (5.33), in the case $k, j \geqslant-3$ we get that

$$
\begin{equation*}
\sum_{k, j=-3}^{+\infty} P_{1}(k, j) \leqslant \sum_{k, j=-3}^{+\infty} \frac{C}{x_{2}} n \varphi_{1}\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} 2^{-k} 2^{-j} \leqslant \frac{C^{\prime}}{x_{2}} n \varphi_{1}\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} \tag{5.35}
\end{equation*}
$$

In the case $k \leqslant-4, j \geqslant-3$ (the case $k \geqslant-3, j \leqslant-4$ is symmetric), we get that

$$
P_{1}(k, j) \leqslant \frac{C}{x_{2}} n \varphi_{1}\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} 2^{-k\left(2+\gamma_{1}\right)} 2^{-j}\left(\left(c 2^{-k}\right)^{-c^{\prime} 2^{-k}}+e^{-c^{\prime} x_{1} / a_{n}^{(1)}}\right)
$$

Hence, we obtain (the calculation is analogous to that in (5.18))

$$
\begin{equation*}
\sum_{k=-\left\lfloor\log _{2}\left(x_{1} / C_{9} a_{n}^{(1)}\right)\right\rfloor}^{-4} \sum_{j=-3}^{+\infty} P_{1}(k, j) \leqslant \frac{C}{x_{2}} n \varphi_{1}\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)}\left(C+\left(\frac{c x_{1}}{a_{n}^{(1)}}\right)^{3+\gamma_{1}} e^{-c x_{1} / a_{n}^{(1)}}\right) . \tag{5.36}
\end{equation*}
$$

In the case $k \leqslant-4, j \leqslant-4$, we get that

$$
\begin{aligned}
P_{1}(k, j) \leqslant \frac{C}{x_{2}} n & \varphi_{1}\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} \\
& \times 2^{-k\left(2+\gamma_{1}\right)} 2^{-j}\left(\left(c 2^{-k}\right)^{-c 2^{-k}}+e^{-c x_{1} / a_{n}^{(1)}}\right)\left(\left(c 2^{-j}\right)^{-c^{\prime} 2^{-j}}+e^{-c^{\prime} x_{1} / a_{n}^{(2)}}\right)
\end{aligned}
$$

and a similar calculation as above gives

$$
\begin{equation*}
\sum_{k=-\left\lfloor\log _{2}\left(x_{1} / C_{9} a_{n}^{(1)}\right)\right\rfloor}^{-4} \sum_{j=-\left\lfloor\log _{2}\left(x_{2} / C_{9} a_{n}^{(1)}\right)\right\rfloor}^{-4} P_{1}(k, j) \leqslant \frac{C}{x_{2}} n \varphi_{1}\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} . \tag{5.37}
\end{equation*}
$$

All together, we obtain that

$$
\begin{equation*}
\sum_{k \geqslant-\log _{2}\left(x_{1} / C_{9} a_{n}^{(1)}\right)} \sum_{j \geqslant-\log _{2}\left(x_{2} / C_{9} a_{n}^{(1)}\right)} P_{1}(k, j) \leqslant \frac{C}{x_{1} x_{2}} n \varphi_{1}\left(x_{1}\right) x_{1}^{-\gamma_{1}} . \tag{5.38}
\end{equation*}
$$

Part 2. It remains to control the contribution when the maxima in $M_{n}^{(1)}, M_{n}^{(2)}$ are
attained by separated increments, i.e.

$$
\begin{align*}
& P_{2}(k, j):=\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, \exists i \neq \ell \in \llbracket 1, n \rrbracket, \text { s. t. } X_{i}^{(1)} \in\left[2^{k} x_{1}, 2^{k+1} x_{1}\right), M_{n}^{(1)} \leqslant 2^{k+1} x_{1},\right.  \tag{5.39}\\
& \left.X_{\ell}^{(2)} \in\left[2^{j} x_{2}, 2^{j+1} x_{2}\right),, M_{n}^{(2)} \leqslant 2^{j+1} x_{2}\right) \\
& \leqslant\binom{ n}{2} \sum_{u=2^{k} x_{1}}^{2^{k+1} x_{1}} \sum_{v \leqslant 2^{j+1} x_{2}} \sum_{s \leqslant 2^{k+1} x_{1}} \sum_{t=2^{j} x_{2}}^{2^{j+1} x_{2}} \mathbf{P}\left(\mathbf{X}_{1}=(u, v)\right) \mathbf{P}\left(\mathbf{X}_{1}=(s, t)\right) \\
& \times \mathbf{P}\left(\hat{\mathbf{S}}_{n-2}=\mathbf{x}-(u, v)-(s, t)+\boldsymbol{\delta}_{n}+\boldsymbol{\delta}_{n-1}, M_{n-2}^{(1)} \leqslant 2^{k+1} x_{1}, M_{n-2}^{(2)} \leqslant 2^{j+1} x_{2}\right) .
\end{align*}
$$

Again, we use Assumption 2.2 to bound the first two probabilities: for the ranges of $u, v$ and $s, t$ considered, using item (iii) in (2.5), we have

$$
\begin{align*}
& \mathbf{P}\left(\mathbf{X}_{1}=(u, v)\right) \leqslant c 2^{-k\left(1+\gamma_{1}\right)+\eta|k|} \varphi_{1}\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} \times \frac{h_{2^{k} x_{1}}^{(1)}(|v|)}{1+|v|} \\
& \mathbf{P}\left(\mathbf{X}_{1}=(s, t)\right) \leqslant c 2^{-j\left(1+\gamma_{1}\right)+\eta|j|} \varphi_{2}\left(x_{2}\right) x_{2}^{-\left(1+\gamma_{2}\right)} \times \frac{h_{2^{k} x_{2}}^{(2)}(|s|)}{1+|s|} \tag{5.40}
\end{align*}
$$

Then, we may sum the last probability in (5.39) over $u$ and $t$ in the range considered, and get after summation (using also that for the range of $v$ and $s$ considered we have $\left.v \leqslant 2^{j+1} x_{2}, s \leqslant 2^{k+1} x_{1}\right)$

```
if \(k \leqslant-4, j \leqslant-4, \quad \mathbf{P}\left(\hat{S}_{n-2}^{(1)} \geqslant x_{1} / 2, \hat{S}_{n-2}^{(2)} \geqslant x_{2} / 2, M_{n-2}^{(1)} \leqslant 2^{k+1} x_{1}, M_{n-2} \leqslant 2^{j+1} x_{2}\right)\),
if \(k \leqslant-4, j \geqslant-3, \quad \mathbf{P}\left(\hat{S}_{n-2}^{(1)} \geqslant x_{1} / 2, M_{n-2}^{(1)} \leqslant 2^{k+1} x_{1}\right)\),
if \(k \geqslant-3, j \leqslant-4, \quad \mathbf{P}\left(\hat{S}_{n-2}^{(2)} \geqslant x_{2} / 2, M_{n-2} \leqslant 2^{j+1} x_{2}\right)\),
if \(k \geqslant-3, j \geqslant-3, \quad 1\),
```

and to treat these terms, we can again use Theorem 5.1, in the same way as for (5.34). Then we can sum over $v$ and $s$ and use item (ii) in (2.5) to get that $\sum_{v} h_{2^{k} x_{1}}^{(i)}(|v|) /(1+|v|)$

Going back to (5.39), and starting with the case $k, j \geqslant-3$, we get

$$
\begin{aligned}
\sum_{k, j=-3}^{+\infty} P_{2}(k, j) & \leqslant \sum_{k, j=-3}^{+\infty} C\binom{n}{2} \varphi_{1}\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} \varphi_{2}\left(x_{2}\right) x_{2}^{-\left(1+\gamma_{2}\right)} 2^{-k} 2^{-j} \\
& \leqslant C^{\prime} n \varphi_{1}\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} n \varphi_{2}\left(x_{2}\right) x_{2}^{-\left(1+\gamma_{2}\right)}
\end{aligned}
$$

Similarly, and using (5.34), we get that if $k \leqslant-4, j \geqslant-3$ (the case $k \geqslant-3, j \leqslant-4$ is symmetric)

$$
P_{2}(k, j) \leqslant C^{\prime} n \varphi_{1}\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} n \varphi_{2}\left(x_{2}\right) x_{2}^{-\left(1+\gamma_{2}\right)} \times 2^{-k\left(2+\gamma_{1}\right)} 2^{-j}\left(\left(c 2^{-k}\right)^{-c^{\prime} 2^{-k}}+e^{-c^{\prime} x_{1} / a_{n}^{(1)}}\right)
$$

As above (with the same argument as in (5.36)), we therefore get that

$$
\sum_{k=-\left\lfloor\log _{2}\left(x_{1} / C_{9} a_{n}^{(1)}\right)\right\rfloor}^{-4} \sum_{j=-3}^{+\infty} P_{2}(k, j) \leqslant C n \varphi_{1}\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} n \varphi_{2}\left(x_{2}\right) x_{2}^{-\left(1+\gamma_{2}\right)} .
$$

An identical argument holds in the case $k \leqslant-4, j \leqslant-4$, and we end up with

$$
\begin{align*}
\sum_{k \geqslant-\log _{2}\left(x_{1} / C_{9} a_{n}^{(1)}\right)} \sum_{j \geqslant-\log _{2}\left(x_{2} / C_{9} a_{n}^{(1)}\right)} P_{2}(k, j) & \leqslant C n \varphi_{1}\left(x_{1}\right) x_{1}^{-\left(1+\gamma_{1}\right)} n \varphi_{2}\left(x_{2}\right) x_{2}^{-\left(1+\gamma_{2}\right)}  \tag{5.41}\\
& \leqslant \frac{C}{x_{1} x_{2}} n \varphi_{1}\left(x_{1}\right) x_{1}^{-\gamma_{1}} .
\end{align*}
$$

For the last inequality, we used that $x_{2} \geqslant C_{8} a_{n}^{(2)}$, so that $n \varphi_{2}\left(x_{2}\right) x_{2}^{-\gamma_{2}}$ is bounded by a constant, thanks to the definition (1.4) of $a_{n}^{(2)}$.

Therefore, going back to (5.31), and using (5.38)-(5.41), we obtain that

$$
\begin{equation*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \geqslant C_{9} a_{n}^{(1)}, M_{n}^{(2)} \geqslant C_{9} a_{n}^{(2)}\right) \leqslant \frac{C}{x_{1} x_{2}} n \varphi_{1}\left(x_{1}\right) x_{1}^{-\gamma_{1}} . \tag{5.42}
\end{equation*}
$$

Conclusion. Let us collect the estimates (5.22), (5.28), (5.30) and (5.42): plugged into (5.21), we finally obtain

$$
\begin{equation*}
\mathbf{P}\left(\hat{S}_{n}=\mathbf{x}\right) \leqslant \frac{C}{x_{1} x_{2}}\left(n \varphi_{1}\left(x_{1}\right) x_{1}^{-\gamma_{1}}+e^{-c\left(x_{1} / a_{n}^{(1)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}\right) . \tag{5.43}
\end{equation*}
$$

This concludes the proof of Theorem 2.5, since the same bound applies to any coordinate.

### 5.4 Proof of Theorem 2.6

Again, we prove only the case of the dimension $d=2$ for simplicity. Recall that we work in the balanced case, so we write $a_{n} \equiv a_{n}^{(i)}$ and $\alpha \equiv \alpha_{i}$. Let us assume that $\left|x_{1}\right| \geqslant\left|x_{2}\right|$, so that $c\left|x_{1}\right| \geqslant\|\mathbf{x}\| \geqslant\left|x_{1}\right|$ (the other case is symmetric). Suppose also for simplicity that $x_{1}$ is positive (so we can drop the absolute value), and $x_{1}>C_{8} a_{n}$. We write

$$
\begin{align*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}\right)=\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \geqslant x_{1} / 8\right)+\mathbf{P}\left(\hat{\mathbf{S}}_{n}\right. & \left.=\mathbf{x}, M_{n}^{(1)} \in\left(C_{9} a_{n}, x_{1} / 8\right)\right)  \tag{5.44}\\
& +\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \leqslant C_{9} a_{n}\right)
\end{align*}
$$

The last term in (5.44) can be bounded using Proposition 5.4, together with (5.2)

$$
\begin{align*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \leqslant C_{9} a_{n}\right) & \leqslant \frac{C}{\left(a_{n}\right)^{2}}\left(n \varphi\left(x_{1}\right) x_{1}^{-\gamma_{1}} e^{-c x_{1} / a_{n}}+e^{-c\left(x_{1} / a_{n}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}\right) \\
& \leqslant C n \varphi\left(x_{1}\right) x_{1}^{-\left(2+\gamma_{1}\right)}+\frac{C}{\left(a_{n}\right)^{2}} e^{-c\left(x_{1} / a_{n}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}, \tag{5.45}
\end{align*}
$$

where we used that $e^{-c x_{1} / a_{n}} \leqslant\left(a_{n} / x_{1}\right)^{2}$ provided that $x_{1} / a_{n} \geqslant C_{8}$ with $C_{8}$ large enough.
For the first term in (5.44), because of the exchangeability of the $\mathbf{X}_{i}$ and thanks to a union bound, we get

$$
\begin{align*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}\right. & \left.=\mathbf{x}, M_{n}^{(1)} \geqslant x_{1} / 8\right) \leqslant \sum_{\mathbf{y} \in \mathbb{Z}^{2}, y_{1} \geqslant x_{1} / 8} n \mathbf{P}\left(\mathbf{X}_{1}=\mathbf{y}\right) \mathbf{P}\left(\hat{\mathbf{S}}_{n-1}=\mathbf{x}-\mathbf{y}+\boldsymbol{\delta}_{n}\right)  \tag{5.46}\\
& \leqslant C \varphi\left(x_{1}\right) x_{1}^{-(2+\gamma)} \sum_{\mathbf{y} \in \mathbb{Z}^{2}, y_{1} \geqslant x_{1} / 8} \mathbf{P}\left(\hat{\mathbf{S}}_{n-1}=\mathbf{x}-\mathbf{y}+\boldsymbol{\delta}_{n}\right) \leqslant C \varphi\left(x_{1}\right) x_{1}^{-(2+\gamma)} .
\end{align*}
$$

Here, we used (2.8): $\mathbf{P}\left(\mathbf{X}_{1}=\mathbf{y}\right)$ is bounded by a constant times $\varphi(\|\mathbf{y}\|)\|\mathbf{y}\|^{-(2+\gamma)}$ for the range of $\mathbf{y}$ under summation (and it is bounded by a constant times $\varphi\left(x_{1}\right) x_{1}^{-(2+\gamma)}$ ).

It remains to control the middle term in (5.44). We write

$$
\begin{align*}
& \mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \in\left(C_{9} a_{n}, x_{1} / 8\right)\right)=\sum_{j=3}^{\left\lfloor\log _{2}\left(x_{1} / C_{9} a_{n}\right)\right\rfloor} \mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \in\left[2^{-(j+1)} x_{1}, 2^{-j} x_{1}\right)\right) \\
& \leqslant \sum_{j=3}^{\left\lfloor\log _{2}\left(x_{1} / C_{9} a_{n}\right)\right\rfloor} \sum_{\mathbf{y} \in \mathbb{Z}^{2}, y_{1} \geqslant 2^{-(j+1)} x_{1}} n \mathbf{P}\left(\mathbf{X}_{1}=\mathbf{y}\right) \mathbf{P}\left(\hat{\mathbf{S}}_{n-1}=\mathbf{x}-\mathbf{y}+\boldsymbol{\delta}_{n}, M_{n}^{(1)} \leqslant 2^{-j} x_{1}\right) \\
& \leqslant n \varphi\left(x_{1}\right) x_{1}^{-(2+\gamma)} \sum_{j=3}^{\left\lfloor\log _{2}\left(x_{1} / C_{9} a_{n}\right)\right\rfloor} 2^{(d+1+\gamma)(j+1)} \mathbf{P}\left(\hat{S}_{n-1}^{(1)} \geqslant x_{1} / 2, M_{n}^{(1)} \leqslant 2^{-j} x_{1}\right) . \tag{5.47}
\end{align*}
$$

For the last inequality, we used (2.8) to bound $\mathbf{P}\left(\mathbf{X}_{1}=\mathbf{y}\right) \leqslant c \varphi(\|y\|)\|y\|^{-(2+\gamma)}$ : this is bounded, for $\mathbf{y}$ with $y_{1} \geqslant 2^{-(j+1)} x_{1}$, by a constant times $\varphi\left(2^{-(j+1)} x_{1}\right) x_{1}^{-(d+\gamma)} 2^{(j+1)(2+\gamma)}$, with $\varphi\left(2^{-(j+1)} x_{1}\right) \leqslant 2^{j} \varphi\left(x_{1}\right)$ thanks to Potter's bound. Then, for every $j$, the sum over $\mathbf{y}$ with $y_{1} \geqslant 2^{-(j+1)} x_{1}$ gives rise to $\mathbf{P}\left(\hat{S}_{n-1}^{(1)} \geqslant x_{1} / 2, M_{n}^{(1)} \leqslant 2^{-j} x_{1}\right)$ (recall that $\delta_{n}^{(1)}$ is bounded by a constant).

Then, it remains to use Theorem 5.1, more precisely (5.17), to get that the last sum in (5.47) is bounded by

$$
\sum_{j=3}^{\left\lfloor\log _{2}\left(x_{1} / C_{9} a_{n}\right)\right\rfloor} 2^{(d+1+\gamma)(j+1)}\left(\left(c 2^{j}\right)^{-c^{\prime} 2^{j}}+e^{-c^{\prime} x_{1} / a_{n}}\right)
$$

which is bounded by a constant. Therefore we have that

$$
\begin{equation*}
\mathbf{P}\left(\hat{\mathbf{S}}_{n}=\mathbf{x}, M_{n}^{(1)} \in\left(C_{9} a_{n}, x_{1} / 8\right)\right) \leqslant C n \varphi\left(x_{1}\right) x_{1}^{-(2+\gamma)} \tag{5.48}
\end{equation*}
$$

Together with (5.45)-(5.46), this conclude the proof of Theorem 2.6 (recall $x_{1} \geqslant c\|x\|$ ).

## 6 Proof for case I (centered): $\mathbf{b}_{n} \equiv \mathbf{0}$

In this section, we prove Theorem 3.1 (and Theorem 3.2 in Section 6.4 below). Recall the definition (3.1) of $n_{i}$ : it verifies $a_{n_{i}}^{(i)} \sim\left|x_{i}\right|$. We work along the favorite direction, that is we assume that $x_{i} / a_{n_{1}}^{(i)} \rightarrow t_{i} \in \mathbb{R}^{*}$ for any $i \in\{1, \ldots, d\}$, which is equivalent to having $n_{i} \sim\left|t_{i}\right|^{\alpha_{i}} n_{1}$ (the reference coordinate is the first one, but this is only for commodity). We fix $\varepsilon>0$, and we decompose $G(\mathbf{x})$ into three subparts:

$$
\begin{equation*}
G(\mathbf{x})=\left(\sum_{n=1}^{\varepsilon n_{1}-1}+\sum_{n=\varepsilon n_{1}}^{\varepsilon^{-1} n_{1}}+\sum_{n=\varepsilon^{-1} n_{1}+1}^{+\infty}\right) \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \tag{6.1}
\end{equation*}
$$

The middle part gives the main contribution: we treat it first, before we show that the other two parts are negligible. In this section and in the rest of the paper, we often omit the integer part: for instance, we do as if $\varepsilon n_{1}$ and $\varepsilon^{-1} n_{1}$ were integers.

### 6.1 Main contribution

Because $n_{i} \sim\left|t_{i}\right|^{\alpha_{i}} n_{1}$, we have that for $\varepsilon \leqslant n / n_{1} \leqslant \varepsilon^{-1}$ the probability $\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)$ falls into the range of application of the local limit theorem (2.1). We have

$$
\sum_{n=\varepsilon n_{1}}^{\varepsilon^{-1} n_{1}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)=\sum_{n=\varepsilon n_{1}}^{\varepsilon^{-1} n_{1}} \frac{1}{a_{n}^{(1)} \cdots a_{n}^{(d)}} g_{\boldsymbol{\alpha}}\left(\frac{x_{1}}{a_{n}^{(1)}}, \ldots, \frac{x_{d}}{a_{n}^{(d)}}\right)+\sum_{n=\varepsilon n_{1}}^{\varepsilon^{-1} n_{1}} \frac{o(1)}{a_{n}^{(1)} \cdots a_{n}^{(d)}}
$$

The second term is negligible compared to the first one, so we focus on the first term. Since $a_{n}^{(i)}$ is regularly varying with exponent $1 / \alpha_{i}$, we get that uniformly for $n / n_{1} \in$ $\left(\varepsilon, \varepsilon^{-1}\right)$,

$$
a_{n}^{(i)}=(1+o(1)) a_{n_{1}}^{(i)} \times\left(n / n_{1}\right)^{1 / \alpha_{i}}
$$

Using also that $x_{i} / a_{n_{1}}^{(i)}=t_{i}+o(1)$ as $n_{1} \rightarrow \infty$, and since $g_{\boldsymbol{\alpha}}(\cdot)$ is continuous, we get that

$$
\begin{aligned}
\sum_{n=\varepsilon n_{1}}^{\varepsilon^{-1} n_{1}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) & =\frac{(1+o(1))}{a_{n_{1}}^{(1)} \cdots a_{n_{1}}^{(d)}} \sum_{n=\varepsilon n_{1}}^{\varepsilon^{-1} n_{1}}\left(\frac{n_{1}}{n}\right)^{\frac{1}{\alpha_{1}}+\cdots+\frac{1}{\alpha_{d}}} g_{\boldsymbol{\alpha}}\left(t_{1}\left(\frac{n_{1}}{n}\right)^{1 / \alpha_{1}}, \ldots, t_{d}\left(\frac{n_{1}}{n}\right)^{1 / \alpha_{d}}\right) \\
& =(1+o(1)) \frac{n_{1}}{a_{n_{1}}^{(1)} \cdots a_{n_{1}}^{(d)}} \int_{\varepsilon}^{\varepsilon^{-1}} u^{-\sum_{i=1}^{d} \alpha_{i}^{-1}} g_{\boldsymbol{\alpha}}\left(t_{1} u^{-1 / \alpha_{1}}, \ldots, t_{d} u^{-1 / \alpha_{d}}\right) \mathrm{d} u
\end{aligned}
$$

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By a change of variable, we therefore get that the first term in (6.1) is

$$
\begin{equation*}
(1+o(1)) \frac{n_{1}}{a_{n_{1}}^{(1)} \ldots a_{n_{1}}^{(d)}} \int_{\varepsilon}^{\varepsilon^{-1}} v^{-2+\sum_{i=1}^{d} \alpha_{i}^{-1}} g_{\boldsymbol{\alpha}}\left(t_{1} v^{1 / \alpha_{1}}, \ldots, t_{d} v^{1 / \alpha_{d}}\right) \mathrm{d} v \tag{6.2}
\end{equation*}
$$

### 6.2 Third part in (6.1)

Using the local limit theorem (2.1), we have that there is a constant $C$ such that

$$
\sum_{n>\varepsilon^{-1} n_{1}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant C \sum_{n>\varepsilon^{-1} n_{1}} \frac{1}{a_{n}^{(1)} \cdots a_{n}^{(d)}} \leqslant C^{\prime} \frac{\varepsilon^{-1} n_{1}}{a_{\varepsilon^{-1} n_{1}}^{(1)} \cdots a_{\varepsilon^{-1} n_{1}}^{(d)}}
$$

For the last inequality we used that $\left(a_{n}^{(1)} \cdots a_{n}^{(d)}\right)^{-1}$ is regularly varying with exponent $-\sum_{i=1}^{d} \alpha_{i}^{-1}<-1$. Using again the regular variation of $a_{n}^{(i)}$, we get that there is a constant $c$ such that the second term in (6.1) is

$$
\begin{equation*}
\sum_{n>\varepsilon^{-1} n_{1}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant c \varepsilon^{\sum_{i=1}^{d} \alpha_{i}^{-1}-1} \times \frac{n_{1}}{a_{n_{1}}^{(1)} \cdots a_{n_{1}}^{(d)}} . \tag{6.3}
\end{equation*}
$$

### 6.3 First part in (6.1)

Let us first consider the case $\sum_{i=1}^{d} \alpha_{i}^{-1}<2$. By Theorem 2.1, and since $\left|x_{1}\right| \geqslant a_{n}^{(1)}$ for $n \leqslant \varepsilon n_{1}$, we get that

$$
\begin{aligned}
\sum_{n=1}^{\varepsilon n_{1}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) & \leqslant \sum_{n=1}^{\varepsilon n_{1}} \frac{C}{a_{n}^{(1)} \cdots a_{n}^{(d)}}\left(n \varphi_{1}\left(\left|x_{1}\right|\right)\left|x_{1}\right|^{-\gamma_{1}}+e^{-c\left(\left|x_{1}\right| / a_{n}^{(1)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}\right) \\
& \leqslant \frac{C^{\prime}\left(\varepsilon n_{1}\right)^{2}}{a_{\varepsilon n_{1}}^{(1)} \cdots a_{\varepsilon n_{1}}^{(d)}} \varphi_{1}\left(\left|x_{1}\right|\right)\left|x_{1}\right|^{-\gamma_{1}}+\frac{C \varepsilon n_{1}}{a_{n_{1}}^{(1)} \cdots a_{n_{1}}^{(d)}} e^{-c^{\prime}\left(\left|x_{1}\right| / a_{\varepsilon n_{1}}^{(1)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}} .
\end{aligned}
$$

Here, we used for the first term that $n /\left(a_{n}^{(1)} \cdots a_{n}^{(d)}\right)$ is regularly varying with exponent $1-\sum_{i=1}^{d} \alpha_{i}^{-1}>-1$. For the second term, we bounded $\left(a_{n}^{(1)} \cdots a_{n}^{(d)}\right)^{-1}$ by $\left(a_{n_{1}}^{(1)} \cdots a_{n_{1}}^{(d)}\right)^{-1}\left(n_{1} / n\right)^{2}$ and also $e^{-c\left(\left|x_{1}\right| / a_{n}^{(1)}\right)^{2}}$ by $\left(n / n_{1}\right)^{2} e^{-c^{\prime}\left(\left|x_{1}\right| / a_{n}^{(1)}\right)^{2}}$ provided that $n_{1} / n$ is large enough (i.e. $\varepsilon$ small enough). Then, we use that $\left(a_{n}^{(1)} \cdots a_{n}^{(d)}\right)^{-1}$ is regularly varying with exponent $-\sum_{i=1}^{d} \alpha_{i}^{-1}$, that $n_{1} \varphi_{1}\left(\left|x_{1}\right|\right)\left|x_{1}\right|^{-\gamma_{1}}$ is bounded above by a constant (thanks to the definition (1.4) of $a_{n}^{(1)}$, together with $\left|x_{1}\right|=a_{n_{1}}^{(1)}$ ), and that $\left|x_{1}\right| / a_{\varepsilon n_{1}}^{(1)} \leqslant c \varepsilon^{-1 / 2 \alpha_{1}}$ (by Potter's bound): we finally end up with

$$
\begin{equation*}
\sum_{n=1}^{\varepsilon n_{1}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C^{\prime \prime} n_{1}}{a_{n_{1}}^{(1)} \cdots a_{n_{1}}^{(d)}} \times\left(\varepsilon^{2-\sum_{i=1}^{d} \alpha_{i}^{-1}}+\varepsilon e^{-c \varepsilon^{-1 / \alpha_{1}}}\right) \tag{6.4}
\end{equation*}
$$

In the case where $\sum_{i=1}^{d} \alpha_{i}^{-1} \geqslant 2$, we need to use Assumption 2.2. For $n \leqslant \varepsilon n_{1}$ we have that $\left|x_{i}\right| \geqslant c\left|t_{i}\right| a_{n_{1}}^{(i)} \geqslant c^{\prime} t_{i} \varepsilon^{1 / \alpha_{i}} a_{n}^{(i)}$, and we get that for $n_{1}$ sufficiently large, by Theorem 2.5

$$
\begin{aligned}
\sum_{n=1}^{\varepsilon n_{1}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) & \leqslant \sum_{n=1}^{\varepsilon n_{1}} \frac{c}{\left|x_{1}\right| \cdots\left|x_{d}\right|}\left(n \varphi_{1}\left(\left|x_{1}\right|\right)\left|x_{1}\right|^{-\gamma_{1}}+e^{-c\left(\left|x_{1}\right| / a_{n}^{(1)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}\right) \\
& \leqslant \frac{c^{\prime}}{a_{n_{1}}^{(1)} \cdots a_{n_{1}}^{(d)}}\left(\left(\varepsilon n_{1}\right)^{2} \varphi_{1}\left(\left|x_{1}\right|\right)\left|x_{1}\right|^{-\gamma_{1}}+\varepsilon n_{1} e^{-c^{\prime} \varepsilon^{-1 / \alpha_{1}}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}\right) .
\end{aligned}
$$

For the second inequality, we used that $\left|x_{i}\right| \geqslant c a_{n_{1}}^{(i)}$ for all $i$ (since $\left|t_{i}\right|>0$ ), and that $\left|x_{1}\right| / a_{n}^{(1)} \geqslant c\left|x_{1}\right| / a_{\varepsilon n_{1}}^{(1)} \geqslant c^{\prime} \varepsilon^{1 / 2 \alpha_{1}}$ for all $n \leqslant \varepsilon n_{1}$ (thanks to Potter's bound). Then, since that $n_{1} \varphi_{1}\left(\left|x_{1}\right|\right)\left|x_{1}\right|^{-\gamma_{1}}$ is bounded by a constant, we get that

$$
\begin{equation*}
\sum_{n=1}^{\varepsilon n_{1}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{c^{\prime} n_{1}}{a_{n_{1}}^{(1)} \cdots a_{n_{1}}^{(d)}}\left(\varepsilon^{2}+\varepsilon e^{-c \varepsilon^{-1 / \alpha_{1}}}\right) \tag{6.5}
\end{equation*}
$$

## Conclusion

Collecting (6.2) together with (6.3) and (6.4)-(6.5), then letting $n \rightarrow+\infty$ and finally $\varepsilon \downarrow 0$, we obtain (3.3).
6.4 Proof in the marginal case $d=2, \boldsymbol{\alpha}=(2,2)$

Again, we work along the favorite direction (see (3.2)), so that in particular we have $n_{1} \sim \lambda n_{2}$ for some constant $\lambda>0$ (recall the definition (3.1) of $n_{i}$ ). For $\varepsilon>0$ fixed, we split the Green function as

$$
\begin{equation*}
G(\mathbf{x})=\sum_{n=1}^{\varepsilon^{-1} n_{1}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)+\sum_{n=\varepsilon^{-1} n_{1}+1}^{+\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) . \tag{6.6}
\end{equation*}
$$

The main contribution comes from the second sum. Thanks to the local limit theorem (2.1), we get that for $n$ sufficiently large

$$
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)=\frac{1}{a_{n}^{(1)} a_{n}^{(2)}}\left(g_{\boldsymbol{\alpha}}\left(\frac{x_{1}}{a_{n}^{(1)}}, \frac{x_{2}}{a_{n}^{(2)}}\right)+o(1)\right) .
$$

For $n>\varepsilon^{-1} n_{1}$ we have that $\left|x_{1}\right| / a_{n}^{(1)} \leqslant 2 \varepsilon^{1 / 2}$ for $\left|x_{1}\right|$ large enough (thanks to the definition of $n_{1}$ and the fact that $a_{n}^{(1)}$ is regularly varying with exponent $1 / 2$ ), and also we have $n \geqslant \frac{1}{2} \lambda \varepsilon^{-1} n_{2}$ so that $\left|x_{2}\right| / a_{n}^{(2)} \leqslant 2 \lambda^{-1 / 2} \varepsilon^{1 / 2}$. All together, and since $g_{\boldsymbol{\alpha}}$ is continuous at 0 , for every $\eta>0$ we can choose $\varepsilon$ small enough so that for large enough $\left|x_{1}\right| \asymp\left|x_{2}\right|$

$$
\frac{g_{\boldsymbol{\alpha}}(0,0)-\eta}{a_{n}^{(1)} a_{n}^{(2)}} \leqslant \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{g_{\boldsymbol{\alpha}}(0,0)+\eta}{a_{n}^{(1)} a_{n}^{(2)}} .
$$

Then, since $v \mapsto \sum_{n \geqslant v}\left(a_{n}^{(1)} a_{n}^{(2)}\right)^{-1}$ is slowly varying, we get that for $n_{1}$ large enough

$$
\begin{equation*}
\sum_{n=\varepsilon^{-1} n_{1}+1}^{+\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant\left(g_{\boldsymbol{\alpha}}(0,0)+2 \eta\right) \sum_{n \geqslant n_{1}} \frac{1}{a_{n}^{(1)} a_{n}^{(2)}}, \tag{6.7}
\end{equation*}
$$

and a similar lower bound holds, with $2 \eta$ replaced by $-2 \eta$.
We now treat the first sum in (6.6). First, for $n \leqslant \varepsilon n_{1}$, Theorem 2.1 gives that

$$
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C}{a_{n}^{(1)} a_{n}^{(2)}} \times\left(n \varphi_{1}\left(x_{1}\right) x_{1}^{-\gamma}+e^{-c\left(x_{1} / a_{n}^{(1)}\right)^{2}}\right)
$$

Exactly as what is done above to obtain (6.5), we obtain that

$$
\sum_{n=1}^{\varepsilon n_{1}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{c^{\prime}}{a_{n_{1}}^{(1)} a_{n_{1}}^{(2)}}\left(\left(\varepsilon n_{1}\right)^{2} \varphi_{1}\left(x_{1}\right)^{-\gamma}+\varepsilon n_{1} e^{-c \varepsilon^{-1}\left(x_{1} / a_{n_{1}}^{(1)}\right)^{2}}\right) \leqslant \frac{c_{\varepsilon} n_{1}}{a_{n_{1}}^{(1)} a_{n_{1}}^{(2)}} .
$$

For $\varepsilon n_{1} \leqslant n \leqslant \varepsilon^{-1} n_{1}$, the local limit theorem (2.1) gives that there is a constant $C>0$ such that

$$
\begin{equation*}
\sum_{n=\varepsilon n_{1}}^{\varepsilon^{-1} n_{1}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \sum_{n=\varepsilon n_{1}}^{\varepsilon^{-1} n_{1}} \frac{C}{a_{n}^{(1)} a_{n}^{(2)}} \leqslant \frac{C_{\varepsilon} n_{1}}{a_{n_{1}}^{(1)} a_{n_{1}}^{(2)}} \tag{6.8}
\end{equation*}
$$

As a conclusion, we get that there is some constant $C_{\varepsilon}^{\prime}$ such that

$$
\begin{equation*}
\sum_{n=1}^{\varepsilon^{-1} n_{1}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant C_{\varepsilon}^{\prime} \frac{n_{1}}{a_{n_{1}}^{(1)} a_{n_{1}}^{(2)}}=o\left(\sum_{n \geqslant n_{1}} \frac{1}{a_{n}^{(1)} a_{n}^{(2)}}\right) \tag{6.9}
\end{equation*}
$$

The last identity comes from [6, Prop. 1.5.9.a.], since $\left(a_{n}^{(1)} a_{n}^{(2)}\right)^{-1}$ is regularly varying with exponent -1 , and summable. This concludes the proof of Theorem 3.2.

## About the balanced case

Let us write $a_{n} \equiv a_{n}^{(i)}$, and $L(\cdot), \sigma(\cdot)$ in place of $L_{i}(\cdot), \sigma_{i}(\cdot)$. The walk $\mathbf{S}_{n}$ is transient if and only if $\sum_{n=1}^{+\infty}\left(a_{n}\right)^{-2}<+\infty$. We may compare the sum to the integral $\int_{1}^{+\infty}\left(a_{t}\right)^{-2} \mathrm{~d} t$ which by a change of variable $u=a_{t}$ (by definition of $a_{t}$ we have $t \sim\left(a_{t}\right)^{2} / \sigma\left(a_{t}\right)$ ), $\mathrm{d} t \sim 2 u \mathrm{~d} u / \sigma(u)$ : we get that $\int_{1}^{+\infty}\left(a_{t}\right)^{-2} \mathrm{~d} t<+\infty$ if and only if $\int_{1}^{+\infty} \frac{2 \mathrm{~d} u}{u \sigma(u)}<+\infty$. With the same change of variable, and using that $a_{n_{1}} \sim\left|x_{1}\right|$, we get that

$$
\sum_{n \geqslant n_{1}}^{+\infty} \frac{1}{\left(a_{n}\right)^{2}} \sim \int_{n_{1}}^{+\infty} \frac{\mathrm{d} t}{\left(a_{t}\right)^{2}} \sim \int_{\left|x_{1}\right|}^{+\infty} \frac{2 \mathrm{~d} u}{u \sigma(u)}
$$

which gives the announced result.

## 7 Proof for case II (non-zero mean): $\mu_{i} \neq 0$ for some $i$ with $\alpha_{i}>1$

In this section, we prove Theorem 3.3. Recall our notations: we have $\mu_{i}=0\left(b_{n}^{(i)} \equiv 0\right)$ for $i>i_{0}$, and $\mu_{i_{0}} \neq 0$ with $\alpha_{i_{0}}>1$. We set $n_{i_{0}}:=x_{i_{0}} / \mu_{i_{0}}\left(x_{i_{0}}\right.$ and $\mu_{i_{0}}$ need to have the same sign), and we also denote $m_{i_{0}}:=a_{n_{i_{0}}}^{\left(i_{0}\right)}$, so that the typical number of steps for the $i_{0}$-th coordinate to visit $x_{i_{0}}$ is $n_{i_{0}}+O\left(m_{i_{0}}\right)$. For simplicity, we work with $\mu_{i_{0}}, x_{i_{0}}>0$. We consider the case where $\|\mathbf{x}\| \rightarrow+\infty$ along the favorite direction, recall (3.5).

We fix $\varepsilon>0$, and decompose $G(\mathbf{x})$ into three subparts:

$$
\begin{equation*}
G(\mathbf{x})=\left(\sum_{n<n_{i_{0}}-\varepsilon^{-1} m_{i_{0}}}+\sum_{n=n_{i_{0}}-\varepsilon^{-1} m_{i_{0}}}^{n_{i_{0}}+\varepsilon^{-1} m_{i_{0}}}+\sum_{n>n_{i_{0}}+\varepsilon^{-1} m_{i_{0}}}\right) \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) . \tag{7.1}
\end{equation*}
$$

The main contribution is the second part, that we treat first, before we show that the two other parts are negligible.

### 7.1 Main contribution

Since the summation index ranges from $n_{i_{0}}-\varepsilon^{-1} m_{i_{0}}$ to $n_{i_{0}}+\varepsilon^{-1} m_{i_{0}}$ and because we work in the favorite direction, we obtain that $\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)$ falls into the range of application of the local limit theorem (2.1). We have that

$$
\begin{array}{r}
\sum_{n=n_{i_{0}}-\varepsilon^{-1} m_{i_{0}}}^{n_{i_{0}}+\varepsilon^{-1} m_{i_{0}}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)= \\
\sum_{n=n_{i_{0}}-\varepsilon^{-1} m_{i_{0}}}^{n_{i_{0}}+\varepsilon^{-1} m_{i_{0}}} \frac{1}{a_{n}^{(1)} \cdots a_{n}^{(d)}} g_{\boldsymbol{\alpha}}\left(\frac{x_{1}-b_{n}^{(1)}}{a_{n}^{(1)}}, \ldots, \frac{x_{d}-b_{n}^{(d)}}{a_{n}^{(d)}}\right) \\
\\
+\sum_{n=n_{i_{0}}-\varepsilon^{-1} m_{i_{0}}}^{n_{i_{0}}+\varepsilon^{-1} m_{i_{0}}} \frac{o(1)}{a_{n}^{(1)} \cdots a_{n}^{(d)}} .
\end{array}
$$

The second term is negligible compared to $a_{n_{i_{0}}}^{\left(i_{0}\right)} /\left(a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}\right)$, so we focus on the first term. Let us consider the different terms $\left(x_{i}-b_{n}^{(i)}\right) / a_{n}^{(i)}$ for the range considered. First of all, notice that $n=(1+o(1)) n_{i_{0}}$, since $m_{i_{0}}=o\left(n_{i_{0}}\right)$ : it gives in particular that $a_{n}^{(i)}=(1+o(1)) a_{n_{i_{0}}}^{(i)}$.

* If $\alpha_{i}>1$, then $b_{n}^{(i)}=\mu_{i} n=b_{n_{i_{0}}}^{(i)}+\mu_{i}\left(n-n_{i_{0}}\right)$, and hence

$$
\begin{equation*}
\frac{x_{i}-b_{n}^{(i)}}{a_{n}^{(i)}}=(1+o(1)) \frac{x_{i}-b_{n_{i_{0}}}^{(i)}}{a_{n_{i_{0}}}^{(i)}}+(1+o(1)) \mu_{i} \frac{n-n_{i_{0}}}{a_{n_{i_{0}}}^{(i)}}=t_{i}+\mu_{i} \frac{n-n_{i_{0}}}{a_{n_{i_{0}}}^{(i)}}+o(1) \tag{7.2}
\end{equation*}
$$

uniformly for $\left|n-n_{i_{0}}\right| \leqslant \varepsilon^{-1} m_{i_{0}}$. We used that $\left(x_{i}-b_{n_{i_{0}}}^{(i)}\right) / a_{n_{i_{0}}}^{(i)} \rightarrow t_{i} \in \mathbb{R}$, cf. (3.5).

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* If $b_{n}^{(i)} \equiv 0$ (in particular if $i>i_{0}$ ), then more directly, using (3.5)

$$
\begin{equation*}
\frac{x_{i}-b_{n}^{(i)}}{a_{n}^{(i)}}=(1+o(1)) \frac{x_{i}}{a_{n_{i_{0}}}^{(i)}}=t_{i}+o(1) \tag{7.3}
\end{equation*}
$$

* The last case we need to consider is when $\alpha_{i}=1$. Then $b_{n}^{(i)}=n \mu_{i}\left(a_{n}^{(i)}\right)$ and we have

$$
\frac{1}{a_{n_{i_{0}}}^{(i)}}\left|b_{n}^{(i)}-b_{n_{i_{0}}}^{(i)}\right| \leqslant \frac{\left|n-n_{i_{0}}\right|}{a_{n_{i_{0}}}^{(i)}}\left|\mu_{i}\left(a_{n_{i_{0}}}^{(i)}\right)\right|+\frac{n}{a_{n_{i_{0}}}^{(i)}}\left|\mu_{i}\left(a_{n}^{(i)}\right)-\mu_{i}\left(a_{n_{i_{0}}}^{(i)}\right)\right| .
$$

The first term goes to 0 as $n_{i_{0}} \rightarrow+\infty$, uniformly for $\left|n-n_{i_{0}}\right| \leqslant \varepsilon^{-1} m_{i_{0}}$ : indeed, $m_{i_{0}}$ is regularly varying in $n_{i_{0}}$ with exponent $1 / \alpha_{i_{0}}<1$, in contrast with $a_{n_{i_{0}}}^{(i)}$ which is regularly varying with exponent 1 (and $\left|\mu_{i}\left(a_{n_{i_{0}}}^{(i)}\right)\right|$ is a slowly varying function). For the second term, we have $n \sim n_{i_{0}}$, and $n_{i_{0}} / a_{n_{i_{0}}}^{(i)} \sim L_{i}\left(a_{n_{i_{0}}}^{(i)}\right)^{-1}$ : we can use Claim 5.3 to get that the ratio $\left|\mu_{i}\left(a_{n}^{(i)}\right)-\mu_{i}\left(a_{n_{i_{0}}}^{(i)}\right)\right| / L_{i}\left(a_{n_{i_{0}}}^{(i)}\right)$ goes to 0 , since $a_{n}^{(i)} / a_{n_{i_{0}}}^{(i)}$ goes to 1 . We therefore obtain that uniformly for $\left|n-n_{i_{0}}\right| \leqslant \varepsilon^{-1} m_{i_{0}}$, using again (3.5),

$$
\begin{equation*}
\frac{x_{i}-b_{n}^{(i)}}{a_{n}^{(i)}}=\frac{x_{i}-b_{n_{i_{0}}}^{(i)}}{a_{n_{i_{0}}}^{\left(i_{0}\right)}}+o(1)=t_{i}+o(1) \tag{7.4}
\end{equation*}
$$

Combining all the possibles cases in (7.2)-(7.3)-(7.4), and recalling Section 2.4 (i.e. $a_{n_{i_{0}}}^{(i)} \sim a_{i, i_{0}} a_{n_{i_{0}}}^{\left(i_{0}\right)}$, we get that for all $i$, uniformly for $\left|n-n_{i_{0}}\right| \leqslant \varepsilon^{-1} m_{i_{0}}$,

$$
\begin{equation*}
\frac{x_{i}-b_{n}^{(i)}}{a_{n}^{(i)}}=t_{i}+\kappa_{i} \frac{n-n_{i_{0}}}{m_{i_{0}}}+o(1) \quad \text { with } \kappa_{i}=a_{i, i_{0}} \mu_{i} \mathbb{1}_{\left\{i \geqslant i_{0}\right\}} \tag{7.5}
\end{equation*}
$$

(recall $m_{i_{0}}=a_{n_{i_{0}}}^{\left(i_{0}\right)}$ and that if $\alpha_{i}<\alpha_{i_{0}}$ we have $a_{i, i_{0}}=0$ ). Because $g_{\alpha}$ is continuous, we obtain that

$$
\begin{align*}
\sum_{n=n_{i_{0}}-\varepsilon^{-1} m_{i_{0}}}^{n_{i_{0}}+\varepsilon^{-1} m_{i_{0}}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) & =\frac{1+o(1)}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \sum_{n=n_{i_{0}}-\varepsilon^{-1} m_{i_{0}}}^{n_{i_{0}}+\varepsilon^{-1} m_{i_{0}}} g_{\boldsymbol{\alpha}}\left(t_{1}+\kappa_{1} \frac{n-n_{i_{0}}}{m_{i_{0}}}, \ldots, t_{d}+\kappa_{d} \frac{n-n_{i_{0}}}{m_{i_{0}}}\right) \\
& =(1+o(1)) \frac{a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} g_{\boldsymbol{\alpha}}\left(t_{1}+\kappa_{1} u, \ldots, t_{d}+\kappa_{d} u\right) \mathrm{d} u . \tag{7.6}
\end{align*}
$$

The last identity holds thanks to a Riemann sum approximation, as $m_{i_{0}}=a_{n_{i_{0}}}^{\left(i_{0}\right)} \rightarrow+\infty$.

### 7.2 Last part in (7.1)

We prove that there are constants $C_{10}, C_{11}$ such that for every $r \geqslant C_{10} m_{i_{0}}$

$$
\begin{equation*}
\sum_{n=n_{i_{0}}+r}^{+\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C_{11} a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}}\left(\frac{r}{m_{i_{0}}} n_{i_{0}} \varphi_{i_{0}}(r) r^{-\gamma_{i_{0}}}+e^{-c\left(r / m_{i_{0}}\right)^{2}} \mathbb{1}_{\left\{\alpha_{i_{0}}=2\right\}}\right) . \tag{7.7}
\end{equation*}
$$

For $r_{\varepsilon}:=\varepsilon^{-1} m_{i_{0}}=\varepsilon^{-1} a_{n_{i_{0}}}^{\left(i_{0}\right)}$, we have $n_{i_{0}} \varphi_{i_{0}}\left(r_{\varepsilon}\right) r_{\varepsilon}^{-\gamma_{i_{0}}} \leqslant c \varepsilon^{\gamma_{i_{0}}}$ thanks to the definition (1.4) of $a_{n_{i_{0}}}^{\left(i_{0}\right)}$ : we therefore get that

$$
\begin{equation*}
\sum_{n=n_{i_{0}}+\varepsilon^{-1} m_{i_{0}}}^{+\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C_{11}^{\prime} a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}}\left(\varepsilon^{\gamma_{i_{0}}-1}+e^{-c \varepsilon^{-2}}\right) \tag{7.8}
\end{equation*}
$$

Let us now prove (7.7). For any $n \geqslant n_{i_{0}}+r$ with $r \geqslant C_{10} m_{i_{0}}$ and $C_{10}$ large enough, we have that $x_{i_{0}}-b_{n}^{\left(i_{0}\right)}=\left(n_{i_{0}}-n\right) \mu_{i_{0}} \leqslant-a_{n}^{\left(i_{0}\right)}$ : we can use Theorem 2.1 to get that

$$
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C}{a_{n}^{(1)} \cdots a_{n}^{(d)}}\left(n \varphi_{i_{0}}\left(n-n_{i_{0}}\right)\left(n-n_{i_{0}}\right)^{-\gamma_{i_{0}}}+e^{-c\left(n-n_{i_{0}}\right)^{2} /\left(a_{n}^{\left(i_{0}\right)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{i_{0}}=2\right\}}\right) .
$$

(We used that $a_{n}^{\left(i_{0}\right)} \leqslant c^{\prime} m_{i_{0}}$ for $n \geqslant n_{i_{0}}$.) We therefore get

$$
\begin{align*}
\sum_{n=n_{i_{0}}+r}^{+\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \sum_{n=n_{i_{0}}+r}^{+\infty} & \frac{C n}{a_{n}^{(1)} \cdots a_{n}^{(d)}} \varphi_{i_{0}}\left(n-n_{i_{0}}\right)\left(n-n_{i_{0}}\right)^{-\gamma_{i_{0}}} \\
& +\sum_{n=n_{i_{0}}+r}^{+\infty} \frac{C}{a_{n}^{(1)} \cdots a_{n}^{(d)}} e^{-c\left(n-n_{i_{0}}\right)^{2} /\left(a_{n}^{\left(i_{0}\right)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{i_{0}}=2\right\}} . \tag{7.9}
\end{align*}
$$

Let us deal with the first sum. If $r \leqslant n_{i_{0}}$, it is bounded above by a constant times

$$
\begin{align*}
\sum_{n=n_{i_{0}}+r}^{2 n_{i_{0}}} & \frac{n}{a_{n}^{(1)} \cdots a_{n}^{(d)}} \varphi_{i_{0}}\left(n-n_{i_{0}}\right)\left(n-n_{i_{0}}\right)^{-\gamma_{i_{0}}}+\sum_{n>2 n_{i_{0}}} \frac{n}{a_{n}^{(1)} \cdots a_{n}^{(d)}} \varphi_{i_{0}}(n) n^{-\gamma_{i_{0}}} \\
& \leqslant \frac{c n_{i_{0}}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \varphi_{i_{0}}(r) r^{1-\gamma_{i_{0}}}+c \varphi_{i_{0}}\left(n_{i_{0}}\right) \frac{n_{i_{0}}^{2-\gamma_{i_{0}}}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \leqslant \frac{c^{\prime} n_{i_{0}}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \varphi_{i_{0}}(r) r^{1-\gamma_{i_{0}}} \tag{7.10}
\end{align*}
$$

For the first sum we used that $\gamma_{i_{0}}>1$, and for the second sum that the sequence under summation is regularly varying with index $1-\gamma_{i_{0}}-\sum_{i=1}^{d} \alpha_{i_{0}}^{-1}<-1$. In the case $r \geqslant n_{i_{0}}$, the first term in (7.9) is bounded by a constant times

$$
c \varphi_{i_{0}}(r) \frac{r^{2-\gamma_{i_{0}}}}{a_{r}^{(1)} \cdots a_{r}^{(d)}} \leqslant \frac{c^{\prime} n_{i_{0}}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \varphi_{i_{0}}(r) r^{1-\gamma_{i_{0}}}
$$

For the second sum in (7.9) ( $\alpha_{i_{0}}=2$ ), we get in the case $r \leqslant n_{i_{0}}$ that it is bounded by

$$
\begin{aligned}
& \frac{C}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}}\left(\sum_{n=n_{i_{0}}+r}^{2 n_{i_{0}}} e^{-c\left(\left(n-n_{i_{0}}\right) / m_{i_{0}}\right)^{2}}+\sum_{n>2 n_{i_{0}}} e^{-c\left(n / a_{n}^{\left(i_{0}\right)}\right)^{2}}\right) \\
& \quad \leqslant \frac{C}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}}\left(m_{i_{0}} e^{-c\left(r / m_{i_{0}}\right)^{2}}+m_{i_{0}} e^{-c^{\prime}\left(n_{i_{0}} / m_{i_{0}}\right)^{2}}\right) \leqslant \frac{C a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} e^{-c\left(r / m_{i_{0}}\right)^{2}} .
\end{aligned}
$$

For $r \geqslant n_{i_{0}}$ the same bound holds with $a_{n_{i_{0}}}^{\left(i_{0}\right)} e^{-c\left(r / m_{i_{0}}\right)^{2}}$ replaced by $a_{r}^{\left(i_{0}\right)} e^{-c\left(r / a_{r}^{\left(i_{0}\right)}\right)^{2}}$, which is bounded by $a_{n_{i_{0}}}^{\left(i_{0}\right)} e^{-c\left(r / a_{r}^{\left(i_{0}\right)}\right)^{2}}$. This term is therefore negligible compared to (7.10) (in the case $r \geqslant n_{i_{0}}$ ). This concludes the proof of (7.7).

### 7.3 First part in (7.1)

We again split the sum into two parts:

$$
\begin{equation*}
\sum_{n=1}^{n_{i_{0}}-\varepsilon^{-1} m_{i_{0}}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)=\sum_{n=1}^{n_{i_{0}} / 2} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)+\sum_{n=n_{i_{0}} / 2+1}^{\varepsilon^{-1} m_{i_{0}}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \tag{7.11}
\end{equation*}
$$

The second part in (7.11) can be treated in the same manner as for (7.7): we have for any $C_{10} m_{i_{0}} \leqslant r \leqslant n_{i_{0}} / 2$

$$
\begin{equation*}
\sum_{n=n_{i_{0}} / 2+1}^{n_{i_{0}}-r} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}}\left(\frac{r}{m_{i_{0}}} n_{i_{0}} \varphi_{i_{0}}(r) r^{-\gamma_{i_{0}}}+e^{-c\left(r / m_{i_{0}}\right)^{2}} \mathbb{1}_{\left\{\alpha_{i_{0}}=2\right\}}\right) . \tag{7.12}
\end{equation*}
$$

Indeed, this comes from the same argument as for (7.9)-(7.10)-we are able to use Theorem 2.1 since $x_{i_{0}}-b_{n}^{\left(i_{0}\right)} \geqslant a_{n}^{\left(i_{0}\right)}$ for $n \leqslant n_{i_{0}}-C_{10} m_{i_{0}}$. Then, using (7.12) with $r_{\varepsilon}=\varepsilon^{-1} m_{i_{0}}$, we get as for (7.8) that

$$
\begin{equation*}
\sum_{n=n_{i_{0}} / 2+1}^{n_{i_{0}}-\varepsilon^{-1} m_{i_{0}}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}}\left(\varepsilon^{\gamma_{1}-1}+e^{-c \varepsilon^{-2}}\right) . \tag{7.13}
\end{equation*}
$$

It remains to control the first term in (7.11), and this is where we use one of our assumptions in Theorem 3.3.
(i) If $\sum_{i=1}^{d} \alpha_{i}^{-1}<2$ or (ii) $\gamma_{i_{0}}>\sum_{i \neq i_{0}} \alpha_{i}^{-1}$ We invoke Theorem 2.1: there is a constant $C$ such that uniformly for $n \leqslant n_{i_{0}} / 2$ (so $x_{i_{0}}-b_{n}^{\left(i_{0}\right)} \geqslant c n_{i_{0}} \geqslant a_{n}^{\left(i_{0}\right)}$ ) we have

$$
\begin{equation*}
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C}{a_{n}^{(1)} \cdots a_{n}^{(d)}}\left(n \varphi_{i_{0}}\left(n_{i_{0}}\right) n_{i_{0}}^{-\gamma_{i_{0}}}+e^{-c\left(n_{i_{0}} / a_{n}^{\left(i_{0}\right)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}\right) \tag{7.14}
\end{equation*}
$$

First of all, bounding below $a_{n}^{(i)}$ by a constant, we have

$$
\begin{equation*}
\sum_{n=1}^{n_{i_{0}} / 2} \frac{1}{a_{n}^{(1)} \cdots a_{n}^{(d)}} e^{-c\left(n_{i_{0}} / a_{n}^{\left(i_{0}\right)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}} \leqslant C n_{i_{0}} e^{-c\left(n_{i_{0}} / m_{i_{0}}\right)^{2}}=o(1) \frac{a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}}, \tag{7.15}
\end{equation*}
$$

the last identity being valid since $e^{-c\left(n_{i_{0}} / m_{i_{0}}\right)^{2}}$ decays faster than any power of $n_{i_{0}}$.
$*$ If $\sum_{i=1}^{d} \alpha_{i}^{-1}<2$ we get that $n /\left(a_{n}^{(1)} \cdots a_{n}^{(d)}\right)$ is regularly varying with exponent larger than -1 , so that

$$
\begin{equation*}
\sum_{n=1}^{n_{i_{0}} / 2} \frac{C}{a_{n}^{(1)} \cdots a_{n}^{(d)}} n \varphi_{i_{0}}\left(n_{i_{0}}\right) n_{i_{0}}^{-\gamma_{i_{0}}} \leqslant \frac{C n_{i_{0}}^{2-\gamma_{i_{0}}} \varphi_{i_{0}}\left(n_{i_{0}}\right)}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}}=o(1) \frac{a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} . \tag{7.16}
\end{equation*}
$$

To obtain the $o(1)$, we used that $\varphi_{i_{0}}\left(n_{i_{0}}\right) n_{i_{0}}^{2-\gamma_{i_{0}}} / a_{n_{i_{0}}}^{\left(i_{0}\right)}$ is regularly varying with exponent $2-\gamma_{i_{0}}-1 / \alpha_{i_{0}}<0$ (recall that $\gamma_{i_{0}} \geqslant \alpha_{i_{0}}>1$ ).
$*$ In the case where $\sum_{i=1}^{d} \alpha_{i}^{-1} \geqslant 2$ with $\gamma_{i_{0}}>\sum_{i \neq i_{0}} \alpha_{i}^{-1}$, then

$$
\begin{equation*}
\sum_{n=1}^{n_{i_{0}} / 2} \frac{n}{a_{n}^{(1)} \cdots a_{n}^{(d)}} n \varphi_{i_{0}}\left(n_{i_{0}}\right) n_{i_{0}}^{-\gamma_{i_{0}}} \leqslant \tilde{\varphi}\left(n_{i_{0}}\right) n_{i_{0}}^{-\gamma_{i_{0}}}=o(1) \frac{a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \tag{7.17}
\end{equation*}
$$

since the sum grows like a slowly varying function $\tilde{\varphi}$ (or remains bounded). The $o(1)$ comes from the fact that $a_{n_{i_{0}}}^{\left(i_{0}\right)} /\left(a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}\right)$ is regularly varying with exponent larger than $-\gamma_{i_{0}}$.
(iii) If Assumption 2.2 holds For $n \leqslant n_{i_{0}} / 2$, we have $x_{i_{0}}-b_{n}^{\left(i_{0}\right)} \geqslant c n_{i_{0}} \geqslant a_{n}^{\left(i_{0}\right)}$ : Theorem 2.5 gives that

$$
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{c}{\prod_{i=1}^{d}\left(\left|x_{i}-b_{n}^{(i)}\right| \vee a_{n}^{(i)}\right)}\left(n \varphi\left(n_{i_{0}}\right) n_{i_{0}}^{-\gamma_{i_{0}}}+e^{-c n_{i_{0}} / a_{n}^{\left(i_{0}\right)}} \mathbb{1}_{\left\{\alpha_{i_{0}}=2\right\}}\right) .
$$

We notice that if $b_{n}^{(i)} \equiv 0$, then since we assumed that $t_{i} \neq 0$, we have that $\left|x_{i}\right| \geqslant c a_{n_{i_{0}}}^{(i)}$ for all $n \leqslant n_{i_{0}} / 2$, provided $n_{i_{0}}$ is large. Otherwise, we write $\left|x_{i}-b_{n}^{(i)}\right| \geqslant\left|b_{n_{i_{0}}}^{(i)}-b_{n}^{(i)}\right|-$ $\left|x_{i}-b_{n_{i_{0}}}^{(i)}\right|$. Since we work along the favorite direction we have $\left|x_{i}-b_{n_{i_{0}}}^{(i)}\right| \leqslant c a_{n_{i_{0}}}^{(i)}$. Also, using that $b_{n}^{(i)}$ is regularly varying with exponent 1 we have that $\left|b_{n_{i_{0}}}^{(i)}-b_{n}^{(i)}\right| \geqslant c\left|b_{n_{i_{0}}}^{(i)}\right|$
uniformly for $n \leqslant n_{i_{0}} / 2$. Therefore, using that $\left|b_{n}^{(i)}\right| / a_{n}^{(i)} \rightarrow+\infty$, for $n \leqslant n_{i_{0}} / 2$ we have $\left|x_{i}-b_{n}^{(i)}\right| \geqslant c^{\prime} b_{n_{i_{0}}}^{\left(i_{0}\right)} \geqslant a_{n_{i_{0}}}^{\left(i_{0}\right)}$ provided that $n_{i_{0}}$ is large enough. All together, recalling also that $\left|x_{i_{0}}-b_{n_{i_{0}}}^{\left(i_{0}\right)}\right| \geqslant c n_{i_{0}}$, we get that

$$
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{c^{\prime} a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} n \varphi\left(n_{i_{0}}\right) n_{i_{0}}^{-\left(1+\gamma_{i_{0}}\right)}+\frac{c^{\prime}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} e^{-c\left(n_{i_{0}} / m_{i_{0}}\right)^{2}} \mathbb{1}_{\left\{\alpha_{i_{0}}=2\right\}} .
$$

Summing the second term over $n \leqslant n_{i_{0}} / 2$, (7.15) already gives that it is negligible compared to (7.6). For the other term, we have

$$
\begin{equation*}
\sum_{n=1}^{n_{i_{0}} / 2} \frac{a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} n \varphi\left(n_{i_{0}}\right) n_{i_{0}}^{-\left(1+\gamma_{i_{0}}\right)} \leqslant \frac{C a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} n_{i_{0}}^{1-\gamma_{i_{0}}} \varphi\left(n_{i_{0}}\right)=o(1) \frac{a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}}, \tag{7.18}
\end{equation*}
$$

the last identity holding since $\gamma_{i_{0}}>1$.
As a conclusion, we obtain in all cases (i)-(ii)-(iii) that

$$
\begin{equation*}
\sum_{n=1}^{n_{i_{0}} / 2} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)=o(1) \frac{a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \tag{7.19}
\end{equation*}
$$

## Conclusion

Combining (7.6) with (7.8), (7.13) and (7.19), then letting $n \rightarrow+\infty$ and finally $\varepsilon \downarrow 0$, we get the conclusion (3.6).

## 8 Proof for case III: $\alpha_{i_{0}}=1$

In this section, we prove Theorem 3.4.

### 8.1 Preliminaries

Recall that $b_{n}^{(i)} \equiv 0$ for all $i<i_{0}$, and that $\alpha_{i_{0}}=1$ with $b_{n}^{\left(i_{0}\right)} \not \equiv 0$ (and $\left|b_{n}^{\left(i_{0}\right)}\right| / a_{n}^{\left(i_{0}\right)} \rightarrow+\infty$ ). Recall that $n_{i_{0}}$ is such that $b_{n_{i_{0}}}^{\left(i_{0}\right)}=x_{i_{0}}\left(b_{n_{i_{0}}}^{\left(i_{0}\right)}\right.$ and $x_{i_{0}}$ need to have the same sign). First, we stress that $\mu_{i_{0}}\left(a_{n}^{\left(i_{0}\right)}\right) \sim \mu_{i_{0}}\left(\left|b_{n}^{\left(i_{0}\right)}\right|\right)$ (this is trivial if $\mu_{i_{0}} \in \mathbb{R}^{*}$, and we refer to Lemma 4.3 in [3] in the case $\left|\mu_{i_{0}}\right|=+\infty$ or 0 with $\left.p_{i_{0}} \neq q_{i_{0}}\right)$ : we get that $\mu_{i_{0}}\left(a_{n}^{\left(i_{0}\right)}\right) \sim \mu_{i_{0}}\left(\left|x_{i_{0}}\right|\right)$, and hence $x_{i_{0}}=b_{n_{i_{0}}}^{\left(i_{0}\right)} \sim n_{i_{0}} \mu_{i_{0}}\left(\left|x_{i_{0}}\right|\right)$. We therefore conclude that $n_{i_{0}} \sim x_{i_{0}} / \mu_{i_{0}}\left(\left|x_{i_{0}}\right|\right)$ as $\left|x_{i_{0}}\right| \rightarrow+\infty$ (provided that $x_{i_{0}}$ and $\mu_{i_{0}}\left(\left|x_{i_{0}}\right|\right)$ have the same sign).

We also define

$$
\begin{equation*}
m_{i_{0}}:=\frac{a_{n_{0}}^{\left(i_{0}\right)}}{\left|\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\right|} \sim n_{i_{0}} \frac{L_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)}{\left|\mu_{i_{0}}\left(a_{\left.n_{i_{0}}\right)}^{\left(i_{0}\right)}\right)\right|}=o\left(n_{i_{0}}\right) . \tag{8.1}
\end{equation*}
$$

We used the definition (1.4) of $a_{n}^{\left(i_{0}\right)}$ for the asymptotic equivalence, and then that $L_{i_{0}}(x)=o\left(\mu_{i_{0}}(x)\right)$ (both if $\mu_{i_{0}} \in \mathbb{R}^{*}$ or $p_{i_{0}}=q_{i_{0}}$, thanks to [6, Prop. 1.5.9.a]). We stress that the typical number of steps for the $i_{0}$-th coordinate to reach $x_{i_{0}}$ is $n_{i_{0}}+O\left(m_{i_{0}}\right)$. The intuition is that, when looking for which $n$ we have that $x_{i_{0}}-b_{n}^{\left(i_{0}\right)}\left(=b_{n_{i_{0}}}^{\left(i_{0}\right)}-b_{n}^{\left(i_{0}\right)}\right)$ is of order $a_{n}^{\left(i_{0}\right)} \sim a_{n_{i_{0}}}^{\left(i_{0}\right)}$, and using that $\left|b_{n}-b_{n_{i_{0}}}\right|$ is roughly of the order of $\left(n-n_{i_{0}}\right)\left|\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\right|$ (see (8.4) below for details) we find that $n-n_{i_{0}}$ has to be of the order of $a_{n_{i_{0}}}^{\left(i_{0}\right)} /\left|\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\right|=: m_{i_{0}}$. This intuition is confirmed in [2, 21], where it is shown that $\left(N\left(x_{i_{0}}\right)-n_{i_{0}}\right) / m_{i_{0}}$ converges in distribution as $\left|x_{i_{0}}\right| \rightarrow+\infty$, where $N\left(x_{i_{0}}\right)=\inf \left\{n, S_{n}^{\left(i_{0}\right)}>x_{i_{0}}\right\}$ is the first passage time to $x_{i_{0}}\left(\right.$ if $\left.x_{i_{0}}>0\right)$.

Then we fix $\varepsilon>0$, and again split $G(\mathbf{x})$ into three parts:

$$
\begin{equation*}
G(\mathbf{x})=\left(\sum_{n<n_{i_{0}}-\varepsilon^{-1} m_{i_{0}}}+\sum_{n=n_{i_{0}}-\varepsilon^{-1} m_{i_{0}}}^{n_{i_{0}}+\varepsilon^{-1} m_{i_{0}}}+\sum_{n>n_{i_{0}}+\varepsilon^{-1} m_{i_{0}}}\right) \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) . \tag{8.2}
\end{equation*}
$$

As suggested above, the main contribution is the middle sum. In the following, we work with $x_{i_{0}}$ and $\mu_{i_{0}}\left(x_{i_{0}}\right)$ positive, simply to avoid the use of absolute values.

### 8.2 Main contribution

For the middle sum in (8.2), the fact that we work along the favorite scaling tells that we can use the local limit theorem (2.1), and get that

$$
\begin{equation*}
\sum_{n=n_{i_{0}}-\varepsilon^{-1} m_{i_{0}}}^{n_{i_{0}}+\varepsilon^{-1} m_{i_{0}}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)=\sum_{n=n_{i_{0}}-\varepsilon^{-1} m_{i_{0}}}^{n_{i_{0}}+\varepsilon^{-1} m_{i_{0}}} \frac{1+o(1)}{a_{n}^{(1)} \cdots a_{n}^{(d)}} g_{\boldsymbol{\alpha}}\left(\frac{x_{i_{0}}-b_{n}^{(1)}}{a_{n}^{(1)}}, \ldots, \frac{x_{d}-b_{n}^{(d)}}{a_{n}^{(d)}}\right) . \tag{8.3}
\end{equation*}
$$

Note that, for the range of $n$ considered, we have that $n=(1+o(1)) n_{i_{0}}$ (recall (8.1)), so that $a_{n}^{(i)}=(1+o(1)) a_{n_{i_{0}}}^{(i)}$.

* if $i>i_{0}$ or $\alpha_{i}<1$ then $b_{n}^{(i)} \equiv 0$ : thanks to our assumption (3.9), we get that

$$
\frac{x_{i}}{a_{n_{i_{0}}}^{(i)}}=(1+o(1)) t_{i}
$$

* if $\alpha_{i}=1$, then we get that for $\left|n-n_{i_{0}}\right| \leqslant \varepsilon^{-1} m_{i_{0}}$

$$
\begin{equation*}
\frac{x_{i}-b_{n}^{(i)}}{a_{n_{i_{0}}}^{(i)}}=\frac{x_{i}-b_{n_{i_{0}}}^{(i)}}{a_{n_{i_{0}}}^{(i)}}+\frac{\left(n_{i_{0}}-n\right) \mu_{i}\left(a_{n_{i_{0}}}^{(i)}\right)}{a_{n_{i_{0}}}^{(i)}}+\frac{n_{i_{0}}}{a_{n_{i_{0}}}^{(i)}}\left(\mu_{i}\left(a_{n_{i_{0}}}^{(i)}\right)-\mu_{i}\left(a_{n}^{(i)}\right)\right) . \tag{8.4}
\end{equation*}
$$

The first part goes to $t_{i} \in \mathbb{R}$, thanks to (3.9). The last part goes to 0 thanks to Claim 5.3 (recall $n_{i_{0}} \sim a_{n_{i_{0}}}^{(i)} L_{i}\left(a_{n_{i_{0}}}^{(i)}\right)^{-1}$ ), since $a_{n}^{(i)} / a_{n_{i_{0}}}^{(i)}$ goes to 1 . For the middle part, we use assumption (3.8) to get that $\mu_{i}\left(a_{n_{i_{0}}}^{(i)}\right) / a_{n_{i_{0}}}^{(i)} \sim \tilde{a}_{i, i_{0}} m_{i_{0}}$. In the end, we obtain that uniformly for $\left|n-n_{i_{0}}\right| \leqslant \varepsilon^{-1} m_{i_{0}}$

$$
\begin{equation*}
\frac{x_{i}-b_{n}^{(i)}}{a_{n_{i_{0}}}^{(i)}}=t_{i}+\frac{\tilde{a}_{i, i_{0}}}{m_{i_{0}}}\left(n_{i_{0}}-n\right)+o(1) \tag{8.5}
\end{equation*}
$$

Using this in the sum (8.3), and with the definition $\tilde{\kappa}_{i}=\tilde{a}_{i, i_{0}} \mathbb{1}_{\left\{i \geqslant i_{0}\right\}}$ (with $\tilde{a}_{i, i_{0}}=0$ if $\alpha_{i}<1$ ), we get thanks to the continuity of $g_{\boldsymbol{\alpha}}$ that the right-hand side of (8.3) is

$$
\begin{align*}
\frac{1+o(1)}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} & \sum_{n=n_{i_{0}}-\varepsilon^{-1} m_{i_{0}}}^{n_{i_{0}}+\varepsilon^{-1} m_{i_{0}}} g_{\boldsymbol{\alpha}}\left(t_{1}+\frac{\tilde{\kappa}_{1}}{m_{i_{0}}}\left(n_{i_{0}}-n\right), \ldots, t_{d}+\frac{\tilde{\kappa}_{d}}{m_{i_{0}}}\left(n_{i_{0}}-n\right)\right) \\
& =(1+o(1)) \frac{m_{i_{0}}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} g_{\boldsymbol{\alpha}}\left(t_{1}+\tilde{\kappa}_{1} u, \ldots, t_{d}+\tilde{\kappa}_{d} u\right) \mathrm{d} u \tag{8.6}
\end{align*}
$$

where we used a Riemann-sum approximation to obtain the last integral.

### 8.3 Third term in (8.2)

First of all, let us stress that there is a constant $C_{12}$ such that for $2 n_{i_{0}} \geqslant n \geqslant n_{i_{0}}+$ $C_{12} m_{i_{0}}$, we have

$$
\begin{align*}
x_{i_{0}}-b_{n}^{\left(i_{0}\right)} & =\left(n_{i_{0}}-n\right) \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)+n\left(\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)-\mu_{i_{0}}\left(a_{n}^{\left(i_{0}\right)}\right)\right) \\
& \leqslant c\left(n_{i_{0}}-n\right) \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right) \leqslant-a_{n}^{\left(i_{0}\right)} . \tag{8.7}
\end{align*}
$$

We used that $\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)-\mu_{i_{0}}\left(a_{n}^{\left(i_{0}\right)}\right)=O\left(L_{i_{0}}\left(a_{n}^{\left(i_{0}\right)}\right)\right)$ for $n_{i_{0}} \leqslant n \leqslant 2 n_{i_{0}}$ (see Claim 5.3), and the fact that $n_{i_{0}} L_{i_{0}}\left(a_{n}^{\left(i_{0}\right)}\right) \leqslant c a_{n_{i_{0}}}^{\left(i_{0}\right)}$. For $n \geqslant 2 n_{i_{0}}$, we have that $x_{i_{0}}-b_{n}^{\left(i_{0}\right)} \leqslant-c b_{n_{i_{0}}}^{\left(i_{0}\right)} \leqslant-a_{n_{i_{0}}}^{\left(i_{0}\right)}$ : the first inequality comes from the fact that $b_{n}^{\left(i_{0}\right)}$ is regularly varying with exponent 1 , the second from the fact that $b_{n}^{\left(i_{0}\right)} / a_{n}^{\left(i_{0}\right)} \rightarrow+\infty$.

Therefore, we can use Theorem 2.5 (recall $\alpha_{i_{0}}=1$ ) to get that for all $n \geqslant n_{i_{0}}+C_{12} m_{i_{0}}$,

$$
\begin{equation*}
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C a_{n}^{\left(i_{0}\right)}}{a_{n}^{(1)} \cdots a_{n}^{(d)}} n L_{i_{0}}\left(\left|x_{i_{0}}-b_{n}^{\left(i_{0}\right)}\right|\right)\left|x_{i_{0}}-b_{n}^{\left(i_{0}\right)}\right|^{-2} . \tag{8.8}
\end{equation*}
$$

Then, for any $n_{i_{0}} \geqslant r \geqslant C_{12} m_{i_{0}}$, setting $j=n-n_{i_{0}}$ so $\left|x_{i_{0}}-b_{n}^{\left(i_{0}\right)}\right| \geqslant c j \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)$ (see (8.7)),

$$
\begin{align*}
\sum_{n=n_{i_{0}}+r}^{2 n_{i_{0}}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) & \leqslant \frac{C a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \sum_{j=r}^{2 n_{i_{0}}} n_{i_{0}} L_{i_{0}}\left(j \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\right)\left(j \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\right)^{-2} \\
& \leqslant \frac{C^{\prime} a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \times n_{i_{0}} \frac{1}{\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)} L_{i_{0}}\left(r \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\right)\left(r \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\right)^{-1} . \tag{8.9}
\end{align*}
$$

Then, setting $r=\varepsilon^{-1} m_{i_{0}}$, and using the definition (1.4) of $a_{n}^{\left(i_{0}\right)}$, we get that

$$
\begin{equation*}
\sum_{n=n_{i_{0}}+\varepsilon^{-1} m_{i_{0}}}^{2 n_{i_{0}}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C^{\prime \prime} \varepsilon m_{i_{0}}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} . \tag{8.10}
\end{equation*}
$$

For the sum with $n \geqslant 2 n_{i_{0}}$, we use (8.8) with the fact that $x_{i_{0}}-b_{n}^{\left(i_{0}\right)} \leqslant-c b_{n}^{\left(i_{0}\right)}$ :

$$
\begin{align*}
\sum_{n=2 n_{i_{0}}}^{+\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) & \leqslant \sum_{n=2 n_{i_{0}}}^{+\infty} \frac{C a_{n}^{\left(i_{0}\right)}}{a_{n}^{(1)} \cdots a_{n}^{(d)}} \times n L_{i_{0}}\left(b_{n}^{\left(i_{0}\right)}\right)\left(b_{n}^{\left(i_{0}\right)}\right)^{-2} \\
& \leqslant \frac{C^{\prime} a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \times n_{i_{0}}^{2} L_{i_{0}}\left(b_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\left(b_{n_{i_{0}}}^{\left(i_{0}\right)}\right)^{-2}=\frac{o(1) m_{i_{0}}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} . \tag{8.11}
\end{align*}
$$

For the summation, we used the sequence under summation is regularly varying with exponent smaller than -1 . For the last $o(1)$, we used that $b_{n_{i_{0}}}^{\left(i_{0}\right)} \sim n_{i_{0}} \mu_{i_{0}}\left(a_{n_{0}}^{\left(i_{0}\right)}\right) \sim n_{i_{0}} \mu_{i_{0}}\left(b_{n_{i_{0}}}^{\left(i_{0}\right)}\right)$, and that $L_{i_{0}}\left(b_{n_{i_{0}}}^{\left(i_{0}\right)}\right) / \mu_{i_{0}}\left(b_{n_{i_{0}}}^{\left(i_{0}\right)}\right) \rightarrow 0$.

### 8.4 First term in (8.2)

As in (8.7), we have $x_{i_{0}}-b_{n}^{\left(i_{0}\right)} \geqslant a_{n}^{\left(i_{0}\right)}$ for $n_{i_{0}} / 2 \leqslant n \leqslant n_{i_{0}}-C_{12} m_{i_{0}}$. Hence Theorem 2.5 gives the same bound as (8.8): for any $r \geqslant C_{12} m_{i_{0}}$, setting $j=n_{i_{0}}-n\left(x_{i_{0}}-b_{n}^{\left(i_{0}\right)} \leqslant-\right.$ $c j \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)$, we have as in (8.9)

$$
\begin{align*}
\sum_{n=n_{i_{0}} / 2}^{n_{i_{0}}-r} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) & \leqslant \frac{C a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \sum_{j=r}^{n_{i_{0}} / 2} n_{i_{0}} L_{i_{0}}\left(j \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\right)\left(j \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)^{-2}\right. \\
& \leqslant \frac{C m_{i_{0}}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \times n_{i_{0}} L_{i_{0}}\left(r \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\right)\left(r \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\right)^{-1} \tag{8.12}
\end{align*}
$$

With $r=\varepsilon^{-1} m_{i_{0}}$, we obtain the same upper bound as in (8.10).
For the term $n \leqslant n_{i_{0}} / 2$, we use that $x_{i_{0}}-b_{n}^{\left(i_{0}\right)} \geqslant c b_{n_{i_{0}}}^{\left(i_{0}\right)}$ provided that $n_{i_{0}}$ is large enough. Since we work along the favorite direction (3.9), we have that $\left|x_{i}-b_{n}^{(i)}\right| \geqslant c a_{n_{i_{0}}}^{(i)}$
for all $n \leqslant n_{i_{0}} / 2$ (recall we assumed $t_{i} \neq 0$ if $b_{n}^{(i)} \equiv 0$ ): Theorem 2.5 gives that

$$
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} n L_{i_{0}}\left(b_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\left(b_{n_{i_{0}}}^{\left(i_{0}\right)}\right)^{-2}
$$

Hence, similarly to (8.11), we get that

$$
\begin{equation*}
\sum_{n=1}^{n_{i_{0}} / 2} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C^{\prime} a_{n_{i_{0}}}^{\left(i_{0}\right)}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \times n_{i_{0}}^{2} L_{i_{0}}\left(b_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\left(b_{n_{i_{0}}}^{\left(i_{0}\right)}\right)^{-1}=\frac{o(1) m_{i_{0}}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \tag{8.13}
\end{equation*}
$$

## Conclusion

As for the previous sections, collecting (8.6) together with (8.10)-(8.11) and (8.12)(8.13), then letting $n \rightarrow+\infty$ and finally $\varepsilon \downarrow 0$, we get the conclusion (3.10).

## 9 Proofs when $x$ is away from the favorite direction or scaling

In this section, we prove the renewal estimates when away from the favorite direction or scaling, i.e. we prove Theorems 4.1 (in Section 9.1), 4.2 (in Section 9.2), and 4.3 (in Section 9.3). Again, let us work with all $x_{i}$ 's positive in this section, to avoid the use of absolute values. Recall also that we work in dimension $d=2$ with $\boldsymbol{\alpha} \neq(2,2)$ and under Assumption 2.2.

### 9.1 Case I, proof of Theorem 4.1

Recall that $n_{i}$ is defined up to asymptotic equivalence by $a_{n_{i}}^{(i)} \sim x_{i}$, and $i_{0}, i_{1}$ are such that $n_{i_{0}}=\min \left\{n_{1}, n_{2}\right\}, n_{i_{1}}=\max \left\{n_{1}, n_{2}\right\}$. In such a way, we have that $x_{i} / a_{n_{i_{0}}}^{(i)} \geqslant c$ for $i=1,2$. Let us work in the case where $x_{i_{1}} / a_{n_{i_{0}}}^{\left(i_{1}\right)} \geqslant C_{13}$ for some large constant $C_{13}$ (otherwise one falls in the favorite scaling (3.2)): it is equivalent to having $n_{i_{1}} / n_{i_{0}}$ larger than some large constant $C_{13}^{\prime}$. We let $n_{i_{0}} \leqslant m \leqslant n_{i_{1}}$ (we optimize its value below), and decompose $G(\mathbf{x})$ into two parts

$$
\begin{equation*}
G(\mathbf{x})=\sum_{n=1}^{m} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)+\sum_{n=m+1}^{\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \tag{9.1}
\end{equation*}
$$

For the first part, since $x_{i_{1}} \geqslant c a_{n}^{\left(i_{1}\right)}$ for $n \leqslant m \leqslant n_{i_{1}}$, Theorem 2.5 gives us that

$$
\begin{aligned}
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) & \leqslant \frac{C}{\left(x_{1} \vee a_{n}^{(1)}\right)\left(x_{2} \vee a_{n}^{(2)}\right)}\left(n \varphi_{i_{1}}\left(x_{i_{1}}\right) x_{i_{1}}^{-\gamma_{i_{1}}}+e^{-c\left(x_{i_{1}} / a_{n}^{\left(i_{1}\right)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{i_{1}}=2\right\}}\right) \\
& \leqslant \frac{C}{a_{n_{i_{0}}}^{\left(i_{0}\right)} x_{i_{1}}}\left(\frac{n}{n_{i_{1}}}+e^{-c\left(a_{n_{i_{1}}}^{\left(i_{1}\right)} / a_{n}^{\left(i_{1}\right)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{i_{1}}=2\right\}}\right) \leqslant \frac{C}{a_{n_{i_{0}}}^{(1)} a_{n_{i_{0}}}^{(2)}} \cdot \frac{a_{n_{i_{0}}}^{\left(i_{1}\right)}}{a_{n_{i_{1}}}^{\left(i_{1}\right)}} \frac{n}{n_{i_{1}}} .
\end{aligned}
$$

For the second inequality, we used that $x_{i_{0}} \vee a_{n}^{\left(i_{0}\right)} \geqslant a_{n_{i_{0}}}^{\left(i_{0}\right)}$, that $x_{i_{1}} \geqslant a_{n}^{\left(i_{1}\right)}$ for all $n \leqslant n_{i_{0}}$, and also that $x_{i_{1}} \sim a_{n_{i_{1}}}^{\left(i_{1}\right)}$, so that in particular $\varphi_{i_{1}}\left(x_{i_{1}}\right) x_{i_{1}}^{-\gamma_{i_{1}}} \leqslant c / n_{i_{1}}$ by definition of $a_{n_{i_{1}}}^{\left(i_{1}\right)}$. Therefore, we get that

$$
\begin{equation*}
\sum_{n=1}^{m} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C^{\prime} n_{i_{0}}}{a_{n_{i_{0}}}^{(1)} a_{n_{i_{0}}}^{(2)}} \times \frac{a_{n_{i_{0}}}^{\left(i_{1}\right)}}{a_{n_{i_{1}}}^{\left(i_{1}\right)}} \frac{m}{n_{i_{1}}} \frac{m}{n_{i_{0}}} \tag{9.2}
\end{equation*}
$$

For the second part in (9.1), we fix some $\delta>0$ (small), and we use the local limit
theorem (2.1) to get that (using that $\alpha_{1}^{-1}+\alpha_{2}^{-1}>1$ )

$$
\begin{align*}
\sum_{n=m+1}^{+\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) & \leqslant \sum_{n=m+1}^{+\infty} \frac{C}{a_{n}^{(1)} a_{n}^{(2)}} \leqslant \frac{c m}{a_{m}^{(1)} a_{m}^{(2)}} \\
& \leqslant \frac{c_{\delta} n_{i_{0}}}{a_{n_{i_{0}}}^{(1)} a_{n_{i_{0}}}^{(2)}} \times\left(\frac{m}{n_{i_{0}}}\right)^{-\kappa_{\delta}} \tag{9.3}
\end{align*}
$$

with $\kappa_{\delta}=\alpha_{1}^{-1}+\alpha_{2}^{-1}-1-\delta>0$. We used in the last inequality that $n /\left(a_{n}^{(1)} a_{n}^{(2)}\right)$ is regularly varying with exponent $1-\alpha_{1}^{-1}-\alpha_{2}^{-1}$, together with Potter's bound.

Then, it remains to optimize our choice of $m$ : combining (9.3) with (9.2), and using that there is a constant $c_{\delta}$ such that $a_{n_{i_{0}}}^{\left(i_{1}\right)} / a_{n_{i_{1}}}^{\left(i_{1}\right)} \leqslant c_{\delta}\left(n_{i_{1}} / n_{i_{0}}\right)^{-1 / \alpha_{i_{1}}+\delta}$, we get that

$$
\begin{align*}
G(\mathbf{x}) & \leqslant \frac{C n_{i_{0}}}{a_{n_{i_{0}}}^{(1)} a_{n_{i_{0}}}^{(2)}}\left(\left(\frac{n_{i_{1}}}{n_{i_{0}}}\right)^{-1-\frac{1}{\alpha_{i_{1}}}+\delta}\left(\frac{m}{n_{i_{0}}}\right)^{2}+\left(\frac{m}{n_{i_{0}}}\right)^{-\kappa \delta}\right)  \tag{9.4}\\
& \leqslant \frac{C n_{i_{0}}}{a_{n_{i_{0}}}^{(1)} \cdots a_{n_{i_{0}}}^{(d)}} \times\left(\frac{n_{i_{1}}}{n_{i_{0}}}\right)^{-\kappa_{\delta}\left(1+1 / \alpha_{i}+\delta\right) /\left(2+\kappa_{\delta}\right)} .
\end{align*}
$$

For the last inequality, we optimized in $m$, by choosing $m / n_{i_{0}}=\left(n_{i_{1}} / n_{i_{0}}\right)^{\left(1+1 / \alpha_{i_{1}}-\delta\right) /\left(2+\kappa_{\delta}\right)}$. This gives the first part of the statement, i.e. (4.1).

In the case where $\left(\mathbf{S}_{n}\right)_{n} \geqslant 0$ is a renewal process, i.e. $\mathbf{X}_{1} \geqslant 0$, (in particular $\alpha_{i} \in(0,1)$ ), then one has a much sharper estimate than (9.3). Indeed, we have the following large deviation result: for all $n \geqslant C_{13}^{\prime} n_{1}$ (so that $x_{i_{0}} \leqslant a_{n}^{\left(i_{0}\right)} / 4$ ),

$$
\begin{equation*}
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C}{a_{n}^{(1)} a_{n}^{(2)}} \mathbf{P}\left(S_{\lfloor n / 2\rfloor}^{\left(i_{0}\right)} \leqslant x_{i_{0}} / 2\right) \leqslant \frac{C}{a_{n}^{(1)} a_{n}^{(2)}} e^{-c\left(x_{i_{0}} / a_{n}^{\left(i_{0}\right)}\right)^{-\zeta_{i_{0}}}} \tag{9.5}
\end{equation*}
$$

for some exponent $\zeta_{i_{0}}$ (whose value depends on $\alpha_{i_{0}}$ ). The first inequality follows from the same argument as for (5.3)-(5.4), we leave the details to the reader. The second inequality is a large deviation result, see for instance [1, Lemmas A.3]. As a consequence, we have the analogous bound as in (9.3),

$$
\sum_{n=m}^{+\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{c_{\delta} n_{i_{0}}}{a_{n_{i_{0}}}^{(1)} a_{n_{i_{0}}}^{(2)}} e^{-c\left(a_{n_{i_{0}}}^{\left(i_{0}\right)} / a_{m}^{\left(i_{0}\right)}\right)^{-\zeta_{i_{0}}}}
$$

The last part decays faster than any power of $m / n_{i_{0}}$, so that (9.4) holds with $\kappa_{\delta}$ replaced by an arbitrarily large constant. This gives the second statement of Theorem 4.1.

### 9.2 Case II-III (a), proof of Theorem 4.2

Here, we assume $\alpha_{1}, \alpha_{2} \geqslant 1$ and for $i=1,2$ : either $\mu_{i} \in \mathbb{R}^{*}$, or $\alpha_{i}=1$ with $p_{i} \neq q_{i}$. Recall that $n_{i}$ is such that $b_{n_{i}}^{(i)}=x_{i}$ (we work in the case where they are both positive), and $m_{i}=a_{n_{i}}^{(i)} / \mu_{i}\left(a_{n_{i}}^{(i)}\right)$ (one can replace $\mu_{i}\left(a_{n_{i}}^{(i)}\right)$ by $\mu_{i}$ if $\alpha_{i}>1$ ). Note that in any case we have $m_{i}=o\left(n_{i}\right)$ as $n_{i} \rightarrow+\infty$.

### 9.2.1 A preliminary estimate

We prove first the following result, that will be useful in this section and the next one: for $i=1,2$

$$
\begin{align*}
\text { if } m \in\left[m_{i}, n_{i}\right], & \sum_{n=n_{i}-m}^{n_{i}+m} \frac{1}{a_{n}^{(i)} \vee\left|x_{i}-b_{n}^{(i)}\right|} \leqslant \frac{C}{\mu_{i}\left(a_{n_{i}}^{(i)}\right)} \log \left(1+\frac{m}{m_{i}}\right) ; \\
\text { if } m \geqslant 2 n_{i}, & \sum_{n=1}^{m} \frac{1}{a_{n}^{(i)} \vee\left|x_{i}-b_{n}^{(i)}\right|} \leqslant \frac{C}{\mu_{i}\left(a_{n_{i}}^{(i)}\right) \wedge \mu_{i}\left(a_{m}^{(i)}\right)} \log m . \tag{9.6}
\end{align*}
$$

To show this, recall that $\left|x_{i}-b_{n}^{(i)}\right| \leqslant c a_{n_{i}}^{(i)}$ for all $n$ between $n_{i}-m_{i}$ and $n_{i}+m_{i}$ (see (7.5) and (8.5)). Therefore, we have that

$$
\sum_{n=n_{i}-m_{i}}^{n_{i}+m_{i}} \frac{1}{a_{n}^{(i)} \vee\left|x_{i}-b_{n}^{(i)}\right|} \leqslant \frac{c m_{i}}{a_{n}^{(i)}} \leqslant \frac{c}{\mu_{i}\left(a_{n_{i}}^{(i)}\right)} .
$$

Then, for $n_{i}+m_{i} \leqslant n \leqslant 2 n_{i}$ (so $\left|x_{i}-b_{n}^{(i)}\right| \geqslant c^{\prime} a_{n_{i}}^{(i)}$, see (8.7) in the case $\alpha_{i}=1$ ), setting $j=n-n_{i}$ we have $\left|x_{i}-b_{n}^{(i)}\right| \geqslant c j \mu_{i}\left(a_{n_{i}}^{(i)}\right)$ (this is trivial if $\alpha_{i}>1$ ): we obtain for $m \in\left[m_{i}, n_{i}\right]$

$$
\sum_{n=n_{i}+m_{i}}^{n_{i}+m} \frac{1}{a_{n}^{(i)} \vee\left|x_{i}-b_{n}^{(i)}\right|} \leqslant \sum_{j=m_{i}}^{m} \frac{c}{j \mu_{i}\left(a_{n_{i}}^{(i)}\right)} \leqslant \frac{c}{\mu_{i}\left(a_{n_{i}}^{(i)}\right)} \log \left(1+\frac{m}{m_{i}}\right) .
$$

A similar argument holds for the sum between $n-m$ and $n-n_{i}$, and this concludes the proof of the first line of (9.6).

For $n \geqslant 2 n_{i}$, we have $\left|x_{i}-b_{n}^{(i)}\right| \geqslant c b_{n}^{(i)}$ (which is larger than $a_{n}^{(i)}$, recall $b_{n}^{(i)} / a_{n}^{(i)} \rightarrow+\infty$ ). Therefore, using $b_{n}^{(i)}=n \mu_{i}\left(a_{n}^{(i)}\right)$, we get that

$$
\sum_{n=2 n_{i}}^{m} \frac{1}{a_{n}^{(i)} \vee\left|x_{i}-b_{n}^{(i)}\right|} \leqslant \sum_{n=2 n_{i}}^{m} \frac{c}{n \mu_{i}\left(a_{n}^{(i)}\right)} \leqslant \frac{C}{\mu_{i}\left(x_{1}\right) \wedge \mu_{i}\left(b_{m}^{(i)}\right)} \log \left(m / n_{i}\right)
$$

The last inequality comes from the fact that: either $\mu_{i}$ is a positive constant (so the bound is trivial); or $\mu_{i}=+\infty$ (if $\alpha_{i}=1, p_{i}>q_{i}$ ) so $\mu_{i}(x) \sim\left(p_{i}-q_{i}\right) \ell(x)$ with $\ell(\cdot)$ non-decreasing, so $\mu_{i}\left(a_{n}^{(i)}\right) \geqslant c \mu_{i}\left(a_{n_{i}}^{(i)}\right)$; or $\mu_{i}=0$ (if $\left.\alpha_{i}=1, p_{i}<q_{i}\right)$ so $\mu_{i}(x) \sim\left(q_{i}-p_{i}\right) \ell^{\star}(x)$ with $\ell^{\star}(\cdot)$ non-increasing, so $\mu_{i}\left(a_{n}^{(i)}\right) \geqslant c \mu_{i}\left(a_{m}^{(i)}\right)$.

### 9.2.2 The case $n_{1} \leqslant n_{2} \leqslant 2 n_{1}$

Let us mention that the case $n_{2} \leqslant n_{1} \leqslant 2 n_{2}$ would be treated symmetrically. We let $m=\left(n_{2}-n_{1}\right) / 2$ so that $n_{1}+m=n_{2}-m$, and we assume that $m \geqslant m_{1} \vee m_{2}$ (otherwise we are in the favorite direction). We write

$$
\begin{equation*}
G(\mathbf{x})=\left(\sum_{n=0}^{n_{1}+m}+\sum_{n=n_{2}-m+1}^{+\infty}\right) \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) . \tag{9.7}
\end{equation*}
$$

For the first term, we split it according to whether $n<n_{1}-m$ or $n \geqslant n_{1}-m$. For $n \in\left(n_{1}-m, n_{1}+m\right)$, and since $n \leqslant n_{2}-m_{2}$, we have that $\left|x_{2}-b_{n}^{(2)}\right| \geqslant c m \mu_{2}\left(a_{n_{2}}^{(2)}\right)$ (see (8.7)): Theorem 2.5 gives that

$$
\begin{aligned}
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant & \frac{C}{a_{n}^{(1)} \vee\left|x_{1}-b_{n}^{(1)}\right|} \\
& \times\left(n_{2} \varphi_{2}\left(m \mu_{2}\left(a_{n_{2}}^{(2)}\right)\right)\left(m \mu_{2}\left(a_{n_{2}}^{(2)}\right)\right)^{-\left(1+\gamma_{2}\right)}+\frac{1}{a_{n}^{(2)}} e^{-c\left(m \mu_{2}\left(a_{n_{2}}^{(2)}\right) / a_{n}^{(2)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}\right) .
\end{aligned}
$$

Then, by (9.6), and using that $n_{2} \varphi\left(a_{n_{2}}^{(2)}\right)\left(a_{n_{2}}^{(2)}\right)^{-\left(1+\alpha_{2}\right)} \leqslant C / a_{n_{2}}^{(2)}$ (together with Potter's bound), we obtain for any $\delta>0$

$$
\begin{aligned}
& \sum_{n=n_{1}-m}^{n_{1}+m} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \\
& \\
& \quad \leqslant \frac{C_{\delta} \log \left(m / m_{1}\right)}{a_{n_{2}}^{(2)} \mu_{1}\left(a_{n_{1}}^{(1)}\right)} \times\left(\left(\frac{m}{m_{2}}\right)^{-\left(1+\alpha_{2}\right)+\delta}\left(m \mu_{2}\left(a_{n_{2}}^{(2)}\right)\right)^{\alpha_{2}-\gamma_{2}}+e^{-c\left(m / m_{2}\right)^{2}} \mathbb{1}_{\left\{\alpha_{2}=2\right\}}\right)
\end{aligned}
$$

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Notice that $\alpha_{2}=\gamma_{2}$ if $\alpha_{2} \in(0,2)$ : we can rewrite the above as

$$
\begin{equation*}
\sum_{n=n_{1}-m}^{n_{1}+m} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C_{\delta}^{\prime} \log \left(m / m_{1}\right)}{a_{n_{2}}^{(2)} \mu_{1}\left(a_{n_{1}}^{(1)}\right)} \times\left(\frac{m}{m_{2}}\right)^{-\left(1+\alpha_{2}\right)+\delta} \times R^{(2)}(m) \tag{9.8}
\end{equation*}
$$

where we set $R^{(i)}(m)=1$ if $\alpha_{i} \in(0,2)$ and $R^{(i)}(m)=m^{2-\gamma_{i}}+e^{-c^{\prime}\left(m / m_{i}\right)^{2}}$ if $\alpha_{i}=2$ (in which case $\mu_{i}\left(a_{n_{i}}^{(i)}\right)$ is a constant).

For the terms with $n \leqslant n_{1}-m$, we use again Theorem 2.5: since $\left|x_{1}-b_{n}^{(1)}\right| \geqslant c m \mu_{1}\left(a_{n_{1}}^{(1)}\right)$ (see (8.7)), and setting $j=n_{2}-n$ so that $\left|x_{2}-b_{n}^{(2)}\right| \geqslant c j \mu_{2}\left(a_{n_{2}}^{(2)}\right)$, we have

$$
\begin{aligned}
& \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \\
& \quad \leqslant \frac{C}{m \mu_{2}\left(a_{n_{2}}^{(2)}\right)} \times\left(n \varphi_{1}\left(j \mu_{1}\left(a_{n_{1}}^{(1)}\right)\right)\left(j \mu_{1}\left(a_{n_{1}}^{(1)}\right)\right)^{-\left(1+\gamma_{1}\right)}+\frac{1}{a_{n}^{(1)}} e^{-c\left(j \mu_{1}\left(a_{n_{1}}^{(1)}\right) / a_{n}^{(1)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}\right) .
\end{aligned}
$$

Therefore, summing over $j$ as done in (8.12)-(8.13) (in the case $\alpha_{1}=1$, the case $\alpha_{1}>1$ is identical), we obtain as in (9.8)

$$
\begin{align*}
\sum_{n=1}^{n_{1}-m} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) & \leqslant \frac{C}{m \mu_{2}\left(a_{n_{2}}^{(2)}\right)}\left(\frac{n_{1}}{\mu_{1}\left(a_{n_{1}}^{(1)}\right)} \varphi_{1}\left(m \mu_{1}\left(a_{n_{1}}^{(1)}\right)\right)\left(m \mu_{1}\left(a_{n_{1}}^{(1)}\right)\right)^{-\gamma_{1}}+e^{-c\left(m / m_{1}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}\right) \\
& \leqslant \frac{C_{\delta}}{a_{n_{2}}^{(2)} \mu_{1}\left(a_{n_{1}}^{(1)}\right)}\left(\frac{m}{m_{2}}\right)^{-1}\left(\frac{m}{m_{1}}\right)^{-\alpha_{1}+\delta} \times R^{(1)}(m) . \tag{9.9}
\end{align*}
$$

Similarly, we can treat the cases $n_{2}-m \leqslant n \leqslant n_{2}+m$ and $n \geqslant n_{2}+m$, and we get that

$$
\begin{aligned}
& \sum_{n=n_{2}-m}^{n_{2}+m} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C_{\delta}^{\prime} \log \left(m / m_{2}\right)}{a_{n_{1}}^{(1)} \mu_{2}\left(a_{n_{2}}^{(2)}\right)}\left(\frac{m}{m_{1}}\right)^{-\left(1+\alpha_{1}\right)+\delta} \times R^{(1)}(m), \\
& \sum_{n=n_{2}+m}^{+\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C_{\delta}^{\prime}}{a_{n_{1}}^{(1)} \mu_{2}\left(a_{n_{2}}^{(2)}\right)}\left(\frac{m}{m_{1}}\right)^{-1}\left(\frac{m}{m_{2}}\right)^{-\alpha_{2}+\delta} \times R^{(2)}(m) .
\end{aligned}
$$

Notice that $a_{n_{1}}^{(1)} \mu_{2}\left(a_{n_{2}}^{(2)}\right)=a_{n_{2}}^{(2)} \mu_{1}\left(a_{n_{1}}^{(1)}\right) \times\left(m_{1} / m_{2}\right)$, so that we can re-write the above as

$$
\begin{align*}
& \sum_{n=n_{2}-m}^{n_{2}+m} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C \log \left(m / m_{2}\right)}{a_{n_{2}}^{(2)} \mu_{1}\left(a_{n_{1}}^{(1)}\right)}\left(\frac{m}{m_{2}}\right)^{-1}\left(\frac{m}{m_{1}}\right)^{-\alpha_{1}+\delta} R^{(1)}(m),  \tag{9.10}\\
& \sum_{n=n_{2}+m}^{+\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C}{a_{n_{2}}^{(2)} \mu_{1}\left(a_{n_{1}}^{(1)}\right)}\left(\frac{m}{m_{2}}\right)^{-\left(1+\alpha_{2}\right)+\delta} R^{(2)}(m) . \tag{9.11}
\end{align*}
$$

All together, combining (9.8)-(9.9) and (9.10)-(9.11), and bounding $\log \left(m / m_{i}\right)$ by a constant times $\left(m / m_{i}\right)^{\delta}$, we can conclude that

$$
G(\mathbf{x}) \leqslant \frac{C_{\delta}^{\prime \prime}}{a_{n}^{(2)} \mu_{1}\left(a_{n_{1}}^{(1)}\right)}\left(\frac{m}{m_{2}}\right)^{-1+\delta}\left(\frac{m}{m_{1}}\right)^{\delta} \times\left\{\left(\frac{m}{m_{1}}\right)^{-\alpha_{1}} R^{(1)}(m)+\left(\frac{m}{m_{2}}\right)^{-\alpha_{2}} R^{(2)}(m)\right\}
$$

which concludes the proof of (4.3), recalling that we chose $m=\left(n_{2}-n_{1}\right) / 2$.

### 9.2.3 The case $n_{2} \geqslant 2 n_{1}$

Again, the case $n_{2} \leqslant n_{1} / 2$ would be treated symmetrically. When $n_{2} \geqslant 2 n_{1}$, we choose $m=3 n_{2} / 4$ if $\gamma_{1} \leqslant \gamma_{2}$, and $m=n_{2}^{\gamma_{2} / \gamma_{1}} \vee\left(3 n_{1} / 2\right)$ if $\gamma_{1}>\gamma_{2}$. We write

$$
\begin{equation*}
G(\mathbf{x})=\left(\sum_{n=1}^{m}+\sum_{n=m+1}^{+\infty}\right) \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) . \tag{9.12}
\end{equation*}
$$

For the first term, since $n \leqslant m \leqslant 3 n_{2} / 4$, for any $\delta^{\prime}$ there is a constant $c_{\delta^{\prime}}$ such that $\left|x_{2}-b_{n}^{(2)}\right| \geqslant c_{\delta^{\prime}} n_{2}^{1-\delta^{\prime}}$, so that Theorem 2.5 gives that for any $\delta>0$ we have a constant $c_{\delta}$ such that, for $n \leqslant m$,

$$
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C_{\delta}}{a_{n}^{(1)} \vee\left|x_{1}-b_{n}^{(1)}\right|} \times\left(m \times\left(n_{2}\right)^{-\left(1+\gamma_{2}\right)+\delta}+\frac{1}{a_{n_{2}}^{(2)}} e^{-c\left(n_{2}^{1-\delta^{\prime}} / a_{n_{2}}^{(2)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{2}=2\right\}}\right) .
$$

Here, the second term decays faster than any power of $n_{2}$ (if the term is present, $a_{n_{2}}^{(2)}$ is regularly varying with exponent $1 / 2$ ), so it is negligible compared to the first term. Then summing over $n \leqslant m$ and using (9.6), we obtain that

$$
\begin{equation*}
\sum_{n=1}^{m} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant C_{\delta}^{\prime} m^{1+\delta}\left(n_{2}\right)^{-\left(1+\gamma_{2}\right)+\delta} \tag{9.13}
\end{equation*}
$$

For the second term in (9.12), we divide it into three parts: $m<n<3 n_{2} / 4$; $3 n_{2} / 4 \leqslant n \leqslant 2 n_{2}$; and $n>2 n_{2}$. For $m<n \leqslant 3 n_{2} / 4$, we use again that $\left|x_{2}-b_{n}^{(2)}\right| \geqslant c_{\delta^{\prime}} n_{2}^{1-\delta^{\prime}}$ and $\left|x_{1}-b_{n}^{(1)}\right| \geqslant c b_{n}^{(1)}$ (recall $m \geqslant 3 n_{1} / 2$ ), so that Theorem 2.5 again gives that for any $\delta>0$
$\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant C_{\delta} n_{2}^{-1+\delta} \times\left(n\left(b_{n}^{(1)}\right)^{-\left(1+\gamma_{1}\right)+\delta}+\frac{1}{a_{n}^{(1)}} e^{-c\left(b_{n}^{(1)} / a_{n}^{(1)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{1}=2\right\}}\right) \leqslant C_{\delta} n_{2}^{-1+\bar{\delta}} \times n^{-\gamma_{1}-\delta}$.
For the last inequality, we discarded the exponential term since it decays faster than any power in $n$, and used that $n \geqslant m \geqslant n_{2}^{1 \wedge\left(\gamma_{2} / \gamma_{1}\right)}$ to bound $n_{2}^{\delta} n^{2 \delta}$ by $n_{2}^{\bar{\delta}} n^{-\delta}$ (by picking $\left.\bar{\delta}=\delta+3 \delta\left(1 \wedge\left(\gamma_{2} / \gamma_{1}\right)\right)\right)$. Summing over $n \geqslant m$, and since $\gamma_{1}+\delta>1$, we get that

$$
\begin{equation*}
\sum_{n=m}^{3 n_{2} / 4} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant C_{\delta}^{\prime} n_{2}^{-1+\bar{\delta}} m^{1-\gamma_{1}} \tag{9.14}
\end{equation*}
$$

For $3 n_{2} / 4 \leqslant n \leqslant 2 n_{2}$, we use that $\left|x_{1}-b_{n}^{(1)}\right| \geqslant c n_{2}^{1-\delta}$ to get by Theorem 2.5 that (discarding the exponential term as above)

$$
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C_{\delta}}{a_{n}^{(1)} \vee\left|x_{2}-b_{n}^{(2)}\right|} \times n_{2}\left(n_{2}\right)^{-\left(1+\gamma_{1}\right)+\delta},
$$

so that, summing over $n$, (9.6) gives that $\sum_{n=3 n_{2} / 4}^{2 n_{2}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant C_{\delta}^{\prime} n_{2}^{-\gamma_{1}+2 \delta}$ which is smaller than a constant times $n_{2}^{-1+2 \delta} m^{1-\gamma_{1}}$ since $m \leqslant n_{2}$ (and $\gamma_{1} \geqslant 1$ ).

For $n>2 n_{2}$, we use that both $\left|x_{1}-b_{n}^{(1)}\right|$ and $\left|x_{2}-b_{n}^{(2)}\right|$ are larger than $c_{\delta^{\prime}} n^{1-\delta^{\prime}}$ : Theorem 2.5 gives that $\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant C n^{-\left(1+\gamma_{2}\right)+\delta}$. Then, summing over $n \geqslant 2 n_{2}$ gives that $\sum_{n>2 n_{2}} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant C_{\delta} n_{2}^{-\gamma_{2}+2 \delta} \leqslant C_{\delta}^{\prime} n_{2}^{-1+2 \delta} m^{1-\gamma_{1}}$.

To conclude the argument, we get that $G(\mathbf{x})$ is bounded by a constant times

$$
m^{1+\delta} n_{2}^{-\left(1+\gamma_{2}\right)+\delta}+n_{2}^{-1+\delta} m^{1-\gamma_{1}} \leqslant C \begin{cases}n_{1} \times n_{2}^{-\left(1+\gamma_{2}\right)+\delta^{\prime}} & \text { if } n_{2}^{\gamma_{2} / \gamma_{1}} \leqslant n_{1} \\ n_{2}^{-\gamma_{2}+\left(\frac{\gamma_{2}}{\gamma_{1}}-1\right)+\delta^{\prime}} & \text { if } \gamma_{1}>\gamma_{2}, n_{2}^{\gamma_{2} / \gamma_{1}}>n_{1} \\ n_{2}^{-\gamma_{2}+\delta^{\prime}} & \text { if } \gamma_{1} \leqslant \gamma_{2},\end{cases}
$$

where we used the definition of $m$ in the last inequality (indeed, the optimal $m$ is $n_{2}^{\gamma_{2} / \gamma_{1}}$ but we have the additional conditions that $m \geqslant n_{1}$ and $n \leqslant n_{2}$ ). This concludes the proof of (4.4).

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## About Assumption (2.8)

Let us explain briefly why assuming (2.8) does not improve much the bounds (4.5)-(4.6)we consider the case when $n_{1} \leqslant n_{2} \leqslant 2 n_{1}$. Set again $m=n_{2}-n_{1}$. For $n \leqslant n_{2}-m$ $\left(=n_{1}+m\right)$ we have that $\left|b_{n}^{(2)}-x_{2}\right| \geqslant c\left(n-n_{2}\right) \mu_{2}\left(a_{n_{2}}\right)$ so that Theorem 2.6 gives that $\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)$ is bounded by a constant times

$$
n_{2} \varphi\left(\left(n-n_{2}\right) \mu_{2}\left(a_{n_{2}}\right)\right)\left(\left(n-n_{2}\right) \mu_{2}\left(a_{n_{2}}\right)\right)^{-(2+\gamma)}+\frac{1}{\left(a_{n}\right)^{2}} e^{-c\left(\left(n-n_{2}\right) \mu_{2}\left(a_{n_{2}}\right) / a_{n}\right)^{2}} \mathbb{1}_{\{\alpha=2\}} .
$$

Hence, summing over $j=n_{2}-n$ between $m$ and $n_{2}$, and using that $n_{2} \leqslant \varphi\left(a_{n_{2}}\right)\left(a_{n_{2}}\right)^{-\alpha}$ (and $m_{2}=a_{n_{2}} / \mu_{2}\left(a_{n_{2}}\right)$ ) we get that

$$
\begin{aligned}
\sum_{n=0}^{n_{1}+m} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) & \leqslant \frac{C}{\mu_{2}\left(a_{n_{2}}\right)} n_{2} \varphi\left(m \mu_{2}\left(x_{2}\right)\right)\left(m \mu_{2}\left(a_{n_{2}}\right)\right)^{-(1+\gamma)}+\frac{C}{a_{n_{2}} \mu_{2}\left(a_{n_{2}}\right)} e^{-c\left(m / a_{n_{2}}\right)^{2}} \mathbb{1}_{\{\alpha=2\}} \\
& \leqslant \frac{C}{a_{n_{2}} \mu_{2}\left(a_{n_{2}}\right)}\left(\left(\frac{m}{m_{2}}\right)^{-(1+\alpha)+\delta}(m)^{\alpha-\gamma}+e^{-c\left(m / m_{2}\right)^{2}} \mathbb{1}_{\{\alpha=2\}}\right)
\end{aligned}
$$

which gives the upper bound in (9.8) up to the factor $\log \left(m / m_{1}\right)$. Similar bounds hold for the other terms, showing that the use of Theorem 2.6 does not bring any real improvement.

### 9.3 Case II-III (b), proof of Theorem 4.3

Here, we assume that $b_{n}^{\left(i_{1}\right)} \equiv 0$ and that either $\alpha_{i_{1}} \geqslant 1$ and $\mu_{i_{0}} \in \mathbb{R}^{*}$, or $\alpha_{i_{1}}=1$ with $p_{i_{0}} \neq q_{i_{0}}$. Recall the definitions (3.1) of $n_{i}$ : in particular, $n_{i_{0}} \sim x_{i_{0}} / \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)$ (we work in the case where both $x_{i_{0}}$ and $\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)$ are positive), and $a_{n_{i_{1}}}^{\left(i_{1}\right)} \sim\left|x_{i_{1}}\right|$. We also define $m_{i_{0}}:=a_{n_{i_{0}}}^{\left(i_{0}\right)} / \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)$ as above. The case where $n_{i_{1}} / n_{i_{0}}$ is bounded away from 0 and $+\infty$ corresponds to the favorite scaling (3.5).

### 9.3.1 The case $n_{i_{1}} / n_{i_{0}}$ small

Let us start with the case when $n_{i_{1}} \leqslant c n_{i_{0}}$ with $c$ a small constant. We split $G(\mathbf{x})$ into

$$
\begin{equation*}
G(\mathbf{x})=\left(\sum_{n=1}^{n_{i_{0}} / 2}+\sum_{n=n_{i_{0}} / 2+1}^{+\infty}\right) \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) . \tag{9.15}
\end{equation*}
$$

For the second term, we can use the results of Section 7 for the case $\alpha_{i_{0}}>1$ (combine (7.6)-(7.8)-(7.13)) and of Section 8 for the case $\alpha_{i_{0}}=1$ (combine (8.6)-(8.10)-(8.11)(8.12)). We get

$$
\begin{equation*}
\sum_{n=n_{i_{0}} / 2+1}^{+\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C}{\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right) a_{n_{i_{0}}}^{\left(i_{1}\right)}} . \tag{9.16}
\end{equation*}
$$

Recall that if $\mu_{i_{0}} \in \mathbb{R}_{+}^{*}$, then we can replace $\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)$ by a constant.
For the first term in (9.15), we use that for $n \leqslant n_{i_{0}} / 2$ then $x_{i_{0}}-b_{n}^{\left(i_{0}\right)} \geqslant x_{i_{0}} / 4 \geqslant a_{n}^{\left(i_{0}\right)}$ so that Theorem 2.5 gives

$$
\begin{align*}
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) & \leqslant \frac{C}{\left|x_{i_{1}}\right| \vee a_{n}^{\left(i_{1}\right)}}\left(n \varphi_{i_{0}}\left(x_{i_{0}}\right) x_{i_{0}}^{-\left(1+\gamma_{i_{0}}\right)}+\frac{1}{a_{n}^{\left(i_{0}\right)}} e^{-c x_{i_{0}} / a_{n}^{\left(i_{0}\right)}} \mathbb{1}_{\left\{\alpha_{i_{0}}=2\right\}}\right) \\
& \leqslant \frac{C^{\prime}}{\left|x_{i_{1}}\right| \vee a_{n}^{\left(i_{1}\right)}} \times n \varphi_{i_{0}}\left(x_{i_{0}}\right) x_{i_{0}}^{-\left(1+\gamma_{i_{0}}\right)} \tag{9.17}
\end{align*}
$$

We discarded the second (exponential) term since it decays faster than any power of $n_{i_{0}}$. Summing over $n$, we get that (recall $\left|x_{i_{1}}\right|=a_{n_{i_{1}}}^{\left(i_{1}\right)}$ )

$$
\begin{align*}
\sum_{n=1}^{n_{i_{0}} / 2} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) & \leqslant C \varphi_{i_{0}}\left(x_{i_{0}}\right) x_{i_{0}}^{-\left(1+\gamma_{i_{0}}\right)}\left(\sum_{n=1}^{n_{i_{1}}} \frac{n}{\left|x_{i_{1}}\right|}+\sum_{n=n_{i_{1}}+1}^{n_{i_{0}}} \frac{n}{a_{n}^{\left(i_{1}\right)}}\right) \\
& \leqslant \varphi_{i_{0}}\left(x_{i_{0}}\right) x_{i_{0}}^{-\left(1+\gamma_{i_{0}}\right)} \times \begin{cases}C \delta n_{i_{1}}^{2-1 / \alpha_{i_{1}}} n_{i_{0}}^{\delta} & \text { if } \alpha_{i_{1}} \leqslant 1 / 2 \\
C n_{i_{0}}^{2} / a_{n_{i_{0}}}^{\left(i_{1}\right)} & \text { if } \alpha_{i_{1}}>1 / 2\end{cases} \tag{9.18}
\end{align*}
$$

We used that $n / a_{n}^{\left(i_{1}\right)}$ is regularly varying with exponent $1-1 / \alpha_{i_{1}}$, which is smaller (resp. larger) than -1 if $\alpha_{i_{1}}>1 / 2$ (resp. if $\alpha_{i_{1}}<1 / 2$ ).

Notice that since $n_{i_{0}} \sim x_{i_{0}} / \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)$, we get that

$$
\begin{equation*}
\varphi_{i_{0}}\left(x_{i_{0}}\right) x_{i_{0}}^{-\left(1+\gamma_{i_{0}}\right)} \frac{n_{i_{0}}^{2}}{a_{n_{i_{0}}}^{\left(i_{0}\right)}} \leqslant \frac{C}{a_{n_{i_{0}}}^{\left(i_{1}\right)} \mu_{i_{0}}\left(x_{i_{0}}\right)} \frac{\varphi_{i_{0}}\left(x_{i_{0}}\right)}{\mu_{i_{0}}\left(a_{\left.n_{i_{0}}\right)}^{\left(i_{0}\right)}\right.}\left(x_{i_{0}}\right)^{1-\gamma_{i_{0}}} \leqslant \frac{C}{\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right) a_{n_{i_{0}}}^{\left(i_{1}\right)}} . \tag{9.19}
\end{equation*}
$$

Indeed, either $\gamma_{i_{0}}>1$, or $\gamma_{i_{0}}=1$ in which case $\varphi_{i_{0}}\left(x_{i_{0}}\right) / \mu_{i_{0}}\left(x_{i_{0}}\right) \rightarrow 0$ as $x_{i_{0}} \rightarrow+\infty$. Together with (9.16), this gives (4.7).

## The case of a renewal process

If $\mathbf{S}_{n}$ is a renewal process (necessarily $\alpha_{i_{1}}<1$ ), we can improve the bound (4.7). For a fixed $\delta>0$, we set $m:=n_{i_{1}} \times\left(n_{i_{0}} / n_{i_{1}}\right)^{\delta}$, which is larger than $n_{i_{1}}$, but smaller than $n_{i_{0}} / 2$. We write

$$
G(\mathbf{x})=\left(\sum_{n=1}^{m}+\sum_{n=m+1}^{+\infty}\right) \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)
$$

The first term is treated as above, using (9.17) (note also that $x_{i_{1}} \vee a_{n}^{\left(i_{1}\right)} \geqslant c a_{n_{i_{1}}}^{\left(i_{1}\right)}$ ):

$$
\sum_{n=1}^{m} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant C \frac{m^{2}}{a_{n_{i_{1}}}^{(1)}} \varphi_{i_{0}}\left(x_{i_{0}}\right) x_{i_{0}}^{-\left(1+\gamma_{i_{0}}\right)} \leqslant C_{\delta}\left(n_{i_{1}}\right)^{2-\frac{1}{\alpha_{i_{1}}}} x_{i_{0}}^{-\left(1+\gamma_{i_{0}}\right)+2 \delta}
$$

where we used that $m:=n_{i_{1}}\left(n_{i_{0}} / n_{i_{1}}\right)^{\delta}$, together with $n_{i_{0}} \leqslant x_{i_{0}}^{1+\delta}$, and also $a_{n}^{\left(i_{1}\right)} \geqslant n_{i_{1}}^{1 / \alpha_{i_{1}}-\delta}$.
For the remaining term, we use that,

$$
\begin{aligned}
\sum_{n>m} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) & =\mathbf{P}\left(\exists n>m \text { such that } \mathbf{S}_{n}=\mathbf{x}\right) \leqslant \mathbf{P}\left(S_{m}^{\left(i_{1}\right)} \leqslant x_{i_{1}}\right) \\
& \leqslant \exp \left(-c\left(a_{m}^{\left(i_{1}\right)} / a_{n_{i_{1}}}^{\left(i_{1}\right)}\right)^{-\zeta_{i_{1}}}\right) \leqslant \exp \left(-\left(n_{i_{0}} / n_{i_{1}}\right)^{\zeta_{\delta}}\right)
\end{aligned}
$$

for some $\zeta_{\delta}>0$. The second line follows from the large deviation result [1, Lemmas A.3] (we have $x_{i_{1}} \geqslant a_{m}^{\left(i_{1}\right)} / 2$ ), with an exponent $\zeta_{i_{1}}$ that depends on $\alpha_{i_{1}}<1$. The second inequality comes from the definition of $m=n_{i_{1}} \times\left(n_{i_{0}} / n_{i_{1}}\right)^{\delta}$.

### 9.3.2 The case $n_{i_{1}} / n_{i_{0}}$ large

Let us take some $m \in\left(2 n_{i_{0}}, n_{i_{1}} / 2\right)$ (the choice is optimized below), and we write

$$
\begin{equation*}
G(\mathbf{x})=\left(\sum_{n=1}^{m}+\sum_{n=m+1}^{+\infty}\right) \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) . \tag{9.20}
\end{equation*}
$$

For the first term, we us that $x_{i_{1}}-b_{n}^{\left(i_{1}\right)} \geqslant c x_{i_{1}} \geqslant a_{n}^{\left(i_{1}\right)}\left(n \leqslant n_{i_{1}} / 2\right)$, so that thanks to Theorem 2.5 we have

$$
\begin{equation*}
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C}{\left|x_{i_{0}}-b_{n}^{\left(i_{0}\right)}\right| \vee a_{n}^{\left(i_{0}\right)}} \times \frac{1}{x_{i_{1}}}\left(m \varphi_{i_{1}}\left(x_{i_{1}}\right) x_{i_{1}}^{-\gamma_{i_{1}}}+e^{-c\left(x_{i_{1}} / a_{n}^{\left(i_{1}\right)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{i_{1}}=2\right\}}\right) . \tag{9.21}
\end{equation*}
$$

Therefore, summing over $n \leqslant m$ and using (9.6) above, we get that

$$
\begin{equation*}
\sum_{n=1}^{m} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C \log m}{\bar{\mu}_{i_{0}}\left(n_{i_{0}}, m\right)} \frac{1}{x_{i_{1}}}\left(m \varphi_{i_{1}}\left(x_{i_{1}}\right) x_{i_{1}}^{-\gamma_{i_{1}}}+e^{-c\left(x_{i_{1}} / a_{m}^{\left(i_{1}\right)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{i_{1}}=2\right\}}\right) \tag{9.22}
\end{equation*}
$$

with $\bar{\mu}_{i_{0}}\left(n_{i_{0}}, m\right):=\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right) \wedge \mu_{i_{0}}\left(a_{m}^{\left(i_{0}\right)}\right) \geqslant n_{i_{0}}^{-\delta} m^{-\delta / 2}$.
For the second term in (9.20), we use that $\left|x_{i_{0}}-b_{n}^{\left(i_{0}\right)}\right| \geqslant c b_{n}^{\left(i_{0}\right)}$ for $n>2 n_{i_{0}}$, so that Theorem 2.5 gives us

$$
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C}{x_{i_{1}} \vee a_{n}^{\left(i_{1}\right)}} n \varphi_{i_{0}}\left(b_{n}^{\left(i_{0}\right)}\right)\left(b_{n}^{\left(i_{0}\right)}\right)^{-\left(1+\gamma_{i_{0}}\right)}
$$

As in (9.17), we discarded the exponential term appearing in the case $\alpha_{i_{0}}=2$, since it decays faster than any power of $n$. Therefore, in the case $\gamma_{i_{0}}>1$, we have

$$
\begin{equation*}
\sum_{n=m+1}^{+\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C}{a_{n_{i_{1}}}^{\left(i_{1}\right)}} \sum_{n \geqslant m} \varphi_{i_{0}}(n) n^{-\gamma_{i_{0}}} \leqslant \frac{C}{a_{n_{i_{1}}}^{\left(i_{1}\right)}} \varphi_{i_{0}}(m) m^{1-\gamma_{i_{0}}} \tag{9.23}
\end{equation*}
$$

For the case $\gamma_{i_{0}}=1$, we refer to the bound (9.26) below: we find that the sum is bounded by a constant times $\left(a_{n_{i_{1}}}^{\left(i_{1}\right)} \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)\right)^{-1}$.

Combining (9.22) and (9.23), and using $x_{i_{1}}=a_{n_{i_{1}}}^{\left(i_{1}\right)}$, we get that in the case $\gamma_{i_{0}}>1$

$$
\begin{equation*}
G(\mathbf{x}) \leqslant \frac{C_{\delta}}{a_{n_{i_{1}}}^{\left(i_{1}\right)}} \times\left(m^{1-\gamma_{i_{0}}+\delta}+m^{\delta} n_{i_{0}}^{\delta}\left(m x_{i_{1}}^{-\gamma_{i_{1}}+\delta}+e^{-c\left(x_{i_{1}} / a_{m}^{\left(i_{1}\right)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{i_{1}}=2\right\}}\right)\right) . \tag{9.24}
\end{equation*}
$$

Then, we are left with the choice of $m$ : we take $m \leqslant n_{i_{1}}^{1-\delta}$ so that the last term decays faster than any power of $m$, and we only have to optimize $m^{1+\delta} \times\left(m^{-\gamma_{i_{0}}}+x_{i_{1}}^{-\gamma_{1}}\left(n_{i_{0}} x_{i_{1}}\right)^{\delta}\right)$. We therefore choose $m \geqslant\left(x_{i_{1}}\right)^{\gamma_{i_{1}} / \gamma_{i_{0}}} \vee n_{i_{0}}$. In the end, we obtain that the parenthesis in (9.24) is bounded by $m^{1+\delta}\left(n_{i_{0}} x_{i_{1}}\right)^{\delta} x_{i_{1}}^{-\gamma_{i_{1}}} \leqslant m_{\delta^{\prime}} x_{i_{1}}^{-\gamma_{i_{1}}}$, where we have put $m_{\delta^{\prime}}=\left(\left|x_{i_{1}}\right| \gamma_{i_{1} / \gamma_{i_{0}}+\delta^{\prime}} \vee\left(n_{i_{0}}\right)^{1+\delta^{\prime}}\right) \wedge\left(n_{i_{1}}\right)^{1-\delta^{\prime}}$ for some $\delta^{\prime}$ slightly larger than $\delta$ (if $\left(n_{i_{1}}\right)^{1-\delta^{\prime}}<$ $\left(n_{i_{0}}\right)^{1+\delta^{\prime}}$ then take $m_{\delta}=+\infty$ ). This gives (4.8).

Alternatively, we also obtain from Theorem 2.5 that for any $n$ (using $x_{i_{1}} \vee a_{n}^{\left(i_{1}\right)} \geqslant a_{n_{i_{1}}}^{\left(i_{1}\right)}$ )

$$
\begin{align*}
\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) & \leqslant \frac{C}{a_{n_{i_{1}}}^{\left(i_{1}\right)}} \times \frac{1}{a_{n}^{\left(i_{0}\right)} \vee\left|x_{i_{0}}-b_{n}^{\left(i_{0}\right)}\right|} \\
& \times\left(n \varphi_{i_{0}}\left(\left|x_{i_{0}}-b_{n}^{\left(i_{0}\right)}\right|\right)\left|x_{i_{0}}-b_{n}^{\left(i_{0}\right)}\right|^{-\gamma_{i_{0}}}+e^{-c\left(\left|x_{i_{0}}-b_{n}^{\left(i_{0}\right)}\right| / a_{n}^{\left(i_{0}\right)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{i_{0}}=2\right\}}\right) \tag{9.25}
\end{align*}
$$

Summing over $n$, and treating the different parts of the sum according to whether $n \leqslant n_{i_{0}}-m_{i_{0}}, n \in\left(n_{i_{0}}-m_{i_{0}}, n_{i_{0}}+m_{i_{0}}\right)$ (in which range $\left|x_{i_{0}}-b_{n}^{\left(i_{0}\right)}\right| \leqslant a_{n}^{\left(i_{0}\right)}$ ) or $n \geqslant n+m_{i_{0}}$, we obtain as in Sections 7-8 that

$$
\begin{equation*}
\sum_{n=1}^{+\infty} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C}{a_{n_{i_{1}}}^{\left(i_{1}\right)}} \times \frac{1}{\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)} \tag{9.26}
\end{equation*}
$$

This gives a general bound, in case the above (9.24) does not give a satisfactory bound (for instance if $m_{\delta^{\prime}}=+\infty$ ).

## The case of a renewal process

In that case, we have that $\mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right)=0$ for all $n \geqslant x_{i_{0}} \sim n_{i_{0}} \mu\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)$ (notice that $\left.\mu_{i_{0}}(x) \geqslant 1\right)$. Hence, we have that
$G(\mathbf{x})=\sum_{n=1}^{2 n_{i_{0}} \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)} \mathbf{P}\left(\mathbf{S}_{n}=\mathbf{x}\right) \leqslant \frac{C \log n_{i_{0}}}{\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right) a_{n_{i_{1}}}^{\left(i_{1}\right)}}\left(n_{i_{0}} \varphi_{i_{1}}\left(x_{i_{1}}\right) x_{i_{1}}^{-\gamma_{i_{1}}}+e^{-c\left(a_{n_{i_{1}}}^{\left(i_{1}\right)} / a_{n_{i_{0}}}^{\left(i_{1}\right)}\right)^{2}} \mathbb{1}_{\left\{\alpha_{i_{1}}=2\right\}}\right)$,
where we used (9.22) with $m=2 n_{i_{0}} \mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right)$, and also that $\mu_{i_{0}}\left(a_{n_{i_{0}}}^{\left(i_{0}\right)}\right) \geqslant \mu_{i_{0}}\left(a_{m}^{\left(i_{0}\right)}\right)$ since $\mu$ is non-decreasing. Noting that $a_{n_{i_{1}}}^{\left(i_{1}\right)} / a_{n_{i_{0}}}^{\left(i_{1}\right)}$ is bounded below by $\left(n_{i_{1}} / n_{i_{2}}\right)^{(1-\delta) / 2}$ in the case $\alpha_{i_{1}}=2$, this gives the bound in Theorem 4.3.

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## A Generalized domains of attraction and multivariate regular variation

## A. 1 A few words on generalized domains of attraction

We stress that the convergence (1.1) is a special case of generalized domains of attractions (called operator-stable distributions): in general, the renormalization matrix $A_{n}$ is invertible, and does not need to be diagonal as in our case. A few relevant and historical references are Sharpe [37], Hudson [24], Hahn and Klass [22, 23], and a comprehensive overview of the subject can be found in [29]. We stress that a local limit theorem exists in general, see [19]. We also mention that when $A_{n}$ is diagonal, all marginals $X^{(i)}$ are in the domain of attraction of an $\alpha_{i}$-stable distribution, which is not necessarily the case in operator-stable distributions, cf. [30].

Sharpe [37] found that one can decompose a multivariate (operator-)stable distribution into the product of two marginals, one normal and one strictly non-normal. In our setting, it means that if we set $d_{0}=\max \left\{i ; \alpha_{i}=2\right\}$, then the stable law $\mathbf{Z}$ in (1.1) has two independent components: $\left(Z_{1}, \ldots, Z_{d_{0}}\right)$ normal, and $\left(Z_{d_{0}+1}, \ldots, Z_{d}\right)$ strictly non-normal, the convergence of these two marginals being enough for the joint convergence see [35, 28]. Then, we refer to [29] for a characterization of the convergence to an operatorstable distribution (either normal or strictly non-normal), in terms of regular variation in $\mathbb{R}^{d}$ of the distribution of $\mathbf{X}_{1}$ (this is a generalization of Feller conditions [15, § IX.8], i.e. (1.2), to the multivariate case).

In the simpler case we are interested in, that is when the matrix $A_{n}$ is diagonal, Resnick and Greenwood [35] (resp. Haan Omey and Resnick [20]) first gave a characterization of the domains of attraction in dimension 2 (resp. $d$ ), with the help of a (simpler) theory of regular variation in $\mathbb{R}^{d}$. We summarize it below, but we first recall the definition of regularly varying function in $\mathbb{R}^{d}$, as introduced in $[35,20]$.

## A. 2 About regular variation in $\mathbb{R}^{d}$, and convergence to stable distributions

The theory of regular variations in $\mathbb{R}$ is well established, and an exhaustive and seminal reference is [6]. The study of regular variation in $\mathbb{R}^{d}$ turns out to be very rich, and has also been extensively studied, starting with [35, 20, 27]: we refer to [29, Part II] and references therein for more details. Here we give a brief (simplified) definition in the special case we are interested in.

First, a function $r: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}^{d}$ is said to be regularly varying with exponent $\beta=$ $\left(\beta_{1}, \ldots, \beta_{d}\right)$ if the components $r_{i}(\cdot)$ are (one-dimensional) regularly varying functions with
respective indices $\beta_{i}$. Then, we say that the function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is regularly varying at $+\infty$ (resp. 0), if there exists a regularly varying function $r: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}^{d}$ (called auxiliary function) with index $\beta \in \mathbb{R}_{+}^{d}$ (resp. $\beta \in \mathbb{R}_{-}^{d}$ ), and $\varepsilon \in\{-1,+1\}$ such that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \frac{f(r(t) \mathbf{x})}{t^{\varepsilon}}=\phi(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^{d} \backslash\{0\} \tag{A.1}
\end{equation*}
$$

The function $\phi: \mathbb{R}^{d} \backslash\{0\} \rightarrow \mathbb{R}^{d}$ verifies $\phi\left(\lambda^{\beta} \mathbf{x}\right)=\lambda^{\varepsilon} \phi(\mathbf{x})$ for all $\mathbf{x} \neq \mathbf{0}$ and $\lambda>0$, where we denoted $\lambda^{a}=\left(\lambda^{a_{1}}, \ldots, \lambda^{a_{d}}\right)$. Then, $\rho=\varepsilon \beta^{-1}$ is called the index of regular variation of $f$.

Similarly, a measure $\pi$ is regularly varying at $+\infty$ (resp. 0) if there exists a regularly varying function $r: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}^{d}$ with index $\beta \in \mathbb{R}_{+}^{d}$ (resp. $\beta \in \mathbb{R}_{-}^{d}$ ), and $\varepsilon \in\{-1,+1\}$ such that,

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \frac{\pi(r(t) \mathrm{d} \mathbf{x})}{t^{\varepsilon}}=\varpi(\mathrm{d} \mathbf{x}) \tag{A.2}
\end{equation*}
$$

for some measure $\varpi$ which cannot be supported on any proper subspace of $\mathbb{R}^{d}$, and which verifies $\varpi\left(\lambda^{\beta} \mathrm{d} \mathbf{x}\right)=\lambda^{\varepsilon} \varpi(\mathrm{d} \mathbf{x})$. Then $\rho=\varepsilon \beta^{-1}$ is called the index of regular variations of $\pi$.

We are now ready to give a necessary and sufficient condition for $\mathbf{S}$ to be in the domain of attraction of an $\boldsymbol{\alpha}$-stable distribution (in the case of a diagonal $A_{n}$ ), as stated in [35, 20]. Since the convergence of the two marginals $\left(Z_{1}, \ldots, Z_{d_{0}}\right)$ (to a normal law) and $\left(Z_{d_{0}+1}, \ldots, Z_{d}\right)$ (to a strictly non-normal law) is enough, we state the results in the case where $d_{0}=d$ or $d_{0}=0$. First, $\mathbf{S}$ is in the domain of attraction of a non-degenerate normal law if and only if the truncated second moment function $\mathfrak{S}(\mathbf{x}):=\mathbf{E}\left[\left\langle\mathbf{X}_{1}, \mathbf{x}\right\rangle^{2} \mathbb{1}_{\left\{\left|\left\langle\mathbf{X}_{1}, \mathbf{x}\right\rangle\right|<1\right\}}\right]$ is regularly varying at $+\infty$ with index $(2, \ldots, 2)$. On the other hand, $\mathbf{S}$ is in the domain of attraction of a strictly non-normal law if and only if $\mathbf{P}_{\mathbf{X}_{1}}(\cdot)$ is regularly varying at infinity with index $\left(\rho_{1}, \ldots, \rho_{d}\right) \in(-2,0)^{d}$ (with $\rho_{i}=-\alpha_{i}$, the scaling sequences being $a_{n}^{(i)}=r_{i}(n)$, with $r_{i}(\cdot)$ defined by (A.2)). We stress that having $\mathbf{P}\left(\mathbf{X}_{1}>\mathbf{x}\right)$ regularly varying at $+\infty$ with index $-\left(\gamma_{1}, \ldots, \gamma_{d}\right)$ is a sufficient condition sufficient condition for being in the domain of attraction of an $\boldsymbol{\alpha}$-stable distribution, with $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{d}\right), \alpha_{i}=\gamma_{i} \wedge 2$.

Finally, let us give two examples of regularly varying distribution of $\mathbf{X}_{1}$ ( $\mathbb{N}^{d}$-valued) we have in mind, that can be thought as "fully independent" and "fully dependent" cases-one can easily think about other, intermediate, cases.
Example A.1. There are exponents $\left(\gamma_{i}\right)_{1 \leqslant i \leqslant d}$ and regularly varying functions $\left(\varphi_{i}\right)_{1 \leqslant i \leqslant d}$ such that

$$
\begin{equation*}
\mathbf{P}\left(\mathbf{X}_{1} \geqslant \mathbf{x}\right)=\prod_{i=1}^{d} \varphi_{i}\left(x_{i}\right) x_{i}^{-\gamma_{i}}, \quad \text { for } \mathbf{x} \in \mathbb{N}^{d} \tag{A.3}
\end{equation*}
$$

We have $\bar{F}_{i}(x) \sim \varphi_{i}\left(x_{i}\right) x_{i}^{-\gamma_{i}}$, and $\mathbf{P}\left(\mathbf{X}_{1}>\mathbf{x}\right)$ is regularly varying at $+\infty$ with index $-\left(\gamma_{1}, \ldots, \gamma_{d}\right)$.

Indeed, if we set $r_{i}(t)$ such that $\bar{F}_{i}\left(r_{i}(t)\right) \sim t^{-1}$ as $t \rightarrow+\infty$, we have that $r(t)=$ $\left(r_{1}(t), \ldots, r_{d}(t)\right)$ is regularly varying with index $\left(\gamma_{1}^{-1}, \ldots, \gamma_{d}^{-1}\right)$. Then, for any $\mathbf{x} \in \mathbb{R}^{d} \backslash\{0\}$, we get

$$
\lim _{t \rightarrow \infty} t \mathbf{P}\left(\mathbf{X}_{1}>r(t) \mathbf{x}\right)=\phi(\mathbf{x}) \quad \text { with } \quad \phi(\mathbf{x})=\sum_{i=1}^{d} x_{i}^{-\gamma_{i}} \mathbb{1}_{\left\{x_{j}=0 \forall j \neq i\right\}}
$$

This shows that $\mathbb{P}\left(\mathbf{X}_{1}>\mathbf{x}\right)$ is regularly varying with index $-\left(\gamma_{1}, \ldots, \gamma_{d}\right)$.
Example A.2. There exist positive exponents $\beta$, $\left(\beta_{i}\right)_{1 \leqslant i \leqslant d}$, and slowly varying functions $\psi,\left(\psi_{i}\right)_{1 \leqslant i \leqslant d}$ such that

$$
\begin{equation*}
\mathbf{P}\left(\mathbf{X}_{1} \geqslant \mathbf{x}\right)=\psi\left(\sum_{i=1}^{d} \psi_{i}\left(x_{i}\right) x_{i}^{\beta_{i}}\right) \times\left(\sum_{i=1}^{d} \psi_{i}\left(x_{i}\right) x_{i}^{\beta_{i}}\right)^{-\beta}, \quad \text { for } \mathbf{x} \in \mathbb{N}^{d} \tag{A.4}
\end{equation*}
$$

We have that $\bar{F}_{i}(x)$ is regularly varying with index $-\gamma_{i}:=-\beta \beta_{i}$, and $\mathbf{P}\left(\mathbf{X}_{1} \geqslant \mathbf{x}\right)$ is regularly varying at $+\infty$ with index $-\left(\gamma_{1}, \ldots, \gamma_{d}\right)$.

Again, setting $r_{i}(t)$ such that $\bar{F}_{i}\left(r_{i}(t)\right) \sim t^{-1}$ as $t \rightarrow+\infty$, we have that $r(t)$ is regularly varying with index $\left(\gamma_{1}^{-1}, \ldots, \gamma_{d}^{-1}\right)$. For any $\mathbf{x} \in \mathbb{R}^{d} \backslash\{0\}$, we then have that

$$
\lim _{t \rightarrow \infty} t \mathbf{P}\left(\mathbf{X}_{1}>r(t) \mathbf{x}\right)=\phi(\mathbf{x}) \quad \text { with } \phi(\mathbf{x})=\left(\sum_{i=1}^{d} x_{i}^{\beta_{i}}\right)^{-\beta}
$$

This shows that $\mathbb{P}\left(\mathbf{X}_{1}>\mathbf{x}\right)$ is regularly varying with index $-\left(\gamma_{1}, \ldots, \gamma_{d}\right)$.

## B Comments on Assumption 2.2

Let us comment on conditions (2.5) in Assumption 2.2, and in particular on the summability of $h_{x_{i}}^{(i)}\left(\left|x_{j}\right|\right) /\left(1+\left|x_{j}\right|\right)$. Indeed, in view of the discussion below Assumption 2.2, a natural idea would be to simply bound $\mathbf{P}\left(X_{1}^{(j)} \in\left[x_{j}, 2 x_{j}\right] \forall j \neq i \mid X_{1}^{(i)} \in\left[x_{i}, 2 x_{i}\right]\right)$ by 1, so items (ii)-(iii) in (2.5) would not be necessary. However, by doing so, one would not be able to derive the bound (2.6) for each coordinate (there would be an extra factor $\left(\log x_{i}\right)^{d-1}$ ), but having (2.6) turns out to be essential in our study. Similarly, condition (iii) in (2.5) may appear artificial, but it is here for technical purposes, in order to be able to bound $h_{u}^{(i)}(v)$ uniformly for $u$ in some interval (recall the proof of Theorem 2.5 in Section 5.3 ). We want to stress here that the condition (2.5) is actually very weak, and is verified in natural examples that come to mind.

## About Example 2.4

We now show that Assumption 2.2 is verified in the case of Example 2.4. For simplicity, we present calculations in the case $d=2$ without slowly varying function ( $\psi$ is a constant): for $\mathbf{x} \in \mathbb{N}^{2}, \mathbf{P}\left(\mathbf{X}_{1}=\mathbf{x}\right)=c_{0}\left(x_{1}^{\beta_{1}}+x_{2}^{\beta_{2}}\right)^{-\beta}$, with $\beta_{1}, \beta_{2}>0$ and $\beta>\beta_{1}^{-1}+\beta_{2}^{-1}$, for some constant $c_{0}$. We can write, setting $\gamma_{1}=\beta_{1}\left(\beta-\beta_{1}^{-1}-\beta_{2}^{-1}\right)$,

$$
\begin{align*}
\mathbf{P}\left(\mathbf{X}_{1}=\left(x_{1}, x_{2}\right)\right) & =c_{0} x_{1}^{-\left(1+\gamma_{1}\right)} x_{1}^{-\beta_{1} / \beta_{2}}\left(1+\frac{x_{2}^{\beta_{2}}}{x_{1}^{\beta_{1}}}\right)^{-\beta}  \tag{B.1}\\
& =\frac{c_{0}}{x_{2}} x_{1}^{-\left(1+\gamma_{1}\right)}\left(\frac{x_{2}^{\beta_{2}}}{x_{1}^{\beta_{1}}}\right)^{1 / \beta_{2}}\left(1+\frac{x_{2}^{\beta_{2}}}{x_{1}^{\beta_{1}}}\right)^{-\beta}
\end{align*}
$$

Hence, we have the bound (2.4), with

$$
\begin{equation*}
h_{u}^{(1)}(v)=\frac{v}{u^{\beta_{1} / \beta_{2}}}\left(1+\left(\frac{v}{u^{\beta_{1} / \beta_{2}}}\right)^{\beta_{2}}\right)^{-\beta} \leqslant \min \left\{\frac{v}{u^{\beta_{1} / \beta_{2}}},\left(\frac{v}{u^{\beta_{1} / \beta_{2}}}\right)^{1-\beta \beta_{2}}\right\}, \tag{B.2}
\end{equation*}
$$

the last inequality coming from considering whether $v$ is smaller or larger than $u^{\beta_{1} / \beta_{2}}$. It remains to verify that $h_{u}^{(1)}(v)$ verify (2.5): for item (i), one easily verifies that $h_{u}^{(1)}(v)$ is bounded by 1 , since $\beta>1 / \beta_{2}$; item (iii) is also trivial. For item (ii), we sum over $v$ depending on whether $v \leqslant u^{\beta_{1} / \beta_{2}}$ or $v>u^{\beta_{1} / \beta_{2}}$

$$
\sum_{v \geqslant 1} \frac{h_{u}^{(1)}(v)}{v} \leqslant \sum_{v=1}^{u^{\beta_{1} / \beta_{2}}} \frac{1}{u^{\beta_{1} / \beta_{2}}}+\frac{1}{u^{\left(1-\beta \beta_{2}\right) \beta_{1} / \beta_{2}}} \sum_{v>u^{\beta_{1} / \beta_{2}}} v^{-\beta \beta_{2}} \leqslant 1+c
$$

and the constant does not depend on $u$ (we have that $\sum_{v>u^{\beta_{1} / \beta_{2}}} v^{-\beta \beta_{2}} \leqslant c\left(u^{\beta_{1} / \beta_{2}}\right)^{1-\beta \beta_{2}}$ since $\left.\beta \beta_{2}>1\right)$. Similarly, we have $\mathbf{P}\left(\mathbf{X}_{1}=\left(x_{1}, x_{2}\right)\right)=\frac{c_{0}}{x_{1}} x_{2}^{-\left(1+\gamma_{2}\right)} h_{x_{2}}^{(2)}\left(x_{1}\right)$, with $\gamma_{2}:=$ $\beta_{2}\left(\beta-\beta_{1}^{-1}-\beta_{2}^{-1}\right)$ and $h_{u}^{(2)}(v)$ as defined in (B.2) but with $\beta_{1}$ and $\beta_{2}$ swapped.

Moreover, using a Riemann sum approximation, we get that $\sum_{v \geqslant 1} h_{u}^{(1)}(v) / v$ converges to $\int_{0}^{\infty} t\left(1+t^{\beta_{2}}\right)^{-\beta} \mathrm{d} t$ as $u \rightarrow+\infty$ (recall (B.2)). Going back to (B.1) and summing over $x_{2}$, we therefore get that $\mathbf{P}\left(X_{1}^{(1)}=x_{1}\right) \sim c_{1} x_{1}^{-\left(1+\gamma_{1}\right)}$ as $x_{1} \rightarrow+\infty$.

This can be generalized to the setting of Example 2.4: we get that Assumption 2.2 is verified, and we find that there is a constant $c_{2}$ (depending only on $\beta, \beta_{i}$ ) such that

$$
\begin{equation*}
\mathbf{P}\left(X_{1}^{(i)}=x_{i}\right) \sim c_{2} \psi\left(x_{i}^{\beta_{i}}\right) \times\left(x_{i}\right)^{-\left(1+\gamma_{i}\right)} \quad \text { as } x_{i} \rightarrow+\infty, \tag{B.3}
\end{equation*}
$$

with $\gamma_{i}:=\beta_{i}\left(\beta-\sum_{i=1}^{d} \beta_{i}^{-1}\right)$. Details are left to the reader.

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