HAL
open science

On matrix-valued log-concavity and related Prékopa and Brascamp-Lieb inequalities

Dario Cordero-Erausquin

To cite this version:

Dario Cordero-Erausquin. On matrix-valued log-concavity and related Prékopa and Brascamp-Lieb inequalities. Advances in Mathematics, 2019, 351, pp.96-116. 10.1016/j.aim.2019.04.046 . hal-02274566

HAL Id: hal-02274566

https://hal.sorbonne-universite.fr/hal-02274566

Submitted on 30 Aug 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON MATRIX-VALUED LOG-CONCAVITY AND RELATED PRÉKOPA AND BRASCAMP-LIEB INEQUALITIES

DARIO CORDERO-ERAUSQUIN

Abstract. We propose a new, self-contained, approach to H. Raufi's extension of Prékopa's theorem for matrix-valued log-concave functions. Along the way, new related inequalities are established, in particular a Brascamp-Lieb variance inequality for matrix weights.

Contents

1. Introduction 1
2. N-log-concave matrix valued functions 4
3. Brascamp-Lieb variance inequality for matrix-valued N-log-concave weights 9
4. Proof of the matrix valued Prékopa's inequality 14
References 17

1. Introduction

The present note is motivated by Raufi's paper [9] on matrix-valued log-concavity. One of our goals is to give a proof of Raufi's extension of Prékopa's inequality that does *not* use complex variables or complex geometry.

Prékopa's theorem [8] is a fundamental result about convexity and integration that gives a functional form of the Brunn-Minkowski inequality for convex sets. It says that marginals of log-concave functions are log-concave. A function $\alpha : \mathbb{R}^n \to \mathbb{R}^+$ is log-concave if $\log(\alpha)$ is concave, that is if we can write $\alpha = e^{-\varphi}$ with φ convex on $\mathbb{R}^n$. Prékopa's theorem asserts that given a log-concave function $g : \mathbb{R}^{n_0+n_1} \to \mathbb{R}^+$, the function $\alpha : \mathbb{R}^{n_0} \to \mathbb{R}^+$ defined by

$$\alpha(t) := \int_{\mathbb{R}^{n_1}} g(t, y)\, dy$$

is again log-concave.

If we want to extend this result to functions taking their values in the cone $\mathcal{M}_d(\mathbb{R})^+$ of positive operators on $\mathbb{R}^d$, or equivalently of $d \times d$ positive symmetric matrices, instead of $\mathbb{R}^+$, that is moving from $d = 1$ to $d > 1$, we need first to provide a notion of log-concavity for such functions. Let us mention that we can also work exactly in the same way with $\mathcal{M}_d(\mathbb{C})^+$, the cone of Hermitian positive operators.

This is something known in the complex setting (with plurisubharmonicity in place of convexity). A simple situation in dimension $n = 1$ appears when doing complex interpolation of families of Hilbert space: a function on the unit disc $g : \mathbb{D} \to \mathcal{M}_d(\mathbb{C})^+$ will be an

interpolating family of Hilbert spaces $(\mathbb{C}^d, g)$ if

$$\mathbf{\Theta}^g(z) := \partial_{\overline{z}}(g^{-1}\partial_z g) = 0,$$

which is an extremal situation for the condition $g(z)\,\mathbf{\Theta}^g(z) \leq 0$ for the symmetric matrix $g(z)\,\mathbf{\Theta}^g(z)$; if $d = 1$ and if we write $g(z) = e^{-u(z)}$, then we have $\mathbf{\Theta}^g(z) = -\frac{1}{4}\Delta u(z)$, and so we recover that $u = -\log(g)$ is sub-harmonic.

In complex geometry, similar notions for $n, d > 1$ have been investigated in connection with notions of curvature for metrics on vector bundles, see [6]. This amounts to notions of positivity for an operator $\mathbf{\Theta}^g$ constructed from the n^2 operators $\theta^g_{j,k} := \partial_{\overline{z}_j}(g^{-1}\partial_{z_k} g)$.

In analogy with these notions from complex geometry, Raufi [9] introduces for a smooth function $g : \mathbb{R}^n \to \mathcal{M}_d(\mathbb{R})^+$ a notion of "Log-concave in the sense of Nakano". We favour the terminology "N-log-concave" for simplicity. We will present in details this notion and the corresponding operator $\mathbf{\Theta}^g$ in the next section. If $d = 1$, it amounts to $\mathbf{\Theta}^g = \mathrm{Hess}(\log(g)) \leq 0$ as expected. Although examples of N-log-concave functions are for the moment limited in the real world, there is (at least!) a remarkable result.

Theorem 1 (Raufi [9])**.** *Let $g : \mathbb{R}^{n_0+n_1} \to \mathcal{M}_d(\mathbb{R})^+$ be a C^2 function with values in the symmetric positive $d\times d$ matrices. Assume that for every $t \in \mathbb{R}^{n_0}$, we have $\int_{\mathbb{R}^{n_1}} |g(t,y)|\,dy < +\infty$ where $|\cdot|$ is a norm on $\mathcal{M}_d(\mathbb{R})$ and introduce*

$$\alpha(t) := \int_{\mathbb{R}^{n_1}} g(t,y)\,dy \in \mathcal{M}_d(\mathbb{R})^+.$$

If g is N*-log-concave on $\mathbb{R}^{n_0+n_1}$, then α is* N*-log-concave on $\mathbb{R}^{n_0}$.*

For $d = 1$, it is the result of Prékopa we discussed above. The assumptions in Raufi's theorem are slightly different than the one we put.

The proof by Raufi is rather sinuous and builds upon several tools from complex analysis and complex geometry. First he complexifies the problem, using $\mathbb{C}^n$ in place of $\mathbb{R}^n$; eventually he will use only functions that depend on the real parts of the variables. He introduces some complex "manifold", which is an (infinite dimensional) hermitian vector bundle (the fibers are weighted Bergman spaces) and reduces the problem to showing that this bundle has a negative Nakano curvature, extending to higher dimensions a result by Berndtsson [1]. He then introduces another bundle, with the help of Fourier transform , for which he needs to establish a vector valued description of Bergman spaces with log-concave weights over tube domains through Fourier transform; this step is probably of independent interest. The new bundle is isometric to the previous one, and finally, for this bundle, he proves the desired curvature estimate using vector valued versions of Hörmander's $\overline{\partial}$-estimates. This long way brings several technical difficulties.

The fact that some Hörmander type L^2 estimate could be useful was to be expected, starting from the author's curvature computation in [3] and from later deep generalizations by Berndtsson (for instance in [1]). But the rest of Raufi's proof, and in particular the use of Fourier transform, is more surprising from the point of view of convex analysis and classical approaches to Brunn-Minkowski type inequalities.

One of our goals is to provide a different, somehow "classical" and direct approach to the problem and to Theorem 1, inspired by Brascamp-Lieb's approach [2] to Prékopa's theorem (see also the survey [5]), in particular without using complex variables. Namely, we we want to:

i) Compute the second derivatives $\mathbf{\Theta}^\alpha$ by "direct" differentiation.
ii) Find a way to relate this "second derivative in time" (variables t) to "derivatives in space" (variables y), using log-concavity. This reduces the problem to variance/spectral type inequality with respect to the density $g(t,y)\,dy$ on $\mathbb{R}^{n_1}$.
iii) Establish and prove a general variance/spectral type inequality on $\mathbb{R}^{n_1}$ that applies to i)-ii) for $t \in \mathbb{R}^{n_0}$ fixed.

In the classical case case where $d = 1$ and we have a log-concave function

$$g(t,y) = e^{-\varphi(t,y)}$$

where φ is a convex function on $\mathbb{R} \times \mathbb{R}^n$ (i.e. $n_0 = 1$, $n_1 = n$), the first step is easy. A straightforward computation gives, for $\alpha(t) = \int_{\mathbb{R}^n} e^{\varphi(t,y)}\,dy$, that

$$\begin{aligned}(1)\quad & -\alpha(t)\partial^2_{tt}\log\alpha(t) = -\partial^2_{tt}\alpha(t) + \frac{1}{\alpha(t)}(\partial_t\alpha(t))^2 \\ &= \int_{\mathbb{R}^n} \partial^2_{tt}\varphi((t,y)\,e^{-\varphi(t,y)} - \int \Big(\partial_t\varphi(t,y) - \tfrac{1}{\int_{\mathbb{R}^n} e^{-\varphi(t,\cdot)}}\int_{\mathbb{R}^n}\partial_t\varphi(t,\cdot)\,e^{-\varphi(t,\cdot)}\Big)^2 e^{-\varphi(t,y)}\,dy\end{aligned}$$

For the second step, because the determinant of the Hessian of φ, in all the variables (t,y), is nonnegative, we have that

$$\partial^2_{tt}\varphi \geq (\mathrm{Hess}_y\,\varphi)^{-1}\nabla_y\partial_t\varphi\cdot\nabla_y\partial_t\varphi. \tag{2}$$

This seems to require that $\mathrm{Hess}_y\,\varphi > 0$, which is actually harmless by approximation. However, there is a way around this difficulty using the associated quadratic form and its polar, as we will show later (this will avoid to discuss approximation or extra assumptions). Property (2) is very much related to the homogeneous (real) Monge-Ampère equation; indeed, if instead of $\geq$ we put $=$, this corresponds to a solution of the HRMA equation in the $(n+1)$ variables. Finally, the third step *iii*) corresponds, when $d = 1$, to the Brascamp-Lieb(-Hörmander) variance inequality, that can be stated as follows: given a convex function φ on $\mathbb{R}^n$, with $\int_{\mathbb{R}^n} e^{-\varphi} < +\infty$, then for every smooth $u \in L^2(e^{-\varphi})$ we have

$$\int_{\mathbb{R}^n}\Big(u(y) - \frac{1}{\int_{\mathbb{R}^n} e^{-\varphi}}\int_{\mathbb{R}^n} ue^{-\varphi}\Big)^2 e^{-\varphi(y)}\,dy \leq \int_{\mathbb{R}^n}(\mathrm{Hess}_y\,\varphi)^{-1}\nabla u(y)\cdot\nabla u(y)\,e^{-\varphi(y)}\,dy \tag{3}$$

A direct combination of (1)-(2) and (3) applied at fixed t, to $\varphi(y) = \varphi(t,y)$ and to $u(y) = \partial_t\varphi(t,y)$ gives that $-\partial^2_{tt}\alpha \geq 0$, as wanted.

As alluded above, let us mention that both in (2) and (3), we can replace $(\mathrm{Hess}_y\,\varphi)^{-1}(v)$, $v \in \mathbb{R}^n$, by the $Q^\circ_y(v) = \sup\big\{v\cdot w\,;\ Q(w) \leq 1\big\}$, the polar of quadratic form $Q_y(w) := (\mathrm{Hess}_y\,\varphi)w\cdot w$, so that everything is well defined only with the assumption that $\mathrm{Hess}_y\,\varphi$ is nonnegative.

The matrix situation where $d > 1$ brings some complications and requires to establish some new properties of log-concave functions, but the principle of proof works exactly the

same. In the last Section §4 we will explain in Fact 11 how to organise the computation of $\mathbf{\Theta}^\alpha$ and we will establish in Proposition 10 an analogue of (2) for N-log-concave functions. We note that unlike the $d = 1$ situation, we cannot assume that $n_0 = 1$ because N-log-concavity does not reduce to a one-dimensional property.

In Section §3, we will establish independently a Brascamp-Lieb variance inequality for matrix weights, which is maybe the main new result of the present paper (and gives some justification for the operator $\mathbf{\Theta}^g$ used in the definitions below). It relies on an L^2-analysis of the Laplace operator associated with a N-log-concave potential. It will then be used in Section §4 to conclude the proof of Theorem 1.

2. N-log-concave matrix valued functions

We start with some elementary notation from linear algebra.

Let $(E, \langle\cdot\,|\,\cdot\rangle, \|\cdot\|)$ be a (finite dimensional) real Hilbert space (later $E = \mathbb{R}^d$ or $E = \mathbb{R}^N \otimes \mathbb{R}^d = \mathcal{M}_{d,N}(\mathbb{R})$ the space of $d \times N$ matrices). Given a symmetric positive operator g on E, we denote by $\langle\cdot\,|\,\cdot\rangle_g$ the associated scalar product on E, that is

$$\forall u, v \in E, \qquad \langle u\,|\,v\rangle_g = \langle gu\,|\,v\rangle, \qquad \|u\|_g := \sqrt{\langle gu\,|\,u\rangle},$$

If we denote by $\succeq$ (resp. $\succeq_g$) the order on nonnegative operators (resp. associated with the scalar product g) on E, then we have, for operator C on E,

$$\begin{aligned}\Big(C\ g\text{-symmetric with}\ \ C \succeq_g 0\Big) \ &\iff\ \Big(gC \text{ symmetric with }\ gC \succeq 0\Big)\\ &\iff\ \langle gCu\,|\,v\rangle = \langle gu\,|\,Cv\rangle, \text{ and } \langle gCu\,|\,u\rangle \geq 0, \quad \forall u, v \in E.\end{aligned}$$

With such C, we can associate the nonnegative quadratic form,

$$\forall u \in E, \qquad Q_{g,C}(u) := \langle Cu\,|\,u\rangle_g = \langle gCu\,|\,u\rangle.$$

The corresponding *g-polar* quadratic form is given by

$$\forall v \in E, \qquad Q^\circ_{g,C}(v) := \sup_{Q_{g,C}(u)\leq 1} \langle u\,|\,v\rangle_g^2.$$

By 2-homogeneity, the g-polar form can also be defined in terms of Legendre's transform, as

$$\frac{1}{2}Q^\circ_{g,C}(v) = \sup_{u\in E}\big\{\langle u\,|\,v\rangle_g - \frac{1}{2}Q_{g,C}(u)\big\}. \tag{4}$$

When C is moreover invertible, that is when gC is a positive symmetric operator on E, which means that $Q_{g,C}$ is an Euclidean norm, then gC^{-1} is a positive symmetric operator and the g-polar quadratic form $Q^\circ_{g,C}$ satisfies

$$\forall v \in E, \qquad Q^\circ_{g,C}(v) = \langle C^{-1}v\,|\,v\rangle_g = \langle gC^{-1}v\,|\,v\rangle$$

or equivalently, $Q^\circ_{g,C} = Q_{g,C^{-1}}$. This can be seen by diagonalizing C in a g-orthonormal basis, for instance.

In the sequel, we will denote by $\langle\cdot\,|\,\cdot\rangle$ the standard scalar product on $\mathbb{R}^N$ for any $N \in \mathbb{N}^*$, and on $\mathcal{M}_{d,N}(\mathbb{R})$.

On $\mathbb{R}^n$ and $\mathbb{R}^d$ we use the canonical basis, and in particular we will identify through it operators on $\mathbb{R}^d$ and $d \times d$ matrices. We denote by $\mathcal{M}_d(\mathbb{R})^+$ the positive cone of positive operators on $\mathbb{R}^d$, that is of $d \times d$ positive symmetric matrices. This will be the fixed range of our densities g.

Let $n, d \geq 1$. Our setting is the following. We are given a matrix-valued function

$$g : \mathbb{R}^n \to \mathcal{M}_d(\mathbb{R})^+ \subset \mathbb{R}^{\frac{n(n+1)}{2}}$$

Log-concavity of g will be defined locally in terms of "second derivatives", so we will assume that g is C^2-smooth. We introduce, for $j, k = 1, \ldots, n$, and $x \in \mathbb{R}^n$,

$$\begin{aligned} \theta^g_{j,k}(x) &:= \partial_{x_k}(g^{-1}\partial_{x_j} g) \in \mathcal{M}_d(\mathbb{R}) \\ &= g^{-1}\Big[\partial^2_{x_k x_j} g - (\partial_{x_k} g) g^{-1} (\partial_{x_j} g)\Big] \in \mathcal{M}_d(\mathbb{R}). \end{aligned}$$

Derivatives are performed on each entry of the matrix, and so the result remains indeed a matrix. Note that for fixed $x \in \mathbb{R}^d$, for $u, v \in \mathbb{R}^d$ and $1 \leq k, j \leq n$, we have

$$\begin{aligned} \langle \theta^g_{j,k} u \,|\, v\rangle_g &= \langle (\partial^2_{x_k x_j} g) u \,|\, v\rangle - \langle (\partial_{x_k} g) u \,|\, g^{-1}(\partial_{x_j} g) v\rangle \\ &= \langle (\partial^2_{x_k x_j} g) u \,|\, v\rangle - \langle g^{-1}(\partial_{x_j} g) u \,|\, g^{-1}(\partial_{x_k} g) v\rangle_g. \end{aligned} \tag{5}$$

In particular the g-adjoint of $\theta^g_{j,k}$ is $\theta^g_{k,j}$:

$$\langle \theta^g_{j,k} u \,|\, v\rangle_g = \langle u \,|\, \theta^g_{k,j} v\rangle_g \tag{6}$$

We have d^2 functions $\theta^g_{j,k} : \mathbb{R}^n \to \mathcal{M}_d(\mathbb{R})$. But here and later we will often omit to write the dependence in x so that $\theta^g_{i,j}$ refers at the same time to a $\mathcal{M}_d(\mathbb{R})$-valued function and to an element of $\mathcal{M}_d(\mathbb{R})$. We next collect all these matrices and form

$$\mathbf{\Theta}^g = [\theta^g_{j,k}]_{1 \leq j,k \leq n}.$$

Again, we omit the dependence in x, so below we are really discussing $\mathbf{\Theta}^g(x) = [\theta^g_{i,j}(x)]_{i,j \leq n}$ for $x \in \mathbb{R}^d$ fixed and omitted, and not the function $x \to \mathbf{\Theta}^g(x)$. Note that if $d = 1$, then $\mathbf{\Theta}^g$ is the Hessian $n \times n$ matrix of $\log(g)$; we shall come back to this later.

There are several possible equivalent ways to see this operator $\mathbf{\Theta}^g$ for fixed $x \in \mathbb{R}^n$. By construction, we have

$$\mathbf{\Theta}^g \in \mathcal{M}_n(\mathbb{R}) \otimes \mathcal{M}_d(\mathbb{R}).$$

If we interpret $\mathbf{\Theta}^g$ as an element of $\mathcal{M}_n(\mathcal{M}_d(\mathbb{R}))$, then we could ask that for very $Y \in \mathbb{R}^n$, the element $\mathbf{\Theta}^g Y \cdot Y := \sum_{i,j}^n Y_i Y_j \theta^g_{i,j} \in \mathcal{M}_d(\mathbb{R})$ is a nonnegative operator on $(\mathbb{R}^d, \langle \cdot \,|\, \cdot \rangle_g)$. This is a rather weak requirement, which corresponds to what is known as Griffith's curvature condition in complex geometry.

There is a stronger and natural notion, which amounts to work with the following canonical identifications that we shall use in the rest of the paper,

$$\mathbf{\Theta}^g \in \mathcal{M}_n(\mathbb{R}) \otimes \mathcal{M}_d(\mathbb{R}) \simeq L(\mathbb{R}^n \otimes \mathbb{R}^d) \simeq L(\mathcal{M}_{d,n}(\mathbb{R})).$$

So we will interpret $\mathbf{\Theta}^g$ as an operator on $\mathbb{R}^n \otimes \mathbb{R}^d \simeq \mathcal{M}_{d,n}(\mathbb{R})$.

An element U of $\mathbb{R}^n \otimes \mathbb{R}^d \simeq \mathcal{M}_{d,n}(\mathbb{R})$ will conveniently be described in term of its columns, which are vectors of $\mathbb{R}^d$,

$$U = [u_1, \dots, u_n], \qquad u_1, \dots, u_n \in \mathbb{R}^d.$$

The action of the operator $\mathbf{\Theta}^g$ is as follows: for $y \in \mathbb{R}^n$ and $u \in \mathbb{R}^d$, $\mathbf{\Theta}^g(y \otimes u) = \sum_{j,k=1}^n y_j e_k \otimes (\theta^g_{j,k} u)$, where $(e_1, \dots, e_n)$ is the canonical basis of $\mathbb{R}^n$. Equivalently, if $U = [u_1, \dots, u_n] \in \mathcal{M}_{d,n}(\mathbb{R})$, then

$$\mathbf{\Theta}^g U = \Big[\sum_{j=1}^n \theta^g_{j,k} u_j \Big]_{k=1,\dots,n} \in \mathcal{M}_{d,n}(\mathbb{R}). \tag{7}$$

We will use the scalar product on $\mathbb{R}^n \otimes \mathbb{R}^d \simeq \mathcal{M}_{d,n}(\mathbb{R})$ induced by the scalar product g on $\mathbb{R}^d$ (at $x \in \mathbb{R}^n$ fixed), that is by the action of g on the columns. It is consistent to denote it by $\langle \cdot \,|\, \cdot \rangle_{\mathrm{Id}_n \otimes g}$, if we introduce the positive symmetric operator

$$(\mathrm{Id}_n \otimes g)U = gU = [gu_1, \dots, gu_n], \quad \text{for } \; U = [u_1, \dots, u_n] \in \mathbb{R}^n \otimes \mathbb{R}^d \simeq \mathcal{M}_{d,n}(\mathbb{R}).$$

Namely, for two matrices $U = [u_1, \dots, u_n], V = [v_1, \dots, v_n] \in \mathcal{M}_{d,n}(\mathbb{R})$ we have

$$\boxed{\langle U \,|\, V \rangle_{\mathrm{Id}_n \otimes g} = \langle (\mathrm{Id}_n \otimes g)U \,|\, U \rangle = \mathrm{tr}((gU)^T V) = \sum_{k=1}^n \langle g u_k \,|\, v_k \rangle = \sum_{k=1}^n \langle u_k \,|\, v_k \rangle_g.}$$

If $d = 1$, then for two vectors $u, v \in \mathbb{R}^n = \mathcal{M}_{1,n}(\mathbb{R})$ we have $\langle u \,|\, v \rangle_{\mathrm{Id}_n \otimes g} = g(x) \langle u \,|\, v \rangle$. We emphasise that in general our scalar products incorporate the weight $g(x)$ (we will later integrate in x). Denote also

$$\|U\|^2_{\mathrm{Id}_n \otimes g} := \langle U \,|\, U \rangle_{\mathrm{Id}_n \otimes g} = \sum_{k=1}^n \langle g u_k \,|\, u_k \rangle$$

the associated Euclidean norm. Since the action of $\mathrm{Id}_n \otimes g$ on $\mathbb{R}^n$ is trivial, the notation $\langle \cdot \,|\, \cdot \rangle_g$ would have been lighter; we favour $\langle \cdot \,|\, \cdot \rangle_{\mathrm{Id}_n \otimes g}$ not only for consistency, but also because it reminds us of the size of the matrices we are working with (later n will vary).

In view of (6), we note that $\mathbf{\Theta}^g$ is a $\langle \cdot \,|\, \cdot \rangle_{\mathrm{Id}_n \otimes g}$-symmetric operator, and that for two matrices $U = [u_1, \dots, u_n], V = [v_1, \dots, v_n] \in \mathcal{M}_{d,n}(\mathbb{R})$ we have

$$\langle \mathbf{\Theta}^g U \,|\, V \rangle_{\mathrm{Id}_n \otimes g} = \sum_{j,k=1}^n \langle \theta^g_{j,k} u_j \,|\, v_k \rangle_g = \sum_{j,k=1}^n \langle u_j \,|\, \theta^g_{k,j} v_k \rangle_g = \langle U \,|\, \mathbf{\Theta}^g V \rangle_{\mathrm{Id}_n \otimes g}. \tag{8}$$

For later reference, let us explicitly write this quantity, in view of (5), as

$$\langle \mathbf{\Theta}^g U \,|\, V \rangle_{\mathrm{Id}_n \otimes g} = \sum_{j,k=1}^n \Big[\langle (\partial^2_{x_k x_j} g) u_j \,|\, v_k \rangle - \langle g^{-1} (\partial_{x_j} g) u_j \,|\, g^{-1} (\partial_{x_k} g) v_k \rangle_g \Big] \tag{9}$$

This can also be taken as a definition of the operator $\mathbf{\Theta}^g$.

We shall also explain below how to interpret the operator $\mathbf{\Theta}^g$ on $(\mathcal{M}_{d,n}(\mathbb{R}), \langle \cdot \,|\, \cdot \rangle_{\mathrm{Id}_n \otimes g})$ as as $dn \times dn$ matrix, acting on $(\mathbb{R}^d)^n$, with the usual scalar product.

Note that in the classical (but now confusing!) case where $d = 1$ and $g \in \mathbb{R}^+$ (we should write $g(x) \in \mathbb{R}^+$ but again x is fixed and omitted), we have that $\langle a \,|\, b\rangle_g = gab$ for $a, b \in \mathbb{R}$ is therefore trivial (we just are just multiplying by the weight $g(x) > 0$ which is irrelevant as far as positivity is concerned) and that $\mathbf{\Theta}^g = \mathrm{Hess}(\log(g)) \in L(\mathbb{R}^n)$. So let us emphasize that:

$$\text{if } d = 1 \text{ and } g = e^{-\varphi}, \text{ then } \mathbf{\Theta}^g = -\operatorname{Hess}\varphi \in L(\mathbb{R}^n), \tag{10}$$

and for $u \in \mathbb{R}^n = \mathcal{M}_{1,n}(\mathbb{R})$,

$$\langle \mathbf{\Theta}^g\, u \,|\, u\rangle_{\mathrm{Id}_n \otimes g} = \langle \mathbf{\Theta}^g\, u \,|\, u\rangle_{g\mathrm{Id}_n} = \big[-(\operatorname{Hess}\varphi)u \cdot u\big]\, e^{-\varphi}.$$

The minus sign is a bit unfortunate from the point of view of convex analysis, but it is too well entrenched in complex geometry to change it. So throughout the paper we will have to consider $-\mathbf{\Theta}^g$ to get nonnegative operators.

The next notion is called "Log-concave in the sense of Nakano" by Raufi [9], by analogy with the situation in complex geometry. It is a possible extension to $d > 1$ of the situation we just described, and also of similar notions we encounter in complex analysis (for instance when doing complex interpolation of Hilbert spaces). We will favour the terminology "N-log-concave" for simplicity.

Definition 1 (N-log-concavity)**.** *Let $g : \mathbb{R}^n \to \mathcal{M}_d(\mathbb{R})^+$ be a C^2 function with values in the symmetric positive matrices. We form at every x the operator $\mathbf{\Theta}^g$ as before. We say that g is* N-log-concave *if at every $x \in \mathbb{R}^n$, $\mathbf{\Theta}^g$ is a nonpositive symmetric operator on $(\mathbb{R}^n \otimes \mathbb{R}^d, \langle\cdot\,|\,\cdot\rangle_{\mathrm{Id}_n\otimes g}) = (M_{d,n}(\mathbb{R}), \langle\cdot\,|\,\cdot\rangle_{\mathrm{Id}_n\otimes g})$, that is, if*

$$\langle \mathbf{\Theta}^g\, U \,|\, U\rangle_{\mathrm{Id}_n\otimes g} \le 0, \qquad \textit{for every matrix } U \in \mathcal{M}_{d,n}(\mathbb{R}). \tag{11}$$

Let us rewrite the previous condition in terms of the canonical structure. If we introduce

$$\widetilde{\mathbf{\Theta}}^g := [\tilde\theta^g_{j,k}]_{j,k\le n}, \quad \text{where } \tilde\theta^g_{j,k} := g\theta^g_{j,k} = \partial^2_{x_kx_j} g - (\partial_{x_k} g) g^{-1} (\partial_{x_j} g) \in \mathcal{M}_d(\mathbb{R}),$$

that we see as an operator on $(\mathbb{R}^d)^n = \mathbb{R}^{dn}$, that is as an $(dn \times dn)$ matrix, acting on $X = (u_1, \dots, u_d) \in (\mathbb{R}^d)^n$ as

$$\widetilde{\mathbf{\Theta}}^g X = \Big(\sum_{i=1}^n \tilde\theta^g_{i,j} u_j\Big)_{i=1,\dots,n} \in (\mathbb{R}^d)^n,$$

then condition (11) is equivalent to the requirement that

$$-\widetilde{\mathbf{\Theta}}^g \succeq 0$$

in the usual sense, as a symmetric $(dn \times dn)$ matrix.

We defined N-log-concavity on the whole $\mathbb{R}^n$, but actually if g is defined only on a domain $\Omega \subset \mathbb{R}^n$, that is if $g : \Omega \to \mathcal{M}_d(\mathbb{R})^+$ is a C^2 function on Ω, we can say that g is N-log-concave on Ω if $\mathbf{\Theta}^g(x)$ is a nonpositive operator at every $x \in \Omega$.

If we impose (11) only on rank one matrices $y \otimes u$, then we arrive to the weaker notion of log-concavity "in the sense of Griffith" that we mentioned above. Note that if $n = 1$ (or $d = 1$, but we want to work with matrices) these two notions coincide. But if $n, d > 1$,

they differ (see [9]), and unlike the case of Griffith log-concavity, it is not sufficient to be N-log-concave on any one-dimensional affine line of $\mathbb{R}^n$ to be N-log-concave on $\mathbb{R}^n$.

An obvious observation that will be used several times is that given a N-log-concave function $g : \mathbb{R}^{n_0+n_1} \to \mathcal{M}_d(\mathbb{R})^+$, if we freeze some coordinates $t \in \mathbb{R}^{n_0}$, then the function $y \to g(t, y)$ remains an N-log-concave function from $\mathbb{R}^{n_1}$ to $\mathcal{M}_d(\mathbb{R})^+$. We stress that although the dimension of the variables might vary, the dimension d will remain fixed throughout the paper.

Let us also mention that if $P \in \mathcal{O}_d(\mathbb{R})$ is a a fixed isometry (for the usual Euclidean structure) of $\mathbb{R}^d$, then the function

$$\tilde{g} := P^{-1} g P : \mathbb{R}^n \to \mathcal{M}_d(\mathbb{R})^+$$

will be such that for $u, v \in \mathbb{R}^d$ and $1 \le k, j \le n$, at any fixed $x \in \mathbb{R}^n$,

$$\langle \theta^{\tilde{g}}_{j,k} u \,|\, v \rangle_{\tilde{g}} = \langle \theta^{g}_{j,k} P u \,|\, P v \rangle_g.$$

So $\tilde{g}$ is log-concave if and only if g is.

Raufi discusses in his paper several other properties of N-log-concave functions, and abstract ways to construct such functions. It is not easy though to give explicit non-trivial examples of N-log concave functions for $n, d \ge 2$.

Example 2. *Let $n = d = 2$. We will define a function $g : \Omega \to \mathcal{M}_2(\mathbb{R}^2)^+ \subset \mathbb{R}^3$ that is* N*-log-concave in $\Omega \subset \mathbb{R}^2$ a neighbourhood of zero. Let $s \ge 0$ be a paramater to be fixed later. For $x = (x_1, x_2) \in \mathbb{R}^2$ define*

$$g(x) = g(x_2, x_2) = \mathrm{Id}_2 - \begin{pmatrix} s x_1^2 + x_2^2 & x_1 x_2 \\ x_1 x_2 & s x_1^2 + x_2^2 \end{pmatrix} = \begin{pmatrix} 1 - s x_1^2 - x_2^2 & -x_1 x_2 \\ -x_1 x_2 & 1 - s x_1^2 - x_2^2 \end{pmatrix}.$$

Note that g has indeed values in $\mathcal{M}_2(\mathbb{R})^+$ when (x_1, x_2) is close to zero. For $s = 0$, this corresponds to the function $g(x) = \mathrm{Id}_2 - x^T x$; when $d = 1$ this is the function $1 - x^2$, which is concave and therefore log-concave on $\{|x| < 1\}$. This unfortunately does not work when $d > 1$, as we will see.

Computation of $\widetilde{\mathbf{\Theta}}^g$ at $x = 0$ gives

$$\widetilde{\mathbf{\Theta}}^g(0) = \begin{pmatrix} -2s & 0 & 0 & -1 \\ 0 & -2 & -1 & 0 \\ 0 & -1 & -2 & 0 \\ -1 & 0 & 0 & -2s \end{pmatrix}$$

whose spectrum is

$$\{-3, -1, -1 - 2s, 1 - 2s\}$$

which belongs to $(-\infty, 0]$ exactly when $s \ge \frac{1}{2}$. If we fix $s > \frac{1}{2}$, then the spectrum of $\widetilde{\mathbf{\Theta}}^g(x)$ will remain in $(-\infty, 0)$ for x small enough, by continuity, and g will be N*-log-concave.*

Finally, let us mention that all what we say remains true for functions with values in the Hermitian positive $d \times$d matrices, after obvious adaptions.

Remark 3 (Hermitian valued matrices)**.** *It is possible to define, in a complete analogous way,* N*-log-concavity for a function*

$$g : \mathbb{R}^n \to \mathcal{M}_d(\mathbb{C})^+$$

with values in the set of Hermitian positive $d \times d$ matrices. At fixed x, g is now a Hermitian product on $\mathbb{C}^d$, denoted again by $\langle \cdot \,|\, \cdot \rangle_g$ and it induces also an hermitian product $\langle \cdot \,|\, \cdot \rangle_{\mathrm{Id}_n \otimes g}$ on $\mathcal{M}_{d,n}(\mathbb{C})$. It is maybe not a good idea to insist too much on the fact that $\mathbb{R}^n \otimes \mathbb{C}^d = \mathbb{C}^n \otimes \mathbb{C}^d \simeq \mathcal{M}_{d,n}(\mathbb{C})$: we have fixed an orthonormal basis on $\mathbb{R}^n$, so complexification should be transparent (as is the complexification of Id_n). We can then define $\mathbf{\Theta}^g$ by the same formulas; it is again a $\langle \cdot \,|\, \cdot \rangle_{\mathrm{Id}_n \otimes g}$ hermitian operator on $\mathcal{M}_{d,n}(\mathbb{C})$, and N*-log-concavity is the requirement that it is a nonpositive one.*

Theorem 1 was proved by Raufi in in this setting, actually.

This is obviously more general than the case we considered. We have chosen to work with real symmetric matrices for notational reasons only (because we prefer $\mathbb{R}^n \otimes \mathbb{R}^d$ to $\mathbb{R}^n \otimes \mathbb{C}^d$), but all our arguments work of course in the positive hermitian case as well.

3. Brascamp-Lieb variance inequality for matrix-valued N-log-concave weights

Given a function $g : \mathbb{R}^n \to \mathcal{M}_d(\mathbb{R})^+$, we denote by $L^2(g)$ the space of Borel functions $F : \mathbb{R}^n \to \mathbb{R}^d$ for which the quantity

$$\int_{\mathbb{R}^n} \|F\|_g^2 = \int_{\mathbb{R}^n} \|F(x)\|_{g(x)}^2 = \int \langle g(x)F(x) \,|\, F(x) \rangle \, dx$$

is finite. It is a Hilbert space over functions from $\mathbb{R}^n$ to $\mathbb{R}^d$ with scalar product $\int_{\mathbb{R}^n} \langle F \,|\, H \rangle_g$. If $d = 1$ and $g = e^{-\varphi}$, it is the usual weighted space $L^2(e^{-\varphi})$. Note that if $F \in L^2(g)$ and $\int |g| < +\infty$ (where $|\cdot|$ is a norm on the matrices, for instance, the operator norm), then the vector

$$\int_{\mathbb{R}^n} gF = \int_{\mathbb{R}^n} g(x)F(x) \, dx \in \mathbb{R}^d$$

is well defined since $\int_{\mathbb{R}^n} \|gF\| \le \int \sqrt{|g|} \, \|\sqrt{g}F\| \le \sqrt{\int_{\mathbb{R}^n} |g| \int_{\mathbb{R}^n} \|F\|_g^2} < +\infty$.

Given a differentiable function $F : \mathbb{R}^n \to \mathbb{R}^d$ we write, at $x \in \mathbb{R}^n$,

$$\nabla_x F = \nabla_x F(x) = [\partial_{x_1} F(x), \dots, \partial_{x_n} F(x)] \in \mathcal{M}_{d,n}(\mathbb{R}).$$

The next result is a generalization of the Brascamp-Lieb variance inequality [2] (anticipated in the complex setting by Hörmander [7]) to matrix-valued potentials and vector valued functions.

Theorem 4 (Brascamp-Lieb variance inequality for matrix N-log-concave weights)**.** *Let $g : \mathbb{R}^n \to \mathcal{M}_d(\mathbb{R})^+$ be a* N*-log-concave function. Assume that $\int_{\mathbb{R}^n} |g| < +\infty$ and set $\mathcal{Z} := \int_{\mathbb{R}^n} g \in \mathcal{M}_d(\mathbb{R})^+$.*

For any vector valued C^1 function $F : \mathbb{R}^n \to \mathbb{R}^d$ belonging to $L^2(g)$ we have

$$\int_{\mathbb{R}^n} \|F(x) - \mathcal{Z}^{-1} \textstyle\int_{\mathbb{R}^n} gF \,\|_{g(x)}^2 \, dx \le \int_{\mathbb{R}^n} Q^\circ_{\mathrm{Id}_n \otimes g, -\mathbf{\Theta}^g}(\nabla_x F) \, dx,$$

where $Q^\circ_{\mathrm{Id}_n\otimes g,-\mathbf{\Theta}^g}$ *is, at fixed* $x\in\mathbb{R}^n$*, the* $\langle\cdot\,|\,\cdot\rangle_{\mathrm{Id}_n\otimes g(x)}$*-polar of the quadratic form* $\mathcal{U}\to -\langle\mathbf{\Theta}^g(x)U\,|\,U\rangle_{\mathrm{Id}_n\otimes g(x)}$ *on* $M_{d,n}(\mathbb{R})$ *associated to* $-\mathbf{\Theta}^g(x)$*. If* $\mathbf{\Theta}^g$ *is almost everywhere invertible as an operator on* $\mathcal{M}_{d,n}(\mathbb{R})$*, then we can also write*

$$\int_{\mathbb{R}^n}\|F(x)-\mathcal{Z}^{-1}\textstyle\int_{\mathbb{R}^n}gF\,\|^2_{g(x)}\,dx\le\int_{\mathbb{R}^n}\langle(-\mathbf{\Theta}^g(x))^{-1}\nabla_xF\,|\,\nabla_xF\rangle_{\mathrm{Id}_n\otimes g(x)}\,dx.$$

When $d=1$, we recover the classical Brascamp-Lieb variance inequality (3). In some sense, the previous theorem gives some extra justification of the relevance of the operator $-\mathbf{\Theta}^g$ and to its non-negativity (and therefore to the associated notion of N-log-concavity).

Remark 5 (Hermitian valued weights)**.** *If* $g:\mathbb{R}^n\to\mathcal{M}_d(\mathbb{C})^+$ *is a* C^2 *function with values in the positive hermitian* $d\times d$ *matrices that is* N*-log-concave as discussed in Remark 3, then the previous Theorem also applies with the obvious adaptations; that is, the inequalities will hold for any* C^1 *function* $F:\mathbb{R}^n\to\mathbb{C}^d$ *belonging to* $L^2(g)$*.*

Indeed, the proof below is completely identical in this (more general) case.

The term on the left in the inequalities of Theorem 4 is a variance, that is the square of the L^2 norm of the orthogonal projection onto the orthogonal of constant functions, and we also have, using linearity of integration,

$$\begin{aligned}\int_{\mathbb{R}^n}\|F(x)-\mathcal{Z}^{-1}\textstyle\int gF\,\|^2_{g(x)}\,dx &= \int_{\mathbb{R}^n}\langle g(x)F(x)-g(x)\mathcal{Z}^{-1}\textstyle\int gF\,|\,F(x)-\mathcal{Z}^{-1}\int gF\rangle\\ &= \int_{\mathbb{R}^n}\langle gF\,|\,F\rangle\,dx-2\int_{\mathbb{R}^n}\langle g(x)F(x)\,|\,\mathcal{Z}^{-1}\textstyle\int gF\rangle\,dx\\ &\qquad+\int_{\mathbb{R}^n}\langle g(x)\mathcal{Z}^{-1}\textstyle\int gF\,|\,\mathcal{Z}^{-1}\int gF\rangle\,dx\\ &= \int_{\mathbb{R}^n}\langle gF\,|\,F\rangle\,dx-\langle\textstyle\int gF\,|\,\mathcal{Z}^{-1}\int gF\rangle\\ &= \int_{\mathbb{R}^n}\|F\|^2_g\,dx-\Big\|\mathcal{Z}^{-1}\textstyle\int_{\mathbb{R}^n}gF\,dx\Big\|^2_{\mathcal{Z}}.\end{aligned}$$

Remark 6 ($\mathcal{M}_d(\mathbb{R})$-operator form)**.** *It is possible to restate the operator inequality for functions with values on* $\mathcal{M}_d(\mathbb{R})$*. N Given a smooth function*

$$A:\mathbb{R}^n\to\mathcal{M}_d(\mathbb{R})$$

we denote $\nabla A\in\mathbb{R}^n\otimes\mathbb{R}^d\otimes\mathbb{R}^d\simeq L(\mathbb{R}^d,\mathcal{M}_{d,n}(\mathbb{R}))$ *the operator defined, at* $x\in\mathbb{R}^n$*, by*

$$\forall a_0\in\mathbb{R}^d,\qquad(\nabla_xA)a_0:=\nabla_x\big(A(x)a_0\big)\in\mathcal{M}_{d,n}(\mathbb{R}).$$

At fixed $x\in\mathbb{R}^n$*, this is an operator between Hilbert spaces that admits an adjoint, that we some abuse of notation we denote simply by* $\mathrm{Ad}_g(\nabla_xA)\in L(\mathcal{M}_{d,n}(\mathbb{R}),\mathbb{R}^d)$*,*

$$\forall a_0\in\mathbb{R}^d,\forall U\in\mathcal{M}_{d,n}(\mathbb{R}),\quad\langle\nabla_xAa_0\,|\,U\rangle_{\mathrm{Id}_n\otimes g(x)}=\langle a_0\,|\,\mathrm{Ad}_{g(x)}(\nabla_xA)U\rangle_{g(x)}.$$

We will also denote by Ad_g *the adjoint with respect to* g *for operators on* $\mathbb{R}^d$*, that is, for* $A\in\mathcal{M}_d(\mathbb{R})$*,*

$$\forall a_0,b_0\in\mathbb{R}^d,\quad\langle Aa_0\,|\,b_0\rangle_{g(x)}=\langle a_0\,|\,\mathrm{Ad}_{g(x)}(A)b_0\rangle_{g(x)},$$

or equivalently, $\mathrm{Ad}_g(A) = g^{-1}A^T g$. *Then the Brascamp-Lieb inequality is equivalent to the following operator valued inequality with respect to the matrix weight* g. *For any smooth function* $A : \mathbb{R}^n \to \mathcal{M}_d(\mathbb{R})$, *we have the following inequality between symmetric* $d \times d$ *matrices*

$$\int_{\mathbb{R}^n} g(x)\,\mathrm{Ad}_{g(x)}\left[A(x) - \mathcal{Z}^{-1}\textstyle\int_{\mathbb{R}^n} g\ A\right]\left(A(x) - \mathcal{Z}^{-1}\textstyle\int_{\mathbb{R}^n} g\ A\right)dx$$
$$\succeq \int_{\mathbb{R}^n} g(x)\,\mathrm{Ad}_{g(x)}\left[\nabla_x A\right](-\mathbf{\Theta}^g(x))^{-1}\nabla_x A\,dx$$

We pass from one form to the other by testing this matrix inequality on a fixed vector a_0 *and considering* $F = Aa_0 : \mathbb{R}^n \to \mathbb{R}^d$.

Proof. Introduce the differential operator L given for $F : \mathbb{R}^n \to \mathbb{R}^d$ of class C^2 by

$$LF := \Delta F - \sum_{k=1}^n (g^{-1}\partial_{x_k} g)\,\partial_{x_k} F$$

which means, coordinate-wise, for $F = (F_1, \dots, F_d)$, with $F_\ell : \mathbb{R}^n \to \mathbb{R}$, and $x \in \mathbb{R}^n$,

$$\forall \ell \le d, \qquad (LF(x))_\ell = \Delta_x F_\ell(x) + \sum_{k=1}^n \sum_{r=1}^d \left(g(x)^{-1}\partial_{x_k} g(x)\right)_{\ell,r} \partial_{x_k} F_r(x).$$

Fact 7. *If* $F, G : \mathbb{R}^n \to \mathbb{R}^d$ *are* C^2 *and compactly supported, then*

$$\int_{\mathbb{R}^n} \langle LF(x) \,|\, G(x)\rangle_{g(x)}\,dx = -\int_{\mathbb{R}^n} \langle \nabla_x F(x) \,|\, \nabla_x G(x)\rangle_{\mathrm{Id}_n \otimes g(x)}\,dx = -\sum_{k=1}^n \int_{\mathbb{R}^n} \langle \partial_{x_k} F \,|\, \partial_{x_k} G\rangle_{g(x)}\,dx.$$

Proof of the Fact. We have

$$\int \langle g(x)\Delta F(x) \,|\, G(x)\rangle = \sum_{k=1}^n \int \langle g(x)\partial^2_{x_k x_k} F(x) \,|\, G(x)\rangle\,dx = \sum_{k=1}^n \sum_{\ell,r=1}^d \int (g(x))_{\ell,r}\partial^2_{x_k x_k} F_r(x) G_\ell(x)\,dx.$$

Integration by parts gives

$$\int_{\mathbb{R}^n} (g(x))_{\ell,r}\partial^2_{x_k x_k} F_r(x) G_\ell(x)\,dx = -\int (\partial_{x_k} g(x))_{\ell,r}\partial_{x_k} F_r(x) G_\ell(x)\,dx - \int_{\mathbb{R}^n} (g(x))_{\ell,r}\partial_{x_k} F_r(x)\partial_{x_k} G_\ell(x)\,dx,$$

and therefore

$$\begin{aligned}\int_{\mathbb{R}^n} \langle g(x)\Delta F(x) \,|\, G(x)\rangle &= -\sum_{k=1}^n \int_{\mathbb{R}^n} \langle \partial_{x_k} g \partial_{x_k} F \,|\, G\rangle - \sum_{k=1}^n \int_{\mathbb{R}^d} \langle g\partial_{x_k} F \,|\, \partial_{x_k} G\rangle \\ &= -\sum_{k=1}^n \int_{\mathbb{R}^n} \langle g^{-1}\partial_{x_k} g \partial_{x_k} F \,|\, G\rangle_g - \int_{\mathbb{R}^n} \langle \nabla F \,|\, \nabla G\rangle_{\mathrm{Id}_n \otimes g},\end{aligned}$$

as claimed. □

The next useful observation is a Bochner type integration by parts formula which explicitates the connection between L and the curvature operator $\mathbf{\Theta}^g$.

Fact 8. *If $F:\mathbb{R}^n\to\mathbb{R}^d$ is a C^2 compactly supported function, then*

$$\int_{\mathbb{R}^n}\|LF(x)\|^2_{g(x)}\,dx=\int_{\mathbb{R}^n}\langle-\,\mathbf{\Theta}^g_x\,\nabla_xF(x)\,|\,\nabla_xF(x)\rangle_{\mathrm{Id}_n\otimes g(x)}\,dx+\int_{\mathbb{R}^n}\sum_{j,k=1}^n\|\partial^2_{x_jx_k}F\|^2_{g(x)}\,dx.$$

Proof of the Fact. Using the previous Fact we can write

$$\int_{\mathbb{R}^n}\langle LF(x)\,|\,LF(x)\rangle_{g(x)}\,dx=-\int_{\mathbb{R}^n}\langle\nabla_xF(x)\,|\,\nabla_xLF(x)\rangle_{\mathrm{Id}_n\otimes g(x)}\,dx.$$

Next, we need to understand the commutation between ∇ and L. We have, for $j=1,\dots,n$

$$\partial_{x_j}LF=\Delta\partial_{x_j}F+\sum_{k=1}^n\partial_{x_j}(g^{-1}\partial_{x_k}g)\partial_{x_k}F+\sum_{k=1}^n(g^{-1}\partial_{x_k}g)\partial^2_{x_jx_k}F=L\partial_{x_j}F+\sum_{k=1}^n\theta^g_{j,k}\partial_{x_k}F.$$

So we have, at any fixed $x\in\mathbb{R}^n$,

$$\langle\nabla_xF\,|\,\nabla_xLF(x)\rangle_{\mathrm{Id}_n\otimes g}=\sum_{j=1}^n\langle\partial_{x_j}F\,|\,\partial_{x_j}LF\rangle_g=\sum_{j=1}^n\langle\partial_{x_j}F\,|\,L\partial_{x_j}F\rangle_g+\sum_{k,j=1}^n\langle\partial_{x_j}F\,|\,\theta^g_{j,k}\partial_{x_k}F\rangle_g,$$

that is

$$\langle\nabla_xF\,|\,\nabla_xLF(x)\rangle_{\mathrm{Id}_n\otimes g}=\langle\mathbf{\Theta}^g\,\nabla F\,|\,\nabla F\rangle_{\mathrm{Id}_n\otimes g}+\sum_{j=1}^n\langle\partial_{x_j}F\,|\,L\partial_{x_j}F\rangle_g.$$

After integration we we find, using again the previous Fact,

$$\begin{aligned}\int_{\mathbb{R}^n}\|LF\|^2_g&=-\int_{\mathbb{R}^n}\langle\mathbf{\Theta}^g\,\nabla F\,|\,\nabla F\rangle_{\mathrm{Id}_n\otimes g}+\int_{\mathbb{R}^n}\sum_{j=1}^n\langle\nabla\partial_{x_j}F\,|\,\nabla\partial_{x_j}F\rangle_{\mathrm{Id}_n\otimes g}\\&=-\int_{\mathbb{R}^n}\langle\mathbf{\Theta}^g\,\nabla F\,|\,\nabla F\rangle_{\mathrm{Id}_n\otimes g}+\int_{\mathbb{R}^n}\sum_{j,k=1}^n\langle\partial^2_{x_j,x_k}F\,|\,\partial^2_{x_j,x_k}F\rangle_g.\end{aligned}$$

□

The proof of the Brascamp-Lieb can be done by introducing, for given F a function Ψ such that $F=L\Psi$. By Fact 7, a necessary condition on F is that $\int_{\mathbb{R}^d}gF=0$ and this will be achieved by considering $F-\mathcal{Z}^{-1}\int gF$. Proving existence and regularity of Ψ requires some work, and for our purposes it is sufficient to establish a classical approximation procedure by "nice" functions.

Fact 9. *In $L^2(g)$, the space*

$$\mathcal{H}:=\big\{L\Psi\ ;\ \Psi:\mathbb{R}^n\to\mathbb{R}^d\ \textit{of class } C^2\ \textit{and compactly supported}\big\}$$

is dense in

$$\Big\{F\in L^2(g)\ ;\ \int_{\mathbb{R}^n}gF=0\Big\}.$$

Proof. The proof is an adaptation of a classical argument (recalled for instance in [4]). Let L^2 be the standard space of square-integrable functions from $\mathbb{R}^n$ to $\mathbb{R}^d$ equipped the standard Euclidean structure $\langle\,|\,\rangle$. The space $L^2(g)$ is isometric to L^2 through the isometry $U : L^2(g) \to L^2$ given by

$$UF = \sqrt{g}F,$$

since

$$\int_{\mathbb{R}^n} \|UF\|^2\,dx \int_{\mathbb{R}^n} \langle UF \,|\, UF\rangle\,dx = \int_{\mathbb{R}^n} \langle gF \,|\, F\rangle\,dx = \int_{\mathbb{R}^n} \langle F \,|\, F\rangle_g\,dx = \int_{\mathbb{R}^n} \|F\|_g^2\,dx.$$

The inverse of U is its adjoint, given by $U^*H = \sqrt{g}^{-1}H$. The corresponding linear differential operator on L^2,

$$L_0 = ULU^*$$

is of the form, for $H : \mathbb{R}^n \to \mathbb{R}^d$,

$$L_0H = \Delta H - B\nabla H - cH,$$

where $B : \mathbb{R}^n \to \mathcal{M}_d(\mathbb{R})$ incorporates derivatives of g and thus is of class C^1 and $c : \mathbb{R}^n \to \mathbb{R}$ incorporates second derivatives of g and is thus of class C^0. Note the presence of B, which does not appear when $d = 1$. So we really have a (non-diagonal) system of differential operators. However regularity matches classical regularization properties of the elliptic operator Δ on vector valued functions. That is, if H is a function of L^2 that verifies in the sense of distributions $L_0H = 0$, then H is $H^2_{loc} = W^{2,2}_{loc}$. Note that it is useful that B is of class C^1 and therefore preserves distributions in H^{-1}, which will then fell in H^1_{loc} by Δ^{-1}, and by repeating the argument, we arrive to H^2_{loc} as a pre-image by Δ of an element of L^2.

So, let $F : \mathbb{R}^n \to \mathbb{R}^d$ be a function in $L^2(g)$ that is orthogonal to $\mathcal{H}$. Our goal is to prove that F is constant. If we denote $H = UF$, we will have $L_0H = 0$ in the sense of distributions on $\mathbb{R}^n$. By the previous discussion, H, and therefore F, will be in $H^2_{loc} = W^{2,2}_{loc}$.

If $\theta : \mathbb{R}^n \to \mathbb{R}^+$ is a smooth compactly supported function we have, setting $\nabla\theta \otimes F = [(\partial_1\theta)F, \dots, (\partial_n\theta)F] \in \mathcal{M}_{d,n}(\mathbb{R})$,

$$\|\nabla(\theta F)\|^2_{\mathrm{Id}_n\otimes g} = \|\nabla\theta\otimes F+\theta\nabla F\|^2_{\mathrm{Id}_n\otimes g} = \|\nabla\theta\otimes F\|^2_{\mathrm{Id}_n\otimes g}+2\theta\langle\nabla\theta\otimes F \,|\, \nabla F\rangle_{\mathrm{Id}_n\otimes g}+\|\nabla F\|^2_{\mathrm{Id}_n\otimes g}\theta^2$$

But by integration by parts,

$$\int \|\nabla F\|^2_{\mathrm{Id}_n\otimes g}\theta^2 = \int \langle \nabla F \,|\, \theta^2 g\nabla F\rangle_{\mathrm{Id}_n\otimes g} = -\int \langle F \,|\, LF\rangle_g\theta^2 - 2\int \theta\langle \nabla\theta\otimes F \,|\, \nabla F\rangle_{\mathrm{Id}_n\otimes g},$$

so that, since $LF = 0$ almost everywhere,

$$\int \|\nabla(\theta F)\|^2_{\mathrm{Id}_n\otimes g} = \int \|\nabla\theta\otimes F\|^2_{\mathrm{Id}_n\otimes g} = \int \|\nabla\theta\|^2\,\|F\|_g^2.$$

If θ is smooth compactly supported function that is equal to 1 in a neighborhood of zero, we set $\theta_k(x) = \theta(x/k)$, then

$$\int \|\nabla(\theta F)\|^2_{\mathrm{Id}_n\otimes g} = \int \|\nabla\theta_k\|\,\|F\|_g^2 \to 0$$

and therefore $\|\nabla F\|_g = 0$ a.e. and F is constant. □

We can now do the proof of Theorem 4. For our $F:\mathbb{R}^n\to\mathbb{R}^d$, introduce $H:\mathbb{R}^n\to\mathbb{R}^d$ given by

$$H(x)=F(x)-\mathcal{Z}^{-1}\int_{\mathbb{R}^n} gF,$$

so that $\int_{\mathbb{R}^n} g(x)H(x)\,dx=0$ and $\nabla H=\nabla F$. We write

$$\begin{aligned}(12)\quad &\int \|F(x)-\mathcal{Z}^{-1}\textstyle\int gF\|^2_{g(x)}\,dx\\ &\qquad=\int_{\mathbb{R}^n}\|H\|_g^2=2\int_{\mathbb{R}^n}\langle H\mid L\Psi\rangle_g-\int_{\mathbb{R}^n}\|L\Psi\|_g^2+\int_{\mathbb{R}^n}\|H-L\Psi\|_g^2\end{aligned}$$

for any given Ψ of class C^2 and compactly supported. We will concentrate on the two first term of the last expression, which would have been the only one to appear if we could have chosen directly $L\Psi=H$; we have an extra-term, which can be chosen arbitrary small.

Using Fact 7 and Fact 8 for Ψ, we can write

$$\begin{aligned}&2\int_{\mathbb{R}^n}\langle H\mid L\Psi\rangle_g-\int_{\mathbb{R}^n}\|L\Psi\|_g^2\\ &=-2\int_{\mathbb{R}^n}\langle\nabla_x H\mid\nabla_X\Psi\rangle_{\mathrm{Id}_n\otimes g(x)}\,dx-\int_{\mathbb{R}^n}\langle\mathbf{\Theta}^g_x\,\nabla_x\Psi(x)\mid\nabla_x\Psi(x)\rangle_{\mathrm{Id}_n\otimes g(x)}\,dx-\int_{\mathbb{R}^n}\sum_{j,k=1}^n\|\partial^2_{x_jx_k}\Psi\|^2_{g(x)}\,dx.\\ &\qquad\le-2\int_{\mathbb{R}^n}\langle\nabla_x H\mid\nabla_X\Psi\rangle_{g(x)}\,dx-\int_{\mathbb{R}^n}\langle\mathbf{\Theta}^g_x\,\nabla_x\Psi(x)\mid\nabla_x\Psi(x)\rangle_{\mathrm{Id}_n\otimes g(x)}\,dx\\ &\qquad\qquad\le\int_{\mathbb{R}^n}Q^\circ_{\mathrm{Id}_n\otimes g(x),\mathbf{\Theta}^g(x)}(\nabla_x H)\,dx=\int_{\mathbb{R}^n}Q^\circ_{\mathrm{Id}_n\otimes g(x),\mathbf{\Theta}^g(x)}(\nabla_x F)\,dx..\end{aligned}$$

where we used the characterization of the $g(x)$-polar form in terms of Legendre's transform (4). So from (12) we get, for any function F and any C^2 compactly supported function Ψ,

$$\int\|F(x)-\mathcal{Z}^{-1}\textstyle\int gF\|^2_{g(x)}\,dx\le\int_{\mathbb{R}^n}Q^\circ_{\mathrm{Id}_n\otimes g(x),\mathbf{\Theta}^g(x)}(\nabla_x F)\,dx+\int_{\mathbb{R}^n}\|H-L\Psi\|_g^2$$

Taking the infimum over Ψ, we conclude thanks to Fact 9. □

4. Proof of the matrix valued Prékopa's inequality

Besides the matrix valued Brascamp-Lieb inequality that we have established above, the proof of Prékopa's inequality relies on the following property that clarifies the rôle of N-log-concavity.

Proposition 10. *Let $g:\mathbb{R}^{n_0+n_1}\to\mathcal{M}_d(\mathbb{R})^+$ be a C^2 function with values in the symmetric positive matrices and assume that g is* N*-log-concave. At any given $x\in\mathbb{R}^{n_0+n_1}$, divide the operator $\mathbf{\Theta}^g$ in blocks:*

$$\mathbf{\Theta}^g_{0,0}:=[\theta^g_{j,k}(x)]_{1\le j,k\le n_0}\in L(\mathbb{R}^{n_0}\otimes\mathbb{R}^d)\simeq L(\mathcal{M}_{d,n_0}(\mathbb{R}))$$

and

$$\mathbf{\Theta}^g_{1,1} := [\theta^g_{i,j}(x)]_{n_0<j,k\le n_0+n_1} \in L(\mathbb{R}^{n_1}\otimes\mathbb{R}^d) \simeq L(\mathcal{M}_{d,n_1}(\mathbb{R})),$$

which are defined according to (7)-(8)*, together with the mixed-derivatives operator*

$$\mathbf{\Theta}^g_{0,1} := \Big[\theta^g_{j,k}(x)\Big]_{1\le j\le n_0, n_0<k\le n_0+n_1} \in L(\mathcal{M}_{d,n_0},\mathcal{M}_{d,n_1})$$

whose action on $V_0=[v_1,\dots,v_{n_0}]\in\mathcal{M}_{d,n_0}$ *is, as expected, given by*

$$\mathbf{\Theta}^g_{0,1}\,V_0 := \Big[\sum_{j=1}^{n_0}\theta^g_{j,n_0+k}v_j\Big]_{k=1,\dots,n_1} \in \mathcal{M}_{d,n_1}$$

Then, for any given $V_0\in\mathcal{M}_{n_0,d}$ *and* $x\in\mathbb{R}^{n_0+n_1}$*, if we denote by* $Q^\circ_{1,1}$ *the* $\langle\cdot\,|\,\cdot\rangle_{\mathrm{Id}_{n_1}\otimes g}$*-polar of the quadratic form* $W\to-\langle\mathbf{\Theta}^g_{1,1}\,W\,|\,W\rangle_{\mathrm{Id}_{n_1}\otimes g}$ *associated to* $-\,\mathbf{\Theta}^g_{1,1}$*, we have*

$$\langle-\,\mathbf{\Theta}^g_{0,0}\,V_0\,|\,V_0\rangle_{\mathrm{Id}_{n_0}\otimes g}\ge Q^\circ_{1,1}(\mathbf{\Theta}^g_{0,1}\,V_0).$$

In case $\mathbf{\Theta}^g_{1,1}$ *is invertible at* x*, this is equivalent to*

$$\langle-\,\mathbf{\Theta}^g_{0,0}\,V_0\,|\,V_0\rangle_{\mathrm{Id}_{n_0}\otimes g}\ge\langle(-\,\mathbf{\Theta}^g_{1,1})^{-1}\,\mathbf{\Theta}^g_{0,1}\,V_0\,|\,\mathbf{\Theta}^g_{0,1}\,V_0\rangle_{\mathrm{Id}_{n_1}\otimes g}.$$

As discussed in the introduction, this extension of (2) is very much related to the homogeneous Monge-Ampère equation.

Proof. For fixed $V_0\in\mathcal{M}_{n_0,d}$ and $x\in\mathbb{R}^{n_0+n_1}$, we will test the positivity of the operator $-\,\mathbf{\Theta}^g$ on the matrix $[V_0,W]\in M_{n_0+n_1,d}$ for an arbitrary matrix $W\in\mathcal{M}_{n_1,d}$. The optimal choice would be $W=(-\,\mathbf{\Theta}^g_{1,1})^{-1}\,\mathbf{\Theta}^g_{0,1}\,V_0$, but we want to proceed without assuming the invertibility of $\mathbf{\Theta}^g_{1,1}$.

Using the notation of the Proposition in the formula (8) and the symmetry property (6), we have

$$\langle\mathbf{\Theta}^g([V_0,W])\,|\,[V_0,W]\rangle_{\mathrm{Id}_{n_0+n_1}\otimes g}=\langle\mathbf{\Theta}^g_{0,0}\,V_0\,|\,V_0\rangle_{\mathrm{Id}_{n_0}\otimes g}+2\langle\mathbf{\Theta}^g_{0,1}\,V_0\,|\,W\rangle_{\mathrm{Id}_{n_1}\otimes g}+\langle\mathbf{\Theta}^g_{1,1}\,W\,|\,W\rangle_{\mathrm{Id}_{n_1}\otimes g}.$$

Since $\langle\mathbf{\Theta}^g([V_0,W])\,|\,[V_0,W]\rangle_{\mathrm{Id}_{n_0+n_1}\otimes g}\le 0$, by strong log-concavity, we have

$$\langle-\,\mathbf{\Theta}^g_{0,0}\,V_0\,|\,V_0\rangle_{\mathrm{Id}_{n_0}\otimes g}\ge 2\langle\!\langle\mathbf{\Theta}^g_{0,1}\,V_0\,|\,W\rangle\!\rangle_{\mathrm{Id}_{n_1}\otimes g}-\langle-\,\mathbf{\Theta}^g_{1,1}\,W\,|\,W\rangle_{\mathrm{Id}_{n_1}\otimes g}.$$

Taking the supremum over W gives the desired inequality. □

We have now all the ingredients to prove the Prékopa-Raufi inequality. With the notation of Theorem 1, we have to estimate $\langle\mathbf{\Theta}^\alpha\,V_0\,|\,V_0\rangle_{\mathrm{Id}_{n_0}\otimes\alpha}$ at some fixed $x\in\mathbb{R}^{n_0}$ for $V_0=[u_1,\dots,u_{n_0}]\in\mathcal{M}_{d,n}(\mathbb{R})$.

Fact 11 (Computing $\mathbf{\Theta}^\alpha$)**.** *With the notation of Theorem 1, at any* $x\in\mathbb{R}^{n_0+n_1}$ *and fixed* $V_0=[v_1,\dots,v_{n_0}]\in\mathcal{M}_{d,n_0}(\mathbb{R})$ *we introduce*

$$F:=\sum_{j=1}^{n_0}(g^{-1}\partial_{x_j}g)v_j\in\mathbb{R}^d. \tag{13}$$

Then, at a fixed $t \in \mathbb{R}^{n_0}$, we have, with the notation $\mathbf{\Theta}^g_{0,0}$ taken from Proposition 10,

$$\begin{aligned}
&\langle \mathbf{\Theta}^\alpha V_0 \,|\, V_0\rangle_{\mathrm{Id}_{n_0}\otimes\alpha} \\
&= \int_{\mathbb{R}^{n_1}} \langle \mathbf{\Theta}^g_{0,0} V_0 \,|\, V_0\rangle_{\mathrm{Id}_{n_0}\otimes g(t,y)}\, dy \\
&\qquad + \int_{\mathbb{R}^{n_1}} \langle F \,|\, F\rangle_{g(t,y)}\, dy - \langle \alpha^{-1}\textstyle\int_{\mathbb{R}^{n_1}} gF\, dy \,|\, \alpha^{-1}\textstyle\int_{\mathbb{R}^{n_1}} gF\, dy\rangle_{\alpha(x)} \\
&= \int_{\mathbb{R}^{n_1}} \langle \mathbf{\Theta}^g_{0,0}(t,y) V_0 \,|\, V_0\rangle_{\mathrm{Id}_{n_0}\otimes g(t,y)}\, dy + \int_{\mathbb{R}^{n_1}} \Big\| F(t,y) - \alpha(y)^{-1}\textstyle\int_{\mathbb{R}^{n_1}} g(t,z)F(x,z)\, dz\Big\|^2_{g(t,y)}\, dy.
\end{aligned}$$

We see that this computation is the generalization of (1).

Proof. We have for $j,k = 1,\dots,n_0$,

$$\begin{aligned}
\langle \theta^\alpha_{j,k}(t) v_j \,|\, v_k\rangle_{\alpha(t)} &= \langle \partial^2_{j,k}\alpha(t) v_j \,|\, v_k\rangle - \langle \partial_j \alpha(t) v_j \,|\, \alpha(t)^{-1}\partial_k \alpha(t) v_k\rangle \\
&= \int_{\mathbb{R}^{n_1}} \langle \partial^2_{j,k} g(t,y) v_j \,|\, v_k\rangle\, dy - \langle \int_{\mathbb{R}^{n_1}} \partial_j g(t,y) v_j\, dy \,|\, \alpha(t)^{-1} \int_{\mathbb{R}^{n_1}} \partial_k g(t,y) v_k\, dy\rangle \\
&= \int_{\mathbb{R}^{n_1}} \langle \theta^g_{j,k}(t,y) v_j \,|\, v_k\rangle_{g(t,y)}\, dy \\
&\qquad + \int_{\mathbb{R}^{n_1}} \langle g(t,y)^{-1}\partial_j g(t,y) v_j \,|\, g(t,y)^{-1}\partial_k g(t,y) v_k\rangle_{g(t,y)}\, dy \\
&\qquad - \langle \int_{\mathbb{R}^{n_1}} \partial_j g(t,y) v_j\, dy \,|\, \alpha(t)^{-1} \int_{\mathbb{R}^{n_1}} \partial_k g(t,y) v_k\, dy\rangle
\end{aligned}$$

Summing over all $j,k \in \{1,\dots,n_0\}$ we find

$$\langle \mathbf{\Theta}^\alpha V_0 \,|\, V_0\rangle_{\mathrm{Id}_{n_0}\otimes\alpha} = \int_{\mathbb{R}^{n_1}} \langle \mathbf{\Theta}^g_{0,0} V_0 \,|\, V_0\rangle_{\mathrm{Id}_{n_0}\otimes g(t,y)}\, dy + \int_{\mathbb{R}^{n_1}} \langle F \,|\, F\rangle_g\, dy - \langle \int_{\mathbb{R}^{n_1}} gF\, dy \,|\, \alpha^{-1} \int_{\mathbb{R}^{n_1}} gF\, dy\rangle$$

For the equality between the expressions, see the computations after Theorem 4. □

We now combine the previous results. The strong log-concavity, in the form given by Proposition 10 combines with Fact 11 and implies that, at our fixed $t \in \mathbb{R}^{n_0}$ and $V_0 = [v_1,\dots,v_{n_0}] \in \mathcal{M}_{d,n_0}(\mathbb{R})$ we have

$$\begin{aligned}
&\langle -\mathbf{\Theta}^\alpha V_0 \,|\, V_0\rangle_{\mathrm{Id}_{n_0}\otimes\alpha} \geq \\
&\qquad \int_{\mathbb{R}^{n_1}} Q^\circ_{1,1}(\mathbf{\Theta}^g_{0,1} V_0)\, dy - \int_{\mathbb{R}^{n_1}} \Big\| F(t,y) - \alpha^{-1}(t)\textstyle\int_{\mathbb{R}^{n_1}} g(t,z)F(t,z)\, dz\Big\|^2_{g(t,y)}\, dy,
\end{aligned}$$

where $Q^\circ_{1,1}$ at (t,y) is the $\langle\cdot\,|\,\cdot\rangle_{\mathrm{Id}_{n_1}\otimes g(t,y)}$-polar of the quadratic form $W \to -\langle \mathbf{\Theta}^g_{1,1} W \,|\, W\rangle_{\mathrm{Id}_{n_1}\otimes g}$ associated to $-\mathbf{\Theta}^g_{1,1}$ at (t,y). The function

$$y \to F(t,y)$$

that we denote by F for convenience, given by (13) at $x = (t,y)$ with t fixed, is such that

$$\nabla_y F = \Big[\sum_{j=1}^{n_0} \partial_{y_1}\big((g^{-1}\partial_{x_j} g)\big) v_j, \dots \sum_{j=1}^{n_0} \partial_{y_{n_1}}\big((g^{-1}\partial_{x_j} g)\big) v_j\Big] \in \mathcal{M}_{d,n_1}(\mathbb{R})$$

and so, with the notation of Proposition 10, we find

$$\nabla_y F = \mathbf{\Theta}^g_{0,1}\, V_0.$$

Writing the previous inequality in terms of F we arrive to

$$\langle -\,\mathbf{\Theta}^\alpha\, V_0 \,|\, V_0\rangle_{\mathrm{Id}_{n_0}\otimes\alpha} \geq \int_{\mathbb{R}^{n_1}} \langle (-\,\mathbf{\Theta}^g_{1,1})^{-1}\nabla_y F \,|\, \nabla_y F\rangle_{\mathrm{Id}_{n_1}\otimes g}\, dy - \int_{\mathbb{R}^{n_1}} \left\| F - \alpha^{-1}\textstyle\int_{\mathbb{R}^{n_1}} gF \right\|_g^2 dy.$$

In some sense the previous inequality is the local form Prékopa's inequality. The conclusion now follows from the Brascamp-Lieb inequality of Theorem 4 on $\mathbb{R}^{n_1}$ applied to the log-concave weight $y \to \tilde{g}(y) := g(t,y) \in \mathcal{M}_d(\mathbb{R})^+$ (for which $\mathbf{\Theta}^{\tilde{g}}$ at y is at $\mathbf{\Theta}^g_{1,1}$ at (t,y)) and to the function $y \to F(t,y)$.

References

[1] B. Berndtsson, *Curvature of vector bundles associated to holomorphic fibrations*, Ann. of Math. (2) **169** (2009), 531–560.

[2] H. J.Brascamp and E. H. Lieb, *On extensions of the Brunn-Minkovski and Prékopa-Leindler theorems, including inequalities for log-concave functions, and with an application to the diffusion equation*, J. Funct. Anal. **22**, 366-389 (1976).

[3] D. Cordero-Erausquin, *On Berndtsson's generalization of Prékopa's theorem*, Math. Z. **249**, no. 2, 401–410 (2005).

[4] D. Cordero-Erausquin, M. Fradelizi and B. Maurey, *The (B)-conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems*, J. Funct. Anal. **214** (2004), 410–427.

[5] D. Cordero-Erausquin and B. Klartag, *Interpolation, convexity and geometric inequalities*, in Geometric Aspects of Functional Analysis, LNM **2050**, Springer, 2012, pp 151–168.

[6] J.P. Demailly, *Complex analytic and differential geometry*, book in open access available at https://www-fourier.ujf-grenoble.fr/ demailly/documents.html.

[7] L. Hörmander, *L^2 estimates and existence theorems for the $\bar{\partial}$ operator*, Acta Math. **113**, 89–152 (1965).

[8] A. Prékopa, *Logarithmic concave measures with application to stochastic programming*, Acta Sci. Math. **32** (1971), 301–315.

[9] H. Raufi, *Log concavity for matrix-valued functions and a matrix-valued Prékopa Theorem*, Preprint available on ArXiv (2013).

Institut de Mathématiques de Jussieu, Sorbonne Université - UPMC (Paris 6), France.
E-mail address: dario.cordero@imj-prg.fr