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PAPER

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Covariance of the running range of a Brownian trajectory

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Abstract

The question how the extremal values of a stochastic process achieved on different time intervals are correlated to each other has been discussed within the last few years on examples of the running maximum of a Brownian motion, of a Brownian bridge and of a Slepian process. Here, we focus on the two-time correlations of the running *range* of Brownian motion (BM)—the maximal extent of a Brownian trajectory on a finite time interval. We calculate exactly the covariance function of the running range and analyse its asymptotic behaviour. Our analysis reveals non-trivial correlations between the value of the largest descent (rise) of a BM from the top to a bottom on some time interval, and the value of this property on a larger time interval.

Keywords: extremal values of Brownian motion, running range, temporal correlations

(Some figures may appear in colour only in the online journal)

1. Introduction

Brownian motion (BM) is a paradigmatic stochastic process encountered in diverse areas of physics and chemistry [1, 2], biology [2, 3], differential evolutionary games [4–6], computer science [7, 8], as well as in mathematical finance [9–12], where BM represents one of the main components in modelling of the dynamics of asset prices.

Figure 1. Panel (a): maximal and minimal displacements of a Brownian trajectory X_t on time intervals $[0, t_1]$ and $[0, t]$. Panel (b): running range of a given BM trajectory X_t depicted in panel (a).

Extreme values of BM, e.g. maximal or minimal displacements, its range (see figure 1), or first passage times to a given domain, are known to play an important role across many disciplines, since they often trigger a particular response of the system [13–19]. Since early seminal works [20–23], much effort has been invested into the analysis of statistical properties of such extreme values. In particular, the distributions of the BM maximal/minimal displacements (including the joint multivariate distributions) [20–24] and of its range at a given moment of time [25–27] have been calculated exactly. Apart from the significance of these results in their own right, knowledge of such distributions is often very useful for a non-perturbative analysis of complicated functionals of BM, permitting construction of convergent bounds and hence, obtaining non-trivial exact results [28–33]. A variety of first-passage phenomena associated with the BM have been analysed using exact approaches [13–18], and there have also been some accurate approximations, which have permitted consideration of rather complicated geometries [19]. A remarkable result has been recently obtained for the distribution of the moment of time when the range of BM, evolving in presence or in absence of boundaries, first achieves some prescribed value [34].

On the other hand, the question how the extreme values of BM are correlated in time has received essentially less attention, although for many basic stochastic processes this issue has been usually addressed at the first place. This problem was approached first only within the past few years, and the two-time correlations of the running maximum of BM [35], of a Brownian bridge (a BM constrained to return to the starting point at a fixed time moment t) [36], and of a Slepian process (the difference of two BM positions taken at different time moments) [37], as well as of some records related to BM [11, 38, 39] have been determined. Here we focus on the two-time correlations (the covariance function) of the running range of Brownian motion (see figure 1)—a maximal extent (span) of a BM on a given time interval. Our analysis reveals non-trivial correlations between the value of the largest descent (rise) of a BM from the top to a bottom on some time interval, and the value of this property on a different (larger) time interval. Our exact results will be useful for constructing bounds on non-local functionals of BM, e.g. for the analysis of two-time correlations of the survival probability of a particle diffusing in presence of immobile traps, and other problems including the efficiency of diffusive foraging processes [40, 41], which depend explicitly on the range of corresponding BM.

This paper is outlined as follows: in section 2 we introduce basic notations and present our main results. In section 3, we present the details of the derivations. Finally, in section 4 we conclude with a brief recapitulation of our results.

2. Notations and the main results

Let X_τ (with $X_{\tau=0} = 0$) denote the position of a one-dimensional BM with diffusion coefficient D at time moment $\tau \in [0, t]$ and let

$$M_t = \max_{0 \leq \tau \leq t} X_\tau \geq 0, m_t = -\min_{0 \leq \tau \leq t} X_\tau \geq 0, \text{ and } R_t = M_t + m_t \geq 0, \quad (1)$$

denote the maximal positive displacement M_t of X_τ within the interval $0 \leq \tau \leq t$, the minimum (i.e. the maximal negative displacement taken with a negative sign) m_t of X_τ within this time interval, and the range R_t of X_τ within this interval, respectively (see figure 1). Our goal is to calculate exactly the covariance function of the running range R_t , which is defined as $\mathbb{E}\{R_{t_1}R_t\}$, $0 \leq t_1 \leq t$, where the symbol $\mathbb{E}\{\dots\}$ here and henceforth denotes averaging with respect to all possible realisations of X_τ .

In virtue of equation (1), the covariance function of the running range can be written down as

$$\begin{aligned} \mathbb{E}\{R_{t_1}R_t\} &= \mathbb{E}\{(M_{t_1} + m_{t_1})(M_t + m_t)\} \\ &= \mathbb{E}\{M_{t_1}M_t\} + \mathbb{E}\{m_{t_1}m_t\} + \mathbb{E}\{M_{t_1}m_t\} + \mathbb{E}\{m_{t_1}M_t\}. \end{aligned} \quad (2)$$

By symmetry, we have that

$$\mathbb{E}\{M_{t_1}M_t\} = \mathbb{E}\{m_{t_1}m_t\}, \quad \mathbb{E}\{M_{t_1}m_t\} = \mathbb{E}\{m_{t_1}M_t\}, \quad (3)$$

such that, eventually,

$$\mathbb{E}\{R_{t_1}R_t\} = 2\left(\mathbb{E}\{M_{t_1}M_t\} + \mathbb{E}\{m_{t_1}M_t\}\right). \quad (4)$$

The first term in the brackets in the latter equation is known [35]. In contrast, the expected value of the product of a minimum and a maximum of a BM attained at two different time intervals has not been determined as yet. Our first main result is the following exact representation

$$\begin{aligned} \frac{\mathbb{E}\{m_{t_1}M_t\}}{Dt} &= \frac{4}{\pi} \sqrt{z(1-z)} - 1 - z + \frac{2}{\pi} \frac{z^{3/2}}{\sqrt{1-z}} g_1\left(\sqrt{\frac{z}{1-z}}\right) \\ &\quad + \frac{1}{\pi} \left[\left(\sqrt{\frac{z}{1-z}} + i\right) g_1\left(\sqrt{\frac{z}{1-z}} + i\right) + \left(\sqrt{\frac{z}{1-z}} - i\right) g_1\left(\sqrt{\frac{z}{1-z}} - i\right) \right], \end{aligned} \quad (5)$$

where $i = \sqrt{-1}$, $z = t_1/t \leq 1$, and the function $g_1(\phi)$ is defined by

$$g_1(\phi) = \frac{1}{\phi^2} \int_0^1 \frac{d\beta}{\beta^2} \left(1 - \frac{\pi\phi\beta}{\sinh(\pi\phi\beta)}\right). \quad (6)$$

Large- and small- ϕ asymptotic behaviour of $g_1(\phi)$ is discussed below (see equations (48) and (53)).

Recalling next the exact expression for $\mathbb{E}\{M_{t_1}M_t\}$ derived in [35], we write down the second main result of our work, which represents the desired covariance function of the running range of BM:

$$\begin{aligned} \frac{\mathbb{E}\{R_{t_1}R_t\}}{2Dt} &= \frac{6}{\pi}\sqrt{z(1-z)} - 1 + \frac{2}{\pi}\arcsin(\sqrt{z}) + \frac{2}{\pi}\frac{z^{3/2}}{\sqrt{1-z}}g_1\left(\sqrt{\frac{z}{1-z}}\right) \\ &+ \frac{1}{\pi}\left[\left(\sqrt{\frac{z}{1-z}} + i\right)g_1\left(\sqrt{\frac{z}{1-z}} + i\right) + \left(\sqrt{\frac{z}{1-z}} - i\right)g_1\left(\sqrt{\frac{z}{1-z}} - i\right)\right]. \end{aligned} \quad (7)$$

It is instructive to consider the asymptotic behaviour of the expression in equation (7) in the limits when $t_1 \ll t$ ($z \rightarrow 0$) and when t_1 is close to t , such that $z \rightarrow 1$. In the former limit (see the expansions in equations (51) and (52) below) we obtain

$$\frac{\mathbb{E}\{R_{t_1}R_t\}}{\mathbb{E}\{R_{t_1}\}\mathbb{E}\{R_t\}} = 1 + \frac{(\pi^2 - 6)}{36}z - \frac{(270 + 7\pi^4 - 90\pi^2)}{10800}z^2 + O(z^3), \quad (8)$$

where $\mathbb{E}\{R_t\} = 2\mathbb{E}\{M_t\} = 4\sqrt{Dt/\pi}$ [25–27]. Equation (8) demonstrates that correlations between R_{t_1} and R_t vanish when $t_1/t \rightarrow 0$ (which happens, namely, when t_1 is fixed and $t \rightarrow \infty$), but this limit is approached via a slow power law, which reveals long-ranged temporal correlations between the values of the running range achieved on two different time intervals.

Within the opposite limit $z \rightarrow 1$, we find, by taking advantage of the asymptotic expansion in equation (53),

$$\frac{\mathbb{E}\{R_{t_1}R_t\}}{\mathbb{E}\{R_t^2\}} = 1 - \frac{(1-z)}{2} - \frac{(1-z)^{3/2}}{3\pi\ln(2)} + O\left((1-z)^{5/2}\right), \quad (9)$$

where $\mathbb{E}\{R_t^2\} = 8\ln(2)Dt$ [25–27].

The asymptotic forms in equations (8) and (9) are presented in figure 2 together with the exact result in equation (7) (solid blue curve). We observe that the small- z asymptotic form in equation (8) provides a very accurate estimate of the covariance function of the running range over the whole domain of variation of z , and only slightly overestimates the actual value of $\mathbb{E}\{R_{t_1}R_t\}$ in the vicinity of $z = 1$. In turn, the asymptotic form in equation (9), which describes the behaviour of $\mathbb{E}\{R_{t_1}R_t\}$ in the vicinity of $z = 1$, is almost indistinguishable from $\mathbb{E}\{R_{t_1}R_t\}$ for z as low as 0.5.

Details of the derivation of the results in equations (5)–(9) are presented in the next section.

3. Details of calculations

We start with the expression for the probability $P(m|M)$ that a BM, commencing at the origin at $t = 0$, has reached on the time interval $[0, t_1]$ a minimum whose *absolute* value $=m > 0$ and on the entire interval $[0, t]$ (with $t_1 \leq t$) a maximum $=M > 0$:

$$\begin{aligned} P(m|M) &= \int_{-m}^M dX \Pi_{t_1}(M|m|X) S_{t-t_1}(M-X) \\ &+ \int_{-m}^M dX S_{t_1}^*(M|m|X) \Pi_{t-t_1}(M-X), \end{aligned} \quad (10)$$

where

- $S_t^*(M|m|X)$ is the conditional ‘survival’ probability, i.e. the probability that a BM, starting at the origin at $t = 0$, attained within the time interval $[0, t]$ a minimum whose absolute value $=m$, stayed below a level $M > 0$ within this interval, and at time moment t appeared at position X (such that $-m < X < M$);

Figure 2. Exact expression for the covariance of the running range of BM in equation (7) (solid blue curve), together with the asymptotic forms in equations (8) (red dotted curve) and (9) (dotted curve). The crosses are the results of numerical simulations.

- $\Pi_t(M|m|X)$ is the trivariate pdf (see [23]) of the event that a BM, starting at the origin at $t = 0$, attained a maximal value $=M > 0$ and a minimal value $=-m$ within the time interval $[0, t]$ and ended up at position X ($-m < X < M$) at time moment t ;
- $S_t(M)$ is the standard ‘survival’ probability (see, e.g. [14]), i.e. the probability that a BM starting at the origin at time moment $t = 0$, did not reach (stayed below) a level $M > 0$ within the time interval $[0, t]$;
- $\Pi_t(M)$ is the pdf (see, e.g. [23]) of the event that a BM, starting at the origin at $t = 0$, attained a maximal value $=M > 0$ on a time interval $[0, t]$.

Let now $S_t(M|m|X)$ denote the standard ‘survival’ probability, i.e. the probability that a BM, starting at the origin, does not leave a stripe $[-m, M]$ within time interval $[0, t]$ and its endpoint at time t is X ($-m < X < M$). Clearly, we have

$$S_{t_1}^*(M|m|X) = \frac{\partial}{\partial m} S_{t_1}(M|m|X) \quad (11)$$

and

$$\Pi_{t_1}(M|m|X) = \frac{\partial^2}{\partial m \partial M} S_{t_1}(M|m|X). \quad (12)$$

Noticing next that

$$\Pi_{t-t_1}(M-X) = \frac{\partial}{\partial M} S_{t-t_1}(M-X), \quad (13)$$

we may rewrite equation (10) formally as

$$\begin{aligned} P(m|M) &= \int_{-m}^M dX \frac{\partial}{\partial M} \left[S_{t_1}^*(M|m|X) S_{t-t_1}(M-X) \right] \\ &= \int_{-m}^M dX \frac{\partial^2}{\partial M \partial m} \left[S_{t_1}(M|m|X) S_{t-t_1}(M-X) \right]. \end{aligned} \quad (14)$$

Further on, recalling that

$$S_{t-t_1}(M-X) \equiv \operatorname{erf}\left(\frac{M-X}{\sqrt{4D(t-t_1)}}\right) = \frac{2}{\pi} \int_0^\infty \frac{dz}{z} e^{-4D(t-t_1)z^2} \sin(2z(M-X)), \quad (15)$$

we cast equation (14) into the form

$$P(m|M) = \frac{2}{\pi} \int_0^\infty \frac{dz}{z} e^{-4D(t-t_1)z^2} \int_{-m}^M dX \frac{\partial^2}{\partial M \partial m} \left[S_{t_1}(M|m|X) \sin(2z(M-X)) \right]. \quad (16)$$

Lastly, to simplify somehow our analysis, we will focus on what follows not $P(m|M)$ itself, but rather the distribution of an auxiliary range $\mathcal{R} = M + m$, where M denotes the maximal displacement achieved on the interval $[0, t]$, while m is the absolute value of the minimal displacement achieved on the subinterval $[0, t_1]$. This distribution is formally defined as

$$P(\mathcal{R}) = \int_0^\infty dM \int_0^\infty dm \delta(M+m-\mathcal{R}) P(m|M). \quad (17)$$

The second moment of this distribution will provide us the desired correlation function $\mathbb{E}\{m_{t_1}M_t\}$ (see equation (4)) through the relation

$$\mathbb{E}\{m_{t_1}M_t\} = \frac{1}{2} (\mathbb{E}\{\mathcal{R}_t^2\} - \mathbb{E}\{M_t^2\} - \mathbb{E}\{m_{t_1}^2\}). \quad (18)$$

The survival probability $S_{t_1}(M|m|X)$ is well-known (see, e.g. [23]) and is given explicitly by

$$S_{t_1}(M|m|X) = \frac{1}{\sqrt{4\pi Dt_1}} \sum_{k=-\infty}^{\infty} \left[\exp\left(-\frac{(X-2k(M+m))^2}{4Dt_1}\right) - \exp\left(-\frac{(X-2M-2k(M+m))^2}{4Dt_1}\right) \right]. \quad (19)$$

For further convenience, it is however more expedient to use not the expression in equation (19), but rather its Laplace-transform $\mathcal{L}_{\lambda,\eta}$ over the ‘variable’ $\eta = 4Dt_1$:

$$S_\lambda(M|m|X) = \mathcal{L}_{\lambda,\eta}[S_{t_1}(M|m|X)] = \int_0^\infty e^{-\lambda\eta} S_{t_1}(M|m|X) d\eta. \quad (20)$$

Performing the Laplace transform and the summation over k , we get

$$S_\lambda(M|m|X) = \frac{2}{\sqrt{\lambda}} \frac{\sinh(2\sqrt{\lambda}m) \sinh(2\sqrt{\lambda}(M-X))}{\sinh(2\sqrt{\lambda}(M+m))}, \quad X \geq 0, \\ S_\lambda(M|m|X) = \frac{2}{\sqrt{\lambda}} \frac{\sinh(2\sqrt{\lambda}M) \sinh(2\sqrt{\lambda}(m+X))}{\sinh(2\sqrt{\lambda}(M+m))}, \quad X \leq 0. \quad (21)$$

Inserting the expressions in equation (21) into equation (17) and performing all the integrations, we get

$$P(\mathcal{R}) = \frac{4}{\pi} \int_0^\infty dz e^{-4D(t-t_1)z^2} \mathcal{L}_{\lambda,\eta}^{-1} \left[Q_1 + Q_2 + Q_3 \right], \quad (22)$$

where

$$\begin{aligned} Q_1 &= \frac{(1 - e^{-2\sqrt{\lambda}\mathcal{R}})(1 - e^{-4\sqrt{\lambda}\mathcal{R}})}{(1 + e^{-2\sqrt{\lambda}\mathcal{R}})^3} \frac{\cos(2z\mathcal{R})}{\lambda + z^2}, \\ Q_2 &= \frac{2e^{-2\sqrt{\lambda}\mathcal{R}}(1 - e^{-2\sqrt{\lambda}\mathcal{R}})}{(1 + e^{-2\sqrt{\lambda}\mathcal{R}})^3} \frac{\sin(2z\mathcal{R})}{\sqrt{\lambda}z}, \\ Q_3 &= \frac{(1 - e^{-2\sqrt{\lambda}\mathcal{R}})(1 + e^{-4\sqrt{\lambda}\mathcal{R}})}{(1 + e^{-2\sqrt{\lambda}\mathcal{R}})^3} \frac{z \sin(2z\mathcal{R})}{\sqrt{\lambda}(\lambda + z^2)}, \end{aligned} \quad (23)$$

and $\mathcal{L}_{\lambda,\eta}^{-1}[\dots]$ denotes the operator of the inverse Laplace transform with respect to λ . We note that one can rather straightforwardly obtain from equation (22) the distribution of the auxiliary range in the limit $t_1 \rightarrow 0$ with t kept fixed (or equivalently, $t \rightarrow \infty$ with t_1 kept fixed) in which $\mathcal{R} \rightarrow M_t$, the latter being the maximum attained by a BM trajectory within the time interval $[0, t]$. In this limit (corresponding to $\lambda \rightarrow \infty$), we find that in the leading in λ order

$$Q_1 \sim \frac{\cos(2z\mathcal{R})}{\lambda + z^2}, \quad (24)$$

while Q_2 and Q_3 vanish. Consequently, here the distribution in equation (22) becomes

$$P(\mathcal{R}) \sim \frac{4}{\pi} \int_0^\infty dz e^{-4Dtz^2} \mathcal{L}_{\lambda,\eta}^{-1} \left[\frac{\cos(2z\mathcal{R})}{\lambda + z^2} \right] \rightarrow \frac{1}{\sqrt{\pi Dt}} \exp\left(-\frac{\mathcal{R}^2}{4Dt}\right), \quad (25)$$

which is the celebrated result due to Lévy [23] for the distribution of the maximum of a BM trajectory within the time interval $[0, t]$. The form of the distribution in the opposite limit when $t_1 \rightarrow t$ (and hence, the auxiliary range tends to the full range, i.e. $\mathcal{R} \rightarrow R_t$), will be verified below.

Finally, we define the moments of the auxiliary range:

$$\mathbb{E}\{\mathcal{R}^n\} = \int_0^\infty d\mathcal{R} \mathcal{R}^n P(\mathcal{R}). \quad (26)$$

Changing the integration variables $z \rightarrow \sqrt{\lambda}\xi$ and $\mathcal{R} \rightarrow \rho/\sqrt{\lambda}$, we have

$$\begin{aligned} \mathbb{E}\{\mathcal{R}^n\} &= \frac{4}{\pi} \int_0^\infty \frac{(1 - e^{-2\rho})}{(1 + e^{-2\rho})^3} \rho^n d\rho \int_0^\infty d\xi \left[(1 - e^{-4\rho}) \frac{\cos(2\xi\rho)}{1 + \xi^2} \right. \\ &\quad \left. + 2e^{-2\rho} \frac{\sin(2\xi\rho)}{\xi} + (1 + e^{-4\rho}) \frac{\xi \sin(2\xi\rho)}{1 + \xi^2} \right] \mathcal{L}_{\lambda,\eta}^{-1} \left[\frac{e^{-4D(t-t_1)\lambda\xi^2}}{\lambda^{1+n/2}} \right]. \end{aligned} \quad (27)$$

The inverse Laplace transform with respect to λ in equation (27) can now be explicitly performed. Recalling that $\eta = 4Dt_1$, we find, explicitly,

$$\mathcal{L}_{\lambda,\eta}^{-1} \left[\frac{e^{-4D(t-t_1)\lambda\xi^2}}{\lambda^{1+n/2}} \right] = \frac{(4Dt_1)^{n/2}}{\Gamma(n/2 + 1)} \left(1 - \frac{(t-t_1)}{t_1} \xi^2 \right)^{n/2} \theta \left(1 - \frac{(t-t_1)}{t_1} \xi^2 \right), \quad (28)$$

where $\theta(x)$ is the Heaviside theta-function such that $\theta(x) = 1$ for $x \geq 0$ and is zero otherwise.

Therefore, the following expression for the moments of the auxiliary range holds:

$$\mathbb{E}\{\mathcal{R}^n\} = \frac{2(Dt_1)^{n/2}}{\pi\Gamma(n/2+1)} \int_0^\infty \frac{(1-e^{-\rho})}{(1+e^{-\rho})^3} \rho^n d\rho \int_0^\infty d\xi \left[(1-e^{-2\rho}) \frac{\cos(\xi\rho)}{1+\xi^2} + 2e^{-\rho} \frac{\sin(\xi\rho)}{\xi} + (1+e^{-2\rho}) \frac{\xi \sin(\xi\rho)}{1+\xi^2} \right] \left(1 - \frac{(t-t_1)}{t_1} \xi^2\right)^{n/2} \theta\left(1 - \frac{(t-t_1)}{t_1} \xi^2\right), \quad (29)$$

which can be formally rewritten, upon an appropriate change of the integration variable as

$$\begin{aligned} \mathbb{E}\{\mathcal{R}^n\} &= \frac{2(Dt_1)^{n/2} \phi}{\pi\Gamma(n/2+1)} \int_0^\infty \frac{(1-e^{-\rho})}{(1+e^{-\rho})^3} \rho^n d\rho \int_0^1 dx (1-x^2)^{n/2} \\ &\quad \times \left[(1-e^{-2\rho}) \frac{\cos(\phi x \rho)}{1+\phi^2 x^2} + 2e^{-\rho} \frac{\sin(\phi x \rho)}{\phi x} + (1+e^{-2\rho}) \frac{\phi x \sin(\phi x \rho)}{1+\phi^2 x^2} \right] \\ &= \frac{4(Dt_1)^{n/2} \phi}{\Gamma(n/2+1)} \int_0^\infty \frac{(1-e^{-\rho})}{(1+e^{-\rho})^3} \exp(-\rho) \rho^n d\rho J_n(\rho), \end{aligned} \quad (30)$$

where $\phi = \sqrt{t_1/(t-t_1)} = \sqrt{z/(1-z)}$ (with $z = t_1/t$) and $J_n(\rho)$ is the following integral:

$$J_n(\rho) = \frac{1}{\pi} \int_0^1 dx (1-x^2)^{n/2} \left[\sinh(\rho) \frac{\cos(\phi x \rho)}{1+\phi^2 x^2} + \frac{\sin(\phi x \rho)}{\phi x} + \cosh(\rho) \frac{\phi x \sin(\phi x \rho)}{1+\phi^2 x^2} \right]. \quad (31)$$

It is expedient next to check the expression in equation (29) in the particular limit when $t_1 \rightarrow t$, i.e. when the auxiliary range $\mathcal{R}$ becomes the full range of BM on the interval $[0, t]$, $\mathcal{R} \equiv R_t$. In this limit $\phi \rightarrow \infty$, the integral in equation (31) becomes

$$J_n(\rho) \sim \frac{1}{2\phi}, \quad (32)$$

in the leading in ϕ order, and hence we recover from equation (29) the standard expression for the moments of the range of a Brownian motion up to time t [25–27]:

$$\begin{aligned} \mathbb{E}\{R_t^n\} &= \frac{4(Dt)^{n/2}}{\Gamma(1+n/2)} \int_0^\infty \frac{(1-e^{-\rho})}{(1+e^{-\rho})^3} \exp(-\rho) \rho^n d\rho \\ &= \frac{4(2^n-4)n!\zeta(n-1)}{2^n\Gamma(n/2+1)} (Dt)^{n/2}, \end{aligned} \quad (33)$$

where $\zeta(n-1)$ is the Riemann zeta-function. Note that for $n=2$ this expression has to be understood as a limit $n \rightarrow 2$. Consequently, the integral over ρ entering equation (29) can be written down as

$$\int_0^\infty \frac{(1-e^{-\rho})}{(1+e^{-\rho})^3} \exp(-\rho) \rho^n d\rho = \frac{\Gamma(n/2+1)}{4(Dt)^{n/2}} \mathbb{E}\{R_t^n\} = \frac{(2^n-4)n!\zeta(n-1)}{2^n}, \quad (34)$$

which will permit us to express the moments of the auxiliary range in terms of the moments of the full range.

To this end, we first notice that the expressions in the integrand in equation (31) admit the following expansion in the Taylor series in powers of ρ :

$$\begin{aligned} \sinh(\rho) \frac{\cos(\phi x \rho)}{1 + \phi^2 x^2} + \cosh(\rho) \frac{\phi x \sin(\phi x \rho)}{1 + \phi^2 x^2} &= \sum_{k=0}^{\infty} \frac{\rho^{2k+1}}{(2k+1)!} \sum_{m=0}^k (-1)^m \binom{2k}{2m} (x\phi)^{2m}, \\ \frac{\sin(\phi x \rho)}{\phi x} &= \sum_{k=0}^{\infty} \frac{(-1)^k \rho^{2k+1} (x\phi)^{2k}}{(2k+1)!}. \end{aligned} \quad (35)$$

Inserting these expansions into equation (31) and integrating over x , we arrive at the following expression for $J_n(\rho)$:

$$\begin{aligned} J_n(\rho) &= \frac{\Gamma(n/2 + 1)}{2\pi} \left[\sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k + 1/2) \rho^{2k+1} \phi^{2k}}{(2k+1)! \Gamma(k + n/2 + 3/2)} \right. \\ &\quad \left. + \sum_{k=0}^{\infty} \frac{\rho^{2k+1}}{(2k+1)!} \sum_{m=0}^k \frac{(-1)^m \Gamma(m + 1/2)}{\Gamma(m + n/2 + 3/2)} \binom{2k}{2m} \phi^{2m} \right]. \end{aligned} \quad (36)$$

Further on, plugging equation (36) into equation (30), and integrating over ρ we have

$$\begin{aligned} \mathbb{E}\{\mathcal{R}^n\} &= \frac{(t_1/t)^{n/2} \phi}{2\pi (Dt)^{3/2}} \left[\sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k + 1/2)}{(2k+1)!} \frac{\mathbb{E}\{R_t^{2k+n+1}\}}{(Dt)^k} \phi^{2k} \right. \\ &\quad \left. + \sum_{k=0}^{\infty} \frac{\Gamma(k + (n+3)/2)}{(2k+1)! (Dt)^k} \mathbb{E}\{R_t^{2k+n+1}\} \sum_{m=0}^k \frac{(-1)^m \Gamma(m + 1/2)}{\Gamma(m + n/2 + 3/2)} \binom{2k}{2m} \phi^{2m} \right], \end{aligned} \quad (37)$$

or, explicitly,

$$\begin{aligned} \mathbb{E}\{\mathcal{R}^n\} &= \frac{2 (Dt_1)^{n/2} \phi}{\pi} \left[\sum_{k=0}^{\infty} \frac{(-1)^k (2^{2k+n} - 2) (2k + n + 1)! \Gamma(k + 1/2)}{2^{2k+n} (2k+1)! \Gamma(k + (n+3)/2)} \zeta(2k + n) \phi^{2k} \right. \\ &\quad \left. + \sum_{k=0}^{\infty} \frac{(2^{2k+n} - 2) (2k + n + 1)!}{2^{2k+n} (2k+1)!} \zeta(2k + n) \sum_{m=0}^k \frac{(-1)^m \Gamma(m + 1/2)}{\Gamma(m + n/2 + 3/2)} \binom{2k}{2m} \phi^{2m} \right]. \end{aligned} \quad (38)$$

We concentrate next on the moments of the zeroth and of the second order, i.e. $n = 0$ and $n = 2$. For $n = 0$ we have

$$\begin{aligned} \mathbb{E}\{\mathcal{R}^0\} &= \frac{2\phi}{\pi} \left[2 + \sum_{k=1}^{\infty} \frac{(-1)^k (1 - 2^{1-2k})}{(k + 1/2)} \zeta(2k) \right. \\ &\quad \left. \times \left(\phi^{2k} + \frac{(\phi - i)}{2\phi} (\phi - i)^{2k} + \frac{(\phi + i)}{2\phi} (\phi + i)^{2k} \right) \right]. \end{aligned} \quad (39)$$

Using the integral representation of the zeta-function,

$$\zeta(s) = \frac{1}{(1 - 2^{1-s}) \Gamma(s)} \int_0^{\infty} \frac{x^{s-1} dx}{e^x + 1}, \quad (40)$$

which strictly holds for $\text{Re } s > 0$, we can perform both the summation in equation (39) and also the integral over dx to get

$$\mathbb{E}\{\mathcal{R}^0\} = \frac{2\phi}{\pi} \left[2 + g_0(\phi) + \frac{(\phi - i)}{2\phi} g_0(\phi - i) + \frac{(\phi + i)}{2\phi} g_0(\phi + i) \right], \quad (41)$$

where

$$g_0(\phi) = \frac{\pi}{4\phi} - 1 - 2 \operatorname{arccoth}(e^{\pi\phi}) + \frac{1}{2\pi\phi} (\operatorname{Li}_2(e^{-2\pi\phi}) - 4 \operatorname{Li}_2(e^{-\pi\phi})), \quad (42)$$

with $\operatorname{arccoth}(\dots)$ being the inverse hyperbolic cotangent and $\operatorname{Li}_2(\dots)$ —the Euler's dilogarithm. One can readily verify that $\mathbb{E}\{\mathcal{R}^0\} \equiv 1$, as it should.

Next, for $n = 2$ we have

$$\begin{aligned} \mathbb{E}\{\mathcal{R}^2\} = & \frac{2Dt_1\phi}{\pi} \left[4 \sum_{k=0}^{\infty} \frac{(-1)^k (1 - 2^{-2k-1}) (k+1)}{(k+1/2)} \zeta(2k+2) \phi^{2k} \right. \\ & + \frac{1}{\phi^3} \sum_{k=0}^{\infty} \frac{(-1)^k (1 - 2^{-2k-1})}{(k+1/2)} \zeta(2k+2) \left((2(k+1)\phi - i)(\phi + i)^{2k+2} \right. \\ & \left. \left. + (2(k+1)\phi + i)(\phi - i)^{2k+2} \right) \right], \end{aligned} \quad (43)$$

or, equivalently,

$$\begin{aligned} \mathbb{E}\{\mathcal{R}^2\} = & \frac{2Dt_1\phi}{\pi} \left(4g_2(\phi) + \frac{2(\phi + i)^2}{\phi^2} g_2(\phi + i) + \frac{2(\phi - i)^2}{\phi^2} g_2(\phi - i) \right. \\ & \left. + \frac{i}{\phi^3} \left((\phi - i)^2 g_1(\phi - i) - (\phi + i)^2 g_1(\phi + i) \right) \right), \end{aligned} \quad (44)$$

where the function $g_1(\phi)$ is defined in equation (6), while

$$g_2(\phi) = \frac{1}{2\phi} \frac{d}{d\phi} (\phi^2 g_1(\phi)) = \frac{1}{2} g_1(\phi) + \frac{1}{2\phi^2} \left(1 - \frac{\pi\phi}{\sinh(\pi\phi)} \right). \quad (45)$$

Expressing $g_2(\phi)$ through $g_1(\phi)$ in equation (44), we find eventually

$$\mathbb{E}\{\mathcal{R}^2\} = \frac{2Dt_1\phi}{\pi} \left(\frac{4}{\phi^2} + 2g_1(\phi) + \frac{(1 + \phi^2)}{\phi^3} [(\phi + i)g_1(\phi + i) + (\phi - i)g_1(\phi - i)] \right). \quad (46)$$

To access the small- ϕ behaviour of $\mathbb{E}\{\mathcal{R}^2\}$, it is necessary to work out a convergent small- ϕ representation of $g_1(\phi)$. This can be found from equation (6) and reads

$$g_1(\phi) = -\frac{2}{\phi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \operatorname{arctg}\left(\frac{\phi}{k}\right). \quad (47)$$

The latter representation in form of the expansion in the inverse tangents permits us to establish the following small- ϕ behaviour:

$$g_1(\phi) = \frac{\pi^2}{6} - \frac{7\pi^4}{1080}\phi^2 + \frac{31\pi^6}{75600}\phi^4 + \mathcal{O}(\phi^6), \quad (48)$$

such that

$$\begin{aligned} (\phi - i)g_1(\phi - i) + (\phi + i)g_1(\phi + i) = & \pi + \left(\frac{4}{3} - \frac{\pi^2}{9} \right) \phi^3 \\ & + \frac{(7\pi^4 + 60\pi^2 - 1440)}{900} \phi^5 + \mathcal{O}(\phi^7). \end{aligned} \quad (49)$$

Consequently, we find that for small values of the parameter ϕ (which corresponds to either small t_1 at a fixed t or large t at a fixed t_1) $\mathbb{E}\{\mathcal{R}^2\}$ obeys

$$\mathbb{E}\{\mathcal{R}^2\} = 2D(t - t_1) \left[1 + \frac{4}{\pi}\phi + \phi^2 + \left(\frac{4}{3\pi} + \frac{\pi}{18} \right) \phi^3 + O(\phi^4) \right]. \quad (50)$$

Note that for $t_1 = 0$ the leading term in the latter equation, i.e. $2Dt$, is precisely the second moment of the running maximum of a Brownian trajectory [23].

In a similar way, we construct a small- z expansion for $\mathbb{E}\{\mathcal{R}^2\}$. For $g_1(\phi)$ we get, for $z \rightarrow 0$,

$$g_1(\phi) = \frac{\pi^2}{6} - \frac{7\pi^4}{1080}z + \left(\frac{31\pi^6}{75600} - \frac{7\pi^4}{1080} \right) z^2 + O(z^3) \quad (51)$$

and

$$\begin{aligned} (\phi - i)g_1(\phi - i) + (\phi + i)g_1(\phi + i) &= 2 \left(\pi - \arctan\left(\frac{2}{\phi}\right) \right) \\ &- 2 \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \left[\arctan\left(\frac{\phi - i}{k}\right) + \arctan\left(\frac{\phi + i}{k}\right) \right] \\ &= \pi + \left(\frac{4}{3} - \frac{\pi^2}{9} \right) z^{3/2} + \left(\frac{21}{80} + \frac{495 - 360\pi^2 + 28\pi^4}{3600} \right) z^{5/2} + O(z^{7/2}). \end{aligned} \quad (52)$$

Equations (51) and (52), together with equation (7), yield the asymptotic expansion in equation (8).

To analyse the large- ϕ behaviour of $\mathbb{E}\{\mathcal{R}^2\}$ (which corresponds to the limit $t_1 \rightarrow t$), we have to proceed in a different way. To get an appropriate large- ϕ representation we formally represent the integral in equation (45) over the interval $[0, 1]$ as the difference of the integral over the semi-infinite interval $[0, \infty)$ and the integral over $[1, \infty)$. The first integral can be performed explicitly and is equal to $\pi \ln(2)/\phi$, while the second one can be represented as an expansion in the incomplete upper gamma functions $\Gamma(0, (2n+1)\pi\phi)$. This gives the following exact representation of the function $g_1(\phi)$:

$$g_1(\phi) = \frac{\pi \ln(2)}{\phi} - \frac{1}{\phi^2} + \frac{2\pi}{\phi} \sum_{n=0}^{\infty} \Gamma(0, (2n+1)\pi\phi). \quad (53)$$

Note that for large values of ϕ (or $n \rightarrow \infty$), the leading behaviour of the incomplete gamma function follows

$$\Gamma(0, (2n+1)\pi\phi) \sim \frac{\exp(-(2n+1)\pi\phi)}{(2n+1)\pi\phi}, \quad (54)$$

which means that the third term in the right-hand side of equation (53) is exponentially small and can be safely neglected. In this limit we obtain

$$\mathbb{E}\{\mathcal{R}^2\} = 8 \ln(2)Dt_1 \left[1 + \frac{1}{2\phi^2} + O(e^{-\phi}) \right]. \quad (55)$$

Note that for $t_1 = t$, i.e. for $\mathcal{R} = R_t$ and $\phi = \infty$, we recover from equation (55) the classical result $\mathbb{E}\{R_t^2\} = 8 \ln(2)Dt$ [25–27]. The expansion in equation (55) gives directly the asymptotic form in equation (9).

4. Conclusions

To conclude, we studied here the covariance function of the running range R_t of a Brownian motion—one of the extreme values of this paradigmatic stochastic process, which defines the maximal extent of a given Brownian trajectory on a time interval $[0, t]$. We have calculated exactly this two-time autocorrelation function and analysed its asymptotic behaviour. Our analysis revealed a very non-trivial form of such a correlation function and also demonstrated that the temporal correlations between the values of the running range achieved on different time intervals are very strong.

As a by-product of our analysis, we have determined the moments of an auxiliary range—the sum of the maximal displacement achieved on the entire time interval $[0, t]$ and of the partial minimum, the minimal displacement achieved on a smaller time interval $[0, t_1]$, $t_1 \leq t$. These complicated expressions and the corresponding probability density function of an auxiliary range will be discussed in our future work.

We note finally that our exact results will be useful for constructing bounds on non-local functionals of BM, e.g. for the analysis of two-time correlations of the survival probability of a particle diffusing in presence of immobile traps, as well as other for problems including, e.g., the efficiency of diffusive foraging processes [40, 41], which depends explicitly on the range of corresponding BM.

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