A note on Sidon sets in bounded orthonormal systems

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Abstract

We give a simple example of an n-tuple of orthonormal elements in L_2 (actually martingale differences) bounded by a fixed constant, and hence subgaussian with a fixed constant but that are Sidon only with constant $\approx \sqrt{n}$. This is optimal. The first example of this kind was given by Bourgain and Lewko, but with constant $\approx \sqrt{\log n}$. We also include the analogous $n \times n$-matrix valued example, for which the optimal constant is $\approx n$. We deduce from our example that there are two n-tuples each Sidon with constant 1, lying in orthogonal linear subspaces and such that their union is Sidon only with constant $\approx \sqrt{n}$. This is again asymptotically optimal. We show that any martingale difference sequence with values in $[-1, 1]$ is "dominated" in a natural sense (related to our results) by any sequence of independent, identically distributed, symmetric $\{-1, 1\}$-valued variables (e.g. the Rademacher functions). We include a self-contained proof that any sequence (φ_n) that is the union of two Sidon sequences lying in orthogonal subspaces is such that $(\varphi_n \otimes \varphi_n \otimes \varphi_n \otimes \varphi_n)$ is Sidon.

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One of the most celebrated results in the theory of Sidon sets in the trigonometric system on the circle (or on a compact Abelian group) is Drury's union theorem that says that the union of two (disjoint) Sidon sets is still a Sidon set. In a recent paper Bourgain and Lewko [2] considered Sidon sets for a general uniformly bounded orthonormal system (φ_n) in L_2 over an arbitrary probability space (T, m). They extended some of the classical results known for systems of characters on compact Abelian groups. We continued on the same theme in [6]. Let us recall the basic definitions. We say that (φ_n) is Sidon if there is a constant C such that for any finitely supported scalar sequence $n \mapsto x_n$

$$\sum |x_n| \le C \|\sum x_n \varphi_n\|_\infty. \tag{1}$$

The smallest such C is called the Sidon constant of (φ_n). The system (φ_n) is called $\otimes^k$-Sidon if the system $(\varphi_n(t_1)\varphi_n(t_2)\cdots\varphi_n(t_k))$ is Sidon in $L_2(T^k, m \times \cdots \times m)$. We say that (φ_n) is subgaussian if there is a constant β such that for any finite scalar sequence (x_n) such that $\sum |x_n|^2 \le 1$ we have

$$\int e^{|\sum x_n \varphi_n|^2/\beta^2} dm \le e.$$

When this holds we say that (φ_n) is β-subgaussian.

Bourgain and Lewko [2] proved that subgaussian does not imply Sidon but does imply $\otimes^5$-Sidon, and the author [6] improved this to $\otimes^2$-Sidon.

Let (g_n) be an i.i.d. sequence of standard Gaussian random variables. We say that (φ_n) is randomly Sidon if there is a constant C such that for any finite scalar sequence (x_n) we have

$$\sum |x_n| \le C\mathbb{E}\|\sum g_n x_n \varphi_n\|_\infty.$$

In [6], we proved that randomly Sidon implies $\otimes^4$-Sidon. It follows as an immediate corollary that the union of two mutually orthogonal Sidon systems is $\otimes^4$-Sidon (see Theorem 15 for a quick outline of a direct proof). This generalizes Drury's celebrated union theorem for sets of characters. Naturally, this last result raises the question whether $\otimes^4$-Sidon can be replaced by $\otimes^k$-Sidon for $k < 4$. While we cannot decide this for $k = 2$ or $k = 3$, the goal of the present note is to settle the question at least for $k = 1$.

We first improve Bourgain and Lewko's example from [2] showing that subgaussian does not imply Sidon for uniformly bounded orthonormal systems. Our example is a (very simple) martingale difference sequence and the constant is asymptotically sharp. As a corollary we show that, not surprisingly, Drury's union theorem does not extend to two mutually orthogonal uniformly bounded orthonormal systems.

Theorem 1. *Fix $\varepsilon > 0$. There is a uniformly bounded real valued orthonormal system (φ_n) with $\|\varphi_n\|_\infty \le 1 + \varepsilon$ for all n that is subgaussian and actually satisfies*

$$\mathbb{E}e^{\sum x_n \varphi_n} \le e^{(1+\varepsilon)^2 \sum x_n^2/2} \tag{2}$$

for any finite sequence of real numbers (x_n), but (φ_n) is not a Sidon system.
More precisely, the smallest constant C_n such that for any scalar coefficients (x_k) we have

$$\sum\nolimits_1^n |x_k| \le C_n \| \sum\nolimits_1^n x_k \varphi_k \|_\infty$$

satisfies

$$\forall n \ge 1 \quad C_n \ge \delta_\varepsilon \sqrt{n}, \tag{3}$$

where $\delta_\varepsilon > 0$ depends only on ε. In addition, (φ_n) is a martingale difference sequence.

Proof. Let (ε_n) be a sequence of independent choices of signs, i.e. independent $\pm$-valued random variables on a probability space $(\Omega, \mathbb{P})$ taking the values ± 1 with probabilility $1/2$. Let $\mathcal{A}_n$ be the σ-algebra generated by $\{\varepsilon_k \mid 0 \le k \le n\}$. Let $0 = a_0 \le \cdots \le a_{n-1} \le a_n \le \cdots$ be a fixed non-decreasing sequence for the moment. Consider $A_0 = \Omega$, $S_0 = 0$, and define inductively $A_n \in \mathcal{A}_n$ and S_n as follows:

$$S_n = S_{n-1} + \varepsilon_n 1_{A_{n-1}} \text{ and } A_n = \{|S_n| \le a_n\}.$$

Assume that $\mathbb{P}(A_n) \ge \delta$ for some fixed $\delta > 0$. Then let

$$f_n = \varepsilon_n 1_{A_{n-1}}. \tag{4}$$

This is a martingale difference sequence with $\|f_n\|_\infty \le 1$, therefore an orthogonal system such that

$$S_n = f_1 + \cdots + f_n$$

and moreover

$$\|f_n\|_2^2 \ge \delta.$$

We claim that the Sidon constant of $\{f_1, \cdots, f_n\}$ is $\ge n/(1 + a_{n-1})$. This follows from the observation that

$$\forall n \quad \|S_n\|_\infty \le 1 + a_{n-1}. \tag{5}$$

Indeed, this is immediate by induction on n (since either $\|S_n\|_\infty \le a_{n-1} + 1$ or $\|S_n\|_\infty \le \|S_{n-1}\|_\infty$ depending whether $\|S_n\|_\infty$ is attained on A_{n-1} or on its complement).

Now by Azuma's inequality (see e.g. [5, p. 501]) we know that (f_n) is subgaussian with a good constant. In fact for any real numbers t and x_n with (x_n) in ℓ_2

$$(6) \qquad \mathbb{E}e^{t\sum x_n f_n} \le e^{t^2\sum |x_n|^2/2}.$$

In particular

$$(7) \qquad \mathbb{P}(\{|S_n| > t\}) \le 2e^{-t^2/2n}.$$

Fix $\varepsilon > 0$. Taking $a_n = c\sqrt{n}$, this gives us

$$\mathbb{P}(\{|S_n| > a_n\}) \le 2e^{-c^2/2},$$

so we can choose a numerical value of c, namely $c = c_\varepsilon$, large enough so that

$$\mathbb{P}(\{|S_n| > a_n\}) \le 1 - (1+\varepsilon)^{-2}.$$

Then we have by what precedes $\|S_n\|_\infty \le a_{n-1} + 1 = c_\varepsilon\sqrt{n-1} + 1$ and

$$\|f_n\|_2 = \mathbb{P}(\{|S_{n-1}| \le a_{n-1}\})^{1/2} \ge (1+\varepsilon)^{-1}$$

for all n. Therefore the Sidon constant of $\{f_1, \cdots, f_n\}$ is $\ge n/(1 + a_{n-1})$. Letting

$$\varphi_n = f_n\|f_n\|_2^{-1}$$

we find $\|\varphi_n\|_\infty \le 1 + \varepsilon$ for all n, (φ_n) is orthonormal and (3) holds. By Azuma's inequality (6) we also have (2). □

Remark 2. I am grateful to B. Maurey for suggesting the following neater example (S'_k). Let us first fix $n \ge 1$, and hence $a_n > 0$ is fixed. Let $M_k = \varepsilon_1 + \cdots + \varepsilon_k$ for all $k \ge 1$. Define the stopping time T_n by $T_n = \inf\{k \ge 0 \mid |M_k| > a_n\}$ and $T_n = \infty$ if $|M_k| \le a_n$ for all $k \ge 0$. Recall the classical inequalities

$$\forall t > 0 \quad \mathbb{P}(\{\sup_{1\le k\le n} |M_k| > t\}) \le 2\mathbb{P}(\{|M_n| > t\}) \le 4e^{-t^2/2n}.$$

The first one goes back to Paul Lévy (see e.g. [5, p. 28]), it is closely related to Désiré André's reflection principle for Brownian motion (see e.g. [4, p. 558]) and the second one follows from (7). We then set for $k \ge 1$ $S'_k = M_{k\wedge T_n}$ and

$$f'_k = S'_k - S'_{k-1} = \varepsilon_k 1_{\{T_n \ge k\}}.$$

In the previous example this corresponds to sets $A'_{k-1} = \{T_n \ge k\} = \{T_n \le k-1\}^c \in \mathcal{A}_{k-1}$. We have clearly $\|S'_k\|_\infty \le a_n + 1$ for all k, and it is easy to check, since $A'_{k-1} = \{\sup_{j<k} |M_j| \le a_n\}$, that we again can choose $a_n = c_\varepsilon\sqrt{n}$ so that for any $1 \le k \le n$ we have

$$\mathbb{P}(A'_{k-1}) \ge \mathbb{P}(A'_n) = \mathbb{P}(\{\sup_{1\le k\le n} |M_k| \le a_n\}) = 1 - \mathbb{P}(\{\sup_{1\le k\le n} |M_k| > a_n\}) \ge (1+\varepsilon)^{-2}.$$

Remark 3. Since (φ_n) is formed of mean zero variables (2) holds iff there is β' such that

$$(8) \qquad \forall p \ge 2\ \forall (x_n) \in \ell_2 \quad \|\sum x_n\varphi_n\|_p \le \beta'\sqrt{p}(\sum |x_n|^2)^{1/2}$$

Remark 4. Let (φ_n) be any orthonormal system. Then for any scalar coefficients (x_k) we have obviously

$$\sum\nolimits_1^n |x_k| \le \sqrt{n}(\sum\nolimits_1^n |x_k|^2)^{1/2} \le \sqrt{n}\|\sum\nolimits_1^n x_k\varphi_k\|_\infty.$$

Thus the order of growth of the Sidon constant in (3) and the next statement are both sharp.

Corollary 5. *There are two orthonormal martingale difference sequences (φ_n^+) and (φ_n^-) with orthogonal linear spans such that each has the same distribution as the Rademacher functions (i.e. each is formed of independent $\pm$-valued random variables with mean zero) but their union is not a Sidon system. More precisely the union of $\{\varphi_k^+ \mid k \le n\}$ and $\{\varphi_k^- \mid k \le n\}$ has a Sidon constant C_n growing like $\sqrt{n}$.*

Proof. Let will modify slightly the preceding proof and construct by induction a sequence S'_n. We wish to choose by induction a set $B_n \subset \Omega$ in $\mathcal{A}_n$ (just like A_n was) and we again set $S'_n = S'_{n-1} + \varepsilon_n 1_{B_{n-1}}$. but we choose B_n satisfying

$$B_n \subset \{|S'_n| \le a_n\} \quad \text{ and } \mathbb{P}(B_n) = 1/2. \tag{9}$$

To be able to make this choice all we need to know is that $\mathbb{P}(\{|S'_n| \le a_n\}) \ge 1/2$. Then the preceding argument, associated to $\varepsilon = \sqrt{2}-1$ still guarantees that $\mathbb{P}(\{|S'_{n-1}| \le a_{n-1}\}) \ge 1/2$. Thus we clearly can select B_n for which (9) holds and we again obtain $\|S'_n\|_\infty \le 1 + \sqrt{n-1}$ for all n.
Then let

$$\varphi_n^\pm = \varepsilon_n(1_{B_{n-1}} \pm 1_{\Omega\setminus B_{n-1}}).$$

Note that since $\mathbb{P}(B_{n-1}) = 1/2$ we have $\varphi_n^+ \perp \varphi_n^-$ for any n and hence $\varphi_n^+ \perp \varphi_k^-$ for any n, k. Then each of the sequences $\{\varphi_k^\pm \mid k \le n\}$ is a martingale difference sequence with values in $\{\pm 1\}$. It is a well known fact (proved by induction as a simple exercise) that this forces each to be distributed uniformly over all choices of signs. Now let $\{\psi_k \mid k \le 2n\}$ denote the union of the two systems $\{\varphi_k^+ \mid k \le n\}$ and $\{\varphi_k^- \mid k \le n\}$. Clearly the Sidon constant of $\{\psi_k \mid k \le 2n\}$ dominates that of $\{(\varphi_k^+ + \varphi_k^-)/2 \mid k \le n\}$. But the latter is the system $\{\varepsilon_k 1_{B_{k-1}} \mid k \le n\}$ as in the preceding proof but with B_k replacing A_k. Since $\|S'_n\|_\infty \le 1 + \sqrt{n-1}$, (3) still holds for this system, so the corollary follows. □

Remark 6. We may clearly replace (ε_n) by an i.i.d. sequence of complex valued variables (z_n) uniformly distributed over the unit circle of $\mathbb{C}$. For those it is still true that for any unimodular sequence (w_n) that is adapted (i.e. w_n is $\mathcal{A}_n$-measurable for each n) the sequence $(z_n w_{n-1})$ is independent and uniformly distributed over the unit circle. Then the corresponding two sequences $(\varphi_n^\pm)$ are Sidon with constant 1, and their union is not Sidon for the same reason as in the preceding corollary.

Problem : In [2] Bourgain and Lewko show that any n-tuple forming a β-subgaussian orthonormal system uniformly bounded by a constant C contains a subset of cardinality $\ge \theta n$ with $\theta = \theta(\beta, C) > 0$ that is Sidon with Sidon constant at most $f(\beta, C)$. They ask whether any such system is actually the union of $k(\beta, C)$ Sidon sequences with Sidon constant at most $f(\beta, C)$.
Is this true for uniformly bounded martingale difference sequences normalized in L_2 ?

Although for the example appearing in the proof of Theorem 1 the answer is affirmative (consider e.g. a partition into odd and even k's), we believe that a more involved one with values in $\{-1, 0, 1\}$ as in (4) but with a more subtle choice of the predictable sets A_{n-1}, should yield a counterexample.

Let M_n be the space of $n \times n$-matrices with complex entries, equipped with the usual operator norm on the n-dimensional Hilbert space. In [6] we consider a non-commutative analogue involving a $n \times n$-matrix valued function $\varphi(t) = [\varphi(t)_{ij}]$ on a probability space (T, m) for which the uniform boundedness condition is replaced by

$$\|\varphi(t)\|_{M_n} \leq C$$

and we assume that $\{\sqrt{n}\varphi(t)_{ij} \mid 1 \leq i, j \leq n\}$ is β-subgaussian and orthonormal. The prototypical example is when φ is uniformly distributed over the unitary group.
In this situation we prove in [6, Prop. 5.4] that there is a constant $\alpha = \alpha(C, \beta)$ such that

$$\forall a \in M_n \quad \mathrm{tr}|a| \leq \alpha \sup_{t_1, t_2 \in T} |\mathrm{tr}(a\varphi(t_1)\varphi(t_2))|.$$

In analogy with Theorem 1 it is natural to wonder what is the best constant C'_n such that in the same situation

$$\forall a \in M_n \quad \mathrm{tr}|a| \leq C'_n \sup_{t \in T} |\mathrm{tr}(a\varphi(t))|.$$

Clearly the orthonormality assumption yields

$$\forall a \in M_n \quad n^{-1}\mathrm{tr}|a| \leq (n^{-1}\mathrm{tr}|a|^2)^{1/2} = \|\mathrm{tr}(a\varphi(t)\|_2 \leq \|\mathrm{tr}(a\varphi(t)\|_\infty = \sup_{t \in T} |\mathrm{tr}(a\varphi(t)|.$$

and hence $C'_n \leq n$.
It is easy to see that this is asymptotically optimal. Indeed, consider the following example. Let $x \mapsto D(x)$ be the mapping taking an $n \times n$ matrix to its diagonal part. Let u denote a random $n \times n$ unitary matrix uniformly distributed over the unitary group. Let $(\varphi_1, \cdots, \varphi_n)$ be the orthonormal n-tuple constructed in the proof of Theorem 1, of which we keep the notation, namely $\varphi_k = f_k\|f_k\|_2^{-1}$. Assuming (T, m) large enough, we define $\varphi : T \to M_n$ so that $\varphi - D(\varphi)$ and $D(\varphi)$ are independent random variables; we make sure that $\varphi - D(\varphi)$ and $u - D(u)$ have the same distribution and we adjust the diagonal entries of $D(\varphi)$ so that they have the same distribution as $(\varphi_1/\sqrt{n}, \cdots, \varphi_n/\sqrt{n})$. Then for a suitable β (independent of n) $\{\sqrt{n}\varphi(t)_{ij} \mid 1 \leq i, j \leq n\}$ is β-subgaussian and orthonormal. However, if a is the diagonal matrix with entries $(\|f_1\|_2, \cdots, \|f_n\|_2)$ we have on one hand by (5) $\|\mathrm{tr}(a\varphi)\|_\infty = \|(f_1 + \cdots + f_n)/\sqrt{n}\|_\infty \leq c_\varepsilon$, and on the other hand $\mathrm{tr}|a| \geq n(1+\varepsilon)^{-1}$. Therefore

$$C'_n \geq n(1+\varepsilon)^{-1}c_\varepsilon^{-1}.$$

Definition 7. Let I be an index set. Let $L_1(m'), L_1(m'')$ be arbitrary L_1-spaces. We say that a family $(f_n)_{n \in I}$ in $L_1(m'')$ is c-dominated by another one $(\psi_n)_{n \in I}$ in $L_1(m')$ if there is a linear map $u : L_1(m') \to L_1(m'')$ with $\|u\| \leq c$ such that $u(\psi_n) = f_n$ for all $n \in I$.

The following criterion due to M. Lévy (see [3] and [6, Prop.1.5]) is very useful: a linear map $v : E \to L_1(m'')$ on a subspace $E \subset L_1(m')$ admits an extension $u : L_1(m') \to L_1(m'')$ with $\|u\| \leq 1$ iff for any finite sequence (η_n) in E we have

$$\|\sup|v(\eta_n)|\|_{L_1(m'')} \leq \|\sup|\eta_n|\|_{L_1(m')}. \tag{10}$$

If we apply this to $E = \mathrm{span}[\psi_n]$ with v defined by $v(\psi_n) = f_n$, this gives us the following criterion: a sequence $(f_n)_{n \in I}$ in $L_1(m'')$ is c-dominated by a sequence $(\psi_n)_{n \in I}$ in $L_1(m')$ iff for any Banach space B and any finite sequence (x_n) in B we have

$$\|\sum f_n x_n\|_{L_1(B)} \leq c\|\sum \psi_n x_n\|_{L_1(B)}. \tag{11}$$

Indeed, it is easy to see that we may restrict consideration to the single space $B = \ell_\infty$, in which case (10) and (11) are identical.

Remark 8. The key fact used in [6] is that, for some numerical constant K, any β-subgaussian sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $X = L_1(T, m)$ is $K\beta$-dominated by a standard i.i.d. sequence of Gaussian normal variables (on a probability space $(\Omega', \mathbb{P}')$), denoted by $(g_n)_{n\in\mathbb{N}}$. This is essentially due to Talagrand; see [6] for detailed references and comments. It would be interesting to have a direct simple proof of this fact.

If we assume moreover that the β-subgaussian sequence $(\varphi_n)_{n\in\mathbb{N}}$ is uniformly bounded, i.e. that $\|\varphi_n\|_\infty \le \alpha$ for all n, then, for some numerical constant K', the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is $K'(\beta + \alpha)$-dominated by (ε_n). This follows from the solution by Bednorz and Latała [1] of Talagrand's Bernoulli conjecture.

We would like to observe that if (f_n) is a martingale difference sequence then a very simple proof is available (with an optimal constant). We start with a special case of the form $f_n = \varepsilon_n\varphi_{n-1}$ with φ_{n-1} depending only on $\varepsilon_1, \cdots, \varepsilon_{n-1}$ satisfying $\|\varphi_{n-1}\|_\infty \le 1$ (which is subgaussian by (6)). This is particularly easy. Indeed, for any $y \in [-1, 1]$ let

$$F(t, y) = (-1)1_{[0,(1-y)/2]}(t) + (1)1_{((1-y)/2,1]}(t),$$

so that $\int_0^1 F(t, y)dt = y$ and $F(t, y) = \pm 1$. Let us consider the sequence of random variables F_n defined on $[0, 1]^{\mathbb{N}} \times \{-1, 1\}^{\mathbb{N}}$ by setting

$$F_n((t_j), (\varepsilon_j)) = \varepsilon_n F(t_{n-1}, \varphi_{n-1}).$$

Let u be the conditional expectation onto the algebra of functions depending on the second variable on $[0, 1]^{\mathbb{N}} \times \{-1, 1\}^{\mathbb{N}}$. Then $u(F_n) = f_n$. Moreover since (F_n) is a martingale with values in ± 1 it has the same distribution as (ε_n) itself. In other words, there is an isometry $v : L_1(\Omega, \mathbb{P}) \to L_1([0, 1]^{\mathbb{N}} \times \{-1, 1\}^{\mathbb{N}})$ such that $v(\varepsilon_n) = F_n$ for all n. Considering the composition uv, this shows that (f_n) is 1-dominated by (ε_n), and the latter is easily shown to be c-dominated by (g_n) (the latter being, say, in $L_1(\Omega', \mathbb{P}')$) for some numerical constant c.

More generally, let $(\Omega', \mathcal{A}', \mathbb{P}')$ be an arbitrary probability space. We have

Lemma 9. *Let $\varphi \in L_1(\Omega', \mathcal{A}', \mathbb{P}')$ be with values in $[-1, 1]$ and such that $\mathbb{E}\varphi = 0$. Then for any Banach space B and any $x_0, x_1 \in B$*

$$\mathbb{E}'\|x_0 + \varphi x_1\| \le \mathbb{E}\|x_0 + \varepsilon_1 x_1\|. \tag{12}$$

More generally, if $\mathcal{B} \subset \mathcal{A}'$ is any σ-subalgebra such that $\mathbb{E}^{\mathcal{B}}\varphi = 0$ we have for any $x_0 \in L_1(\Omega', \mathcal{B}, \mathbb{P}'; B)$

$$\mathbb{E}'\|x_0 + \varphi x_1\| \le \mathbb{E}'\mathbb{E}\|x_0 + \varepsilon_1 x_1\|. \tag{13}$$

Proof. We have

$$x_0 + \varphi x_1 = \int x_0 + F(t, \varphi)x_1 dt.$$

and hence by Jensen

$$\|x_0 + \varphi x_1\| \le \int \|x_0 + F(t, \varphi)x_1\|dt = \|x_0 - x_1\|(1 - \varphi)/2 + \|x_0 + x_1\|(1 + \varphi)/2.$$

After integration, we obtain (12). To prove (13) it suffices to show that

$$\mathbb{E}^{\mathcal{B}}\|x_0 + \varphi x_1\| \le \mathbb{E}^{\mathcal{B}}(\|x_0 + x_1\| + \|x_0 - x_1\|)/2, \tag{14}$$

or equivalently that for any $A \in \mathcal{B}$ with $\mathbb{P}'(A) > 0$ we have

$$\mathbb{P}'(A)^{-1} \int_A \|x_0 + \varphi x_1\| d\mathbb{P}' \le \mathbb{P}'(A)^{-1} \int_A (\|x_0 + x_1\| + \|x_0 - x_1\|)/2 d\mathbb{P}', \tag{15}$$

Assume that $A \in \mathcal{B}$ is an atom of $\mathcal{B}$. Then x_0 is constant on A and $\mathbb{E}^{\mathcal{B}}$ when restricted to A coincides with the average over A. Thus (15) reduces to (13) with $\mathbb{P}'$ replaced by $\mathbb{P}'(A)^{-1}\mathbb{P}'_{|A}$. The case of a general $A \in \mathcal{B}$ can be proved by a routine approximation argument left to the reader. □

We now show that any real valued martingale difference sequence with values in $[-1, 1]$ is 1-dominated by (ε_n).

Lemma 10. *Let (d_n) be a sequence of real valued martingale differences on $(\Omega', \mathcal{A}', \mathbb{P}')$, i.e. there are σ-subalgebras $\mathcal{A}_n \subset \mathcal{A}$ $(n \ge 0)$ forming an increasing filtration such that d_n is $\mathcal{A}_n$-measurable for all $n \ge 0$ and $\mathbb{E}^{\mathcal{A}_{n-1}} d_n = 0$ for all $n \ge 1$. We assume that $\mathcal{A}_0$ is trivial (so that d_0 is constant). If $|d_n| \le 1$ a.s. for any n, then there is an operator $u : L_1(\Omega, \mathcal{A}, \mathbb{P}) \to L_1(\Omega', \mathcal{A}', \mathbb{P}')$ with $\|u\| = 1$ such that $u(1) = 1$ and $u(\varepsilon_n) = d_n$ for all $n \ge 1$.*

Proof. By the above criterion (11) it suffices to show that for any Banach space B and any finite sequence (x_n) in B we have for any k

$$\|d_0 x_0 + \sum\nolimits_1^k d_n x_n\|_{L_1(B)} \le \|d_0 x_0 + \sum\nolimits_1^k \varepsilon_n x_n\|_{L_1(B)}. \tag{16}$$

By (13) with $\mathcal{B} = \mathcal{A}_{k-1}$ and $\varphi = d_k$ we have

$$\|d_0 x_0 + \sum\nolimits_1^k d_n x_n\|_{L_1(B)} \le \|d_0 x_0 + \sum\nolimits_1^{k-1} d_n x_n + \varepsilon_k x_k\|_{L_1(\mathbb{P}' \times \mathbb{P}; B)}.$$

Now working on the product space $(\Omega, \mathcal{A}, \mathbb{P}) \times (\Omega', \mathcal{A}', \mathbb{P}')$ with $\mathcal{B}$ equal to $\sigma(\mathcal{A}_{k-2} \cup \varepsilon_k)$ we find

$$\|d_0 x_0 + \sum\nolimits_1^{k-1} d_n x_n + \varepsilon_k x_k\|_{L_1(\mathbb{P}' \times \mathbb{P}; B)} \le \|d_0 x_0 + \sum\nolimits_1^{k-2} d_n x_n + \varepsilon_{k-1} x_{k-1} + \varepsilon_k x_k\|_{L_1(\mathbb{P}' \times \mathbb{P}; B)}.$$

Continuing in this way we obtain (16). □

Remark 11 (On the complex valued case in Lemma 10). Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the (one dimensional) torus. Consider the sequence $(z_n)_{n \in \mathbb{N}}$ formed of the coordinate functions on $\mathbb{T}^{\mathbb{N}}$ equipped with its normalized Haar measure μ. A priori the complex analogue of the preceding proof, with (z_n) replacing (ε_n), requires to assume that the martingale under consideration is a Hardy martingale in the sense described e.g. in [5, p. 133]. Indeed, the Poisson kernel is the natural analogue of the barycentric argument we use for Lemma 9. Using this, Lemma 10 remains valid, with (z_n) replacing (ε_n), for a martingale difference sequence (d_n) adapted to the usual filtration on $\mathbb{T}^{\mathbb{N}}$ such that for any n the variable $z \mapsto d_n(z_0, \cdots, z_{n-1}, z)$ is either analytic or anti-analytic.
Note that without any additional assumption the complex valued case of Lemma 10 fails, simply because the system $(1, \varepsilon_1)$ is not 1-dominated by $(1, z_1)$. Indeed, by (10) this would imply the inequality $2 = \int \max\{|1 + \varepsilon_1|, |1 - \varepsilon_1|\} d\mathbb{P} \le \int \max\{|1 + z_1|, |1 - z_1|\} d\mu$, which clearly fails.

The next two remarks will be used at the very end of this paper.

Remark 12. Let $(z_n)_{n \in \mathbb{N}}$ and μ on $\mathbb{T}^{\mathbb{N}}$ be as in Remark 11. Consider two sequences (f_n^1) and (f_n^2) in an L_1-space X. We form their "disjoint union" (f_n) by setting $f_{2k} = f_k^2$ and $f_{2k+1} = f_k^1$. We claim that if (f_n^1) (resp. (f_n^2)) is c_1-dominated (resp. c_2-dominated) by (z_n), then (f_n) is $(c_1 + c_2)$-dominated by (z_n). Actually, the same claim is valid for the disjoint union of arbitrary families indexed by sets I_1 and I_2 (using $(z_n)_{n\in I_1\dot\cup I_2}$ on $\mathbb{T}^{I_1\dot\cup I_2}$ instead), but the idea is easier to describe with $I=\mathbb{N}$. Indeed, since (z_n), (z_{2n}) and (z_{2n+1}) all have the same distribution, there is $u_j:\ L_1(\mathbb{T}^{\mathbb{N}},\mu)\to X$ $(j=1,2)$ with $\|u_j\|\le c_j$ such that $u_2(z_{2n})=f_n^2$ and $u_1(z_{2n+1})=f_n^1$. Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be the conditional expectations on $L_1(\mathbb{T}^{\mathbb{N}},\mu)$ with respect to the σ-algebras generated respectively by (z_{2n+1}) and (z_{2n}). Then let $u=u_1\mathbb{E}_1+u_2\mathbb{E}_2$. We have $u(z_n)=f_n$ for all n and $\|u\|\le\|u_1\mathbb{E}_1\|+\|u_2\mathbb{E}_2\|\le c_1+c_2$. This proves our claim.

Remark 13. Let (z_n) be as in Remark 12 on $(\mathbb{T}^{\mathbb{N}},\mu)$. Let (φ_n) be in $L_\infty(T,m)$. We claim that if $\|\varphi_n\|_\infty\le 1$ for all n, then $(\varphi_n\otimes z_n)$ is dominated by (z_n). Assume first $|\varphi_n|=1$ a.e. for all n. Then the translation invariance of the distribution of (z_n) shows that $(\varphi_n\otimes z_n)$ has the same distribution as (z_n), so the claim is obvious in this case. Note that any number $\varphi\in\mathbb{C}$ with $|\varphi|\le 1$ is an average of two points on the unit circle. Using this it is easy to verify the claim. It can also be checked easily using the criterion in (11).

We end this paper by an outline of a proof that the union of two Sidon sequences is $\otimes^4$-Sidon, more direct than the one in [6]. The route we use avoids the consideration of randomly Sidon sequences, it is essentially the commutative analogue of the proof in [7], with the free Abelian group replacing the free group. The key fact for the latter route is still the following:

Lemma 14. *Let (z_n) be in $L_\infty(\mathbb{T}^{\mathbb{N}},\mu)$ as in Remark 12. Let (T,m) be a probability space. Let (f_n) be a sequence in $L_1(T,m)$ that is dominated by (z_n). Then any sequence (ψ_n) in $L_\infty(T,m)$ that is both uniformly bounded and biorthogonal to (f_n) is $\otimes^2$-Sidon. Here biorthogonal means*

$$\forall n,m\qquad\int\psi_n f_m=\delta_{nm}.$$

Proof. Let $u:L_1(\mathbb{T}^{\mathbb{N}},\mu)\to L_1(T,m)$ such that $u(z_n)=f_n$. Elementary considerations show that it suffices to show that the sequence $(u^*(\psi_n))$ is $\otimes^2$-Sidon. By another elementary argument $(u^*(\psi_n))$ is biorthogonal to (z_n). Therefore, it suffices to prove this Lemma for the case $(T,m)=(\mathbb{T}^{\mathbb{N}},\mu)$ and $(\psi_n)=(z_n)$. This is proved in [6] with (z_n) replaced by an i.i.d. gaussian sequence, using the Ornstein-Uhlenbeck (or Mehler) semigroup. Here we may use Riesz products instead.

We claim that for any N and any $z^0\in\mathbb{T}^{\mathbb{N}}$ the function $F=\sum_1^N z_n^0 z_n\otimes z_n$ admits for any $0<\varepsilon\le 1$ a decomposition $F=t_\varepsilon+r_\varepsilon$ in the algebraic tensor product $L_1(\mathbb{T}^{\mathbb{N}})\otimes L_1(\mathbb{T}^{\mathbb{N}})$ with

$$\|t_\varepsilon\|_\wedge=\int|t_\varepsilon(x,y)|d\mu(x)d\mu(y)\le w(\varepsilon)\text{ and }\|r_\varepsilon\|_\vee\le\varepsilon,$$

where we have set

$$\|r_\varepsilon\|_\vee=\sup\nolimits_{a,b\in B_{L_\infty}}\left|\int r_\varepsilon(x,y)a(x)b(y)d\mu(x)d\mu(y)\right|,$$

and where $w(\varepsilon)$ is a function depending only on $0<\varepsilon\le 1$ (and not on N or z^0). To verify this we fix z^0 and consider in $L_1(\mathbb{T}^{\mathbb{N}}\times\mathbb{T}^{\mathbb{N}},\mu\times\mu)$ the Riesz product

$$\nu_\varepsilon(z,z')=\prod\nolimits_1^N(1+\varepsilon\Re(z_n^0 z_n z_n'))=\sum\nolimits_{\alpha\subset[1\dots N]}(\varepsilon/2)^{|\alpha|}\prod_{n\in\alpha}(z_n^0 z_n z_n'+\overline{z_n^0 z_n z_n'}).$$

We will view the tensors in $L_1(\mathbb{T}^{\mathbb{N}})\otimes L_1(\mathbb{T}^{\mathbb{N}})$ as functions of $(z,z')\in\mathbb{T}^{\mathbb{N}}\times\mathbb{T}^{\mathbb{N}}$. Note

$$\nu_\varepsilon(z,z')=\sum\nolimits_{\alpha\subset[1\dots N]}(\varepsilon/2)^{|\alpha|}\sum_{\beta\subset\alpha}\prod_{n\in\beta}z_n^0 z_n z_n'\prod_{n\in[1\dots N]\setminus\beta}\overline{z_n^0 z_n z_n'}.\tag{17}$$

Observe that the terms of the latter sum are orthogonal. Without trying to optimize (see [6] for a discussion of the optimal logarithmic growth for w) we set

$$t'_\varepsilon = (\nu_\varepsilon - \nu_0)/\varepsilon.$$

Note that (since $\nu_\varepsilon \geq 0$ and hence $\|\nu_\varepsilon\|_1 = 1$) we have $\|t'_\varepsilon\|_\wedge \leq 2/\varepsilon$. Let $r'_\varepsilon = \sum_1^N \Re(z_n^0 z_n z'_n)) - t'_\varepsilon$. By the orthogonality in the sum (17) one checks that $\|r'_\varepsilon\|_\vee \leq \varepsilon/2$. This gives us the desired decomposition but, instead of $\sum_1^N z_n^0 z_n z'_n$, we are decomposing the sum

$$\sum\nolimits_1^N \Re(z_n^0 z_n z'_n)) = (1/2)\sum\nolimits_1^N z_n^0 z_n z'_n + (1/2)\sum\nolimits_1^N \overline{z_n^0 z_n z'_n}.$$

To remove the second term we introduce an extra variable $\omega \in \mathbb{T}$ that acts on $\mathbb{T}^{\mathbb{N}}$ by multiplication (i.e. $\omega(z_n) = (\omega z_n)$) and we define (here $m_{\mathbb{T}}$ is normalized Haar measure on $\mathbb{T}$)

$$t_\varepsilon(z, z') = 2\int \bar{\omega} t'_\varepsilon(\omega z, z') dm_{\mathbb{T}}(\omega) \text{ and } r_\varepsilon(z, z') = 2\int \bar{\omega} r'_\varepsilon(\omega z, z') dm_{\mathbb{T}}(\omega).$$

This gives us $\|t_\varepsilon\|_\wedge \leq 4/\varepsilon$ and $\|r_\varepsilon\|_\vee \leq \varepsilon$. Moreover we have

$$(1/2)\sum\nolimits_1^N z_n^0 z_n z'_n = (1/2)t_\varepsilon + (1/2)r_\varepsilon$$

which proves the claim with $w(\varepsilon) \leq 4/\varepsilon$.

We can now complete the proof. Let (a_n) be a scalar sequence. Let $\Psi = \sum_1^N a_n \psi_n \otimes \psi_n$. Choosing z_n^0 so that $z_n^0 a_n = |a_n|$ we have

$$\langle \Psi, F\rangle = \sum z_n^0 a_n = \sum |a_n|,$$

and hence $\sum |a_n| = \langle \Psi, t_\varepsilon\rangle + \langle \Psi, r_\varepsilon\rangle$ which leads to

$$\sum |a_n| \leq \|\Psi\|_\infty w(\varepsilon) + \sum |a_n| \varepsilon (\sup_{1\leq n\leq N} \|\psi_n\|_\infty^2).$$

To conclude, we set $C' = \sup_{n\geq 1} \|\psi_n\|_\infty$ and we choose, say, $\varepsilon = 1/2C'^2$. We have then

$$\sum |a_n| \leq 2w(\varepsilon)\|\Psi\|_\infty.$$

□

Let us say that a bounded set S in $L_\infty(T, m)$ is Sidon with constant C if for any finitely supported function $x : S \to \mathbb{C}$ we have $\sum_{\varphi\in S} |x(\varphi)| \leq C\|\sum x(\varphi)\varphi\|$. If (φ_n) is an enumeration of S, this is the same as $\sum_{n\in\mathbb{N}} |x(n)| \leq C\|\sum_{n\in\mathbb{N}} x(n)\varphi_n\|$. Similarly we extend the term $\otimes^4$-Sidon to sets in $L_\infty(T, m)$.

For the convenience of the reader we give a slightly more direct proof of the following result from [6], which generalizes Drury's theorem.

Theorem 15. *Let $\Lambda_1 = \{\varphi_n^1 \mid n \in I(2)\}$ and $\Lambda_2 = \{\varphi_n^2 \mid n \in I(1)\}$ be two Sidon sets (indexed by sets $I(1), I(2)$) in $L_\infty(T, m)$, with constants C_1, C_2. Assume that $\Lambda_1 \perp \Lambda_2$ in $L_2(m)$ and there are $C'_1, C'_2, \delta > 0$ such that*

$$\forall n \quad \delta \leq \|\varphi_n^1\|_2 \leq \|\varphi_n^1\|_\infty \leq C'_1 \text{ and } \delta \leq \|\varphi_n^2\|_2 \leq \|\varphi_n^2\|_\infty \leq C'_2.$$

Then the union $\Lambda_1 \cup \Lambda_2$ is $\otimes^4$-Sidon with a constant C depending only on $C_1, C_2, C'_1, C'_2, \delta$.

Proof. We assume for simplicity that the sets are sequences indexed by $\mathbb{N}$. By homogeneity (changing C_1', C_2' accordingly) we may assume that $\|\varphi_n^1\|_2 = \|\varphi_n^2\|_2 = 1$ for all n. Let $E_j \subset L_\infty(T,m)$ be the norm closed span of (φ_n^j) $(j = 1,2)$. Consider the linear mapping $T_j : E_j \to L_\infty(\mathbb{T}^{\mathbb{N}})$ such that $T_j(\varphi_n^j) = z_n$. By assumption $\|T_j\| \le C_j$. By the injectivity of L_∞-spaces T_j has an extension $\widetilde{T_j} : L_\infty(T,m) \to L_\infty(\mathbb{T}^{\mathbb{N}})$ such that $\widetilde{T_j}_{|E_j} = T_j$ and $\|\widetilde{T_j}\| = \|T_j\| \le C_j$. We introduce the operator $\mathcal{T} : L_\infty(T,m) \to L_\infty(\mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}})$ defined by

$$\mathcal{T}(f)(z,z') = \widetilde{T_1}(f)(z) + \widetilde{T_2}(f)(z').$$

Then $\|\mathcal{T}\| \le C_1 + C_2$. The operator $\mathcal{T} \otimes id_{L_\infty(T,m)}$ clearly extends to an bounded operator

$$W : L_\infty(T \times T) \to L_\infty(\mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}} \times T),$$

satisfying $\|W\| \le \|\mathcal{T}\| \le C_1 + C_2$.

We claim that the collection

$$\mathcal{U} = \{W(\varphi_n^1 \otimes \varphi_n^1)\} \cup \{W(\varphi_n^2 \otimes \varphi_n^2)\}$$

is biorthogonal to

$$\mathcal{V} = \{\overline{z_n \otimes 1 \otimes \varphi_n^1}\} \cup \{\overline{1 \otimes z_n \otimes \varphi_n^2}\}.$$

Indeed, note $W(\varphi_n^1 \otimes \varphi_n^1) \subset L_\infty(\mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}}) \otimes \varphi_n^1$ and $W(\varphi_n^2 \otimes \varphi_n^2) \subset L_\infty(\mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}}) \otimes \varphi_n^2$. Therefore, by our $L_2(m)$-orthogonality assumption

$$\forall n,m \quad W(\varphi_n^1 \otimes \varphi_n^1) \perp 1 \otimes z_m \otimes \varphi_m^2 \text{ and } W(\varphi_n^2 \otimes \varphi_n^2) \perp z_m \otimes 1 \otimes \varphi_m^2.$$

Moreover, if we set $\xi_n^1 = \widetilde{T_2}(\varphi_n^1)$ we have

$$W(\varphi_n^1 \otimes \varphi_n^1) = \mathcal{T}(\varphi_n^1) \otimes \varphi_n^1 = z_n \otimes 1 \otimes \varphi_n^1 + 1 \otimes \xi_n^1 \otimes \varphi_n^1,$$

which shows that $(W(\varphi_n^1 \otimes \varphi_n^1))$ is biorthogonal to $\{\overline{z_n \otimes 1 \otimes \varphi_n^1}\}$. Similarly $(W(\varphi_n^2 \otimes \varphi_n^2))$ is biorthogonal to $\{\overline{1 \otimes z_n \otimes \varphi_n^2}\}$. This proves the claim.
By Remarks 13 and 12, the family $\mathcal{V} = \{\overline{z_n \otimes 1 \otimes \varphi_n^1}\} \cup \{\overline{1 \otimes z_n \otimes \varphi_n^2}\}$ is dominated in $L_1(\mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}} \times T)$ by the sequence (z_n). By Lemma 14 we conclude that $\mathcal{U}$ is $\otimes^2$-Sidon in $L_\infty(\mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}} \times T)$. Since W is bounded this implies that $\{\varphi_n^1 \otimes \varphi_n^1\} \cup \{\varphi_n^2 \otimes \varphi_n^2\}$ is also $\otimes^2$-Sidon in $L_\infty(T \times T)$. Consequently $\Lambda_1 \cup \Lambda_2$ is $\otimes^4$-Sidon in $L_\infty(T,m)$. The assertion about the constant C is easy to check by going over the various steps. □

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