Reaction-diffusion systems with initial data of low regularity

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Abstract

Models issued from ecology, chemical reactions and several other application fields lead to semi-linear parabolic equations with super-linear growth. Even if, in general, blow-up can occur, these models share the property that mass control is essential. In many circumstances, it is known that this L^1 control is enough to prove the global existence of weak solutions. The theory is based on basic estimates initiated by M. Pierre and collaborators, who have introduced methods to prove L^2 a priori estimates for the solution.

Here, we establish such a key estimate with initial data in L^1 while the usual theory uses L^2. This allows us to greatly simplify the proof of some results. We also establish new existence results of semilinearity which are super-quadratic as they occur in complex chemical reactions. Our method can be extended to semi-linear porous medium equations.

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Introduction

Ecology with Lotka-Volterra systems, chemistry with reaction-rate equations, multi-species diffusion of molecules and many other scientific fields lead to reaction-diffusion systems characterized by different diffusion coefficients. We consider such systems, set in a smooth domain Ω of $\mathbb{R}^N$, under the form

$$\begin{cases} \frac{\partial}{\partial t}u_i(t,x) - d_i\Delta u_i = f_i(u_1, ...u_m), \qquad x \in \Omega,\ t \geq 0,\ i = 1, ..., m, \\ \frac{\partial}{\partial \nu}u_i = 0 \quad \text{ on } \partial\Omega. \end{cases}$$

Even though they are standard, the mathematical understanding of these parabolic systems is still very limited. There are two major obstructions to the construction of weak solutions. Firstly, the

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right hand sides f_i often has quadratic, and possibly faster, growth for large values of the u_i's. A typical example, for three species is for $\{i,j,k\} = \{1,2,3\}$ (see section 1 for precisions)

$$f_i = u_j^\alpha u_k^\beta - u_i^\gamma$$

Secondly, when the diffusion coefficients d_i are very different, there is no maximum principle, and thus a priori estimates are not available besides L^1 control. To circumvent these difficulties, a major L^1 theory has been elaborated, which can be roughly summarized as follows: if an L^1 bound can be proved for the f_i's, then the existence of a weak solution can be proved, see [26, 27, 18]. Such an L^1 bound can itself be derived thanks to general fundamental lemmas which were initiated by Pierre and Schmitt [29, 30]. The most elaborate version is the L^2 lemma of Pierre [27] which we recall below.

Our main contribution is to construct an existence theory based on the following $L^1 \cap H^{-1}$ lemma, which is typically applied to combinations of equations in the above system. Namely, consider a constant $B \in [0,\infty)$, smooth functions $U,\ F : [0,+\infty) \times \Omega \to \mathbb{R}^+$ and $V : [0,+\infty) \times \Omega \to \mathbb{R}^+$such that $\int_\Omega \Delta_x V(t,x)dx = 0$, and satisfying the relations

$$\begin{cases} \frac{\partial}{\partial t} U(t,x) - \Delta_x V = B - F(t,x), \quad t \geq 0,\ x \in \Omega \subset \mathbb{R}^N, \\ \frac{\partial U}{\partial \nu}(t,x) = 0 \quad \text{on } (0,+\infty) \times \partial\Omega, \\ U(t=0) = U^0 \geq 0 \quad \text{in } \Omega. \end{cases} \tag{R}$$

Then, we have

Lemma 1 (First key estimate with L^1 data) *With the notations above, assume $F \geq 0$, $U \geq 0$, $U^0 \in L^1(\Omega)^+ \cap H^{-1}(\Omega)$ and $\int_\Omega \Delta_x V(t,x)dx = 0$. Then, there exists a constant $C > 0$ depending only on Ω, such that*

$$\int_0^T \int_\Omega UV \leq K(T) + \frac{1}{2}\|U^0\|^2_{H^{-1}(\Omega)} \quad \textit{where} \quad K = \int_0^T \left[C\langle F(t)\rangle + \langle V(t)\rangle\right] \left[B|\Omega|t + \int_\Omega U^0\right]dt. \tag{1}$$

The proof of this lemma, and of a more precise version, is given in Appendix A, and is reminiscent of the lifting method introduced in [7] for the Cahn-Hilliard equation. Several variants can be derived, for instance in dimension $N \leq 3$, a variant allows us to relax the assumption on the sign of F, see Appendix B. Also, Lemma 1 will be used for the porous media equation with a little more work.

This result can be compared with the L^2 version of Pierre [27] (we indicate here a simple variant).

Lemma 2 (Pierre's L^2 lemma) *Assume $V = AU$ with $A(t,x) \geq 0$ and bounded i.e ($0 < a \leq A(t,x) \leq b < \infty$), $U \geq 0$, $U^0 \in L^2(\Omega)$ and (R) holds, then*

$$a \int_0^T \int_\Omega U^2 \leq \left\| \int_0^T A(t,x)dt \right\|_{L^\infty(\Omega)} \int_\Omega (U^0)^2. \tag{2}$$

For the ease of the reader, we reproduce the proof in Appendix C.

This *a priori L^2 estimate* (2) was stated in [29, 30] and then widely exploited, see for example [10, 27, 6, 8, 28] and the references therein. Moreover, this estimate has a natural extension to the case where the diffusion operators are degenerate of porous media type, i.e., $\Delta(u_i^{m_i})$ where $m_i > 1$, see [18, Theorem 2.7].

Unfortunately, Estimate (2) heavily depends on the $L^2(\Omega)$-norm of the initial data, and does not hold any longer when the initial data are merely in $L^1(\Omega)$. However, the space $L^1(\Omega)$ is a natural functional setting for the systems at hand, as shown by the a priori $L^1(\Omega)$-uniform estimates (3) and (5) (see section 1 below).

The paper is composed of four sections and four appendices. In section 1, we expose in detail the type of nonlinearities f_i's which have usually been treated so far in the literature, and we present the state of the art about existence theory. Section 2 is devoted to show how Lemma 1 can be used to prove new results, namely, to extend the range of possible nonlinearities f_i, or to lower the required regularity on the initial data. Furthermore, our approach simplifies the existing proofs known for L^1 initial data. The case of porous media type systems is treated in section 3 thanks to an application of Lemma 1. Lemma 1 is proved in appendix A and it is complemented in appendix B with the case where the right hand side F belongs to L^1 with no sign condition. Lemma 2 is proved in appendix C while appendix D is devoted to the case of the porous media system.

We have tried to write the paper in an almost self-contained form, moreover we give precise references for all the points that are not detailed in the work.

1 Assumptions, overview and two motivating examples

Throughout this paper, $\Omega \subset \mathbb{R}^N$ is open bounded with "good enough" boundary, and we denote $Q := (0,+\infty) \times \Omega$, $Q_T := (0,T) \times \Omega$, $\Sigma := (0,+\infty) \times \partial\Omega$, $\Sigma_T := (0,T) \times \partial\Omega$ and, for $p \in [1,+\infty)$

$$\|u(t,.)\|_{L^p(\Omega)} = \left(\int_\Omega |u(t,x)|^p \, dx\right)^{1/p}, \quad \|u\|_{L^p(Q_T)} = \left(\int_0^T \int_\Omega |u(t,x)|^p \, dtdx\right)^{1/p}.$$

1.1 Assumptions

Let us consider a general $m \times m$ reaction-diffusion system

$$\begin{cases} \forall i = 1, ..., m \\ \frac{\partial u_i}{\partial t} - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in} \quad Q_T, \\ \frac{\partial u_i}{\partial \nu} = 0 & \text{on} \quad \Sigma_T, \\ u_i(0,\cdot) = u_i^0(.) \geq 0 & \text{in} \quad \Omega \end{cases} \tag{S}$$

where for all $i = 1, \cdots, m$, $d_i > 0$ and $f_i : [0,+\infty)^m \to \mathbb{R}$ is locally Lipschitz continuous.

Moreover, we assume that the nonlinearities satisfy :

(P) : for all $i = 1, \cdots, m$, f_i is quasi-positive *i.e*

$$\forall \mathbf{r} = (r_1, r_2, \cdots, r_m) \in [0,+\infty)^m, \ f_i(r_1 \cdot, r_{i-1}, 0, r_{i+1}, ..., r_d) \geq 0.$$

(M) : there exists $(a_1, \cdots, a_m) \in (0,+\infty)^m$ such that $\forall \mathbf{r} \in [0,+\infty)^m$, $\sum_{1 \leq i \leq m} a_i f_i(\mathbf{r}) \leq 0.$

Remark 3 *Most of our results can be extended to:*
(M') $\forall \boldsymbol{r} \in [0,+\infty[^m$, $\sum_{1 \leq i \leq m} a_i f_i(\boldsymbol{r}) \leq C[1 + \sum_{1 \leq i \leq m} r_i]$,

or to the more general situation when the nonlinearities f_i depend also on t and x:
(M") $\forall (t,x) \in Q$ *and* $\forall \boldsymbol{r} \in [0,+\infty[^m$, $\displaystyle\sum_{1\le i\le m} a_i f_i(t,x,\boldsymbol{r}) \le H(t,x) + C \sum_{1\le i\le m} r_i$ *where* $H \in L^1(Q)$.

Properties **(P)** and **(M)** (or **(M')**) appear naturally in applications. Indeed, evolution reaction-diffusion systems are mathematical models for evolution phenomena undergoing at the same time spatial diffusion and (bio-)chemical type of reactions. The unknown functions are generally densities, concentrations, temperatures so that their nonnegativity is required. Moreover, often a control of the total mass, sometimes even preservation of the total mass, is naturally guaranteed by the model. Interest has increased recently for these models in particular for applications in biology, ecology and population dynamics. We refer to [27, Section 2] for examples of reaction-diffusion systems with properties **(P)** and **(M)** or **(M')**.

1.2 Overview

Many mathematical results are known about the global weak solutions to System (S) and we recall some of them and refer to [33] for classical solutions.

First of all, let us make precise what we mean by solution to (S) on $Q_T = (0,T)\times\Omega$.

By *classical solution* to (S), we mean that, at least
(*i*) $u = (u_1, \cdots, u_m) \in \mathcal{C}([0,T); L^1(\Omega)^m) \cap L^\infty([0,\tau]\times\Omega)^m, \forall \tau \in (0,T)$;
(*ii*) $\forall k,\ell = 1\dots N$, $\forall p \in (1,+\infty)$

$$\partial_t u_i, \partial_{x_k} u_i, \partial_{x_k x_\ell} u_i \in L^p(Q_T) \quad i = 1, \cdots, m;$$

(*iii*) equations in (S) are satisfied a.e (almost everywhere).

By *weak solution* to (S) on Q_T, we essentially mean solution in the sense of distributions or, equivalently here, solution in the sense of Duhamel's formula with the corresponding semigroups. More precisely, for $1 \le i \le m$, $f_i(u) \in L^1(Q_T)$ and

$$u_i(t,.) \quad = \quad S_{d_i}(t)u_i^0(.) + \int_0^t S_{d_1}(t-s) f_i(u(s,.))\, ds$$

where $S_{d_i}(.)$ is the semigroup generated in $L^1(\Omega)$ by $-d_i\Delta$ with homogeneous Neumann boundary condition.

For initial data $u^0 \in (L^\infty(\Omega))^m$, the local Lipschitz continuity of the nonlinearities implies the existence of a local classical solution to (S) on a maximal interval $[0, T_{\max})$. Moreover, the initial data are nonnegative, so the quasi-positivity **(P)** ensures that the solution stays nonnegative as long as it exists, see [33]. The assumption **(M)** gives an upper bound on the total mass of the system *i.e* for all $t \in (0, T_{\max})$

$$\sum_{i=1}^m \int_\Omega a_i u_i(t,x)\, dx \le \sum_{i=1}^m \int_\Omega a_i u_i^0(x)\, dx. \tag{3}$$

In fact, multiplying each i-th equation by a_i and adding the m equations to obtain

$$\sum_{i=1}^m a_i \partial_t u_i - \sum_{i=1}^m a_i d_i \Delta u_i = \sum_{i=1}^m a_i f_i \le 0. \tag{4}$$

By integrating (4) on $(0,t)\times\Omega$ and taking into account the boundary conditions $\displaystyle\int_\Omega \Delta u_i = \int_{\partial\Omega} \frac{\partial u_i}{\partial \nu} = 0$ and **(M)**, we obtain (3). In other words, the total mass of m components is preserved. Together with the nonnegativity of u_i, estimate (3) implies that

$$\forall t \in (0, T_{\max})\ ,\ \|u_i(t,.)\|_{L^1(\Omega)} \leq \frac{1}{a_i}\|\sum_{j=1}^{m} a_j u_j^0\|_{L^1(\Omega)}. \tag{5}$$

So the $u_i(t,.)$ remain bounded in $L^1(\Omega)$ uniformly in time as long as solution exists.

Let us emphasize that uniform L^∞-bounds, rather than L^1-bounds, would provide global existence in time of *classical* solutions, by the standard theory for reaction-diffusion systems. The point here is that bounds are a priori only in L^1 and one cannot apply the L^∞-approach (see [33]) even if the initial data are regular except in the restrictive case where $d_1 = \cdots = d_m$.
However the situation is quite more complicated if the diffusion coefficients are different from each other. Some additional assumptions on the structure of the source terms are needed for global existence of classical solutions. This question has been widely studied. For some recent results, see e.g [17, 11, 15, 31, 35, 9, 34, 12]. Concerning the results established before 2010, we refer the intrested reader to the exhaustive survey [27] for a general presentation of the problem, further references, and many deep comments on the mathematical difficulties raised by such systems. In fact, the solutions may blow up in $L^\infty(\Omega)$ in finite time as proved in [29, 30] where explicit finite time T^* blow up is given. More precisely, Pierre and Schmitt exhibited a system with two species fulfilling **(P)** and **(M)** with $d_1 \neq d_2$ and strictly superquadratic polynomial nonlinearities f_i such that $T^* < +\infty$ and

$$\lim_{t\nearrow T^*} \|u_1(t,.)\|_{L^\infty(\Omega)} = \lim_{t\nearrow T^*} \|u_2(t,.)\|_{L^\infty(\Omega)} = +\infty.$$

Thus, even in the semi-linear case it is necessary to deal with weak solutions if one expects global existence for more general nonlinearities and initial data of low regularity.

It is worth to pointing out that, while considerable effort has been devoted to the study of systems with initial data in $L^\infty(\Omega)$ or $L^2(\Omega)$ and at most quadratic nonlinearities, relatively little is known in the case of systems with initial data in $L^1(\Omega)$. We refer to [26, 27] for linear diffusion, and [16, 18, 19, 28] for nonlinear diffusion.

Here, we just recall the following three theorems closely related to our present study.

• **L^1-Theorem:**

In order to give the precise statement, let us introduce the following approximation of System (S)

$$\begin{cases} i = 1, ..., m, & & \\ \partial_t u_i^n - \Delta u_i^n = f_i^n(u^n) & \text{in} & Q = (0,+\infty)\times\Omega, \\ \frac{\partial u_i^n}{\partial \nu}(t,.) = 0 & \text{on} & \Sigma = (0,+\infty)\times\partial\Omega, \\ u_i^n(0,.) = u_{i,0}^n \geq 0 & \text{in} & \Omega, \end{cases} \tag{6}$$

where $u_{i,0}^n := \inf\{u_{i,0},\ n\}$ and $f_i^n := \dfrac{f_i}{1+\frac{1}{n}\sum_{1\leq j\leq m}|f_j|}$.

For $i = 1, \cdots, m$, $u_{i,0}^n \in L^\infty(\Omega)$ and converges to $u_{i,0}$ in $L^1(\Omega)$, f_i^n is locally Lipschitz continuous and satisfy **(P)** and **(M)**. Moreover $\|f_i^n\|_{L^\infty(\Omega)} \leq n$. Therefore, the approximate system (6) has a nonnegative classical global solution $u^n = (u_1^n, \cdots, u_m^n)$ (see e.g [20]).

Theorem 4 (Pierre, [27]) *Besides* **(P)+(M)***, assume that the following* ***a priori*** **L^1-*estimate*** *holds: there exists $C(T)$ independent of n such that*

$$\forall i = 1, ..., m, \ \forall T > 0, \ \int_{Q_T} |f_i^n(u_1^n, \cdots, u_m^n)| \leq C(T). \tag{7}$$

Then, as $n \to \infty$, up to a subsequence, u^n converges in $L^1(Q_T)$ for all $T > 0$ to some global weak solution *u of (S) for all $(u_{0,1}, \cdots, u_{0,m}) \in (L^1(\Omega)^+)^m$.*

For a proof, see [27, Theorem 5.9].

The two following theorems are more adequate for cases where the nonlinearities are at most quadratic or super-quadratic respectively.

- **L^2-Theorem with linear diffusion:**

Theorem 5 (Pierre, [27]) *Besides* ***(P)*** *and* ***(M)****, assume that the f_i are at most quadratic i.e theres exists $C > 0$ such that for all $i = 1, \cdots, m$:*

$$\textbf{(QG)} \qquad |f_i(r)| \leq C\big(1 + \sum_{j=1}^{m} r_j^2\big).$$

Then, there exists a global weak solution *to (S) for all $u_0 = (u_{0,1}, \cdots, u_{0,m}) \in (L^2(\Omega)^+)^m$.*

For a proof, see [27, Theorem 5.11].

- **L^2-Theorem with nonlinear diffusion:** Consider the following $m \times m$ reaction-diffusion system

$$(NLDS) \begin{cases} \forall i = 1, ..., m & & \\ \frac{\partial u_i}{\partial t} - d_i \Delta(u_i^{m_i}) = f_i(u_1, u_2, ..., u_m) & \text{in} & Q_T, \\ \frac{\partial u_i}{\partial \nu} = 0 & \text{on} & \Sigma_T, \\ u_i(0, \cdot) = u_i^0(.) \geq 0 & \text{in} & \Omega \end{cases}$$

where for all $i = 1, \cdots, m$, $d_i > 0$, $m_i > 1$ and $f_i : [0, +\infty)^m \to \mathbb{R}$ is locally Lipschitz continuous.

Theorem 6 (Laamri–Pierre, [18]) *Besides* ***(P)*** *and* ***(M)****, assume that the f_i's are at most super-quadratic i.e theres exists $C > 0$ such that for all $i = 1, \cdots, m$:*

$$\textbf{(SQG)} \qquad |f_i(r)| \leq C\big(1 + \sum_{j=1}^{m} r_j^{m_i+1-\varepsilon}\big).$$

Then, there exists a global weak solution *to $(NLDS)$ for all $u_0 = (u_{0,1}, \cdots, u_{0,m}) \in (L^2(\Omega)^+)^m$.*

For a proof, see [18, Theorem 2.7].

1.3 Motivating examples

Now, let us introduce two systems we are considering in this paper. In fact, these systems contain the major difficulties encountered in a large class of similar problems as regards global existence in time of solutions.

• **First system:** Let $(\alpha,\beta,\gamma)\in[1,+\infty)^3$ and the following reaction-diffusion system

$$(S_{\alpha\beta\gamma})\quad\begin{cases}\dfrac{\partial u_1}{\partial t}(t,x)-d_1\Delta u_1 &= \alpha\big(u_3^\gamma-u_1^\alpha u_2^\beta\big) & \text{in} & Q_T,\\ \dfrac{\partial u_2}{\partial t}(t,x)-d_2\Delta u_2 &= \beta\big(u_3^\gamma-u_1^\alpha u_2^\beta\big) & \text{in} & Q_T,\\ \dfrac{\partial u_3}{\partial t}(t,x)-d_3\Delta u_3 &= \gamma\big(-u_3^\gamma+u_1^\alpha u_2^\beta\big) & \text{in} & Q_T,\\ \frac{\partial u_i}{\partial \nu}(t,x) &= 0 & \text{on} & \Sigma_T,\\ u_i(t=0) &= u_i^0\geq 0 & \text{in} & \Omega,\\ i=1,\ 2,\ 3. \end{cases}$$

If α,β and γ are positive integers, system $(S_{\alpha\beta\gamma})$ is intended to describe for example the evolution of a reversible chemical reaction of type

$$\alpha U_1+\beta U_2\rightleftharpoons\gamma U_3$$

where u_1, u_2, u_3 stand for the density of U_1, U_2 and U_3 respectively.
Let us recall the known results about the global existence of solutions to system $(S_{\alpha\beta\gamma})$; for more details see [17] and the references therein.
In the case where the diffusion coefficients are different from each other, global existence is more complicated (it obviously holds if $d_1=d_2=d_3$). It has been studied by several authors in the following cases.
First case $\alpha=\beta=\gamma=1$. In this case, global existence of classical solutions has been obtained by Rothe [32] for dimension $N\leq 5$. Later, it has first been proved by Martin-Pierre [22] for all dimensions N and then by Morgan [24].
Global existence of weak solutions has been proved by Laamri [16] for initial data only in $L^1(\Omega)$.
Second case $\gamma=1$ regardless of α and β. In this case, global existence of classical solutions has been obtained by Feng [13] in all dimensions N and more general boundary conditions.
Third case $\alpha+\beta<\gamma$, or when $1<\gamma<\dfrac{N+6}{N+2}$ regardless of α and β. In these cases, global existence of classical solutions was established by the first author in [17].
Up to our knowledge, in the case $\alpha+\beta\geq\gamma>1$, global existence of classical solutions remain an open question when the diffusivities d_i are away from each other.
Exponential decay towards equilibrium has been studied by Fellner-Laamri [11].
Fourth case $\alpha=\beta=1$ or $\gamma\leq 2$. In this case, Pierre [27] has proved global existence of weak solutions for initial data in $L^2(\Omega)$.

• **Second system:** Let us consider the following system naturally arising in chemical kinetics when modeling the following reversible reaction

$$\alpha U_1+\beta U_3\rightleftharpoons\gamma U_2+\delta U_4$$

$$
(S_{\alpha\beta\gamma\delta})\begin{cases}
\dfrac{\partial u_1}{\partial t}(t,x) - d_1\Delta u_1 &= \alpha\big(u_2^\gamma u_4^\delta - u_1^\alpha u_3^\beta\big) \quad \text{in} \quad Q_T,\\
\dfrac{\partial u_2}{\partial t}(t,x) - d_2\Delta u_2 &= \beta\big(u_1^\alpha u_3^\beta - u_2^\gamma u_4^\delta\big) \quad \text{in} \quad Q_T,\\
\dfrac{\partial u_3}{\partial t}(t,x) - d_3\Delta u_3 &= \gamma\big(u_2^\gamma u_4^\delta - u_1^\alpha u_3^\beta\big) \quad \text{in} \quad Q_T,\\
\dfrac{\partial u_4}{\partial t}(t,x) - d_4\Delta u_4 &= \delta\big(u_1^\alpha u_3^\beta - u_2^\gamma u_4^\delta\big) \quad \text{in} \quad Q_T,\\
\frac{\partial u_i}{\partial \nu}(t,x) &= 0 \quad \text{on} \quad \Sigma_T,\\
u_i(t=0) &= u_i^0 \geq 0 \quad \text{in} \quad \Omega,\\
i = 1,\ 2,\ 3,\ 4.
\end{cases}
$$

Up to our best knowledge, global existence in the **more restrictive case** $\alpha = \beta = \gamma = \delta = 1$ has been studied by many authors. Concerning global existence of weak solution in all dimensions N, it has been first obtained in [10] for initial data u_i^0 such $u_i^0 \log u_i^0 \in L^2(\Omega)$, then in [27] for initial data in $L^2(\Omega)$ and later in [28] for initial data in $L^1(\Omega)$; see also [31, 35] and the references therein. In small dimensions $N \leq 2$, global existence of classical solutions is established in [15, 8, 31]. Recent results in [9, 34, 12] have proved that global classical solutions exist in all dimensions.

1.4 Brief Summary

In this work, we exploit the "good" L^1-framework provided by the two conditions **(P)** and **(M)** on the one hand and our key estimate with L^1 data (1) on the other hand. This allows us to extend known global existence results for initial data in $L^2(\Omega)$ to cases where the initial data only belong to $L^1(\Omega) \cap H^{-1}(\Omega)$.

More precisely, the main results of this paper can be summarized in the following points.

1) We are able to prove global existence of weak solutions

- to system $(S_{\alpha\beta\gamma})$ in the following cases ($\gamma \leq 2$ and whatever are α, β) and ($\alpha = \beta = 1$ and whatever is γ). See subsection 2.2 ;

- to system $(S_{\alpha\beta\gamma\delta})$ in the following cases ($\gamma = \delta = 1$ and whatever are α, β) and ($\alpha = \beta = 1$ and whatever are γ, δ). See subsection 2.3.

2) We are also able to establish that Theorem 5 and Theorem 6 hold for initial data only in $L^1(\Omega) \cap H^{-1}(\Omega)$. See subsection 2.1 and section refsec:porous respectively.

As we will see in more detail in the next sections, our demonstrations are based on two ingredients : our key estimate with L^1 data (1) and L^1-Pierre's theorem, Theorem 4.

Before ending this section, note that we state our theorems for Neumann boundary conditions, but they can easily be adapted to Dirichlet boundary conditions. One must however be careful when choosing two different boundary conditions for u_i and u_j for $i \neq j$, see [5, 23].

2 Applications of the key L^1 estimates

Our first estimate (1) can be exploited for establishing global existence -in- time of weak solutions to a large class of reaction-diffusion systems *with initial data of low regularity.*

2.1 A classical quadratic model with mass dissipation

As a first use of our estimate (1), see also Appendix A, we show here how one may prove global existence of weak solutions for quadratic nonlinearities and initial data of low regularity. In fact, this case is of interest due to its relevance in many applications such as chemical reactions or parabolic Lotka-Volterra type systems, see [25].

We consider the following general system: for $t \geq 0,\ x \in \Omega \subset \mathbb{R}^N$, $1 \leq i \leq m$,

$$\begin{cases} \dfrac{\partial}{\partial t} u_i(t,x) - d_i \Delta u_i = f_i(u_1, \cdots, u_m), \\ \dfrac{\partial u_i}{\partial \nu}(t,x) = 0 \text{ on } (0,+\infty) \times \partial\Omega, \\ u_i(t=0) = u_i^0 \geq 0, \end{cases} \tag{8}$$

where $d_i > 0$ and $f_i : \mathbb{R}^m \to \mathbb{R}$ is locally Lipschitz continuous. Besides **(P)** and **(M)**, assume that the f_i are at most quadratic $i.e$ there exists $C > 0$ such that for all $1 \leq i \leq m$

Theorem 7 *Besides **(P)** and **(M)**, assume that the f_i are at most quadratic i.e there exists $C > 0$ such that for all $1 \leq i \leq m$*

(QG) $\qquad |f_i(r)| \leq C\big(1 + \sum_{j=1}^{m} r_j^2\big).$

Then, for all $u^0 = (u_1^0, \cdots, u_m^0) \in (L^1(\Omega)^+ \cap H^{-1}(\Omega))^m$, System (8) has a non-negative weak solution which satisfies for all $T > 0$

$$u_i \in L^2(Q_T), \qquad u_i \in L^\infty\big((0,T); H^{-1}(\Omega)\big).$$

Proof. We approximate the initial data and right hand side of System (8) with $u_{i,0}^n := \inf\{u_{i,0},\ n\}$ and $f_i^n := \dfrac{f_i}{1 + \frac{1}{n}\sum_{1\leq j\leq m} |f_j|}$ and set

$$\begin{cases} i = 1, ..., m, & & \\ \frac{\partial}{\partial t} u_i^n - \Delta u_i^n = f_i^n(u^n) & \text{in} & Q = (0,+\infty) \times \Omega, \\ \frac{\partial u_i^n}{\partial \nu}(t,.) = 0 & \text{on} & \Sigma = (0,+\infty) \times \partial\Omega, \\ u_i^n(0,.) = u_{i,0}^n \geq 0 & \text{in} & \Omega. \end{cases} \tag{9}$$

For $i = 1, \cdots, m$, $u_{i,0}^n \in L^\infty(\Omega)$ and converges to $u_{i,0}$ in $L^1(\Omega)$, f_i^n is locally Lipschitz continuous and satisfies (P) and (M). Moreover $\|f_i^n\|_{L^\infty(\Omega)} \leq n$. Therefore, the approximate system (9) has a nonnegative classical global solution (see e.g [20]).

Thanks to Theorem 4, it is sufficient to establish the following a priori L^1-estimate

$$\forall i = 1, ..., m,\ \forall T > 0,\ \int_{Q_T} |f_i^n(u_1^n, \cdots, u_m^n)| \leq C(T) \tag{10}$$

where $C(T)$ is independent of n.

Multiplying each i-th equation of System (9) by a_i and adding the m equations to obtain

$$\frac{\partial}{\partial t}[\sum_{i=1}^{m} a_i u_i^n] - \sum_{i=1}^{m} a_i d_i \Delta u_i^n = \sum_{i=1}^{m} a_i f_i^n \leq 0 \tag{11}$$

Applying Lemma 1 with $U = \sum_{i=1}^{m} a_i u_i^n$, $V = \sum_{i=1}^{m} d_i a_i u_i^n$, $B = 0$ and $F = -\sum_{i=1}^{m} a_i f_i^n$. Then, there exists $C(T)$ independent of n such that

$$\int_0^T \int_\Omega [\sum_{i=1}^{m} a_i u_i^n]\, [\sum_{i=1}^{m} d_i a_i u_i^n] dt dx \leq C(T).$$

Thanks to their non-negativity, the u_i^n's are bounded in $L^2(Q_T)$ and then the $f_i^n(u^n)$'s are bounded in $L^1(Q_T)$ independently of n. $\square$

Remark 8 *The above proof seems very simple. But it rests on the very powerful Theorem 4, whose proof is delicate. Moreover, Lemma 1 directly provides the L^1-bound of the nonlinearities f_i.*

Remark 9 *With a very fine analysis, Theorem 7 was proved by Pierre and Rolland in [28] under an* extra assumption *on the growth of the negative part of the reaction terms, namely:*
(H) $\exists \varphi \in \mathcal{C}([0,+\infty),[0,+\infty))$ such that for all $r = (r_1, \cdots, r_m) \in [0,+\infty)^m$ and for all $1 \leq i \leq m$

$$[f_i(r) - f_i(\pi_i(r))]^- \leq \varphi(r_i)\big[1 + \sum_{i=1}^{m} r_i\big]$$

where $\pi_i(r) := (r_1, \cdots, r_{i-1}, 0, r_{i+1}, \cdots, r_m)$.
Also, it was noticed in [28] that the nonlinearities $f_2(r_1,r_2) = -f_1(r_1,r_2) = (r_1^2 + r_2^2)\sin(r_1 r_2)$ do not satisfy (H): for this example, global existence with general $L^1(\Omega)$ initial data had remained so far an open question. But our theorem does apply to this example.

Example: Let us mention that our theorem 7 applies to the famous Lotka-Volterra's system where

$$f_i(u) = \left(e_i + \sum_{j=1}^{m} a_{ij} u_j \right) u_i, \quad 1 \leq i \leq m \tag{12}$$

in (S) and satisfying, with $A := (a_{ij}) \in \mathcal{M}_m(\mathbb{R})$,

$$(e_1, \cdots, e_m) \in]-\infty, 0]^m \text{ and } \langle Au, u \rangle \leq 0 \qquad \forall (u_1, \cdots, u_m) \in [0,+\infty)^m.$$

Let us recall that this result was obtained in [35] under the strong condition

$$^tA = -A$$

and initial data in $(L^2(\Omega)^+)^m$.

2.2 A 3×3 system with nonlinearities of general growth

To show how far our approach may be carried out, we now consider systems with nonlinearities of higher degree of the following form: for $t \geq 0,\ x \in \Omega \subset \mathbb{R}^N$,

$$(S_{\alpha\beta\gamma}) \qquad \begin{cases} \frac{\partial}{\partial t} u_1(t,x) - d_1 \Delta u_1 = \alpha(u_3^\gamma - u_1^\alpha u_2^\beta), \\ \frac{\partial}{\partial t} u_2(t,x) - d_2 \Delta u_2 = \beta(u_3^\gamma - u_1^\alpha u_2^\beta), \\ \frac{\partial}{\partial t} u_3(t,x) - d_3 \Delta u_3 = \gamma(u_1^\alpha u_2^\beta - u_3^\gamma), \\ \frac{\partial u_i}{\partial \nu}(t,x) = 0 \text{ on } (0,+\infty) \times \partial\Omega, \\ u_i(t=0) = u_i^0 \geq 0, \\ 1 \leq i \leq 3 \end{cases} \tag{13}$$

where $(\alpha, \beta, \gamma) \in [1, +\infty[^3$.

Theorem 10 *Assume $u_i^0 \in L^1(\Omega)^+ \cap H^{-1}(\Omega)$, $i = 1,\ 2,\ 3$.*
1) For $\gamma \leq 2$ and any $\alpha,\ \beta$, System (13) has a non-negative weak solution which satisfies, for all $T > 0$,

$$u_i \in L^2(Q_T), \quad u_i \in L^\infty\big((0,T); H^{-1}(\Omega)\big), \quad \int_0^T \int_\Omega u_1^\alpha u_2^\beta dx dt \leq C(T).$$

2) For $\alpha = \beta = 1$ and any γ, System (13) has a non-negative weak solution which satisfies, for all $T > 0$,

$$u_i \in L^2(Q_T), \quad u_i \in L^\infty\big((0,T); H^{-1}(\Omega)\big), \quad \int_0^T \int_\Omega u_3^\gamma dx dt \leq C(T).$$

Proof. For the simplicity of the presentation, we consider the truncated system as before, but we drop the indexation by n. Let $T > 0$. Using the conservation laws for $\gamma u_1 + \alpha u_3$ and $\gamma u_2 + \beta u_3$ and Lemma 1, we conclude that u_i is bounded in $L^2(Q_T)$.
First case $\gamma \leq 2$. We integrate the first (or the second) equation to obtain

$$\int_\Omega u_1(T,x)dx + \alpha \int_0^T \int_\Omega u_1^\alpha u_2^\beta dx dt = \alpha \int_0^T \int_\Omega u_3^\gamma dx dt + \int_\Omega u_1^0(x) dx.$$

All the terms on the right-hand side are bounded, we conclude the L^1-bound on $u_1^\alpha u_2^\beta$.
Second case $\alpha = \beta = 1$. We integrate the third equation to obtain

$$\int_\Omega u_3(T,x)dx + \gamma \int_0^T \int_\Omega u_3^\gamma dx dt = \int_\Omega u_3^0(x)dx + \gamma \int_0^T \int_\Omega u_1 u_2 dx dt.$$

All the terms on the right hand side being bounded, we conclude the L^1-bound on u_3^γ.

In the two cases, $u_1^\alpha u_2^\beta - u_3^\gamma$ is bounded in $L^1(Q_T)$. Therefore, we conclude thanks to Theorem 4. $\square$

Remark 11 *When $\gamma \leq 2$, the specific form $u_1^\alpha u_2^\beta$ does not play any role, and any function $f(u_1, u_2) \geq 0$ satisfying $f(0, u_2) = f(u_1, 0) = 0$ will work.*
In the same way, when $\alpha = \beta = 1$, the specific form u_3^γ does not play any role, and any function $g(u_3) \geq 0$ satisfying $g(0) = 0$ will work.

2.3 A 4×4 system with nonlinearities of general growth

In this subsection, let us consider the following system naturally arising in chemical kinetics when modeling the following reversible reaction

$$\alpha U_1 + \beta U_2 \rightleftharpoons \gamma U_3 + \delta U_4.$$

For $t \geq 0,\ x \in \Omega \subset \mathbb{R}^N$,

$$(S_{\alpha\beta\gamma\delta}) \quad \begin{cases} \frac{\partial}{\partial t}u_1(t,x) - d_1\Delta u_1 = \alpha(u_3^\gamma u_4^\delta - u_1^\alpha u_2^\beta), \\ \frac{\partial}{\partial t}u_2(t,x) - d_2\Delta u_2 = \beta(u_3^\gamma u_4^\delta - u_1^\alpha u_2^\beta), \\ \frac{\partial}{\partial t}u_3(t,x) - d_3\Delta u_3 = \gamma(u_1^\alpha u_2^\beta - u_3^\gamma u_4^\delta), \\ \frac{\partial}{\partial t}u_4(t,x) - d_4\Delta u_4 = \delta(u_1^\alpha u_2^\beta - u_3^\gamma u_4^\delta), \\ \frac{\partial u_i}{\partial \nu}(t,x) = 0 \text{ on } (0,+\infty)\times\partial\Omega, \\ u_i(t=0) = u_i^0 \geq 0, \\ 1 \leq i \leq 4. \end{cases} \tag{14}$$

Theorem 12 *Assume $u_i^0 \in L^1(\Omega)^+ \cap H^{-1}(\Omega),\ i = 1,\ 2,\ 3,\ 4.$*
1) For $\gamma = \delta = 1$ and whatever are $\alpha,\ \beta$, System (14) has a non-negative weak solution which satisfies for all $T > 0$

$$u_i \in L^2(Q_T),\ u_i \in L^\infty\big((0,T); H^{-1}(\Omega)\big),\ \int_0^T\int_\Omega u_1^\alpha u_2^\beta dxdt \leq C(T).$$

2) For $\alpha = \beta = 1$ and whatever are $\gamma,\ \delta$, System (14) has a non-negative weak solution which satisfies for all $T > 0$

$$u_i \in L^2(Q_T),\ u_i \in L^\infty\big((0,T); H^{-1}(\Omega)\big),\ \int_0^T\int_\Omega u_3^\gamma u_4^\delta dxdt \leq C(T).$$

Proof. We proceed in the same way as the previous theorem. Using the conservation laws for $\gamma u_1 + \alpha u_3$ and $\delta u_2 + \beta u_4$ and Lemma 1, we obtain u_i is bounded in $L^2(Q_T)$ independently of n. □

Remark 13 *As in Remark 11, we observe that*
- *when $\gamma = \delta = 1$, the specific form $u_1^\alpha u_2^\beta$ does not play any role, and any function $f(u_1,u_2) \geq 0$ satisfying $f(0,u_2) = f(u_1,0) = 0$ will work ;*
- *when $\alpha = \beta = 1$, the specific form $u_3^\gamma u_4^\delta$ does not play any role, and any function $g(u_3,u_4) \geq 0$ satisfying $g(0,u_4) = g(u_3,0) = 0$ will work*

Remark 14 *As we see, the hypothesis* **(QG)** *is not used in the proofs of these last two theorems. The a priori controls given by the conservation laws allow us to conclude without the assumption of quadratic growth.*

2.4 A classical quadratic model with mass control

Next, we establish global existence in time to another class of reaction-diffusion systems, which includes, among others, Lotka-Volterra systems. Namely, we consider the following system: for $t \geq 0,\ x \in \Omega \subset \mathbb{R}^N$, $1 \leq i \leq m$,

$$\begin{cases} \dfrac{\partial}{\partial t}u_i(t,x) - d_i\Delta u_i = f_i(u_1,\cdots,u_m), \\ \dfrac{\partial u_i}{\partial \nu}(t,x) = 0 \text{ on } (0,+\infty)\times\partial\Omega, \\ u_i(t=0) = u_i^0 \geq 0, \end{cases} \tag{15}$$

where for all $1 \leq i \leq m$, $d_i > 0$ and $f_i : \mathbb{R}^m \to \mathbb{R}$ is locally Lipschitz.

Theorem 15 (Quadratic growth) *Assume the nonlinearities f_i satisfy* **(P)**, **(M')** *and* **(QG)**. *Then, for all $u^0 = (u_1^0, \cdots, u_m^0) \in (L^1(\Omega)^+ \cap H^{-1}(\Omega))^m$, System (15) has a non-negative weak solution which satisfies for all $T > 0$*

$$u_i \in L^2(Q_T), \qquad u_i \in L^\infty\big((0,T); H^{-1}(\Omega)\big).$$

Comment: As we shall see, the crux is that mass control prevents the blow-up which may otherwise occur, for example, when the right-hand side has quadratic growth.

Proof. For the simplicity of the presentation, we consider the truncated system as before, but we drop the indexation by n.

We first prove the mass control. Again multiplying each i-th equation by a_i and adding the m equations gives, by using **(M')**:

$$\frac{\partial}{\partial t}\big[\sum_{i=1}^m a_i u_i\big] - \Delta\big[\sum_{i=1}^m a_i d_i u_i\big] \le C_0 \left(\sum_{i=1}^m a_i + \sum_{i=1}^m a_i u_i\right). \tag{16}$$

Integrating (16) on Ω to obtain, with $B = C_0 \sum_{i=1}^m a_i$,

$$\frac{d}{dt}\int_\Omega \big[\sum_{i=1}^m a_i u_i(t)\big] \le B|\Omega| + C_0 \int_\Omega \big[\sum_{i=1}^m a_i u_i(t)\big] \tag{17}$$

Let $T > 0$. By integrating the previous Gronwall inequality (17), we obtain, thanks to non-negativity, for all $t \in [0,T]$

$$\sum_{1\le i\le m} \|a_i u_i^n(t)\|_{L^1(\Omega)} \le e^{C_0 T}\left[\sum_{1\le i\le m} \|a_i u_i^0\|_{L^1(\Omega)} + B|\Omega|T\right].$$

The next step is to prove the quadratic estimate. We define $U = e^{-C_0 t}\sum_{i=1}^m a_i u_i$, $V = e^{-C_0 t}\sum_{i=1}^m a_i d_i u_i$. From equation (16), we compute

$$\frac{\partial}{\partial t} U - \Delta V = B - F, \quad \text{with} \quad F \ge 0.$$

Therefore, we can apply Lemma 1 and conclude a quadratic estimate (notice that $\int_0^T \langle F \rangle dt$ is controlled thanks to the mass control above)

$$\int_0^T \int_\Omega UV \le K(T) + \|U^0\|_{H^{-1}}$$

from which we immediately deduce that the u_i's are bounded in $L^2(Q_T)$. Thus, the $f_i(u)$'s are bounded in $L^1(Q_T)$ according the assumption **(QG)**. Therefore, we can apply Theorem 4. $\square$

3 System with Porous Media diffusion

We continue our gallery of applications of Lemma 1 with an example based on the porous medium equation. Because most of the literature about porous media uses Dirichlet boundary condition, we also do so. Consider the following system: for $t \geq 0,\ x \in \Omega \subset \mathbb{R}^N$, $1 \leq i \leq m$,

$$\begin{cases} \dfrac{\partial}{\partial t} u_i(t,x) - d_i \Delta u_i^{m_i} = f_i(u_1, \cdots, u_m), \\ u_i(t,x) = 0 \text{ on } (0,+\infty) \times \partial\Omega, \\ u_i(t=0) = u_i^0 \geq 0, \end{cases} \tag{18}$$

where $d_i > 0$, $m_i > 1$ and $f_i : \mathbb{R}^m \to \mathbb{R}$ is locally Lipschitz-continuous.

Moreover, we assume that the nonlinearities f_i satisfy:
(SQG): there exists $C > 0$ and $\varepsilon > 0$ such that $\forall 1 \leq i \leq m$

$$\forall \mathbf{r} = (r_1, \cdots, r_m) \in [0,+\infty)^m, \quad |f_i(\mathbf{r})| \leq C\big(1 + \sum_{j=1}^{m} r_j^{m_j+1-\varepsilon}\big).$$

Remark 16 *As is mentioned in [18], the main point of the "$-\varepsilon$" in the above assumption is that it makes the nonlinearities $f_i^n(u^n)$ not only bounded in $L^1(Q_T)$, but uniformly integrable. This is the main tool to pass to limit in the reactive terms.*

We now define what we mean by solution to our system (18).

Definition 17 *Given $u_{i,0} \in L^1(\Omega)^+$ for $i = 1, \cdots, m$. We say that $u = (u_1, \cdots, u_m) : (0,+\infty) \times \Omega \to (0,+\infty)^m$ is a global weak solution to system (18) if for all $i = 1, \cdots, m$*

$$\begin{cases} \text{for all } T > 0, \\ u_i \in \mathcal{C}([0,T); L^1(\Omega)),\ u_i^{m_i} \in L^1(0,T; W_0^{1,1}(\Omega)),\ f_i(u) \in L^1(Q_T) \text{ and } \ \forall \psi \in \mathcal{C}_T, \\ -\displaystyle\int_\Omega \psi(0) u_i^0 - \int_{Q_T} \left[u_i \partial_t \psi + u_i^{m_i} \Delta\psi\right] = \int_{Q_T} \psi\, f_i, \end{cases} \tag{19}$$

where

$$\mathcal{C}_T := \{\psi : [0,T] \times \overline{\Omega} \to \mathbb{R}\ ;\ \psi,\ \partial_t\psi,\ \partial^2_{x_i x_j}\psi \text{ are continuous},\ \psi = 0 \text{ on } \Sigma_T \text{ and } \psi(T) = 0\}. \tag{20}$$

The definition 17 corresponds to the notion of *very weak solution* in Definition 6.2 of [36].

Theorem 18 *Assume that the nonlinearities f_i satisfy **(P)**, **(M)** and **(SQG)**. Then, for all $u^0 = (u_1^0, \cdots, u_m^0) \in (L^1(\Omega)^+ \cap H^{-1}(\Omega))^m$, System (18) has a non-negative weak solution which satisfies for all $T > 0$*

$$u_i \in L^{m_i+1}(Q_T), \qquad u_i \in L^\infty\big((0,T); H_0^{-1}(\Omega)\big).$$

Preliminary remark about Theorem 18 and its proof: Let us emphasize that, under the assumptions **(P)**, **(M)** and **(SQG)**, the existence of global weak solutions to System (18) subject to homogeneous Dirichlet boundary conditions with initial data in $L^2(\Omega)$ was recently established by the first author and Pierre, see [18, Corollary 2.8]). The proof is based on a dimension-independent $L^{m_i+1}(Q_T)$-estimate which heavily depends on the L^2-norm of the initial data (see [18, Theorem 2.7]), and then it is not valid for initial data in $L^1(\Omega)$. But, we are able to establish the same uniform bound in $L^{m_i+1}(Q_T)$ (see (29)) with ***initial data only in*** $L^1(\Omega)^+ \cap H^{-1}(\Omega)$ thanks to the estimate (1) of Lemma 1. This is a *new result* in this nonlinear degenerate setting and interesting for itself. Nevertheless, the hypothesis **(SQG)** is crucial to pass to the limit.
Once the uniform estimate is established, the demonstration by Laamri-Pierre in [18] applies *per se*. But, for the sake of completeness and for the reader's convenience, we will give below the main steps of the proof.

Proof. We divide the proof in three steps.
- *First step:* Let us consider the following approximation of System (18)

$$\begin{cases} i = 1, ..., m, \\ \frac{\partial u_i^n}{\partial t} - d_i \Delta((u_i^n)^{m_i}) = f_i^n(u^n) & \text{in} \quad Q = (0, +\infty) \times \Omega, \\ u_i^n = 0 & \text{on} \quad \Sigma = (0, +\infty) \times \partial\Omega, \\ u_i^n(0, .) = u_{i,0}^n \geq 0 & \text{in} \quad \Omega, \end{cases} \tag{21}$$

where $u_{i,0}^n := \inf\{u_{i,0},\ n\}$ and $f_i^n := \dfrac{f_i}{1 + \frac{1}{n}\sum_{1\leq j\leq m}|f_j|}$.
For $i = 1, \cdots, m$, $u_{i,0}^n \in L^\infty(\Omega)$ and converges to $u_{i,0}$ in $L^1(\Omega)$, f_i^n is locally Lipchitz continuous and satisfies **(P), (M)** and **(SQG)**. Moreover $\|f_i^n\|_{L^\infty(\Omega)} \leq n$. Therefore, the approximate system (21) has a non-negative bounded global solution (see e.g [18, Lemma 2.3]). Moreover, this solution is regular in the sense that

$$\text{for all } T > 0,\ u_i^n \in L^\infty(Q_T) \cap \mathcal{C}([0, T]; L^1(\Omega)),\ (u_i^n)^{m_i} \in L^2(0, T; H^1(\Omega))$$

and the equations in (21) are satified a.e.
Consequently, we have for $i = 1, \cdots, m$

$$\forall \psi \in \mathcal{C}_T,\ -\int_\Omega \psi(0) u_{i,0}^n - \int_{Q_T} \left[u_i^n \partial_t \psi + (u_i^n)^{m_i} \Delta\psi\right] = \int_{Q_T} \psi\, f_i^n. \tag{22}$$

Our goal is to pass to the limit as $n \to +\infty$ in (22).
- *Second step* : First we will establish a priori estimates and then we prove that the $f_i^n(u^n)$ are uniformly bounded in $L^1(Q_T)$. For this, we multiply each i-th equation by a_i and adding the m equations to obtain

$$\frac{\partial}{\partial t}\Big[\sum_{i=1}^m a_i u_i^n\Big] - \Delta\Big[\sum_{i=1}^m d_i a_i (u_i^n)^{m_i}\Big] = \sum_{i=1}^m a_i f_i^n(u^n). \tag{23}$$

Thanks to assumption **(M)** and the definition of f_i^n, we obtain

$$\frac{\partial}{\partial t} U_n - \Delta V_n \leq 0 \text{ in } Q_T, \quad V_n = 0 \text{ on } \Sigma_T. \tag{24}$$

where we set $U_n := \sum_{i=1}^{m} a_i u_i^n$ and $V_n := \sum_{i=1}^{m} d_i a_i (u_i^n)^{m_i}$. In order to apply Lemma 1, we first prove that $< V_n(t) >$ is bounded.
To this end, introduce θ solution of

$$-\Delta\theta = 1 \text{ in } \Omega, \quad \theta = 0 \text{ in } \partial\Omega. \tag{25}$$

Now, integrating (24) in time leads to

$$U_n(t,x) - \Delta \int_0^t V_n(s,x)ds \leq U_n(0,x) \leq U(0,x). \tag{26}$$

Multilpying (26) by θ and integrating over Ω, to obtain

$$\int_\Omega \theta U_n(t,x)dx + \int_\Omega \theta \left(-\Delta[\int_0^t V_n(s,x)ds] \right) \leq \int_\Omega \theta U(0,x)dx \leq \|\theta\|_{L^\infty(\Omega)} \|U_0\|_{L^1(\Omega)} \tag{27}$$

After integration by parts, we obtain

$$\int_\Omega \int_0^t V_n(s,x)dsdx = \int_\Omega \int_0^t V_n(s,x)(-\Delta\theta)dsdx \leq \|\theta\|_{L^\infty(\Omega)} \|U_0\|_{L^1(\Omega)} \tag{28}$$

which gives the desired estimate.
Thus, according to Lemma 1, there exists $C(T) > 0$ independent of n such that

$$\int_0^T \int_\Omega [\sum_{i=1}^{m} a_i u_i^n] [\sum_{i=1}^{m} a_i d_i (u_i^n)^{m_i}] dtdx \leq C(T).$$

Thanks to the non-negativity, there exists $C > 0$ independent of n such that for all $i = 1, \cdots m$,

$$\|u_i^n\|_{L^{m_i+1}(Q_T)} \leq C. \tag{29}$$

Together with the assumption **(SQG)**, the $f_i^n(u^n)$ are uniformly bounded in $L^1(Q_T)$, more precisely, there exists a constant $C > 0$ independent of n such that

$$\|f_i^n(u^n)\|_{L^1(Q_T)} \leq C. \tag{30}$$

In fact, $\{f_i^n(u^n)\}$ is not only bounded in $L^1(Q_T)$ but is also uniformly integrable on Q_T. Indeed, for all measurable set $E \subset Q_T$ with Lebesgue measure denoted by $|E|$, we have (recall that $|f_i^n| \leq |f_i|$)

$$\int_E |f_i^n(u^n)| \leq C \left[|E| + \int_E (u_i^n)^{m_i+1-\varepsilon} \right] \leq C \left[|E| + \left(\int_{Q_T} (u_i^n)^{m_i+1} \right)^{\frac{m_i+1-\varepsilon}{m_i+1}} |E|^{\frac{\varepsilon}{m_i+1}} \right].$$

Thanks to (29), $\int_E \sum_i |f_i^n(u^n)|$ may be made uniformly small by taking $|E|$ small enough. This is exactly the uniform integrability of the $f_i^n(u^n)$.

• *Third step:* Passage to the limit as $n \to +\infty$ in (22)
First, we apply the following compactness lemma wich allows us to extract a converging subsequence from the u_i^n.

Lemma 19 (Baras, [4]) *Let* $(w_0, H) \in L^1(\Omega) \times L^1(Q_T)$ *and let* w *be the solution of*

$$w_t - d\Delta(w^q) = H \text{ in } Q_T, \quad w = 0 \text{ in } \Sigma_T, \quad w(0,.) = w_0.$$

The mapping $(w_0, H) \mapsto w$ *is compact from* $L^1(\Omega) \times L^1(Q_T)$ *to* $L^1(Q_T)$ *for all* $q > \dfrac{(N-2)^+}{N}$.

Since the $f_i^n(u^n)$ is uniformly bounded in $L^1(Q_T)$, according to Lemma 19, $\{u_i^n\}$ is relatively compact in $L^1(Q_T)$. Therefore, up to a subsequence, $\{u_i^n\}$ converges in $L^1(Q_T)$ and a.e to some limit $u_i \in L^1(Q_T)$. Moreover, $f_i^n(u^n)$ is uniformly integrable and converges a.e $f_i(u)$. Then, Vitali's theorem implies that $f_i(u) \in L^1(Q_T)$ and $f_i^n(u^n)$ converges in $L^1(Q_T)$ to $f_i(u)$.
Second, thanks to the L^{m_i+1}-estimate (29), there exists a subsequence which converges to u_i in $L^p(Q_T)$ for all $p < m_i + 1$.
Now we can pass to the limit as $n \to +\infty$ in the weak formulation (22) to obtain

$$-\int_\Omega \psi(0)u_i^0 - \int_{Q_T} \left[u_i \partial_t \psi + u_i^{m_i} \Delta\psi\right] = \int_{Q_T} \psi\, f_i, \tag{31}$$

for all $\psi \in \mathcal{C}_T$, recall that

$$\mathcal{C}_T := \{\psi : [0,T] \times \overline{\Omega} \to \mathbb{R}\ ;\ \psi,\ \partial_t\psi,\ \partial^2_{x_i x_j}\psi \text{ are continuous},\ \psi = 0 \text{ on } \Sigma_T \textit{ and } \psi(T) = 0\}.$$

To prove that $u = (u_1, \cdots, u_n)$ is solution to System (18) in the sense of definition 17, it remains to establish that for all $T > 0$, $u_i \in \mathcal{C}([0,T]; L^1(\Omega))$ and $u_i^{m_i} \in L^1(0,T; W^{1,1}(\Omega))$; in fact, we will prove that $u_i^{m_i} \in L^\beta(0,T; W_0^{1,\beta}(\Omega))$ for all $\beta \in [1, 1 + \dfrac{1}{1+m_iN})$.

Lemma 20 (Lukkari, [21]) *Let* $(w_0, H) \in L^1(\Omega) \times L^1(Q_T)$ *and let* w *be the solution of*

$$w_t - d\Delta(w^q) = H \text{ in } Q_T, \quad w = 0 \text{ in } \Sigma_T, \quad w(0,.) = w_0. \tag{32}$$

Then, there exists $C > 0$

$$\int_{Q_T} |w|^{q\alpha} \leq C \ \text{ for } \ 0 < \alpha < 1 + \frac{2}{qN}, \tag{33}$$

$$\int_{Q_T} |\nabla w^q|^\beta \leq C \ \text{ for } \ 1 \leq \beta < 1 + \frac{1}{1+qN}, \tag{34}$$

$$\left\| \frac{\partial w}{\partial t} \right\|_{L^1(0,T;W^{-1,1}(\Omega)} \leq C. \tag{35}$$

where $C = C\left(T, \alpha,\ \beta,\ q,\ \|w_0\|_{L^1(\Omega)},\ \|H\|_{L^1(Q_T)}\right)$.

For a proof, see Lukkari [21, Lemma 4.7]. However, in this reference, the proof is given with zero initial data, but with right-hand side a bounded measure. We may use the measure $\delta_{t=0} \otimes w_0\, dx$ to include the case of initial data w_0. We may also use the results in [1, Theorem 2.9]). □

Now let's come back to the proof of theorem.
Let $i \in \{1, \cdots, m\}$. According to the estimate (30), we apply the estimate (34) to i-th equation of (18) to obtain $\nabla(u_i^n)^{m_i}$ is bounded in the space $L^\beta(0,T; W_0^{1,\beta}(\Omega))$ for all $\beta \in [1, 1 + \dfrac{1}{1+m_iN})$. These spaces being reflexive (for $\beta > 1$), it follows that $\nabla(u_i)^{m_i}$ also belongs to these same spaces.
According again (30), we deduce from the estimate (34) that $\dfrac{\partial u_i}{\partial t} \in L^1(0,T; W^{-1,1}(\Omega))$ which implies in particular that $u_i \in \mathcal{C}([0,T]; L^1(\Omega))$. □

Remark 21 *The paper by Gess, Sauer and Tadmor, [14], provides another route for the compactness argument which can be applied directly to the u_i^m. It states, among other results, that solution of equation (32) in the full space, with data in L^1 as in Lemma 20, satisfy*

$$w \in L^q\big((0;T);W^{s,q}(\mathbb{R}^N)\big), \qquad s = \frac{2}{q}.$$

and a similar space-time regularity result in fractional Sobolev spaces.

A First key estimate with L^1 initial data

We prove the key Lemma 1 and, for the sake of completeness, we recall the statement. We define

$$\langle F \rangle = \frac{1}{|\Omega|}\int_\Omega F(x)dx$$

and, the $H^{-1}(\Omega)$ norm of F as

$$\|F\|_{H^{-1}} = \|\nabla W\|_{L^2},$$

where W solves

$$\begin{cases} \Delta W = F - \langle F \rangle, & x \in \Omega, \\ \frac{\partial W}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases}$$

Lemma 22 (First key estimate with L^1 data) *Consider smooth functions F, $U : [0,+\infty)\times\Omega \to \mathbb{R}^+$ and $V : (0,+\infty)\times\Omega \to \mathbb{R}$ such that $\int_\Omega \Delta_x V(t,x)dx = 0$ and B a non-negative constant. Assume that $U^0 = U(t=0) \in L^1(\Omega)\cap H^{-1}(\Omega)$ and that the differential relation holds*

$$\begin{cases} \frac{\partial}{\partial t}U(t,x) - \Delta_x V = B - F(t,x) \le 0, & t \ge 0,\ x \in \Omega \subset \mathbb{R}^N, \\ \frac{\partial U}{\partial \nu}(t,x) = 0 \text{ on } (0,+\infty)\times\partial\Omega. \end{cases} \tag{36}$$

Then, for some constant C depending on Ω, the inequality holds

$$\frac{1}{2}\|U(T)\|^2_{H^{-1}} + \int_0^T\int_\Omega UV \le K(T) + \frac{1}{2}\|U^0\|^2_{H^{-1}} \quad \text{where} \quad K = \int_0^T \big[C\langle F(t)\rangle + \langle V(t)\rangle\big]\ \big[B|\Omega|t + \int_\Omega U^0\big]dt. \tag{37}$$

Notice that, with our assumptions, F is controlled in $L^1(Q_T)$ because

$$0 \le \langle U(T)\rangle = \langle U^0\rangle + BT - \int_0^T \langle F(s)\rangle ds.$$

Proof. The proof is reminiscent from the lifting method introduced in [7]. We write, substracting its average to equation (R),

$$\partial_t(U - \langle U\rangle) - \Delta V = \langle F\rangle - F.$$

Then, we solve with Neumann Boundary Condition

$$-\Delta W = U - \langle U\rangle, \qquad \langle W\rangle = 0.$$

Therefore, we find

$$-\Delta\big[\partial_t W + V\big] \leq \langle F\rangle \qquad \text{(a constant)},$$

and from elliptic theory, see [2, 3], we conclude that

$$\partial_t W + V \leq C\langle F\rangle + \langle V\rangle. \tag{38}$$

Finally, multiplying by U,

$$\int_\Omega \underbrace{\big[-\Delta W + \langle U\rangle\big]}_{=U} \partial_t W + \int_\Omega UV \leq \big[C\langle F\rangle + \langle V\rangle\big]\int_\Omega U$$

and, because $\int_\Omega \partial_t W = 0$, this gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla W(t)|^2 + \int_\Omega UV(t) \leq \big[C\langle F\rangle + \langle V\rangle\big]\big[B|\Omega|t + \int_\Omega U^0\big].$$

The result follows after time integration. $\square$

Comment : An immediate variant of Lemma 22 holds true with the Dirichlet boundary condition and its proof is even simpler because subtracting the averages is useless and the upper bound (38) can be found thanks to explicit super-solutions.

B A key estimate with L^1 source, no sign condition

Lemma 22 can be adapted to the case when the right-hand side F does not have a sign. However, this loss of non-positivity is a major difficulty and this leads to the technical restriction $N \leq 3$. Because elliptic regularity is involved, the situation is more complicated and it is not clear whether the restriction $N \leq 3$ is only technical or due to deeper phenomena. As we will see, it naturally appears in the proof.

Again, for functions $V,\ U,\ F : (0, +\infty)\times\Omega \to \mathbb{R}$ such that $\displaystyle\int_\Omega \Delta_x V = 0$, we consider the differential relation

$$\begin{cases} \frac{\partial}{\partial t}U(t,x) - \Delta V = F \in L^1, \quad t \geq 0,\ x \in \Omega \subset \mathbb{R}^N, \\ \frac{\partial U}{\partial \nu}(t,x) = 0 \text{ on } (0,+\infty)\times\partial\Omega, \\ U(t=0) = U^0 \geq 0. \end{cases} \tag{39}$$

Lemma 23 *Assume $N \leq 3$, $V \geq a\,U \geq 0$ for some $a > 0$, $U^0 \in L^1(\Omega)^+ \cap H^{-1}(\Omega)$, and $\displaystyle\int_\Omega \Delta_x V = 0$. Then, it holds*

$$a\int_0^T \int_\Omega U^2 \leq C\left(\int_0^T [\|F(t)\|_1 + \langle V(t)\rangle]dt\right)^2 + \|U^0\|^2_{H^{-1}(\Omega)}. \tag{40}$$

Proof. As in Appendix A, we first substract its average to equation (39) and find

$$\partial_t (U - \langle U \rangle) - \Delta V = F - \langle F \rangle.$$

Next, we solve with Neumann Boundary Condition

$$-\Delta W = U - \langle U \rangle, \qquad \langle W \rangle = 0.$$

Then, as in Appendix A, we find

$$-\Delta \big[\partial_t W + V \big] = F - \langle F \rangle,$$

and from elliptic theory, [2, 3], since $1 - \frac{2}{N} < \frac{1}{2}$ when $N \le 3$,

$$\Phi := \partial_t W + V, \quad \text{satisfies} \quad \|\Phi\|_2 \le C \|F\|_1 + \langle V \rangle.$$

Therefore, multiplying again by U,

$$\int_\Omega \underbrace{[-\Delta W + \langle U \rangle]}_{=U} \partial_t W + \int UV = \int \Phi U \le \|\Phi(t)\|_2 \|U(t)\|_2.$$

Because $\displaystyle\int \partial_t W = 0$, this gives

$$\frac{1}{2} \frac{d}{dt} \int |\nabla W|^2 + a \int U^2 \le \|\Phi(t)\|_2 \|U(t)\|_2.$$

Therefore, we conclude after time integration. $\square$

C Proof of Pierre's L^2 Lemma

Integrating in time the equation (R) with $V = AU \ge 0$, one finds, setting $W(t,x) = \int_0^t AU(s,x) ds$,

$$U(t,x) - \Delta W(t,x) \le U^0,$$

We multiply by $AU = \frac{\partial W}{\partial t}$ and find successively

$$AU^2(t,x) - \frac{\partial W}{\partial t} \Delta W(t,x) \le AUU^0,$$

$$\int_\Omega AU^2(t,x) + \int_\Omega \frac{\partial \nabla W}{\partial t} \nabla W(t,x) \le \int_\Omega AUU^0,$$

$$\int_\Omega AU^2(t,x) + \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla W(t,x)|^2 \le \int_\Omega AUU^0,$$

$$\int_0^T \int_\Omega AU^2(t,x) + \frac{1}{2} \int_\Omega |\nabla W(T,x)|^2 \le \Big(\int_0^T \int_\Omega A(U^0)^2 \int_0^T \int_\Omega AU^2 \Big)^{1/2}.$$

As a consequence, we arrive at the conclusion that

$$\int_0^T \int_\Omega AU^2(t,x) \le \sqrt{\Big\| \int_0^T A \Big\|_\infty} \Big(\int_\Omega (U^0)^2 \Big)^{1/2} \Big(\int_0^T \int_\Omega AU^2 \Big)^{1/2}.$$

Comment: In this proof, the L^2 bound arises the time derivative, while in Lemma 22 it stems from the Laplacian.

D A priori bound for the porous medium equation

For the semi-linear porous medium equation, we need a consequence of Lemma 22 that we establish here. With the notations of section 3 and for sake of simplicity, we assume $a_1 = \dots = a_m = 1$ and $d_1 = \dots = d_m = 1$. By summing the m equations, we obtain the problem

$$\frac{\partial}{\partial t} \sum_i u_i - \Delta \sum_i u_i^{m_i} = -F \in L^1(Q_T),$$

where $F := -\sum_i f_i \geq 0$.

We prove that for some constant which depends on T, $\|u_i^0\|_{L^1(\Omega)\cap H^{-1}(\Omega)}$ and $\|F\|_{L^1(\Omega)}$, we have

$$\int_0^T \int_\Omega \sum_i u_i \sum_i u^{m_i} dxdt \leq C \qquad \text{and} \quad \int_0^T \int_\Omega \sum_i u^{m_i} dxdt \leq C. \tag{41}$$

To prove these inequalities, we apply formula (1) with $B = 0$ and find

$$\int_0^T \int_\Omega \sum_i u_i \sum_i u^{m_i} \leq C + C \int_0^T \int_\Omega \sum_i u_i^{m_i} dxdt.$$

Therefore, using the Hölder's inequality

$$\int_0^T \int_\Omega \sup_i u_i \sum_i u^{m_i} \leq C + C \sum_i \left(\int_0^T \int_\Omega u_i^{m_i+1} dxdt \right)^{\frac{m_i}{m_i+1}}$$

and with $\alpha = \sup_i \frac{m_i}{m_i+1} < 1$,

$$\begin{aligned} \int_0^T \int_\Omega \sup_i u_i \sum_i u^{m_i} dxdt &\leq C + C \sum_i \left(\int_0^T \int_\Omega u_i^{m_i+1} dxdt \right)^\alpha \\ &\leq C + C \left(\int_0^T \int_\Omega \sum_i u_i^{m_i+1} dxdt \right)^\alpha \\ &\leq C + C \left(\int_0^T \int_\Omega \sup_i u_i \sum_i u_i^{m_i} dxdt \right)^\alpha \end{aligned}$$

This proves that $\int_0^T \int_\Omega \sup_i u_i \sum_i u^{m_i} dxdt$ is bounded and thus the result (41).

References

[1] B. Abdellaoui, I. Peral, M. Walias : "*Some existence and regularity results for porous medium and fast equations with a gradient term*". Trans. Amer. Math. Soc. 367 (2015), no. 7, 4757–4791.

[2] S. Agmon, A. Douglis, L. Nirenberg : "*Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*". Comm. Pure Appl. Math. 12 (1959), 623–727.

[3] S. Agmon, A. Douglis, L. Nirenberg : "*Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II*". Comm. Pure Appl. Math. 17 (1964), 35–92.

[4] P. Baras : "*Compacité de l'opérateur $f \mapsto u$ solution d'une équation non linéaire $\frac{du}{dt} + Au \ni f$*". C.R.A.S. Série A, t. 286 (1978), 1113-1116.

[5] J. Bebernes and A. Lacey: "*Finite-time blow up for semilinear reactive-diffusive systems*". J. Diff. Equ., 95 (1992), 105-129.

[6] M. Bendahmane, Th. Lepoutre, A. Marrocco, B. Perthame : "*Conservative cross diffusions and pattern formation through relaxation*". J. Math. Pures et Appli. **92** 6 (2009), 651-667.

[7] J. F. Blowey and C. M. Elliott : "*The Cahn-Hilliard gradient theory for phase separation with nonsmooth free energy. I. Mathematical analysis*". European J. Appl. Math. 2 (1991), no. 3, 233–280.

[8] J.-A. Canizo, L. Desvillettes, K. Fellner : "*Improved duality estimates and applications to reaction-diffusion equations*". Comm. Partial Differential Equations 39 (2014), 1185-204.

[9] C. Caputo, T. Goudon, A. Vasseur : "*Solutions of the 4-species quadratic reaction-diffusion system are bounded and $\mathcal{C}^\infty$-smooth, in any space dimension*". Anal. PDE 12 (2019), no. 7, 1773–1804.

[10] L. Desvillettes, K. Fellner, M. Pierre, J. Vovelle : "*About Global Existence for Quadratic Systems of Reaction-Diffusion*". Adv. Nonlinear Stud. **7** (2007), 491–511.

[11] K. Fellner, E.-H. Laamri : "*Exponential decay towards equilibrium and global classical solutions for nonlinear reaction-diffusion systems*". J. Evol. Equ. **16** (2016), 681-704.

[12] K. Fellner, J. Morgan, B. Q. Tang : "*Global classical solutions to quadratic systems with mass control in arbitrary dimensions* ". Preprint.

[13] W. Feng : "*Coupled system of reaction-diffusion equations and Applications in carrier facilitated diffusion*". Nonlinear Analysis, Theory, Methods and Applications **17**(3) (1991), 285–311.

[14] B. Gess, J. Sauer, E. Tadmor : "*Optimal regularity in time and space for the porous medium equation*". Preprint arXiv:1902.08632 (2019).

[15] T. Goudon and A. Vasseur : "*Regularity Analysis for Systems of Reaction-Diffusion Equations*", Annales Sci. ENS (4) 43 (2010), no. 1, 117–142.

[16] E.-H. Laamri : "*Existence globale pour des systèmes de réaction-diffusion dans L^1*". Ph.D thesis, université de Nancy 1, France, 1988.

[17] E.-H. Laamri : "*Global existence of classical solutions for a class of reaction-diffusion systems*". Acta Applicandae Mathematicae, **115** 2 (2011), 153–165.

[18] E.-H. Laamri, M. Pierre : " *Global existence for reaction-diffusion systems with nonlinear diffusion and control of mass*". Ann. Inst. H. Poincar Anal. Non Linaire 34 (2017), no. 3, 571–591.

[19] E.-H Laamri, M. Pierre : "*Stationary reaction-diffusion systems in* L^1". Mathematical Models and Methods in Applied Sciences Vol. 28, No. 11 (2018) 2161–2190.

[20] O. A. Ladyzenskaya, V. A. Solonnikov, N. N. Uralceva : "Linear and quasilinear equations of parabolic type". Transl. Math. Monographs, **23**, A.M.S., Providence, R.I. 1968.

[21] T. Lukkari : "*The porous medium equation with measure data*". J. Evol. Equ., **10** (2010), 711-729.

[22] R. H. Martin, M. Pierre : "*Nonlinear reaction-diffusion systems*". In Nonlinear Equations in the Applied Sciences, W.F. Ames and C. Rogers ed., Math. Sci. Eng. 185, Ac. Press, New York, 1991.

[23] R. H. Martin, M. Pierre : "*Influence of mixed boundary conditions in some reaction-diffusion systems*". Proc. Roy. Soc. Edinburgh, section A 127 (1997), 1053-1066 .

[24] J. Morgan : "*Global existence for semilinear parabolic systems*". SIAM J. Math. Ana. 20 (1989), 1128-1144.

[25] B. Perthame : "Parabolic equations in biology: growth, reaction, movement and diffusion". Lecture Notes on Mathematical Modelling in the Life Sciences. Springer, Berlin, 2015.

[26] M. Pierre : "*Weak solutions and supersolutions in* L^1 *for reaction-diffusion systems*". J. Evol. Equ. **3**, (2003), no. 1, 153–168.

[27] M. Pierre : "*Global Existence in Reaction-Diffusion Systems with Control of Mass: a Survey*". Milan J. Math. Vol. 78 (2010), 417-455.

[28] M. Pierre, G. Rolland : "*Global existence for a class of quadratic reaction-diffusion system with nonlinear diffusion and* L^1 *initial data*". J. Nonlinear Analysis TMA. Volume 138 (2017), 369–387.

[29] M. Pierre, D. Schmitt : "*Blow-up in Reaction-Diffusion Systems with Dissipation of Mass*". SIAM J. Math. Ana. 28, No2 (1997), 259-269.

[30] M. Pierre, D. Schmitt : "*Blow-up in reaction-diffusion systems with dissipation of mass*". SIAM Rev. **42**, (2000), pp. 93–106 (electronic).

[31] M. Pierre, T. Suzuki, Y. Yamada : "*Dissipative reaction diffusion systems with quadratic growth*". Indiana Univ. Math. J. 68 (2019), no. 1, 291–322.

[32] F. Rothe : "Global Solutions of Reaction-Diffusion Systems". Lecture Notes in Mathematics, Springer, Berlin, (1984).

[33] P. Quittner, Ph. Souplet : "Superlinear Parabolic Problems: Blow-up, Global Existence and Steady States". Advanced Texts, Birkhäuser, (2007).

[34] Ph. Souplet : "*Global existence for reaction-diffusion systems with dissipation of mass and quadratic growth*". J. Evol. Equ. 18 (2018), no. 4, 1713–1720.

[35] T. Suzuki, Y. Yamada : "*Global-in-time behavior of Lotka-Volterra system with diffusion: skew-symmetric case*". Indiana Univ. Math. J. 64 (2015), no. 1, 181–216.

[36] J. L. Vazquez : "*The porous medium equation Mathematical Theory*". Oxford Mathematical Monographs, 2006.