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More Infinite Products: Thue-Morse and the Gamma function

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Abstract

Letting (t_n) denote the Thue-Morse sequence with values $0, 1$, we note that the Woods-Robbins product

$$\prod_{n \geq 0} \left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_n}} = 2^{-1/2}$$

involves a rational function in n and the ± 1 Thue-Morse sequence $((-1)^{t_n})_{n \geq 0}$. The purpose of this paper is twofold. On the one hand, we try to find other rational functions for which similar infinite products involving the ± 1 Thue-Morse sequence have an expression in terms of known constants. On the other hand, we also try to find (possibly different) rational functions R for which the infinite product $\prod R(n)^{t_n}$ also has an expression in terms of known constants.

1 Introduction

Several infinite products involving the sum of binary digits of the integers were inspired by the discovery of the Woods and Robbins infinite product (see [16, 12]). More precisely, letting t_n denote the sum, modulo 2, of the binary digits of the integer n, the sequence $(t_n)_{n \geq 0} = 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1 \ldots$ is called the Thue-Morse sequence with values 0 and 1 (see, e.g., [5] and the references therein). The Woods-Robbins product identity is

$$\prod_{n \geq 0} \left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_n}} = \frac{1}{\sqrt{2}}. \quad (1)$$

Several infinite products inspired by (1) were discovered later (see, e.g., [3, 4, 1, 6, 10]). They all involve, as exponents, sequences of the form $(-1)^{u_{w,b}(n)}$ where $u_{w,b}(n)$ is the number, reduced modulo 2, of occurrences of the word (the block) w in the b-ary expansion of the integer n. But none of these products are in terms of 0-1-sequences $(u_{w,b}(n))_{n\geq 0}$ alone. In particular, none of them are in terms of the binary sequence $(t_n)_{n\geq 0} = (u_{1,2}(n))_{n\geq 0}$ given above. Furthermore, there has been no attempt up to now to find explicitly-given large classes of rational functions R for which the infinite product $\prod R(n)^{(-1)^{t_n}}$ has an expression in terms of known constants.

The purpose of this paper is thus twofold. First, to find other infinite products of the form $\prod R(n)^{(-1)^{t_n}}$ admitting an expression in terms of known constants. Second, to find infinite products of the form $\prod R(n)^{t_n}$ also having an expression in terms of known constants. Two examples that we find are

$$\prod_{n\geq 0} \left(\frac{4n+1}{4n+3}\right)^{(-1)^{t_n}} = \frac{1}{2}.$$

$$\prod_{n\geq 0} \left(\frac{(4n+1)(4n+4)}{(4n+2)(4n+3)}\right)^{t_n} = \frac{\pi^{3/4}\sqrt{2}}{\Gamma(1/4)}.$$

2 Products of the form $\prod R(n)^{(-1)^{t_n}}$

We start with a lemma about the convergence of infinite products involving the sequence $((-1)^{t_n})$.

Lemma 2.1. *Let t_n be the sum, reduced modulo 2, of the binary digits of the integer n. Let $R \in \mathbb{C}(X)$ be a rational function such that the values $R(n)$ are defined for $n \geq 1$. Then the infinite product $\prod_n R(n)^{(-1)^{t_n}}$ converges if and only if the numerator and the denominator of R have same degree and same leading coefficient.*

Proof. If the infinite product converges, then $R(n)$ must tend to 1 when n tends to infinity. Thus the numerator and the denominator of R have the same degree and the same leading coefficient.

Now suppose that the numerator and the denominator of R have the same leading coefficient and the same degree. Decomposing them into factors of degree 1, it suffices, for proving that the infinite product converges, to show that infinite products of the form $\prod_{n\geq 1} \left(\frac{n+b}{n+c}\right)^{(-1)^{t_n}}$ converge for complex numbers b and c such that $n+b$ and $n+c$ do not vanish for any $n \geq 1$. Since the general factor of such a product tends to 1, this is equivalent, grouping the factors pairwise, to proving that the product

$$\prod_{n\geq 1} \left[\left(\frac{2n+b}{2n+c}\right)^{(-1)^{t_{2n}}} \left(\frac{2n+1+b}{2n+1+c}\right)^{(-1)^{t_{2n+1}}}\right]$$

converges. Since $(-1)^{t_{2n}} = (-1)^{t_n}$ and $(-1)^{t_{2n+1}} = -(-1)^{t_n}$ we only need to prove that the infinite product

$$\prod_{n\geq 1} \left(\frac{(2n+b)(2n+1+c)}{(2n+c)(2n+1+b)} \right)^{(-1)^{t_n}}$$

converges. Taking the (principal determination of the) logarithm, we see that

$$\log \left(\frac{(2n+b)(2n+1+c)}{(2n+c)(2n+1+b)} \right) = \mathcal{O}(1/n^2),$$

which gives the convergence result. $\square$

In order to study the infinite product $\prod_{n\geq 1} R(n)^{(-1)^{t_n}}$, it suffices, using Lemma 2.1 above, to study products of the form $\prod_n \left(\frac{n+a}{n+b}\right)^{(-1)^{t_n}}$ where a and b belong to $\mathbb{C} \setminus \{-1, -2, -3, \ldots\}$.

Theorem 2.2. *Define*

$$f(a,b) := \prod_{n\geq 1} \left(\frac{n+a}{n+b} \right)^{(-1)^{t_n}} \quad \text{and} \quad g(x) := \frac{f(\frac{x}{2}, \frac{x+1}{2})}{x+1}$$

for a, b, x complex numbers that are not negative integers. Then

$$f(a,b) = \frac{g(a)}{g(b)}.$$

Furthermore, g satisfies the functional equation

$$(1+x)g(x) = \frac{g(\frac{x}{2})}{g(\frac{x+1}{2})} \qquad \forall x \in \mathbb{C} \setminus \{-1, -2, -3, \ldots\}.$$

In particular we have $g(1/2) = 1$ and $g(1) = \sqrt{2}/2$.

Proof. Recall that $(-1)^{t_{2n}}$ and $(-1)^{t_{2n+1}} = -(-1)^{t_n}$. Hence

$$\begin{aligned}
f(a,b) = \prod_{n\geq 1} \left(\frac{n+a}{n+b} \right)^{(-1)^{t_n}} &= \prod_{n\geq 1} \left(\frac{2n+a}{2n+b} \right)^{(-1)^{t_{2n}}} \prod_{n\geq 0} \left(\frac{2n+1+a}{2n+1+b} \right)^{(-1)^{t_{2n+1}}} \\
&= \prod_{n\geq 1} \left(\frac{(2n+a)(2n+1+b)}{(2n+b)(2n+1+a)} \right)^{(-1)^{t_n}} \left(\frac{1+b}{1+a} \right) \\
&= \prod_{n\geq 1} \left(\frac{(n+\frac{a}{2})(n+\frac{1+b}{2})}{(n+\frac{a+1}{2})(n+\frac{b}{2})} \right)^{(-1)^{t_n}} \left(\frac{1+b}{1+a} \right) \\
&= \frac{f(\frac{a}{2}, \frac{a+1}{2})}{f(\frac{b}{2}, \frac{b+1}{2})} \left(\frac{1+b}{1+a} \right) = \frac{g(a)}{g(b)}.
\end{aligned}$$

Now taking $a = \frac{x}{2}$ and $b = \frac{x+1}{2}$ in the equality $\frac{g(a)}{g(b)} = f(a,b)$ yields

$$\frac{g(\frac{x}{2})}{g(\frac{x+1}{2})} = f(\tfrac{x}{2}, \tfrac{x+1}{2}) = (x+1)g(x),$$

which is the announced functional equation.

Finally putting $x = 0$ in this functional equation, and noting that $g(0) \neq 0$, yields $g(\frac{1}{2}) = 1$, while putting $x = 1$ gives $g(1)^2 = \frac{1}{2}$, hence $g(1) = \frac{1}{\sqrt{2}}$, since $g(1) > 0$. □

This theorem implies many identities, including the original one of Woods-Robbins (W.-R.).

Corollary 2.3. *Let a and b belong to $\mathbb{C} \setminus \{-1, -2, -3, \ldots\}$. Then the following equalities hold.*

$$(i) \qquad \prod_{n \geq 1} \left(\frac{(n+a)(2n+a+1)(2n+b)}{(2n+a)(n+b)(2n+b+1)} \right)^{(-1)^{t_n}} = \frac{b+1}{a+1}$$

$$(ii) \qquad \prod_{n \geq 1} \left(\frac{(n+a)(2n+a+1)^2}{(2n+a)(2n+a+2)(n+a+1)} \right)^{(-1)^{t_n}} = \frac{a+2}{a+1}$$

$$(iii) \qquad \prod_{n \geq 1} \left(\frac{(2n+2a)(2n+a+1)}{(2n+a)(2n+1)} \right)^{(-1)^{t_n}} = \frac{1}{a+1}.$$

and, for $a \in \mathbb{C} \setminus \{0, -1, -2, -3, \ldots\} \cup \{-1/2, -3/2, -5/2, \ldots\})$,

$$(iv) \qquad \prod_{n \geq 1} \left(\frac{(2n+a+1)(2n+2a-1)}{(2n+a)(2n+4a-2)} \right)^{(-1)^{t_n}} = \frac{2a}{a+1}.$$

Proof. (i) is proved by writing its left side, say A, in terms of values of f and applying Theorem 2.2:

$$A = f(a, \tfrac{a}{2}) f(\tfrac{a+1}{2}, b) f(\tfrac{b}{2}, \tfrac{b+1}{2}) = \frac{g(a)}{g(\frac{a}{2})} \frac{g(\frac{a+1}{2})}{g(b)} \frac{g(\frac{b}{2})}{g(\frac{b+1}{2})} = \frac{b+1}{a+1}.$$

(ii) is obtained from (i) by taking $b = a+1$.
(iii) is obtained from (i) by taking $b = 0$.
(iv) is obtained from (i) by taking $b = 2a - 1$. □

We give examples with particular values of the parameters in the next corollary.

Corollary 2.4. *We have the following equalities.*

$$(a)\quad \prod_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_n}} = \frac{\sqrt{2}}{2}\ \text{(W.-R.)} \qquad (b)\quad \prod_{n\geq 0}\left(\frac{4n+1}{4n+3}\right)^{(-1)^{t_n}} = \frac{1}{2}$$

$$(c)\quad \prod_{n\geq 1}\left(\frac{(2n-1)(4n+1)}{(2n+1)(4n-1)}\right)^{(-1)^{t_n}} = 2 \qquad (d)\quad \prod_{n\geq 0}\left(\frac{(n+1)(2n+1)}{(n+2)(2n+3)}\right)^{(-1)^{t_n}} = \frac{1}{2}$$

$$(e)\quad \prod_{n\geq 0}\left(\frac{(2n+2)(4n+3)}{(2n+3)(4n+5)}\right)^{(-1)^{t_n}} = \frac{\sqrt{2}}{2} \qquad (f)\quad \prod_{n\geq 0}\left(\frac{(n+1)(4n+5)}{(n+2)(4n+3)}\right)^{(-1)^{t_n}} = 1$$

$$(g)\quad \prod_{n\geq 0}\left(\frac{(n+1)(2n+2)}{(n+2)(2n+3)}\right)^{(-1)^{t_n}} = \frac{\sqrt{2}}{2} \qquad (h)\quad \prod_{n\geq 0}\left(\frac{(n+1)(4n+5)}{(n+2)(4n+1)}\right)^{(-1)^{t_n}} = 2$$

$$(i)\quad \prod_{n\geq 0}\left(\frac{(2n+2)(4n+1)}{(2n+3)(4n+5)}\right)^{(-1)^{t_n}} = \frac{\sqrt{2}}{4} \qquad (j)\quad \prod_{n\geq 0}\left(\frac{(2n+1)(4n+1)}{(2n+3)(4n+5)}\right)^{(-1)^{t_n}} = \frac{1}{4}$$

$$(k)\quad \prod_{n\geq 0}\left(\frac{(4n+1)(8n+7)}{(4n+2)(8n+3)}\right)^{(-1)^{t_n}} = 1 \qquad (l)\quad \prod_{n\geq 0}\left(\frac{(8n+1)(8n+7)}{(8n+3)(8n+5)}\right)^{(-1)^{t_n}} = \frac{1}{2}.$$

Proof. Corollary 2.3 (ii) with $a = 0$ yields

$$\prod_{n\geq 1}\left(\frac{(2n+1)^2}{(2n+2)^2}\right)^{(-1)^{t_n}} = 2.$$

Taking the square root, and multiplying by the value of $\frac{2n+1}{2n+2}$ for $n = 0$, we obtain the Woods-Robbins identity (a).

Corollary 2.3 (iii) with $a = \frac{1}{2}$ gives

$$\prod_{n\geq 1}\left(\frac{2n+\frac{3}{2}}{2n+\frac{1}{2}}\right)^{(-1)^{t_n}} = \frac{2}{3}.$$

This implies (b) (note the different range of multiplication again).

Corollary 2.3 (iii) with $a = \frac{1}{2}$ gives

$$\prod_{n\geq 1}\left(\frac{(2n-1)(2n+\frac{1}{2})}{(2n-\frac{1}{2})(2n+1)}\right)^{(-1)^{t_n}} = 2,$$

which implies (c).

Corollary 2.3 (i) with $a = 1$ and $b = 2$ yields

$$\prod_{n\geq 1}\left(\frac{(n+1)(2n+2)^2}{(2n+1)(n+2)(2n+3)}\right)^{(-1)^{t_n}} = \frac{3}{2}.$$

We obtain (d) after multiplying by the factor corresponding to $n = 0$, then by the square of $\prod_{n\geq 0} \left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_n}}$ (this square is equal to $\frac{1}{2}$ from the identity of Woods and Robbins).

Corollary 2.3 (i) with $a = 1$ and $b = \frac{3}{2}$ yields

$$\prod_{n\geq 1} \left(\frac{(n+1)(2n+2)(2n+\frac{3}{2})}{(2n+1)(n+\frac{3}{2})(2n+\frac{5}{2})}\right)^{(-1)^{t_n}} = \frac{5}{4}.$$

Equality (e) is then obtained by multiplying by the factor corresponding to $n = 0$ and then multiplying by $\prod_{n\geq 0} \left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_n}}$, (which again is equal to $\frac{\sqrt{2}}{2}$).

Corollary 2.3 (i) with $a = 2$ and $b = \frac{3}{2}$ gives

$$\prod_{n\geq 1} \left(\frac{(n+2)(2n+3)(2n+\frac{3}{2})}{(2n+2)(n+\frac{3}{2})(2n+\frac{5}{2})}\right)^{(-1)^{t_n}} = \frac{5}{6}.$$

We simplify by $(2n+3)$, multiply by the factor corresponding to $n = 0$, and we obtain (the inverse of) Equality (f).

Corollary 2.3(ii) with $a = 1$ gives (g) with the usual manipulations (multiplying by the factor for $n = 0$ and by $\prod_{n\geq 0} \left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_n}}$). Alternatively (g) can be obtained by multiplying (e) and (f).

Equality (h) is obtained by dividing (f) by (b). The inverse of Equality (i) is obtained by dividing (h) by (g). Equality (j) is obtained by multiplying (i) by $\prod_{n\geq 0} \left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_n}}$, (which is equal to $\frac{\sqrt{2}}{2}$).

Corollary 2.3 (iv) with $a = \frac{3}{4}$ yields

$$\prod_{n\geq 1} \left(\frac{(2n+\frac{7}{4})(2n+\frac{1}{2})}{(2n+\frac{3}{4})(2n+1)}\right)^{(-1)^{t_n}} = \frac{6}{7},$$

which implies (k).

Corollary 2.3 (i) with $a = \frac{3}{4}$ and $b = \frac{1}{4}$ gives (l) (multiply by the factor corresponding to $n = 0$ and use (b)). $\square$

Remark 2.5. The proofs that we give, e.g., in Corollary 2.4, provide infinite products whose values are rational: in the case of the Woods-Robbins infinite product $P = \prod_{n\geq 0} \left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_n}}$ we actually obtain the value of P^2 $(= 1/2)$. We finally get $\frac{\sqrt{2}}{2}$ only because the product we first obtain involves the square of a rational function.

3 More remarks on the function g

As we have seen above, the function g defined by $g(x) := \frac{f(\frac{x}{2},\frac{x+1}{2})}{x+1}$ has the property that $f(a,b) = \frac{g(a)}{g(b)}$. It satisfies the functional equation $\frac{g(\frac{x}{2})}{g(\frac{x+1}{2})} = (1+x)g(x)$ for x not equal to a negative integer. This functional equation has some resemblance with the celebrated *duplication formula* for the Γ function: $\Gamma(\frac{z}{2})\Gamma(\frac{z+1}{2}) = 2^{1-z}\sqrt{\pi}\,\Gamma(z)$.

We also point out the cancellation of $g(0)$ when we computed $g(1/2)$. In particular, we have not been able to give the value of $g(0)$ in terms of known constants. The quantity $g(0) = \prod_{n\geq 1} \left(\frac{2n}{2n+1}\right)^{(-1)^{t_n}}$ already appeared in a paper of Flajolet and Martin [8]: more precisely they are concerned with the constant

$$\varphi := 2^{-1/2}e^{\gamma}\frac{2}{3}R, \text{ where } R := \prod_{n\geq 1}\left(\frac{(4n+1)(4n+2)}{4n(4n+3)}\right)^{(-1)^{t_n}},$$

and it easily follows that

$$\varphi = \frac{2^{-1/2}e^{\gamma}}{g(0)}.$$

Finally, we will prove that the function g is decreasing. Actually we have the stronger result given in Theorem 3.2 below. We first state and slighty extend a lemma (Lemmas 3.2 and 3.3 from [2]).

Lemma 3.1. *For every function G, define the operator $\mathcal{T}$ by $\mathcal{T}G(x) := G(2x) - G(2x+1)$. Let $\mathcal{A} = \{G : \mathbb{R}^+ \to \mathbb{R},\ G \text{ is } C^{\infty},\ \forall x \geq 0,\ (-1)^r G^{(r)}(x) > 0\}$. Then*

- *for all $k \geq 0$, one has $T^k\mathcal{A} \subset \mathcal{A}$. Furthermore, if G belongs to $\mathcal{A}$;*
- *if the series $\sum_{n\geq 0} \mathcal{T}G(n)$ converges, then all the series $\sum_{n\geq 0} T^kG(n)$ converge and $R(n, T^kG) := \sum_{j\geq n}(-1)^{t_j}T^kG(j)$ has the sign of $(-1)^{t_n}$.*

Proof. See [2, Lemmas 3.2 and 3.2] where everything is proved, except that the last assertion about the sign of $R(n, T^kG)$ is stated only for $k = 0$, but clearly holds for all $k \geq 0$. $\square$

Theorem 3.2. *The function $x \to f(\frac{x}{2}, \frac{x+1}{2})$ is decreasing on the nonnegative real numbers.*

Proof. $G(x) = \log\frac{x+a}{x+b}$ for $x \geq 0$. Then, $G^{(r)}(x) = (-1)^{r-1}(r-1)!((x+a)^{-r} - (x+b)^{-r})$ for $r \geq 1$ so that G belongs to $\mathcal{A}$. Now applying Lemma 3.1 to $\mathcal{T}G$ and $n = 1$ yields $\sum_{n\geq 1}(-1)^{t_n}(\log\frac{2n+a}{2n+b} - \log\frac{2n+1+a}{2n+1+b}) < 0$, which is the same as saying that $\frac{f(\frac{a}{2},\frac{a+1}{2})}{f(\frac{b}{2},\frac{b+1}{2})} < 1$. $\square$

4 Products of the form $\prod R(n)^{t_n}$

We let again $(t_n)_{n\geq 0} = 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ldots$ denote the 0-1-Thue-Morse sequence. We have seen that several infinite products of the form $\prod R(n)^{(-1)^{t_n}}$ admit a closed-form expression, but it might seem more natural (or at least desirable) to have results for infinite products of the form $\prod R(n)^{t_n}$. Our first result deals with the convergence of such products.

Lemma 4.1. *Let t_n be the sum, reduced modulo 2, of the binary digits of the integer n. Let $R \in \mathbb{C}(X)$ be a rational function such that the values $R(n)$ are defined for $n \geq 1$. Then the infinite product $\prod_n R(n)^{t_n}$ converges if and only if the numerator and the denominator of R have the same degree, the same leading coefficient, and the same sum of roots (in $\mathbb{C}$).*

Proof. If the infinite product $\prod R(n)^{t_n}$ converges, then $R(n)$ must tend to 1 when n tends to infinity (on the subsequence for which $t_n = 1$). Hence the numerator and denominator of R have the same degree and the same leading coefficient. But then, as we have seen, the product $\prod R(n)^{(-1)^{t_n}}$ converges, and so does the product $\prod R(n)^{2t_n+(-1)^{t_n}}$. But this product is equal to $\prod R(n)$, which is known to converge if and only if the sum of the roots of the numerator is equal to the sum of the roots of the denominator.

Now if the numerator and denominator of R have the same degree, the same leading coefficient, and the same sum of roots, then both infinite products $\prod R(n)^{(-1)^{t_n}}$ and $\prod R(n)$ converge, which implies the convergence of the infinite product $\prod R(n)^{1-2t_n} = \prod R(n)^{(-1)^{t_n}}$. $\square$

Now we give three equalities for products of the form $\prod R(n)^{t_n}$.

Theorem 4.2. *The following three equalities hold:*

$$\prod_{n\geq 0}\left(\frac{(4n+1)(4n+4)}{(4n+2)(4n+3)}\right)^{t_n} = \frac{\pi^{3/4}\sqrt{2}}{\Gamma(1/4)} \tag{2}$$

$$\prod_{n\geq 0}\left(\frac{(n+1)(4n+5)}{(n+2)(4n+1)}\right)^{t_n} = \sqrt{2} \tag{3}$$

$$\prod_{n\geq 0}\left(\frac{(8n+1)(8n+7)}{(8n+3)(8n+5)}\right)^{t_n} = \sqrt{2\sqrt{2}-2}. \tag{4}$$

Proof. As above we have $2t_n = 1 - (-1)^{t_n}$. Then we write

$$\left(\prod_{n\geq 0}\left(\frac{(4n+1)(4n+4)}{(4n+2)(4n+3)}\right)^{t_n}\right)^2 = \frac{\displaystyle\prod_{n\geq 0}\frac{(4n+1)(4n+4)}{(4n+2)(4n+3)}}{\displaystyle\prod_{n\geq 0}\left(\frac{(4n+1)(4n+4)}{(4n+2)(4n+3)}\right)^{(-1)^{t_n}}}.$$

The computation of the numerator is classical (see, e.g., [17, Section 12-13]):

$$\prod_{n\geq 0}\frac{(4n+1)(4n+4)}{(4n+2)(4n+3)} = \prod_{n\geq 0}\frac{(n+1/4)(n+1)}{(n+1/2)(n+3/4)} = \frac{\Gamma(1/2)\Gamma(3/4)}{\Gamma(1/4)\Gamma(1)} = \frac{\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)}$$
$$= \frac{\pi^{3/2}\sqrt{2}}{\Gamma(1/4)^2},$$

where the last equality uses the reflection formula $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$ for $x \notin \mathbb{Z}$.

To compute the denominator, we start from the Woods-Robbins product and split the set of indices into even and odd indices, so that

$$\frac{\sqrt{2}}{2} = \prod_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_n}} = \prod_{n\geq 0}\left(\frac{4n+1}{4n+2}\right)^{(-1)^{t_{2n}}}\left(\frac{4n+3}{4n+4}\right)^{(-1)^{t_{2n+1}}}.$$

Using that $t_{2n} = t_n$ and $t_{2n+1} = 1 - t_n$, we thus have

$$\frac{\sqrt{2}}{2} = \prod_{n\geq 0}\left(\frac{(4n+1)(4n+4)}{(4n+2)(4n+3)}\right)^{(-1)^{t_n}}.$$

Gathering the results for the numerator and for the denominator we deduce

$$\left(\prod_{n\geq 0}\left(\frac{(4n+1)(4n+4)}{(4n+2)(4n+3)}\right)^{t_n}\right)^2 = \frac{2\pi^{3/2}}{\Gamma(1/4)^2},$$

hence the first assertion in our theorem.

The proof of the second assertion goes along the same lines. We start from Equality (h) in Corollary 2.4

$$\prod_{n\geq 0}\left(\frac{(n+1)(4n+5)}{(n+2)(4n+1)}\right)^{(-1)^{t_n}} = 2.$$

Now

$$\prod_{n\geq 0}\left(\frac{(n+1)(4n+5)}{(n+2)(4n+1)}\right) = \prod_{n\geq 0}\left(\frac{(n+1)(n+5/4)}{(n+2)(n+1/4)}\right) = \frac{\Gamma(2)\Gamma(1/4)}{\Gamma(1)\Gamma(5/4)} = 4.$$

Note that this equality can also be obtained by telescopic cancellation in the finite product $\prod_{0\leq n\leq N}\left(\frac{(n+1)(4n+5)}{(n+2)(4n+1)}\right)$.

Thus

$$\prod_{n\geq 0}\left(\left(\frac{(n+1)(4n+5)}{(n+2)(4n+1)}\right)^{t_n}\right)^2 = \prod_{n\geq 0}\left(\frac{(n+1)(4n+5)}{(n+2)(4n+1)}\right)^{1-(-1)^{t_n}} = 2;$$

hence

$$\prod_{n\geq 0}\left(\frac{(n+1)(4n+5)}{(n+2)(4n+1)}\right)^{t_n} = \sqrt{2}.$$

The proof of the third assertion is similar. We start from Equality (l) in Corollary 2.4:

$$\prod_{n\geq 0}\left(\frac{(8n+1)(8n+7)}{(8n+3)(8n+5)}\right)^{(-1)^{t_n}} = \frac{1}{2}.$$

Now, as previously,

$$\prod_{n\geq 0}\left(\frac{(8n+1)(8n+7)}{(8n+3)(8n+5)}\right) = \prod_{n\geq 0}\left(\frac{(n+1/8)(n+7/8)}{(n+3/8)(n+5/8)}\right) = \frac{\Gamma(3/8)\Gamma(5/8)}{\Gamma(1/8)\Gamma(7/8)}.$$

But

$$\frac{\Gamma(3/8)\Gamma(5/8)}{\Gamma(1/8)\Gamma(7/8)} = \frac{\frac{\pi}{\sin(3\pi/8)}}{\frac{\pi}{\sin(\pi/8)}} = \frac{\sin(\pi/8)}{\sin(3\pi/8)} = \frac{\sin(\pi/8)}{\cos(\pi/8)} = \tan(\pi/8) = \sqrt{2}-1.$$

From this we obtain

$$\prod_{n\geq 0}\left(\frac{(8n+1)(8n+7)}{(8n+3)(8n+5)}\right)^{1-(-1)^{t_n}} = 2\sqrt{2}-2.$$

Thus, as claimed in the second assertion of the theorem,

$$\prod_{n\geq 0}\left(\frac{(8n+1)(8n+7)}{(8n+3)(8n+5)}\right)^{t_n} = \sqrt{2\sqrt{2}-2}.$$

$\square$

Remark 4.3. Several other closed-form expressions for infinite products $\prod R(n)^{t_n}$ can be obtained. For example one can use closed-form expressions for infinite products $\prod R(n)^{(-1)^{t_n}}$ where $R(n)$ satisfies the hypotheses of Lemma 4.1 and the classical result about $\prod R(n)$. Another possibility is to start from an already known product $A = \prod_{n\geq 0} S(n)^{(-1)^{t_n}}$ where S satisfies the hypotheses of Lemma 2.1 and note that (splitting the indexes into even and odd)

$$A = \prod_{n\geq 0}(S(2n)^{(-1)^{t_{2n}}} S(2n+1)^{(-1)^{t_{2n+1}}}) = \prod_{n\geq 0}\left(\frac{S(2n)}{S(2n+1)}\right)^{(-1)^{t_n}}.$$

As easily checked the rational function satisfies the hypotheses of Lemma 4.1. The reader can observe that this generalizes the method used to prove the first assertion (i.e., Equality (1)) of Theorem 4.2.

5 Generalization to another block counting sequence

In this section we give an example of two products of the kind of those in Corollary 2.4 and in Theorem 4.2 that involve the Golay-Shapiro sequence (also called the Rudin-Shapiro sequence). Let us recall that this sequence $(v_n)_{n\geq 0}$ (in its binary version) can be defined as follows: $v_0 = 0$, and for all $n \geq 0$, $v_{2n} = v_n$, $v_{4n+1} = v_n$, $v_{4n+3} = 1 - v_{2n+1}$. Another definition is that v_n is the number, reduced modulo 2, of (possibly overlapping) 11's in the binary expansion of the integer n. The $\pm$ version $((-1)^{v_n})_{n\geq 0}$ of this sequence was introduced independently the same year (1951) by Shapiro [14] and by Golay [9], and rediscovered in 1959 by Rudin [13] who acknowledged Shapiro's priority.

An infinite product involving the Golay-Shapiro sequence was given in [3] (also see [4]):

$$\prod_{n\geq 1}\left(\frac{(2n+1)^2}{(n+1)(4n+1)}\right)^{(-1)^{v_n}} = \frac{\sqrt{2}}{2}\cdot$$

Here we prove the following theorem.

Theorem 5.1. *The following two equalities hold.*

$$\prod_{n\geq 0}\left(\frac{4(n+2)(2n+1)^3(2n+3)^3}{(n+3)(n+1)^2(4n+3)^4}\right)^{(-1)^{v_n}} = 1 \tag{5}$$

$$\prod_{n\geq 0}\left(\frac{4(n+2)(2n+1)^3(2n+3)^3}{(n+3)(n+1)^2(4n+3)^4}\right)^{v_n} = \frac{16\Gamma(3/4)^4}{\pi^6}\cdot \tag{6}$$

Proof. Let R be a rational function in $\mathbb{C}(X)$ such that its numerator and denominator have same degree and same leading coefficient. Suppose furthermore that $R(n)$ is defined for any integer $n \geq 1$. Then it is not difficult to see that the infinite product $\prod_{n\geq 1} R(n)^{(-1)^{v_n}}$ converges (summation by part, given the well-known property that the partial sum $\sum_{1\leq k\leq n}(-1)^{v_k}$ is $\mathcal{O}(\sqrt{n})$). Now, using the recursive definition of $(v_n)_{n\geq 0}$ one has

$$\begin{aligned}
\prod_{n\geq 1} R(n)^{(-1)^{v_n}} &= \prod_{n\geq 1} R(2n)^{(-1)^{v_{2n}}} \prod_{n\geq 0} R(2n+1)^{(-1)^{v_{2n+1}}} \\
&= \prod_{n\geq 1} R(2n)^{(-1)^{v_n}} \prod_{n\geq 0} R(4n+1)^{(-1)^{v_{4n+1}}} \prod_{n\geq 0} R(4n+3)^{(-1)^{v_{4n+3}}} \\
&= \prod_{n\geq 1} R(2n)^{(-1)^{v_n}} \prod_{n\geq 0} R(4n+1)^{(-1)^{v_n}} \prod_{n\geq 0} R(4n+3)^{-(-1)^{v_{2n+1}}}.
\end{aligned}$$

Thus

$$\prod_{n\geq 1}\left(\frac{R(n)}{R(2n)R(4n+1)}\right)^{(-1)^{v_n}} \prod_{n\geq 0} R(4n+3)^{(-1)^{v_{2n+1}}} = R(1)^{(-1)^{v_1}} = R(1).$$

But

$$\prod_{n\geq 0} R(4n+3)^{(-1)^{v_{2n+1}}} = \frac{\displaystyle\prod_{n\geq 0} R(2n+1)^{(-1)^{v_n}}}{\displaystyle\prod_{n\geq 0} R(4n+1)^{(-1)^{v_{2n}}}} = \frac{\displaystyle\prod_{n\geq 1} R(2n+1)^{(-1)^{v_n}}}{\displaystyle\prod_{n\geq 1} R(4n+1)^{(-1)^{v_n}}}$$

which finally gives

$$\prod_{n\geq 1} \left(\frac{R(n)R(2n+1)}{R(2n)R(4n+1)^2}\right)^{(-1)^{v_n}} = R(1).$$

Now taking $R(X) = \frac{(X+2)^2}{(X+1)(X+3)}$ we obtain

$$\prod_{n\geq 1} \left(\frac{4(n+2)(2n+1)^3(2n+3)^3}{(n+3)(n+1)^2(4n+3)^4}\right)^{(-1)^{v_n}} = \frac{9}{8}.$$

Thus

$$\prod_{n\geq 0} \left(\frac{4(n+2)(2n+1)^3(2n+3)^3}{(n+3)(n+1)^2(4n+3)^4}\right)^{(-1)^{v_n}} = 1.$$

Now

$$\begin{aligned}\prod_{n\geq 0} \frac{4(n+2)(2n+1)^3(2n+3)^3}{(n+3)(n+1)^2(4n+3)^4} &= \prod_{n\geq 0} \frac{(n+2)(n+\frac{1}{2})^3(n+\frac{3}{2})^3}{(n+3)(n+1)^2(n+\frac{3}{4})^4} \\ &= \frac{\Gamma(3)\Gamma(1)^2\Gamma(3/4)^4}{\Gamma(2)\Gamma(1/2)^3\Gamma(3/2)^3} = \frac{16\Gamma(3/4)^4}{\pi^6}.\end{aligned}$$

□

6 Some questions

The arithmetical properties values of the infinite products that we have obtained can be quite different. Some are rational (e.g., $\prod_{n\geq 0}\left(\frac{(4n+1)(8n+7)}{(4n+2)(8n+3)}\right)^{(-1)^{t_n}} = 1$ in Corollary 2.4 (k)), some are algebraic irrational such as the Woods-Robbins product ($\prod_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_n}} = \frac{\sqrt{2}}{2}$), some are transcendental (e.g., $\prod_{n\geq 0}\left(\frac{(4n+1)(4n+4)}{(4n+2)(4n+3)}\right)^{t_n} = \frac{\pi^{3/4}\sqrt{2}}{\Gamma(1/4)}$ whose transcendency is a consequence of the algebraic independence of π and $\Gamma(1/4)$ proved by Čudnovs'kiĭ; see [7]). As in [1] some of these values could be proved transcendental if one admits the Rohrlich conjecture. A still totally open question is the arithmetical nature of the Flajolet-Martin constant(s) (see the beginning of Section 3), namely φ and R, where $\varphi := 2^{-1/2}e^{\gamma}\frac{2}{3}R$, and $R := \prod_{n\geq 1}\left(\frac{(4n+1)(4n+2)}{4n(4n+3)}\right)^{(-1)^{t_n}} = \frac{3}{2g(0)}$.

Another question is to generalize the results for the Thue-Morse sequence to other sequences counting certain patterns in the base-b expansion of integers: the example given in Section 5 is a first step in this direction.

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This paper is an extended version of [11]. While we were preparing this extended version, we found the paper [15] which has interesting results on finite (and infinite) sums involving the sum of digits of integers in integer bases.

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